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A Class of Acceptable and Practical Social Welfare Orderings with Variable Population: Stepwise Social Welfare Orderings and Their Applications

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A Class of Acceptable and Practical Social Welfare Orderings with Variable Population: Stepwise Social Welfare Orderings and Their Applications

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Abstract

This study proposes a new class of social welfare orderings, a *stepwise rank-dependent social welfare ordering*, which naturally generalizes a rank-dependent utilitarianism in the setting of social choice with variable population size. In fact, a stepwise social welfare ordering is simply designed to have the same value for each proportion of the population, with the obvious advantage that allows functional form to be freely chosen for assessing well-being inequality. We show that a stepwise rank-dependent social welfare ordering is easily characterized by standard axioms: strong Pareto, anonymity, Pigou-Dalton transfer

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A previous title of this paper is "A Class of Acceptable and Practical Social Welfare Functions with Variable Population: A Stepwise Rank-Dependent Utilitarianism and Its Application", and this version is substantially rewritten by generalizing main results and removing issues of ethical pathologies related to so-called repugnant and sadistic conclusions. The authors would like to appreciate kind encouragements and helpful comments from Professors Kotaro Suzumura and Reiko Gotoh in developing practical methods of social welfare function approach. We are grateful to Marcus Pivato, Matthew Adler, John Weymark, Koichi Tadenuma, and Geir Asheim for thoughtful comments and warm encouragements. We also offer our special thanks for the grants from Fostering Joint International Research (B) (JSPS KAKENHI Grant Number: 19K01683, 20KK0036) and the Joint Research Center for this research. Sakamoto makes contributions to all theoretical results and related applications in Sections 3 and 4, and Mori analyzes empirical results in Section 4.

equity, continuity, rank-separability, and consistency for population replication. If additional requirements on standard invariance are imposed on it, its functional form is specified as a well-known rank-weighted social welfare ordering. As some practical applications to measure social welfare, we propose three simple methods such as a *quantile mean comparison method*, which evaluates social welfare by comparing each quantile's average income in an approximate lexicographic ordering or quantile-dependent weighted summation. These applications have obvious advantages in that they can see the whole picture of income distributions compared with standard tools such as the traditional GDP per capita, median, range, and top/bottom ratios. Furthermore, we show a representation theorem that generalizes stepwise social welfare orderings for the problem of optimal population size.

Key Words: Stepwise Social Welfare Orderings, Quantile Mean Comparison, Interval Population-Ratio Comparison, Interval Weighted Mean Comparison

JEL Code: D63, D71, H43, I31, I32, J18, Q56

1. Introduction

Policymakers must often compare total well-being and its distribution between groups with different demographics, such as nations, local areas, and specific groups (e. g. gender, race, ethnicity, persons with disabilities, etc.) in order to implement reasonable policies for public health, economic growth, and preparation for natural disasters. Also, they must evaluate various policies that affect future demographic composition, such as educational policies, immigration control and residency management, and effective preventions for global climate change, taking into account of sustainable environment, biodiversity, and future generations' well-being. If we need to compare social welfare between different population sizes, how should we aggregate individual interests and assess social states with variable population? If a certain degree of interpersonal comparability of well-being were admissible, what kind of aggregation method should be appropriate to use?

At the starting point of welfare economics for assessing social situations with different population sizes, previous studies tried to avoid Arrow's impossibility theorem (Arrow 1951; 1963) and explore theoretical extensions of social choice theory by admitting interpersonal comparability of individual well-being. In their seminal study, Blackorby and Donaldson (1984) proposed a new class of social welfare functions, *critical-level generalized utilitarianism*. The critical-level generalized utilitarianism evaluates social states with different population sizes by comparing their total values obtained by subtracting the given critical level from individual's utility levels. Parfit (1976) pointed out the problem of the *repugnant conclusion* with a simple utilitarianism in the sense that a utilitarian social welfare ordering prefers a situation with many individuals having low utility to that with a few individuals having high utility. In Blackorby and Donaldson's critical-level utilitarianism, since a value obtained by subtracting a critical-level from an extremely low utility level could be negative, a situation with many individuals having low utility would be socially worse than that with relatively few individuals having high utility. Hence, it can avoid Parfit's repugnant conclusion.

The celebrated findings of critical-level generalized utilitarianism by Blackorby and Donaldson (1984) galvanized leading theorists into studies on various axiomatic characterizations and explorations of variants of social welfare orderings with variable population[1], e.g. critical-level leximin, number-dampened critical-level generalized utilitarianism, rank-discounted critical-level generalized utilitarianism, and rank-additive social welfare ordering (Ng 1986; 1989; Blackorby, Bossert and Donaldson 1995; 1996; 1997; 2001; 2005; Asheim and Zuber 2014; Pivato 2020). Almost all social welfare orderings proposed by these studies are variants of *separable* generalized utilitarian rules that satisfy strong independence conditions defined on utility profiles. Furthermore, Blackorby et al. (1999) proved that there is no efficient and equitable social welfare ordering that satisfies strong independence under the assumption of cardinal full comparability of well-being.[2] When the independence condition is slightly weakened, an average utilitarianism could only survive. Of course, the average utilitarianism can satisfy strong Pareto principle, but it is not equitable at all.

Does this result imply that there exists no efficient and equitable social welfare ordering under the assumption of cardinal full comparability? The answer to this question would be quite the opposite if the society could agree to drop the independence condition or separability on utility profiles. In social choice theory with variable population, there is a class of versatile, efficient, and equitable social welfare orderings that can make a consistent judgment for any changes in population size. The basic idea of these desirable social welfare orderings is derived from the generalized Gini index (or rank-weighted

[1] See Blackorby et al. (2002; 2005) for excellent summaries on social choice theory with variable population size. Blackorby et al. (2005) is a must-read textbook written by the pioneers themselves in this field.

[2] This impossibility theorem is directly proved from the classical result in Dechamps and Gevers (1978). They show that under the assumption of cardinal full comparability, social welfare orderings that satisfy strong Pareto, anonymity, separability are weak utilitarianism or leximin. In the framework of social choice with variable population size, social welfare orderings that satisfy the axioms of independence, strong Pareto, and anonymity are weak utilitarianism or leximin with the same population. If a positive affine transformation is applied for both rules in comparing utility profiles with different population, it becomes impossible to make a consistent evaluation.

utilitarianism) that was analyzed and proposed by Weymark (1981). The generalized Gini index is defined by weighted sum of utilities where weights are given in order of relative rank of individual utility profiles. Therefore, this social welfare ordering can satisfy the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and continuity.[3] Obviously, this rule cannot satisfy separability, and may not provide a consistent ordering for population replication. As a result, the rank-weighted utilitarianism has been excluded in the literature by the setting where the independence or separability are required.

There are, however, three exceptional applications of rank-weighted utilitarian rules in social choice theory with variable population. The first one is the *rank-discounted critical-level generalized utilitarianism* proposed by Asheim and Zuber (2014; 2018). Given a *rank-discounted rate* $\beta \in (0, 1)$, this social welfare ordering has a similar form of critical-level generalized utilitarianism, except that each value is rank-discounted by the power of the fixed discounted rate.[4] Obviously, if the critical level is zero, then this social welfare ordering is a kind of rank-weighted utilitarianism whose weights are given by the power of the rank-discounted rate. Also, due to the very similar structure to standard rank-weighted utilitarianism, even if the critical level is not zero, this ordering can satisfy the desirable axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and continuity. However, it violates the consistency condition for any changes of population replication[5]. Furthermore, since the weights are based on the *fixed*

[3] See Weymark (1981) for theoretical properties of the generalized Gini index. Ebert (1988a) shows that this rule is characterized by strong Pareto, anonymity, rank-separability, and continuity under the assumption of cardinal full comparability. Furthermore, when continuity is not required, the generalized leximin can be characterized by the above axioms (Sakamoto 2020).

[4] The definition of rank-discounted critical-level generalized utilitarianism is as follows:
For all utility profiles u_N and v_M, u_N is at least as good as v_M if and only if

$$\sum_{i=1}^{n} \beta^{[i]}[g(u_{[i]}) - g(c)] \geqq \sum_{i=1}^{m} \beta^{[i]}[g(v_{[i]}) - g(c)],$$

where β is a rank discounted rate in (0, 1), g is a concave function, $u_{[i]}$ is an individual well-being with the i-th lowest utility value in the profile u_N, c is a critical value, $|N|= n$ and $|M|= m$.

[5] The fact that this social welfare ordering violates the consistency of population replication can be easily shown in the following example. Suppose that $u_N = (2, 4)$, $v_N = (1, 6)$, the discount rate $\beta = 0.5$, the critical level $c = 0$, and the function g is given by an identity mapping. Then, by using the rank discounted critical level generalized utilitarianism, it holds that $1/2\times2 + 1/4\times4 = 2 = 1/2\times1 + 1/4\times6$. Hence, u_N and v_N are indifferent. On the other

discounted rate, it has a disadvantage that the degree of freedom in selecting weights is low. The second one is the single-parameter Gini social welfare ordering proposed by Donaldson and Weymark (1980). This social welfare ordering is a generalization of social welfare functions which are ordinally equivalent to average incomes multiplied by (1 - Gini coefficients).[6] This ordering is useful because it can satisfy not only the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, continuity, and cardinal full comparability, but also the consistency condition for any population replications. However, even this social welfare ordering has almost no degree of freedom in selecting weights for evaluation of inequality because the weight vector is almost fixed, except for single parameters.[7] The third one is so-called Yaari's *controversial social welfare ordering* and its extension (Yaari 1988; Ebert 1988b; Asheim and Zuber 2018). Although this ordering is a kind of rank-weighted generalized utilitarianism with well-designed weights that satisfy population consistency,[8] it has the disadvantage of non-responsiveness to population growth

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hand, in the evaluation of $(u_N, u_N) = (2, 4, 2, 4)$ and $(v_N, v_N) = (1, 6, 1, 6)$, it holds that $1/2\times2 + 1/4\times2 + 1/8\times4 + 1/16\times4 = 18/8 > 15/8 = 1/2\times1 + 1/4\times1 + 1/8\times6 + 1/16\times6$. Hence, (u_N, u_N) is strictly better than (v_N, v_N). Therefore, the ordering fails to provide a consistent judgment for replication changes.

[6] The single-parameter Gini social welfare ordering is defined as follows:
For all utility profiles u_N and v_M, u_N is at least as good as v_M if and only if

$$\frac{1}{n^\delta}\sum_{i=1}^{n}[\ i^\delta - \ (i-1)^\delta]\,\tilde{u}_i \geqq \ \frac{1}{m^\delta}\sum_{i=1}^{m}[\ i^\delta - \ (i-1)^\delta]\,\tilde{v}_i,$$

where δ is a parameter that is greater than 1, and a tilde of u means a utility profile sorted in descending order. By definition, if δ is 2, this ordering simply judges the profiles following average incomes × (1 - Gini coefficients). Obviously, this social welfare ordering satisfies the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and Dalton's *population principle* (the axiom is called *replication equivalence* in this paper).

[7] The sensitivity of inequality aversion in this ordering is given by the single parameter δ. Although it may be an advantage in terms of saving social decision costs to determine a single parameter, if we would like to evaluate inequalities following various normative aspects, it could be disadvantageous in the sense that a single parameter cannot take various consideration on inequalities.

[8] Donaldson and Weymark (1980) is the first study on a class of weights satisfying population consistency. Yaari's social welfare ordering can be interpreted as a simple application of a rank-weighted utilitarianism with Donaldson and Weymark's weights. In similar to the Yaari's result, Ebert (1988b) shows that any social welfare ordering defined on *the same population* can be represented by a class of rank-weighted Kolm-Pollak, Atkinson, and utilitarian social welfare orderings with Donaldson and Weymark's weights whenever a series of SWOs with the same population satisfies the population consistency. Asheim and Zuber (2018) propose a rank-weighted

and its limited functional form.

The purpose of this study is to avoid the unfreedom problem of functional forms in the above rules, and to generalize and find a new class of versatile, efficient, and equitable social welfare orderings that satisfy both conditions of population consistency and responsiveness to population growth. In fact, if the society requires rank-separability instead of separability, the class of efficient and equitable social welfare orderings extends from a simple generalized utilitarianism to a rank-dependent utilitarianism. In addition, even if the society requires a class of rank-dependent utilitarianism to satisfy population consistency, there are a new class of social welfare orderings, a *stepwise rank-dependent utilitarianism*, which has freedom to choose functional forms for evaluations on inequalities. This rule divides each profile with n individuals into n-quantiles from the bottom utility to the top utility, assigns a value deduced by its share for each quantile, and calculate the transformed sum. To satisfy a consistency condition for population replication, this social welfare ordering maps any population onto the closed interval [0, 1] and computes integral parts of the transformed values of step functions of utility profiles. Intuitively speaking, a stepwise rank-dependent utilitarianism judges the profiles by comparing sums of rank-dependent functions defined on utility quantiles[9]. By definition, this social welfare ordering is a kind of rank-dependent utilitarianism, and is not only compatible with the axioms of efficiency, equity, and population consistency, but it also has an obvious advantage that the functional form can be freely selected. One of the practical applications is a *quantile mean comparison method*, where income or consumption levels are divided into groups of k quantiles and the average income of each group are compared in an approximate lexicographic manner or weighted summation manner. This comparison method makes it possible to compare well-being distributions in an intuitive and easy-to-understand manner, and is a practical method that considers both efficiency and equity more than the traditional comparison of GDP per capita.

generalized utilitarianism with Donaldson and Weymark's weights.

[9] This social welfare ordering divides the utility profile into n quantiles by $1/n$ %. Then, a specific functional form is given to each quantile, and social welfare is judged by its sum. According to this method, the rank-dependent value of top 1% in the 100-individual economy is equal to that of top 1% in 10,000-individual economy.

Furthermore, this paper shows a generalized representation theorem of social welfare orderings that includes a class of stepwise rank-dependent utilitarianism[10] so that policymakers can consider the problem of optimal population size.

The main contributions of this paper can be summarized as follows. First, this study succeeds in finding a new class of efficient, equitable, and consistent social welfare orderings, a *stepwise rank-dependent social welfare ordering*, which is equivalent to one satisfying the desirable axioms: strong Pareto, anonymity, rank-separability, Pigou-Dalton transfer equity, continuity, and population consistency. Second, by combining the results of previous studies and this paper, theoretical correspondence between acceptable social welfare orderings and scale invariance is clarified in the problem of social choice with variable population. Third, general functional forms of stepwise social welfare orderings are obtained for considering the problem of optimal population size. Forth, the paper proposes three practical applications such as a quantile mean comparison method and use actual data to illustrate how the results differ from GDP per capita method.

The structure of this paper is as follows. Section 2 explains the notations, definitions, and axioms in this paper. Section 3 axiomatically characterizes a stepwise rank-dependent utilitarianism and shows a representation theorem of a generalized stepwise rank-dependent utilitarianism for considering the problem of optimal population size. Section 4 proposes three simple applications of our stepwise social welfare orderings. Then, we compare social welfare among eight developed countries by the quantile mean comparison method and show its differences from the GDP per capita method. The last section offers a summary and discusses the remaining issues.

[10] This result is immediately derived from the important representation theorem by Blackorby et al. (2001).

2. Notations and Definitions

This section explains notations, definitions, and axioms in this paper. Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_{++}$ be the sets of natural numbers, real numbers, and positive real numbers, respectively. The sets $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$ included by $\mathbb{N}$ are typical elements with different population sizes. The set of all possible well-being vectors is denoted by $U = \cup_{N\subseteq\mathbb{N}} \mathbb{R}^N$. For all $u_N \in U$, let $u_{[N]} = (u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ be a non-decreasing rearrangement of the well-being vector u_N, that is, $u_{[1]} \leqq u_{[2]} \leqq \ldots \leqq u_{[n]}$. The set of ranks is denoted by $[N] = \{[1], [2], \ldots, [n]\}$. Given $u_N \in U$, for any subset M of N, let u_M and $u_{N\setminus M}$ be $(u_i)_{i\in M}$ and $(u_i)_{i\in N\setminus M}$, respectively. In the similar way, for any subset $[M]$ of $[N]$, let $u_{[M]}$ and $u_{[N]\setminus[M]}$ be $(u_{[i]})_{[i]\in[M]}$ and $(u_{[i]})_{[i]\in[N]\setminus[M]}$, respectively. For an arbitrary set X, a binary relation defined on X is an *ordering* if and only if it satisfies completeness and transitivity.[11] Let a *social welfare ordering* $\succcurlyeq$ be defined on U. For all $u_N, v_M \in U$, $u_N \succcurlyeq v_M$ means that u_N is at least as socially good as v_M. Asymmetric and symmetric parts of $\succcurlyeq$ are given by $\succ$ and $\backsim$, respectively.[12]

Each individual well-being (e. g. income or wealth) is assumed to be intra- and interpersonally comparable and sometimes satisfies the following scale invariance conditions:

Translation-Scale Full Comparability. $\forall u_N, v_N \in U$, $\forall a\in\mathbb{R}$, $u_N \succcurlyeq v_N \leftrightarrow (a+u_i)_{i\in N} \succcurlyeq (a+v_i)_{i\in N}$.

Ratio-Scale Full Comparability. $\forall u_N, v_N \in U$, $\forall b\in\mathbb{R}_{++}$, $u_N \succcurlyeq v_N \leftrightarrow (bu_i)_{i\in N} \succcurlyeq (bv_i)_{i\in N}$.

Cardinal Full Comparability. $\forall u_N, v_N \in U$, $\forall a\in\mathbb{R}$, $\forall b\in\mathbb{R}_{++}$, $u_N \succcurlyeq v_N \leftrightarrow (a+bu_i)_{i\in N} \succcurlyeq (a+bv_i)_{i\in N}$.

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[11] Completeness requires that for all x, y in X, $x \succcurlyeq y$ or $y \succcurlyeq x$. Transitivity requires that for all x, y, z in X, ($x \succcurlyeq y$ & $y \succcurlyeq z$) implies $x \succcurlyeq z$.

[12] This paper implicitly assumes that a social ordering function satisfies *an independence condition* and *Pareto indifference*. The theoretical relationships between welfarism, neutrality, independence, and Pareto principle in the setting of variable population are examined in Blackorby et al. (1999, Theorems 1-3).

Next, let us define a series of traditional axioms in social choice theory. First, as an axiom of efficiency, the paper requires strong Pareto principle.

Strong Pareto. $\forall u_N, v_N \in U$, if $u_N \geqq v_N$, then $u_N \succcurlyeq v_N$. Moreover, if $u_N > v_N$, then $u_N \succ v_N$.[13]

Throughout the paper, all social welfare orderings must treat each individual well-being equally, and this requirement is represented by the following anonymity axiom.

Anonymity. $\forall$ bijections π on N, $\forall u_N \in U$, $u_N \backsim u_{\pi(N)}$.

This paper considers two types of continuity of social welfare orderings. The first continuity axiom demands that both the upper contour set and the lower contour set of social welfare ordering should be closed with *the same* population.

Continuity. $\forall u_N \in U$, both $\{v_N \in U \mid v_N \succcurlyeq u_N\}$ and $\{v_N \in U \mid u_N \succcurlyeq v_N\}$ are closed.

The second continuity is an extended version of the above continuity condition. This continuity axiom demands that both the upper contour set and the lower contour set of social welfare ordering should be closed in the setting of *variable population*.

Extended Continuity. $\forall u_N \in U$, both $\{v_M \in U \mid v_M \succcurlyeq u_N\}$ and $\{v_M \in U \mid u_N \succcurlyeq v_M\}$ are closed.

Separability requires social welfare orderings to ignore well-being information about indifferent

[13] For all $u_N, v_N \in U$, [$u_N \geqq v_N$ iff $u_i \geqq v_i$ for all i] and [$u_N > v_N$ iff $u_i \geqq v_i$ for all i and $u_j > v_j$ for some j].

individuals between two profiles. This axiom plays a central role in the famous joint characterization theorem (Dechamps and Gevers 1978), where Paretian and anonymous social welfare orderings must be weak utilitarian, leximin, or leximax rules with the same population under the assumption of cardinal full comparability. In the setting of variable population size, it is shown that an efficient, separable, and population consistent social welfare ordering must be an average generalized utilitarianism.

Separability. $\forall u_N, v_N, u'_N, v'_N \in U$, if $\exists M \subseteq N, (u_M = u'_M \ \& \ v_M = v'_M) \ \& \ (u_{N\setminus M} = v_{N\setminus M} \ \& \ u'_{N\setminus M} = v'_{N\setminus M})$, then $u_N \succcurlyeq v_N \leftrightarrow u'_N \succcurlyeq v'_N$.

Separability can be compatible with any concerns for poverty of individual well-being, but it ignores relative inequality of well-being distribution.[14] Therefore, let us consider rank-separability as a weaker condition under which relative inequality can be taken into account on measuring social welfare.

Rank-Separability. $\forall u_N, v_N, u'_N, v'_N \in U$, if $\exists [M] \subseteq [N], \left(u_{[M]} = u'_{[M]} \ \& \ v_{[M]} = v'_{[M]}\right) \ \& \ (u_{[N]\setminus[M]} = v_{[N]\setminus[M]} \ \& \ u'_{[N]\setminus[M]} = v'_{[N]\setminus[M]})$, then $u_N \succcurlyeq v_N \leftrightarrow u'_N \succcurlyeq v'_N$.

This axiom requires social welfare orderings to ignore well-being information about the same well-being in the same ranks between two profiles. Obviously, separability implies rank-separability under the assumption of anonymity. The next section shows that simply imposing rank-separability instead of separability yields a versatile class of distribution-sensitive social welfare orderings.

Let us introduce an axiom of equity. Pigou-Dalton transfer equity states the following: Given that

[14] Separability ignores relative inequality in the following profiles, but rank-separability could give different judgments. Consider two well-being profiles: (10, 20, 50, 100) and (8, 30, 50, 100). Then, the third and fourth individuals are indifferent among the two profiles. By definition, separability requires that a ranking on the two profiles (10, 20, 50, 100) and (8, 30, 50, 100) be the same on (10, 20, 3, 1) and (8, 30, 3, 1). However, this may not seem plausible to those who care about relative inequality (especially, the worst individual's well-being).

well-being of other persons is fixed, and there is a well-being gap between two individuals, the same amount of transfer that improves the gap will not at least reduce social welfare.

Pigou-Dalton Transfer Equity. $\forall u_N, v_N \in U$, $\forall \varepsilon \in \mathbb{R}_{++}$, if $\exists i, j \in N$, $v_i - \varepsilon = u_i \geqq u_j = v_j + \varepsilon$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succcurlyeq v_N$.

Finally, consider two consistency conditions of population replication. The first consistency of population replication, *replication equivalence,* requires that social welfare does not change if a well-being distribution remains the same no matter how many times a utility profile is replicated. This requirement is the same as the principle of population proposed by Dalton (1920). To define replication equivalence, let $k * u_N$ denote a k-replica of well-being profile u_N (i.e., $k * u_N = (\underbrace{u_N, \dots, u_N}_{k\ times})$). Then, the axiom is defined as follows.

Replication Equivalence. $\forall k \in \mathbb{N}, \forall u_N \in U$, $u_N \sim k * u_N$.

Since replication equivalence considers only well-being distributions, population growth or decline cannot affect social welfare. Although it may be an appropriate axiom for inequality measurements, it would be too strong for social welfare measurements. The next replication invariance requires that social welfare judgments on any two profiles remain the same with the replicated population. By definition, replication equivalence implies replication invariance.

Replication Invariance. $\forall k \in \mathbb{N}, \forall u_N, v_N \in U$, $u_N \succcurlyeq v_N \leftrightarrow k * u_N \succcurlyeq k * v_N$.

3. Stepwise Rank-dependent Social Welfare Ordering and Its Variations

This section characterizes a *stepwise rank-dependent social welfare ordering* and considers its variations: stepwise rank-weighted utillitarianism, stepwise rank-weighted Atkinson social welfare ordering, stepwise rank-weighted Kolm-Pollak social welfare ordering, stepwise generalized utilitarianism, and stepwise leximin. First, let us consider the most general form of social welfare ordering in this paper, a stepwise rank-dependent utilitarianism. This ordering is easily defined by the following stepwise form of a utility profile consisting of a finite number of individuals.

Definition: A function $u_{[N]} : [0, 1] \rightarrow \mathbb{R}$ is called a rank-dependent step function on u_N if and only if for all u_N, for all t in [0, 1], $u_{[N]}(t) = u_{[i]}$ whenever t in $[\,[i-1]/n, [i]/n\,)$.

Generally speaking, this step function is a simple mapping from discrete utility profiles into continuous utility distributions defined on [0, 1]. Note that any social welfare orderings satisfying replication equivalence can be represented by some orderings defined on the set of all utility distributions on [0, 1], which are obtained by this step function. Using this transformed utility profile by the step function, a stepwise rank-dependent utilitarianism is defined as follows.

Definition: A social welfare ordering $\succcurlyeq^{SRDU}$ is a stepwise rank-dependent utilitarianism if and only if there exists a function $g: \mathbb{R} \times [0,1] \rightarrow \mathbb{R}$ such that $\forall u, u' \in \mathbb{R}$ with $u > u', \forall t, t' \in [0,1]$ with $t > t', [g(u,t) > g(u',t)\ \&\ g(u,t') \geq g(u,t)]$ & $\forall u_N, v_M \in U,\ u_N \succcurlyeq^{SRDU} v_M \leftrightarrow \int_0^1 g(u_{[N]}(t), t)dt \geqq \int_0^1 g(v_{[M]}(t), t)dt.$

A stepwise rank-dependent utilitarianism is a natural generalization of a rank-dependent utilitarianism with the same population.[15] The rank-dependent functional form g can be specified by imposing some assumptions like scale invariance. An interesting case is that the functional form g is given by a rank-weighted generalized utilitarianism.[16] This form is obtained when g satisfies the following proportionality property (Sakamoto 2021). A functional g satisfies *proportionality with respect to relative position* if $\forall u, u' \in \mathbb{R}$, $\forall t, t' \in [0, 1], g(u, t) : g(u, t') = \ g(u', t) : g(u', t')$.[17] That is, a ratio of well-being values at *t* and *t'* is independent from well-being values and depends on only relative positions in [0, 1]. Moreover, as is well known from Blackorby and Donaldson (1982), Ebert (1988b), d'Aspremont and Gevers (2002), when some scale invariance is required to a rank-dependent utilitarianism with the same population, its functional form can be specified by the degree of scale invariance. In fact, translation-scale full comparability makes the functional form a *rank-weighted exponential one* (i. e. rank-weighted Kolm-Pollak type), ratio-scale full comparability makes it a *rank-weighted power one* (i. e. rank-weighted Atkinson type), and cardinal full comparability makes it a *rank-weighted identity one* (i. e. rank-weighted utilitarian type). As shown in the following Theorem 1, these results have a natural extension in the case of stepwise social welfare orderings with variable population.

Theorem 1. *A social welfare ordering* $\succcurlyeq$ *satisfies strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, continuity, and replication equivalence if and only if it is a stepwise rank-dependent utilitarianism. Moreover, it satisfies the following additional scale invariances if and only if its functional*

[15] Note that a stepwise rank-dependent utilitarianism is well-defined for all *finite* natural numbers. In the setting of infinite populations, the rank-dependent step function cannot be defined and there may be no computable algorithm for admissible social welfare orderings (Basu and Mitra 2003; Zame 2007; Lauwers 2010).

[16] In general, a function g in a rank-weighted generalized utilitarianism can take any shape as long as it is a continuous and monotonic concave function. Note that a rank-weighted generalized utilitarianism includes three key social welfare orderings: *Pareto egalitarianism*, *Prioritarianism*, and *Sufficientarianism*, which have been considered important in distributive ethics (Sakamoto 2021). In short, a class of acceptable social welfare orderings seems to be nothing more than a representation of social concerns for tolerable poverty and inequalities.

[17] Asheim and Zuber (2018) characterize a rank-weighted generalized utilitarianism with Donaldson and Weymark (1980)'s weights by using another axiom which has the similar effect of this proportionality condition.

form can be specified as follows: for all t in [0, 1] *and all profiles* u_N,

(1) *if proportionality w. r. t. relative position is satisfied, then* $g(u_{[N]}(t), t) = w(t) g\left(u_{[N]}(t)\right)$;

(2) *if translation-scale full comparability is satisfied,* $g(u_{[N]}(t), t) = -w(t) exp\left(-\alpha u_{[N]}(t)\right)$ *with* $\alpha > 0$ or $g(u_{[N]}(t), t) = w(t) u_{[N]}(t)$;

(3) *if ratio-scale full comparability is satisfied and the domain U is restricted on* $\cup_{N \subseteq \mathbb{N}} \mathbb{R}^N_{++}$, $g(u_{[N]}(t), t) = w(t) \left(u_{[N]}(t)\right)^\alpha$ *with* $\alpha \in (0, 1]$ or $g(u_{[N]}(t), t) = w(t) \ln\left(u_{[N]}(t)\right)$;

(4) *if cardinal full comparability is satisfied,* $g(u_{[N]}(t), t) = w(t) u_{[N]}(t)$;

where $w(t) \geqq w(t') \geqq 0$ for all t, t' in [0, 1] with $t < t'$ and $\int_0^1 w(t) dt = 1$.

[Proof] It is easy to prove that a stepwise rank-dependent utilitarianism satisfies the above axioms. We only prove that a social welfare ordering satisfying the axioms can be represented by a stepwise rank-dependent utilitarianism through the following four claims. Let $\succcurlyeq$ satisfy the above axioms.

[Claim 1] $\forall n \in \mathbb{N}$, $\exists$ a rank-dependent function $g^n_{[i]}$: $\mathbb{R} \to \mathbb{R}$, $\forall u_N, v_N \in U$, $u_N \succcurlyeq v_N \leftrightarrow \sum_{i=1}^n g^n_{[i]}(u_{[i]}) \geqq \sum_{i=1}^n g^n_{[i]}(v_{[i]})$.

By Ebert's characterization theorem (Ebert 1988a, Theorem 1), a social welfare ordering with the same population is a rank-dependent utilitarianism if and only if it satisfies strong Pareto, anonymity, and rank-separability. Hence, Claim 1 holds true. ■

[Claim 2] $\forall n \in \mathbb{N}$, $\exists$ a function g^n: $\mathbb{R} \times [0, 1] \to \mathbb{R}$ with $\frac{1}{n} g^n_{[i]}(u) = \int_{[i-1]/n}^{[i]/n} g^n(u, t) dt$ for all $[i]$ and all u, $\forall u_N, v_N \in U$, $u_N \succcurlyeq^{SRDU} v_N \leftrightarrow \sum_{i=1}^n g^n_{[i]}(u_{[i]}) \geqq \sum_{i=1}^n g^n_{[i]}(v_{[i]})$.

Given $n \in \mathbb{N}$, consider a function g^n such that $\forall [i] \in [N]$, $\forall t \in [[i\text{-}1]/n, [i]/n)$, $g^n(u, t) = g^n_{[i]}(u)$. With this function, it obviously holds that $\frac{1}{n} g^n_{[i]}(u) = \int_{[i-1]/n}^{[i]/n} g^n(u, t) dt$ and $u_N \succcurlyeq^{SRDU} v_N \leftrightarrow$

$\sum_{i=1}^{n} g_{[i]}^{n}(u_{[i]}) \geqq \sum_{i=1}^{n} g_{[i]}^{n}(v_{[i]})$. ■

[Claim 3] $\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow \frac{lcm(n,m)}{n} * u_N \succcurlyeq \frac{lcm(n,m)}{m} * v_M$, where $lcm(n, m)$ is the least common multiple of n and m.

By replication equivalence, $\forall u_N, v_M \in U, \frac{lcm(n,m)}{n} * u_N \sim u_N \succcurlyeq v_M \sim \frac{lcm(n,m)}{m} * v_M$. Transitivity implies that $u_N \succcurlyeq v_M \leftrightarrow \frac{lcm(n,m)}{n} * u_N \succcurlyeq \frac{lcm(n,m)}{m} * v_M$. ■

[Claim 4] $\forall n, m \in \mathbb{N}, \forall [i] \in [N], \forall [j] \in [M], \forall u \in \mathbb{R}, \int_{[i-1]/n}^{[i]/n} g^{lcm(n,m)}(u,t)dt = \int_{[i-1]/n}^{[i]/n} g^{n}(u,t)dt$ and $\int_{[j-1]/m}^{[j]/m} g^{lcm(n,m)}(u,t)dt = \int_{[j-1]/m}^{[j]/m} g^{m}(u,t)dt$.

By Claims 1-3, it immediately follows that $\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow \int_0^1 g^{lcm(n,m)}(u_{[N]}(t),t)dt \geqq \int_0^1 g^{lcm(n,m)}(v_{[M]}(t),t)dt$. Then, if the equations in Claim 4 do not hold, Claim 3 cannot hold. ■

Claim 4 guarantees that any value function induced by some function can be calculated by finer function. Then, for all rational number t in [0, 1], let the $g(u,t) = \lim_{n\to\infty} g^n(u,t)$ If $g^n(u,t)$ does not converge for some rational numbers, there are at most finite discontinuous points in the set of rational numbers belonging to [0, 1], since the functional value of $[i]/n$ must be Riemann-integrable for all natural numbers n and $[i]$. For all discontinuous points, we assume that the value of g is equal to the maximal value of limits. This function is easily extended to the set of real numbers in [0, 1] because there are at most discontinuous points in [0, 1]. Thus, by using g defined as the above, the following equation holds.

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow \int_0^1 g(u_{[N]}(t),t)dt \geqq \int_0^1 g(v_{[M]}(t),t)dt.$$

Moreover, Sakamoto (2021, Theorem 3) and Ebert (1988b, Theorem 5) implies that the specific forms of stepwise rank-dependent utilitarianism are obtained by proportionality and scale invariances. ■

Let us call social welfare orderings induced by functional forms (1) to (4), respectively, (i) stepwise rank-weighted generalized utilitarianism, (ii) stepwise rank-weighted Kolm-Pollak SWO, (iii) stepwise rank-weighted Atkinson SWO, and (iv) stepwise rank-weighted utilitarianism.[18] Furthermore, when separability is imposed instead of rank-separability, the results in Theorem 1 can be easily rewritten as average version of SWOs by applying well-known results of Blackorby and Donaldson (1982). That is, we can obtain an average generalized utilitarianism, average Kolm-Pollak SWO, average Atkinson SWO, and average utilitarianism.[19] By definition, these social welfare orderings imply that a rank-dependent value assigned to each utility always depends on its population ratio. For example, the top 10% among any population always have the same value for any replicated well-being profile.

Note that the stepwise rank-dependent utilitarianism includes the single parameter Gini social welfare ordering proposed by Donaldson and Weymark (1980). In fact, if the weight function specified in (4) of Theorem 1 is defined as $w(t) = (1+\delta)(1-t)^{\delta}$ for all t in [0, 1], then a stepwise rank-weighted utilitarianism with this weight function is equivalent to the single-parameter Gini index. A definition of a stepwise rank-weighted utilitarianism is so simple that the weights for any population size is easy to

[18] Although Yaari's controversial social welfare ordering (Yaari 1988) looks the same form of stepwise rank-weighted utilitarianism, note that there are important differences between them. First, Yaari ignores the famous impossibility theorem on infinite anonymity and Pareto principle in the setting of intergenerational equity (Diamond 1965). In general, a stepwise social welfare ordering is well-defined for any finite population but not infinite population. Second, our stepwise social welfare orderings are easy to understand and easily applied to other functional forms such as leximin, Kolm-Pollak family, Atkinson family, and generalized utilitarianism.

[19] As Morreau and Weymark (2016) argue, scale invariance conditions as normative requirements appear to be less important because scale invariance cannot discriminate *superficial* differences of well-being caused by just scale transformation from *actual* differences of well-being among people. Therefore, there may be no reason to care about scale invariance except for the purpose of specifying a functional form in a rank-dependent utilitarianism. Of course, it is possible to reinterpret ratio-scale full comparability as *homotheticity* of social welfare ordering. Note, however, that when the domain includes negative real numbers, a class of homothetic Atkinson social welfare orderings can be no longer so-called Atkinson type, and only the special version of rank-weighted utilitarianism can survive (Sakamoto 2021).

compute.[20] Furthermore, each weight function is not necessarily required to be continuous and only needs to be a measurable function. In this sense, the degree of freedom in selecting weight vectors is very high.

Theorem 1 shows that a stepwise rank-dependent utilitarianism is an efficient, equitable, consistent, and versatile social welfare ordering in the sense that it satisfies the axioms of strong Pareto, Pigou-Dalton transfer equity, and replication equivalence, and has a high degree of freedom in selecting functional forms. An interesting application of it is a *quantile mean comparison method*. This method divides population into some quantiles and compares each average well-being level for each quantile in an *approximate* lexicographic manner or *quantile-dependent* weighted summations. For example, suppose that the weight of j-th quantile ($j = 1, \ldots, k$-1) is given by $w_{j\text{-th}} = 1 - \Sigma_{\text{h}=1, \ldots, \text{j-1}} w_{h\text{-th}} - 1/1000^{j}$ and $w_{k\text{-th}}$ must be $1/1000^{k-1}$. If $k = 3$, then $w_{\text{1st}} = 0.999$, $w_{\text{2nd}} = 0.000999$, and $w_{\text{3rd}} = 0.000001$. By giving a huge priority to populations with lower well-being, this quantile mean comparison method approximately ignores the levels of well-being in the upper quantiles and is almost the same as a generalized leximin (Sakamoto 2020). If the society does not prefer such an extreme comparison by a lexicographic ordering, by assigning *quantile-dependent weight* to each quantile, social welfare can be obtained by their weighted sum. This method is clearly more distributive-sensitive than the conventional GDP per capita method, and provides an intuitive and simple method for comparing income distributions.

Besides, the stepwise rank-weighted utilitarianism can be easily linked to both Lorentz domination and generalized Lorentz domination.[21] In fact, by using the rank-dependent step function, the generalized Lorentz domination can be written as follows.

Generalized Lorenz Domination. $\forall u_N, v_M \in U$, u_N generally Lorenz dominates v_M if and only if $\forall p \in$

[20] In Donaldson and Weymark (1980), Ebert (1988b), and Bossert (1990), they consider theoretical properties of a series of weight vectors of standard rank-weighted utilitarianism satisfying replication invariance. However, the functional form of weights is given by the differences of specific function defined on ratios of population and is difficult to compute for comparing various situations.

[21] For the same reason, there are also obvious theoretical relationships among Lorenz domination, stochastic dominance, and progressive income transfers. For the basic results in this field, see Dutta (2002, Theorem 3. 3).

$[0, 1], \int_0^p u_{[N]}(t)dt \geqq \int_0^p v_{[M]}(t)dt.$

By the similar logic of the classical results in income inequality (Atkinson 1970, Dasgupta et al. 1973, and Shorrocks 1983), we can immediately show the following result.

Proposition. *For all u_N, $v_M \in U$, u_N generally Lorenz dominates v_M if and only if for any rank-dependent weight function w,* $\int_0^1 w(t)u_{[N]}(t)dt \geqq \int_0^1 w(t)v_{[M]}(t)dt.$

It seems that, however, replication equivalence would be too strong as a condition of consistency for population replication. In fact, a stepwise rank-weighted utilitarianism states that a one-person economy with a high well-being should be better than a 10,000-person economy where all persons have a slightly lower well-being.[22] To avoid such a problem, positive responsiveness of social welfare to population increment must be considered. Fortunately, this difficulty can be solved by requiring extended continuity (a stronger version of continuity) and replication invariance (a weaker version of replication equivalence) due to the celebrated theorem in Blackorby et al. (2001).

Theorem 2. *If a social welfare ordering satisfies the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, extended continuity, and replication invariance, then it is represented by a generalized form of stepwise rank-dependent utilitarianism as follows:*

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow W(n, V(u_N)) \geqq W(m, V(v_M)),$$

where $W: \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and increasing in its second argument and V is a function that represents a stepwise rank-dependent utilitarianism. Moreover, if it satisfies the following additional scale

[22] This problem is called the *reverse repugnant conclusion* and is analyzed by Blackorby et al. (1998).

invariances, then its functional form in a stepwise rank-dependent utilitarianism can be specified as follows: for all t in [0, 1] and all profiles u_N,

(1) *if proportionality w. r. t. relative position is satisfied, then* $g\left(u_{[N]}(t),t\right)=w(t)g\left(u_{[N]}(t)\right)$;

(2) *if translation-scale full comparability is satisfied,* $g\left(u_{[N]}(t),t\right)=-w(t)exp\left(-\alpha u_{[N]}(t)\right)$ *with* $\alpha>0$ or $g\left(u_{[N]}(t),t\right)=w(t)u_{[N]}(t)$;

(3) *if ratio-scale full comparability is satisfied and the domain U is restricted on* $\cup_{N\subseteq\mathbb{N}}\mathbb{R}^N_{++}$, $g\left(u_{[N]}(t),t\right)=w(t)\left(u_{[N]}(t)\right)^{\alpha}$ *with* $\alpha\in(0,1]$ or $g\left(u_{[N]}(t),t\right)=w(t)\ln\left(u_{[N]}(t)\right)$;

(4) *if cardinal full comparability is satisfied,* $g\left(u_{[N]}(t),t\right)=w(t)u_{[N]}(t)$;

where $w(t)\geqq w(t')\geqq 0$ for all t, t' in [0, 1] with $t<t'$ and $\int_0^1 w(t)dt=1$.

[Proof] We prove Theorem 2 by using the following three claims. Let $\succcurlyeq$ satisfy the above axioms.

[Claim 1] $\exists$ a function W: $\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ and $\exists$ a function V^n: $\mathbb{R}^n\to\mathbb{R}$ for all n such that $\forall u_N, v_M\in U$, $u_N\succcurlyeq v_M \leftrightarrow W(n, V^n(u_N))\geqq W(m, V^m(v_M))$, where W is continuous and increasing in its second argument.

By the famous representation theorem in Blackorby et al. (2001), if a social welfare ordering satisfies extended continuity and weak Pareto,[23] then Claim 1 holds true. Since strong Pareto implies weak Pareto, Claim 1 obviously holds. ■

[Claim 2] $\forall n\in\mathbb{N}$, $\exists$ a rank-dependent function $g^n_{[i]}$: $\mathbb{R}\to\mathbb{R}$, $\forall u_N, v_N\in U$, $u_N\succcurlyeq v_N \leftrightarrow \sum_{i=1}^n g^n_{[i]}(u_{[i]})\geqq \sum_{i=1}^n g^n_{[i]}(v_{[i]})$.

Claim 2 is trivial since it is directly shown by Claim 1 in Theorem 1. ■

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[23] Weak Pareto is defined as follows: $\forall u_N, v_N\in U$, if $\forall i\in N$, $u_i>v_i$, then $u_N\succ v_N$.

[Claim 3] $\forall k \in \mathbb{N}, \forall u_N \in U$, $V^n(u_N) = V^{kn}(k * u_N)$.

Given u_N, suppose that $V^n(u_N) = \bar{u}$ where $\sum_{i=1}^{n} g_{[i]}^{n}(\bar{u}) = \sum_{i=1}^{n} g_{[i]}^{n}(u_{[i]})$. By the definition of V^n, it is ordinally equivalent to a rank-dependent utilitarianism with the same population and $u_N \sim n*(\bar{u})$ holds true. Replication invariance implies that $u_N \sim n*(\bar{u})$ iff $k*u_N \sim kn*(\bar{u})$ for all natural numbers k. Hence, $V^n(u_N)$ $= \bar{u} = V^{kn}(k*u_N)$. ■

By Claim 3, if an ordering is represented by a function V, then it satisfies replication equivalence. Hence, this function is ordinally equivalent to a stepwise rank-dependent utilitarianism with a certain rank-dependent function by the proof of Theorem 1. Moreover, by using the similar logic in the proof of Theorem 1, we can show the conditions of proportionality and scale invariances impliy the specific functional forms of g defined in Theorem 2. ■

Note that Theorem 2 is a simple representation theorem based on a stepwise rank-dependent utilitarianism and says nothing on specific functional forms and positive responsiveness to population increment. In fact, a class of social welfare orderings represented in Theorem 2 includes a *reverse number-dampened* social welfare ordering; the form is given by a function $f(n)$ multiplied by a stepwise rank-dependent utilitarianism, where $f(n)$ is a strictly decreasing function with respect to a natural number. If the society would like some properties on positive responsiveness to population increment, the additional axiom should be needed. Moreover, if seeking some type of additive social welfare orderings such as a number-dampened stepwise rank-dependent utilitarianism, some additivity axiom would be required. This paper does not provide a definitive conclusion on the question of which specific functional forms should be used as a class of acceptable social welfare orderings that generalizes a stepwise rank-dependent utilitarianism. In order to solve this ambivalent problem, the society must consider the issues of ethical values of social inequality, degrees of freedom for selecting parameters, and population growth.[24] It may

[24] The so-called *repugnant conclusion* (Parfit 1976; 1984) can be avoided if $f(n)$ is upper bounded and strictly

be possible to solve this difficulty by experimental, empirical, and ethical analyses.

Combining the results in Theorems 1 and 2, theoretical relationships between scale invariance and stepwise forms of acceptable social welfare ordering are easily obtained. If translation-scale full comparability were admissible, a stepwise rank-weighted Kolm-Pollak SWO should be used. If ratio-scale full comparability were admissible, a stepwise rank-weighted Atkinson SWO should be used. If cardinal full comparability were admissible, a stepwise rank-weighted utilitarianism should be used.

Finally, if ordinal full comparability were admissible,[25] one candidate for acceptable social welfare orderings would be the following stepwise leximin.

Definition: A social welfare ordering $\succcurlyeq^{SL}$ is a *stepwise leximin* if and only if $\forall u_N, v_M \in U$, $u_N \succcurlyeq^{SL} v_M \leftrightarrow$ $[\forall t$ in $[0, 1]$, $u_{[N]}(t) = v_{[M]}(t)]$ or $[\exists t'$ in $[0, 1], \forall t < t'$, $u_{[N]}(t) = v_{[M]}(t)$ and $u_{[N]}(t') > v_{[M]}(t')]$.

Obviously, the stepwise leximin is efficient, equitable, and consistent because it satisfies the axioms of strong Pareto, Pigou-Dalton transfer equity, and replication equivalence. With a slight modification of the famous Hammond's theorem (Hammond 1976), it is shown that this social welfare ordering is characterized by the axioms of strong Pareto, anonymity, Hammond equity, separability, ordinal full comparability, and replication equivalence.[26] If the society would like the stepwise leximin to

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increasing function in number-dampened stepwise SWOs. Also, the *sadistic conclusion* (Arrhenius 2000), which is regarded as a problem of another impossibility, seems to have some important ethical flaws of its own. Furthermore, this study succeeds to provide substantial solutions to the tyranny of aggregation and non-aggregation shown by Fleurbaey and Tungodden (2010). These issues are discussed in Section 4 of the previous version of this paper, but they will not be discussed in detail in this paper.

[25] Hammond equity and ordinal full comparability are defined as follows.

Hammond Equity. $\forall u_N, v_N \in U$, if $\exists i, j \in N$, $v_i > u_i \geqq u_j > v_j$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succcurlyeq v_N$.

Ordinal Full Comparability. $\forall u_N, v_N \in U$, $\forall$ an increasing function φ, $u_N \succcurlyeq v_N \leftrightarrow (\varphi(u_i))_{i\in N} \succcurlyeq (\varphi(v_i))_{i\in N}$.

[26] Note that the stepwise leximin and the number-dampened stepwise leximin can be defined in the discrete

have positive responsiveness to population increment and replication invariance, then the following number-dampened stepwise leximin could be a candidate for acceptable social welfare orderings under the assumption of ordinal full comparability.

Definition: A social welfare ordering $\succcurlyeq^{NSL}$ is a *number-dampened stepwise leximin* if and only if $\exists$ an increasing function $f: \mathbb{N} \rightarrow \mathbb{R}_{++}, \forall u_N, v_M \in U, u_N \succcurlyeq^{NSL} v_M \leftrightarrow f(n) \cdot u_N \succcurlyeq^{SL} f(m) \cdot v_M$.

Hence, theoretical relationships between scale invariance and acceptable social welfare orderings could be clarified. The following table summarizes the relationships.

population manner. Consider the following lexicographic ordering:

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow \frac{lcm(n,m)}{n} * u_N \succcurlyeq^{L} \frac{lcm(n,m)}{m} * v_M,$$

where $\succcurlyeq^{L}$ is a lexicographic ordering with the same population. The above ordering is obviously equivalent to the stepwise leximin. The number-dampened stepwise leximin can be defined in the similar way.

Table 1. Theoretical Relationships of Scale Invariances and Admissible SWOs with Variable Population

Invariance	**Admissible Value Functions with Variable Population**	**Generalized Forms of Admissible Value Functions with Variable Population**
Ordinal Full Comparability (OFC)	Stepwise Leximin (=SP∧A∧SEP∧HE∧RE∧OFC)	Number-Dampened Stepwise Leximin
Translation-Scale Full Comparability (TFC)	Average KP (=SP∧A∧SEP∧C∧ RE∧TFC), Stepwise Rank-Weighted KP (=SP∧A∧R-SEP∧C∧PD∧RE∧TFC)	Number-Dampened Forms of Average KP, Stepwise Rank-Weighted KP, etc.
Ratio-Scale Full Comparability (RFC)	Average A (=SP∧A∧SEP∧C∧RE∧RFC), Stepwise Rank-Weighted A (=SP∧A∧R-SEP∧C∧PD∧RE∧RFC)	Number-Dampened Forms of Average ABD, Stepwise Rank-Weighted ABD, etc.
Cardinal Full Comparability (CFC)	Average U (=SP∧A∧SEP∧C∧RE∧CFC), Stepwise Rank-Weighted U (=SP∧A∧R-SEP∧C∧PD∧RE∧CFC)	Number-Dampened Forms of Average U, Stepwise Rank-Weighted U, etc.
No requirement	Average GU (=SP∧A∧SEP∧C∧RE), Stepwise Rank-Dependent GU (=SP∧A∧R-SEP∧C∧PD∧RE)	Number-Dampened Forms of Average GU, Stepwise Rank-Dependent GU, etc.

Each abbreviation is defined as follows: SP: Strong Pareto; A: Anonymity; SEP: Separability; HE: Hammond Equity; RE: Replication Equivalence, R-SEP: Rank-Separability; C: Continuity; PD: Pigou-Dalton Transfer Equity; KP: Kolm-Pollak Social Welfare Ordering; A: Atkinson Social Welfare Ordering; U: Utilitarianism.

4. Applications and Empirical Results

This section proposes three comparison methods as applications of our stepwise rank-dependent SWOs: a quantile mean comparison, an interval population-ratio comparison, and an interval population-weighted mean comparison. Moreover, we compare the quantile mean comparison method with per capita income comparison and illustrate how our methods have advantages over traditional statistical measures by using actual data.

First, we define our three measures as follows.

Quantile Mean Comparison. A social welfare ordering is a *quantile mean comparison* if and only if $\exists$ quantiles $\{Q_j\}_{j=1}^{k}$, $\exists$ quantile-dependent weight vector $(w_{[1]},\dots,w_{[k]})$, $\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow (w_{[1]},\dots,w_{[k]})^{T}(\bar{u}_{Q_1},\dots,\bar{u}_{Q_k}) \geqq (w_{[1]},\dots,w_{[k]})^{T}(\bar{v}_{Q_1},\dots,\bar{v}_{Q_k})$, where $\bar{u}_{Q_j} = \int_{Q_j} u_{[N]}(t)dt$ and $\bar{v}_{Q_j} = \int_{Q_j} v_{[M]}(t)dt$ for $j = 1, \dots, k$.

Interval Population-Ratio Comparison. A social welfare ordering is an *interval population-ratio comparison* if and only if $\exists$ intervals $\{\mathcal{I}_j\}_{j=1}^{k}$ which is a partition of $\mathbb{R}$, $\exists$ interval-dependent coefficients $(c_{<1>},\dots,c_{<k>})$, $\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow (c_{<1>},\dots,c_{<k>})^{T}(\frac{n_{<1>}}{n},\dots,\frac{n_{<k>}}{n}) \geqq (c_{<1>},\dots,c_{<k>})^{T}(\frac{m_{<1>}}{m},\dots,\frac{m_{<k>}}{m})$, where $n_{<j>} = |\{i \in N|\ u_i \in \mathcal{I}_j\}|$ and $m_{<j>} = |\{i \in M|\ v_i \in \mathcal{I}_j\}|$ for $j = 1, \dots, k$.

Interval Population-Weighted Mean Comparison. A social welfare ordering is an *interval population-weighted mean comparison* if and only if $\exists$ intervals $\{\mathcal{I}_j\}_{j=1}^{k}$ which is a partition of $\mathbb{R}$, $\exists$ interval-dependent coefficients $\{a_{<j>}\}_{j=1}^{k}$ *and* $\{b_{<j>}\}_{j=1}^{k}$, $\forall u_N, v_M \in U, u_N \succcurlyeq v_M \leftrightarrow (\frac{a_{<1>}}{\bar{u}_{\mathcal{I}_1}} + b_{<1>},\dots,\frac{a_{<k>}}{\bar{u}_{\mathcal{I}_k}} + b_{<k>})^{T}(\frac{n_{<1>}}{n}\bar{u}_{\mathcal{I}_1},\dots,\frac{n_{<k>}}{n}\bar{u}_{\mathcal{I}_k}) \geqq (\frac{a_{<1>}}{\bar{v}_{\mathcal{I}_1}} + b_{<1>},\dots,\frac{a_{<k>}}{\bar{v}_{\mathcal{I}_k}} + b_{<k>})^{T}(\frac{m_{<1>}}{m}\bar{v}_{\mathcal{I}_1},\dots,\frac{m_{<k>}}{m}\bar{v}_{\mathcal{I}_k})$, where

$\bar{u}_{\mathcal{I}_j} = \frac{\sum_{\mathcal{I}_j} u_i}{n_{<j>}}$ and $\bar{v}_{\mathcal{I}_j} = \frac{\sum_{\mathcal{I}_j} v_i}{m_{<j>}}$ for $j = 1, \ldots, k$.

The quantile mean comparison method is a simple application of stepwise rank-weighted utilitarianism, where the functional form $g(u_i)$ is given by $w_{[j]}u_i$ if u_i is in quantile j. Since individuals belonging to the same quantile are given the same weights, the quantile mean comparison method judges social welfare by following quantile-dependent weighted sum. That is, each average income for each quantile has its own quantile-dependent weight. As is obvious from the definition, this method simplifies income distributions and allows us to look directly at average income of each quintile, which facilitates an intuitive interpretation of social welfare. It will be useful as a measure for welfare comparisons and policy evaluation in various contexts (e.g., national and regional comparisons, or gender and ethnic comparisons).

The interval population-ratio comparison method can be interpreted as the case where a functional form $g(u_i)$ in an average generalized utilitarianism is given by $a_{<j>} + b_{<j>}(u_i)^{\alpha}$ whenever u_i is in interval j and its inequality aversion α is zero. This method measures social welfare by seeing population-ratio in each rating scale which is defined by income levels in given intervals.[27] If a policymaker would like to consider some interval to be at poverty level, we can assume well-being at the intervals below θ to be poverty and well-being at the intervals above θ to be adequate standard of living for some index θ. Then, let us suppose that the interval-dependent coefficient $c_{<j>} = a_{<j>} + b_{<j>}$ is negative in the poverty intervals (i.e., the existence of individuals in the poverty intervals has a negative impact on social welfare).[28] Under this formulation, note that the usual version of strong Pareto and Pigou-Dalton transfer

[27] If the coefficients are allowed to be extreme like a generalized leximin in this method, the method can be interpreted as a stepwise leximin where the standard of living takes on discrete values. For example, if the standard of living is ranked by seven categories: very poor, poor, lower middle, middle, upper middle, rich, and very rich, the stepwise leximin will socially prefer an income distribution with lower population ratio of the very poor.

[28] If the society would think that the worse the level of poverty gets, the more negative impact on social welfare it has, the coefficient w in poverty interval should be an increasing function of the poverty level.

equity will not be satisfied, but the weaker form of these axioms for cross-interval changes in well-being can be satisfied. If the society can agree that income or well-being intervals that determine each level of living standards could be set appropriately, then all we have to do is just evaluating vectors that explicitly show the ratio of population enjoying each level of living standards, which can be used as one of practical and easy-to-understand welfare measures. Of course, the interval population-ratio comparison method may be changed to the number-dampened version, which considers the population size instead of the population-ratio, to evaluate vectors of population belonging to each interval.[29] The methods have the advantage of providing a method for comparing social welfare that covers well-being of the middle or upper groups as well, compared to traditional poverty indicators that exclusively focus on the poor.

The interval population-weighted mean comparison method is another application of the average generalized utilitarianism. A functional form $g(u_i)$ in it is given by $a_{<j>} + b_{<j>}u_i$ and the coefficient in interval j is given by $\frac{a_{<j>}}{\bar{u}_{J_j}} + b_{<j>}$. Note that the coefficient is variable with respect to the average income. Since the minimum value of j's coefficients is given by the upper bound of interval j, it may be good to evaluate social welfare by using the minimum coefficients for simplification. In this method, we can directly see both population-ratio and mean in each interval. This method also provides a practical and easy-to-understand measure for comparing social welfare between countries and regions.[30] Of course,

[29] The advantage of the interval-based comparison method is that social welfare of the *entire world* can be simply decomposed of the social welfare of *individual nations* due to separability. In the interval population-ratio comparison method, social welfare of the entire world can be represented by a weighted sum of social welfare of all nations, with the weight being the nation's share of the world's population. Therefore, interval-based comparisons can be useful as a simple measure of social welfare in the sense that they allow us to directly look at distributions of population of the very poor, the poor, the middle class (the middle class may be further subdivided), the rich, and the very rich at both global and local levels. Note that, for simplicity, the interval-dependent coefficients are set to constants, but the coefficients could be variable with respect to average income.

[30] In addition, several variations of interval-based comparisons such as an *interval difference-ratio comparison*, which compares weighted sums of average shortfall or excess ratio per interval, and an *interval difference comparison*, which compares weighted sums of average shortfall or excess amount per interval, could easily be considered. However, since these measures satisfy neither separability nor rank separability, they will not be

these interval-based comparisons can be easily applied to multidimensional poverty analysis and the capability approach.

When using the above three methods, note that it is not at all necessary to compare weighted sums with arbitrarily given weights. On the contrary, it would be more practical to dare to compare raw vectors, which is composed of mean per quantile, population ratio per interval, or population-weighted mean per interval. Although we finally need weights to calculate weighted sums of these measures, we cannot expect to figure out *exact* and *moral* weights that should be assigned to various levels of inequality or poverty from any axiomatic studies. It looks far more useful for policymakers to consider only raw information of vector itself, which describes a simple feature of well-being distributions in an easily viewable way, without letting an arbitrary weight set.[31]

In the next subsection, we will illustrate how the quantile mean comparison method is a more convenient and practical measure than traditional measures by using actual data.

Method and Data

This subsection compares social welfare among eight developed countries by the quantile mean comparison method. In the quantile mean comparison method, each country's population is divided into some quantiles, and an average well-being level of each quantile is compared in an approximate lexicographic manner or simple weighted summations. Although it is fully understood that there are major deficits in using income as human well-being, in order to show the advantage of quantile mean comparison method over the traditional comparison based on GDP per capita, we will compare post-tax income levels among developed countries.

The dataset is obtained from the *World Inequality Database* (https://wid.world/) because this group provides highly reliable and finest data including information on the entire 1-100 percentiles of post-tax

considered in this paper.

[31] Another alternative would be to compare the range of weighted sums by computing them within the socially agreed range of weights that seem to be reasonable.

incomes among many countries. We select eight developed countries (Britain, Denmark, France, Germany, Finland, Norway, Sweden, and the United States) for our analysis, since their detailed information on post-tax income are available. Of course, there is no problem in adding other countries, but we decide to compare the post-tax income of the above countries due to the visibility of graphs. Each country's post-tax income is based on a PPP-adjusted real value. In addition, the post-tax income includes the amount of in-kind transfers and is based on the unit of *equal-split adult* (i.e. income divided equally among spouses).

Although countries are compared on their post-tax income distributions, consumption levels seem to be more appropriate than income levels for measuring individual well-being and social welfare. The big problem is that income levels often are seriously affected by seasonality and life cycle. On the contrary, consumption levels are relatively stable from changes in life cycle or seasonality. Hence, consumption levels are interpreted to reflect properly actual well-being of individuals or households. However, household income is easy to collect, but household consumption is difficult to capture. Also, the commonly called *welfare ratio* (the ratio of annual household income over the minimum living cost based on the concept of absolute poverty) may be one of powerful candidates for welfare measurements, but the calculation of the welfare ratio needs too many information so that we cannot find appropriate data source. Therefore, this study uses only information on post-tax income level of each quantile in each country.

Calculation of the Quantile Mean Comparison Method

We compare social welfare among the eight developed countries by the quantile mean comparison method based on their post-tax income distributions. As is clear from the definition of the quantile mean comparison method, both the number and the width of quantiles are free, and there is no restriction on each quantile-dependent weight assigned to each quantile. For example, it is possible to compare quantiles with inhomogeneous widths and weights such as the bottom 10% ($w_{1st} = 0.45$), 11-50% ($w_{2nd} = 0.3$), 51-90% ($w_{3rd} = 0.239$), 91-99% ($w_{4th} = 0.01$), the top 1% ($w_{5th} = 0.001$), and where $w_{i\text{-th}}$ is a weight for i-th quantile. This study simply uses a five-quantile mean comparison method (i.e. each quantile has 20% population) because it does not seem to be a large difference in empirical findings by subdividing quantiles more than necessary. In the case of the simple quantile comparison method, social welfare does not change for any proportional increase or decrease of the population size because of the property of replication equivalence. Hence, there are obstacles when comparing the United States to countries or regions with extremely small populations such as Monaco and Luxembourg (Note that GDP per capita also has the same problem). To deal with this issue, a generalized form of the stepwise rank-dependent utilitarianism must be specified for assessing the effect of each country's population size. However, since the population size of the eight countries is sufficiently large, this study does not consider the problem of the generalized form of quantile mean comparison method. Instead, we simply compute average income for each quantile based on the five-quantile mean comparison method.

Comparison of Average Income in Each Quintile

Figures 1 to 7 shows national income per capita, average income of each quintile, and the top 1% income among the eight countries during 1980-2017. As seen in figure 1, the United States and Norway's national income per capita are higher than those of the other countries. In contrast, a completely different situation can be seen in terms of the quantile mean comparison method. In fact, figure 2 illustrate that the US income level among the bottom 20% is clearly lower than that of the others. While the middle class in the United States is relatively good among developed countries (figures 4-5), the US income in the top 20% is obviously outstanding from the other countries (figure 6). Among the top income group, the US income in the top 1% is 2-3 times higher than that of the other countries (figure 7). In this sense, the quantile mean comparison method can directly show us the whole picture of income inequality in the United States, and it can tell us how the US anomality with distributive injustice is proceeding compared to the other countries. If we compare social welfare in an approximate lexicographic manner, the United States is judged to have the lowest welfare among the eight countries. Of course, in the mild case where a value function is defined as a quantile-dependent weighted sum rather than the approximate lexicographic ordering, social welfare completely depends on the forms of weight functions.

Figure 1. National Income per capita (thousands of USD), 1980-2017

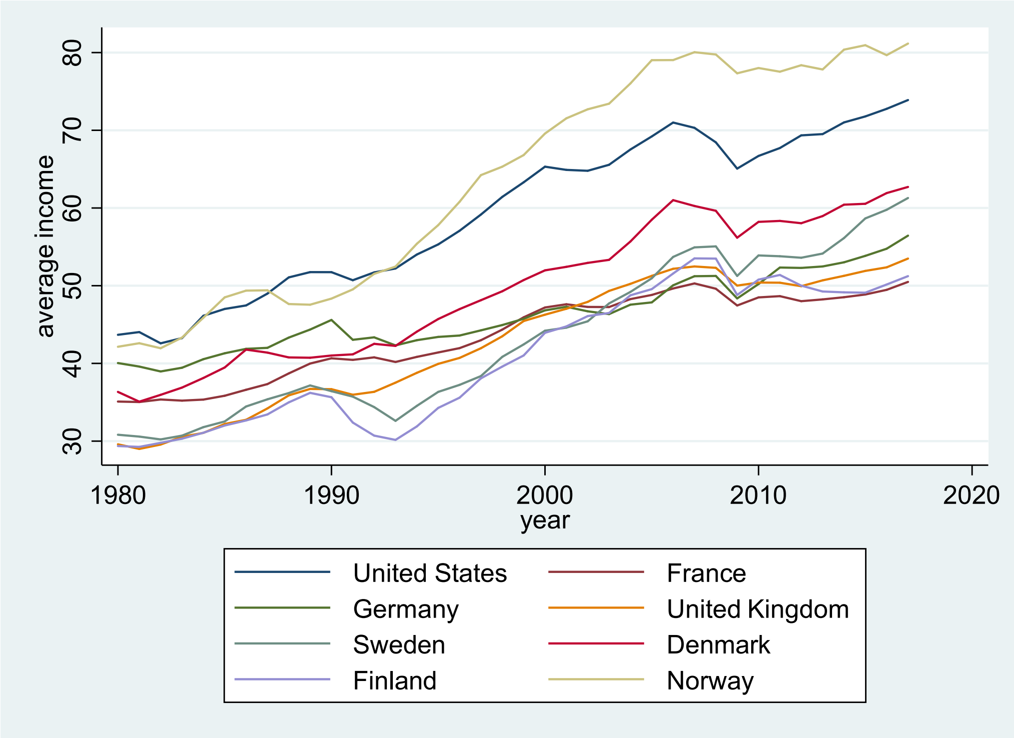


Figure 2. Average Income (thousands of USD) of 1st Quintile (The Bottom 20%), 1980-2017

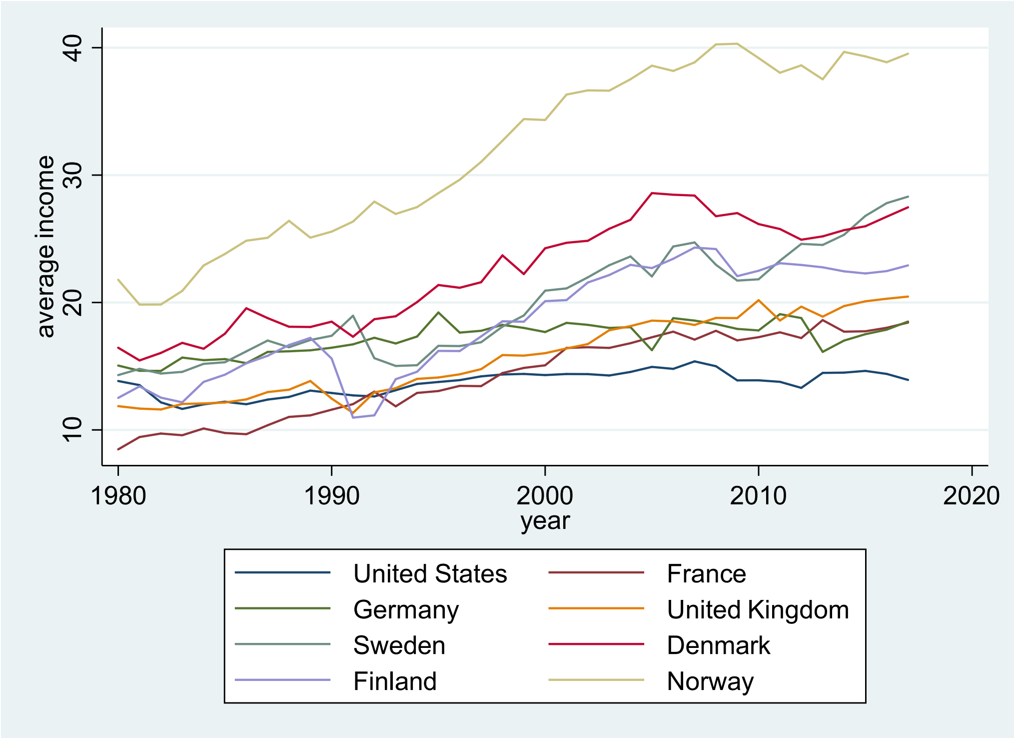


Figure 3. Average Income (thousands of USD) of 2nd Quintile (21-40%), 1980-2017

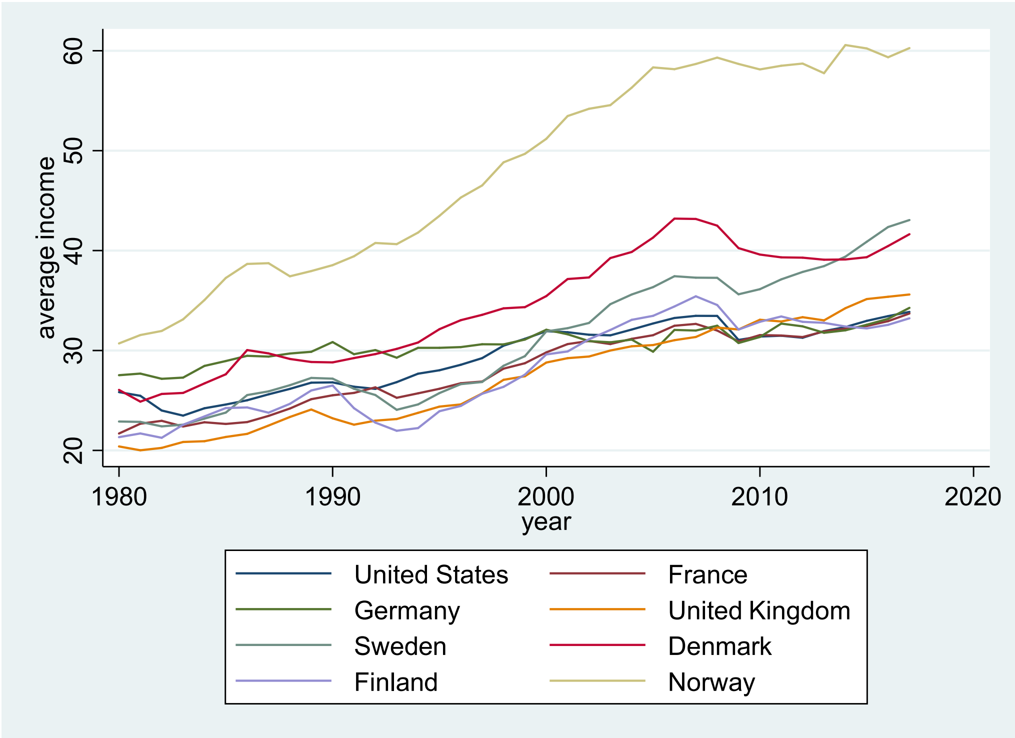


Figure 4. Average Income (thousands of USD) of 3rd Quintile (41-60%), 1980-2017

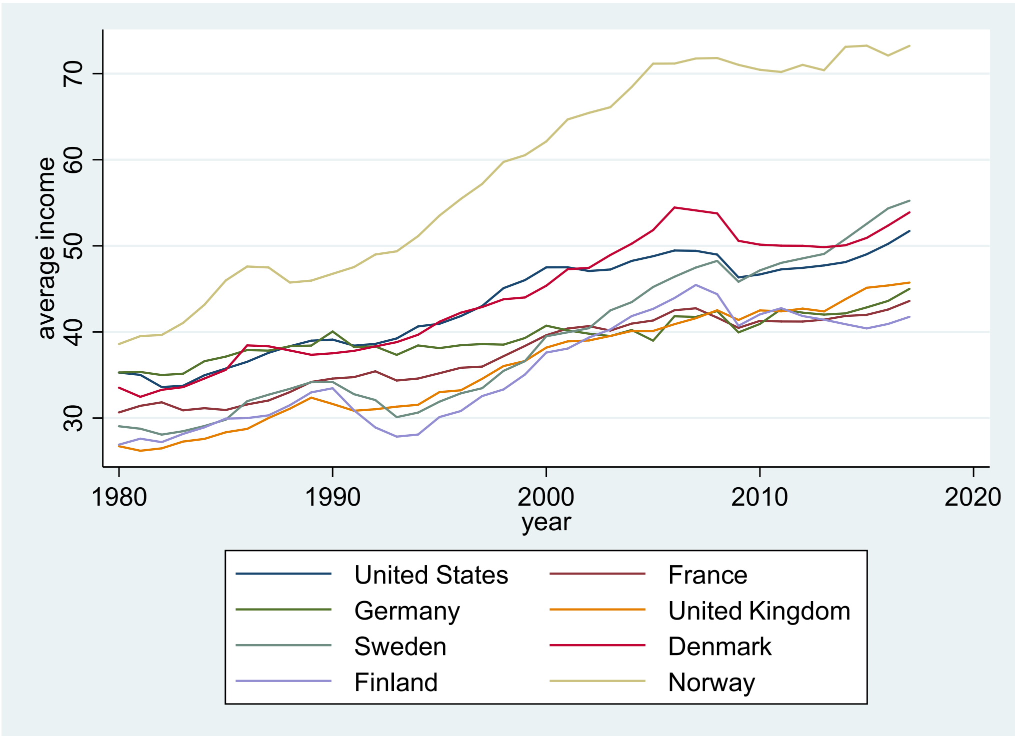


Figure 5. Average Income (thousands of USD) of 4th Quintile (61-80%), 1980-2017

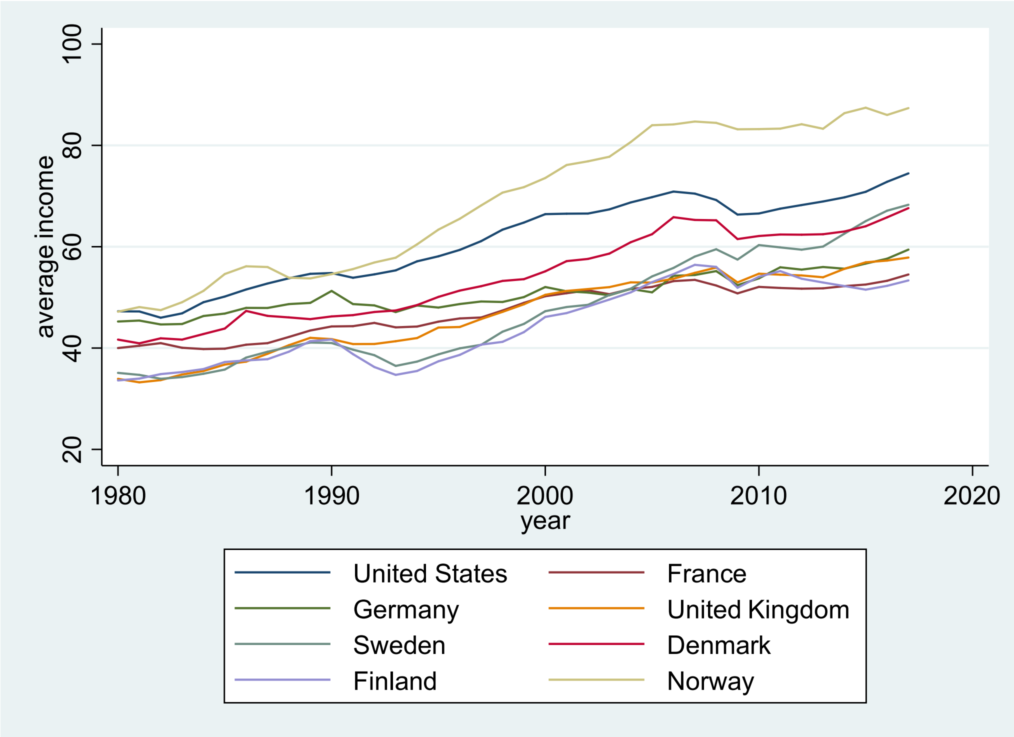


Figure 6. Average Income (thousands of USD) of 5th Quintile (The Top 20%), 1980-2017

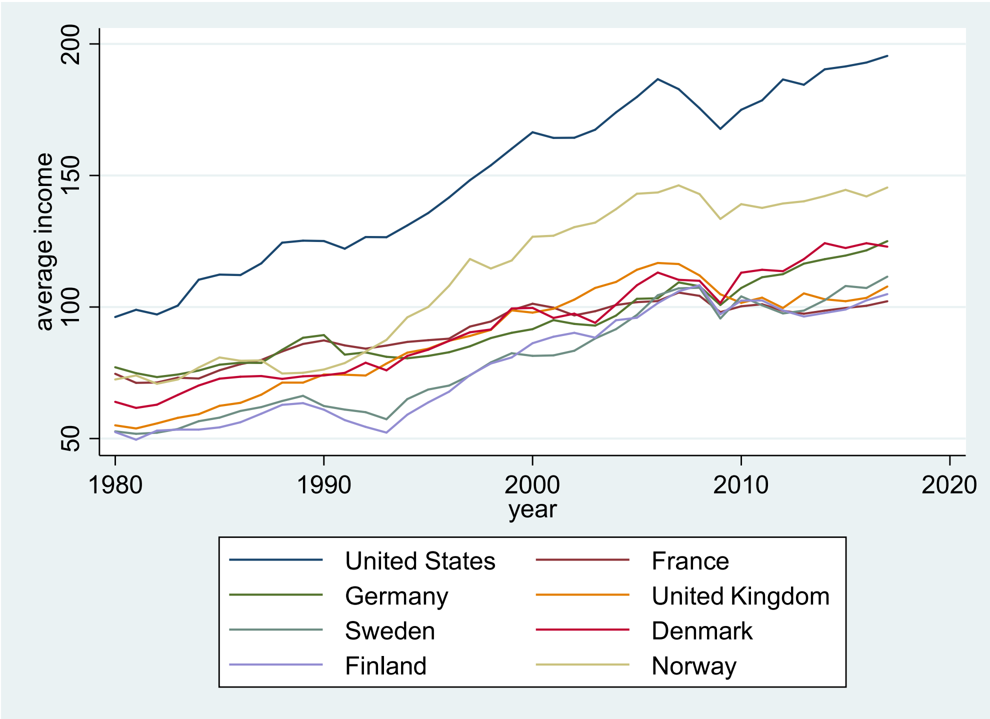
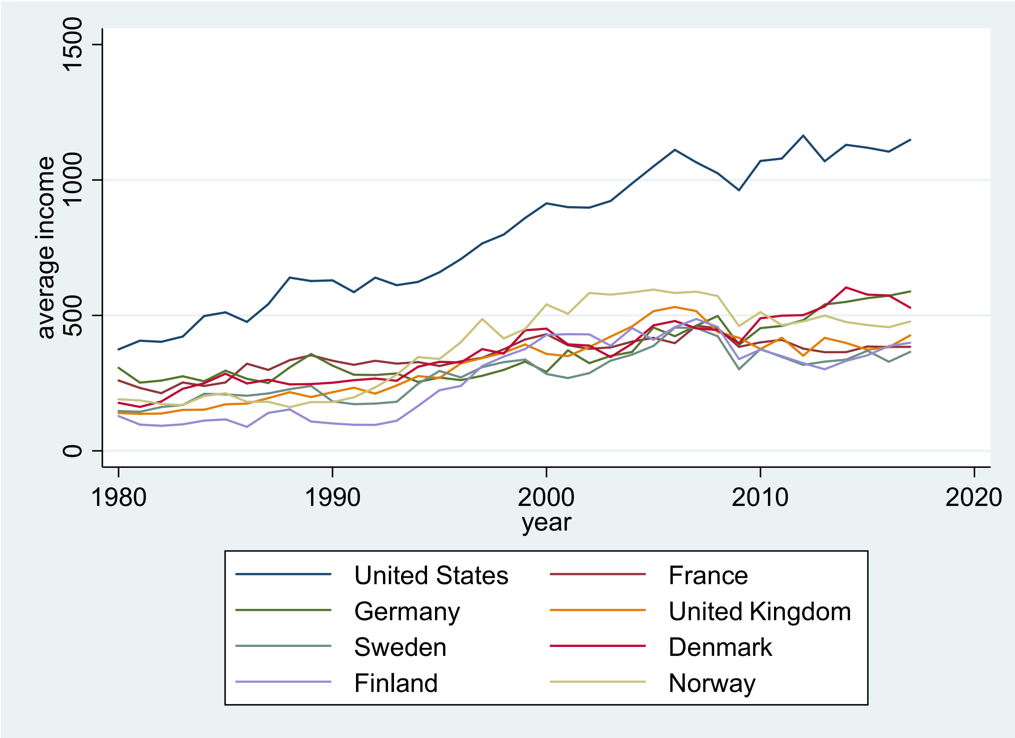


Figure 7. Average Income (thousands of USD) of the Top 1%, 1980-2017



At the end of this section, we summarize differences between traditional practices on policy evaluations and our stepwise rank-dependent utilitarianism. There are at least three popular measures for policy evaluations in applied economics such as labor economics and development economics: range; top/bottom ratio; and median income.[32]

Firstly, the *range* is defined as the difference between the highest and the lowest income. Obviously, it measures the maximum value of income gap. By this definition, it has obvious disadvantages: there is no consideration for intermediate income groups except for the top and the bottom; any improvement of income gap among the middle group cannot increase social welfare. Hence, this method violates strong Pareto and Pigou-Dalton transfer equity.

[32] By the similar logic of the following examples, other leading indicators and descriptive statistics such as mode, moments, entropy, and quantile number (not quantile *mean*) have obvious defects compared to our quantile mean comparison approach.

Secondly, the *top/bottom ratio* is defined as the ratio of the average income of the top $\bar{\alpha}$% and the bottom $\underline{\alpha}$% for some $\bar{\alpha}$ and $\underline{\alpha}$ in (0, 100), It measures social income gap based on a relative scale. By the similar way to the range, this also has major disadvantages: there is no consideration for intermediate income groups except for the top and the bottom; any improvement of income gap among the middle group cannot increase social welfare; it is only a ratio and cannot reflect income growth; social welfare is not invariant with respect to cardinal full comparability. Hence, it violates strong Pareto, Pigou-Dalton transfer equity, and cardinal full comparability.

Finally, a *median income* is just a median in the income distribution. By the definition, this also has the similar disadvantages: there is no consideration for other income groups except for the median; any improvement of income gap among the group cannot increase social welfare. Hence, it violates strong Pareto and Pigou-Dalton transfer equity.

Therefore, there are significant problems with the above three popular social indicators, and the quantile comparison method seems to have great advantages because of its desirable properties. Although there is still a need for social judgment as to how to decide each weight for each quantile in the quantile mean comparison method, the fact that a class of acceptable social welfare orderings is limited to the stepwise rank-dependent utilitarianism and its variants seems to be very important. In other words, for any policy evaluation, it is justified from the axiomatic characterization results to divide the income group into some quantiles and analyze the policy effect on the average income of each income group. Depending on the weight of each quantile, it may be possible that the great benefits of the middle group will offset the small loss of the bottom group. The stepwise rank-dependent utilitarianism allows us to directly investigate the policy impact on each income quantile and this seems to be a step forward in the context of policy evaluation practices and measuring social welfare.

5. Concluding Remarks

This study proposes a new class of versatile, efficient, and equitable social welfare orderings for social choice with variable population. A stepwise rank-dependent utilitarianism is characterized by standard axioms: strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, continuity, and replication equivalence. Furthermore, a class of stepwise social welfare orderings can be extended and generalized in a natural way by demanding extended continuity and positive responsiveness to population increment. Notably, these orderings include various practical comparison methods such as the quantile mean comparison and interval population-ratio comparison. By combining previous results and our findings, theoretical relationships between scale invariances and acceptable aggregation rules would become clear. Of course, this study takes only a small step toward the goal of creating a practical approach to compare social welfare with different population sizes. However, the authors are convinced that our generation will be able to provide some practical social welfare orderings characterized by a system of relatively uncontroversial axioms. Although there are still more issues to consider, our generation has a bright and hopeful future for normative economics originated from Pigou's welfare economics (Pigou 1920) and Arrow's social choice theory (Arrow 1951; 1963).[33]

The following outlines the remaining challenges for further research.

First, it seems sound to apply the stepwise forms of social welfare orderings to intertemporal social choice problems[34], but it may be problematic to apply them to the context of intergenerational equity.

[33] This difficult field in economics has numerous impossibility theorems (Suzumura 2002; Sen 1970, 2017). However, as this study shows, it is still possible to construct efficient, equitable, and consistent social welfare orderings. The authors believe that the theory of social choice is full of not only *impossibilities* but various *possibilities* because of its wide range of applications and its importance as a "*science of evaluation*".

[34] Of course, a class of non-separable social welfare orderings invokes a new difficulty to consider the past utility stream in the context of intertemporal social welfare comparisons. On the other hand, a separable social welfare ordering allows us to ignore the past utilities. However, note that a stepwise rank-dependent utilitarianism could survive and provide a consistent judgment if economies always have continued to improve human well-being,

Although the theory of intergenerational equity compares infinite streams of utilities, it is impossible to predict the streams of utilities over infinite period.[35] Moreover, even if there exists a social welfare ordering extension that satisfies some desirable properties (Bossert, Sprumont and Suzumura 2007), this ordering extension may be inherently incomputable (Zame 2007; Lauwers 2010). In addition, preferences of future generations are essentially unstable, and there are the *non-identity problem* and path-dependency problem in which tastes and values of future generations could be altered by the selections of past generations (Parfit 1984; Suzumura and Tadenuma 2007). In order to deal with the above issues, our era's mathematical tools and models have big problems, so further path-breaking improvement is necessary in all aspects of prediction methods, theoretical models, and intergenerational ethics.

Second, our results can be easily applied to the theory of social choice for assessing risky social situations. Given the celebrated *Harsanyi's aggregation theorem* (Harsanyi 1955), the society must face the trade-off between ex-ante and ex-post equity. As Fleurbaey (2010) shows, if the society could agree to require weak or strict dominance and ex-post Pareto principle (some restricted versions of ex ante Pareto), then the only option would be ex-post types of social welfare orderings. Note that the stepwise forms of social welfare orderings are easily applied to both ex-ante and ex-post types of social welfare orderings with risky situations. However, the problem of whether ex-ante or ex-post types of social evaluation should be preferable requires further consideration.

Third, in order to make our applications such as the quantile mean comparison more practical,

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where the past utilities are below any future utilities and have no influence on ranks between the past and future utilities. In this sense, further discussion is needed on this issue of intertemporal welfare comparison.

[35] Economic models for predicting climate change, economic growth, and financial crises have obvious deficiencies due to vulnerability of model error and the dependency of too-variable parameters. The best model our generation currently knows has many potential errors, even in forecasting for decades. The issues of intergenerational equity should not be analyzed by a model for an infinite period, but by a model for a finite period, and it may be necessary to take measures such as incorporating utilities of future generations after the term period into an objective function as one variable. Alternatively, it may be better to use a method of comparing only the ones that converge to a steady state in possible infinite utility streams.

we need some technical improvements in admissible weighting, appropriateness of the number and the width of quantiles or intervals, handling of measurement errors, a treatment of population inflows and outflows, and incompleteness of value judgments. It is essential to accumulate various verifications on the robust results to guarantee that the analysis is consistent with our normative judgments on social inequalities and injustice. Furthermore, when informational basis of the quantile mean comparison or interval-based comparison simply depends on household incomes adjusted by purchasing power, it seems necessary to reexamine both the standard consumer theory and usual practice on the aggregated price level and the purchasing power parity in economics. In comparing living standards of home and foreign countries, consumption bundles greatly differ among the poor and the rich in all countries. Hence, there will be cognitive and substantial problems in comparing income levels adjusted by purchasing power to judge collective welfare of different groups in different countries.[36] It may be necessary to devise a new method of comparison of purchasing power and calculation of aggregated price levels.

Fourth, the methods of stepwise rank-dependent utilitarianism and stepwise leximin can be easily applied to the other well-being measurements, such as Better Life Index, multi-dimensional poverty indices, and some inequality indices on ordinal variables. For example, in each item's judgment in the Better Life Index, the quantile mean comparison method and stepwise leximin, respectively, are applied to cardinal measures such as household incomes and ordinal scales (categorical variables) such as security. Of course, when summarizing all items and evaluating social welfare instead of comparing each item between countries, it would be necessary to develop and construct a class of desirable aggregation rules

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[36] Although farmer's self-consumption and street vendor's economic transactions often do not appear in official statistics, they are likely to be important sources of income or consumption for low-income groups in developing countries. Hence, comparisons based on simple official statistics would make it difficult to evaluate accurate standards of livings in developing countries. There is no doubt that the capability approach (Sen 1985), which focuses on what people can do or can be, rather than simply comparing incomes or consumptions based on purchasing power, could be the most appropriate method for evaluating human's well-being. However, the formulation of the capability approach and the attempt to practically measure capabilities will face significant difficulties when trying to consider the aspect of choice opportunities rather than the consequences of choice.

that properly reflect the results in comparisons of each item.[37]

Finally, considering the problem of optimal population size in the context of environmental problems and social welfare programs, the behavior of population size in a social objective function is highly important. Generally speaking, the optimal population size can be obtained by maximizing the generalized form of social welfare functional $W(n, V(u_N))$ proposed in this paper. However, this function says nothing on the specific functional form of population size. This implies that *optimality* of population needs further investigations into the extent of whether inequalities of well-being distributions with population increments are *tolerable*. In this sense, social choice theory requires serious consideration in terms of philosophical, empirical, and evolutionary perspectives on the values of human-beings, sustainability, and a desirable society.

[37] Another practical solution is to apply our interval population-ratio comparison on this context. Suppose that human's well-being is ranked by some discrete living standard levels based on multi-dimensional functionings. For example, we will classify living standard into seven categories: very poor, poor, lower middle, middle, upper middle, rich, and very rich. In the interval population-ratio comparison method, all social planners have to do is simply having a concern for population-ratio of each living standard level.

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