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Can Acceptable Social Welfare Orderings Show Compassion for Both Relative Inequality and Poverty? A Reexamination of Interpersonal Comparisons of Well-being and Scale Invariance

Norihito Sakamoto

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Research Center for Normative Economics
Institute of Economic Research
Hitotsubashi University
https://www.ier.hit-u.ac.jp/rcne/

Can Acceptable Social Welfare Orderings Show Compassion for Both Relative Inequality and Poverty? A Reexamination of Interpersonal Comparisons of Well-being and Scale Invariance

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Norihito Sakamoto†

Abstract

This study shows that combining the Pareto principle and continuity with scale invariance, which has usually been interpreted as the requirement of interpersonal comparisons of well-being, imposes a major constraint on a functional form of a social welfare ordering. In fact, if the social welfare ordering is required to satisfy cardinal full comparability of well-being, then it must belong to a class of weighted utilitarianisms with variable weights. However, thorough an appropriate reformulation on interpersonal comparisons of well-being, the class of acceptable social welfare orderings is shown to be a rank-dependent generalized utilitarianism that can show compassion for both relative inequality and poverty. If additional conditions are required, we can obtain some refinements of this social welfare ordering such as a rank-weighted generalized utilitarianism and a rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering. Moreover, the rank-weighted generalized utilitarianism is shown to approximately include the well-known three social welfare orderings that have been proposed in ethics: the Pareto egalitarianism, prioritarianism, and sufficientarianism. Therefore, it makes clear that the theoretical difference between the ideas of distributive justice in ethical theory is simply caused by intensity levels for tolerable inequality and poverty. These results can be easily extended to the context of social choice with variable populations.

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† Tokyo University of Science, E-mail: n-sakamoto@rs.tus.ac.jp
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1. Introduction

In all kinds of public decisions, such as in healthcare, infrastructure development, educational quality, and various poverty reduction projects, we need to aggregate individual well-being and compare social welfare. However, given some measures of individual well-being, how should we aggregate individual well-being to get consistent social value judgments? If interpersonal comparison of well-being somehow could be admissible, what is the appropriate aggregation rule to evaluate social welfare?

As is well known, Arrow's impossibility theorem (Arrow 1951) shows that there is no reasonable way to socially aggregate individual preferences that are ordinally inter-personal and non-comparable into a social welfare ordering (SWO).[1] In contrast to this pessimistic result, Sen and his successors established that allowing for interpersonal comparability of well-being opens the door to a variety of possibilities (Sen 1970; Roberts 1980a; 1980b). These studies propose a class of attractive social welfare orderings that evaluate distributions of well-being, characterized by a set of plausible axioms (anonymity, strong Pareto principle, Pigou-Dalton transfer equity, separability, invariance on some acceptable transformations of well-being, and rationality of social ranking).[2] As a result, the leximin, utilitarianism, Kolm-Pollack SWOs, and Atkinson-Blackorby-Donaldson SWOs are candidates for an acceptable social welfare ordering (Hammond 1976; d'Aspremont and Gevers 1977; Maskin 1978; Blackorby and Donaldson 1982). However, Dechamps and Gevers (1978) proved a surprising result that under the cardinal full comparability of well-being, a social welfare ordering satisfying the axioms of anonymity, the strong Pareto principle,

[1] Even under the assumption of cardinal interpersonal noncomparability of well-being, Arrow's impossibility theorem can easily reemerge (Sen 1970, Ch. 7*). In general, allowing for *intrapersonal* comparability of well-being is not enough to define an acceptable social welfare ordering. See Sakamoto (2020) for a discussion on theoretical relationships among inter- and intra-personal comparability of well-being and equity of social welfare.

[2] What should be considered as plausible axioms is a highly contentious issue in ethics and epistemology. However, a social welfare ordering satisfying continuity and the Pareto principle is known to be represented by an increasing and continuous real-valued function in which its value lies in the minimum and maximum of utilities for any utility profile (Blackorby et al. 2005, Th. 4.1., p. 94). In addition to these axioms, it is no major problem to require the other axioms, such as the Pigou-Dalton transfer equity, which is relatively uncontroversial, and separability, which is computationally easy.

separability, and minimum equity must be either the leximin or the utilitarianism. Since the utilitarianism is not distribution-sensitive at all, while the leximin is too distribution-sensitive to ignore aggregated welfare, their result seems to be a kind of impossibility theorem.[3]

The reasonable way to avoid the repugnant result of Dechamps and Gevers is to abandon separability. In fact, Weymark (1981) proposed a *generalized Gini social welfare ordering* (a rank-weighted utilitarianism) that does not satisfy separability. The rank-weighted utilitarianism judges social welfare by a weighted sum in which the fixed weight reflects relative inequality. This method is clearly desirable because it can reflect both aggregated efficiency and distribution-equity. In addition, this social welfare ordering is shown to be characterized by the plausible axioms such as anonymity,[4] the strong Pareto principle, Pigou-Dalton transfer equity, rank-separability, and continuity under the assumption of cardinal full comparability (Ebert 1988a).[5] Furthermore, Sakamoto (2020) shows that dropping the continuity leads to a *generalized leximin* that includes both the rank-weighted utilitarianism and the leximin.[6]

The generalized leximin is a social welfare ordering that divides a set of individuals following their well-being ranks and lexicographically compares a sequence of weighted sums defined on the divided groups (Sakamoto 2020). This ordering can consider relative inequality of well-being distributions, but it cannot show any compassion for the value of well-being itself because it is invariant to any cardinal transformations of well-being. Under the assumption of cardinal full comparability of well-being, the generalized leximin is characterized by the plausible axioms of anonymity, the strong Pareto principle, rank-separability, and Pigou-Dolton transfer equity. Therefore, it seems difficult to recommend any other candidates but the

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[3] The condition of interpersonal comparability of well-being is formally identical to the condition of invariance on scale transformation in the context of income and wealth inequality measurements. Note, therefore, that the results by Dechamps and Gevers can apply directly to the theory of inequality measurements on income and wealth distributions.

[4] In general, rank-separability implies anonymity. Hence, there is no need to explicitly state the condition of anonymity in the system of axioms used in the characterization result. However, the author dares to mention it for comparison with the results of the other axiomatizations using separability.

[5] A similar axiomatic characterization is provided by Ben-Poraith et al. (1997). They characterize the rank-weighted utilitarianism by the axiom of a variation of Pigou-Dalton equity axiom combined with rank-separability.

[6] See Ebert (1988b) and Sakamoto (2020) for theoretical relationships between rank-separability and scale invariance. Moreover, similar results are obtained in the setting with variable populations (Sakamoto and Mori 2020).

generalized leximin for an efficient and equitable social welfare ordering. For example, both the sufficientarian social welfare ordering, which prioritizes a total well-being of the poor below a poverty line (Adler 2019; Bossert et al. 2020), and critical-level generalized utilitarianism (Blackorby and Donaldson 1984) can satisfy axioms such as the Pareto principle and anonymity but they violate an invariance condition for scale transformation of well-being. Hence, they are excluded from the class of social welfare orderings that satisfy the above plausible axioms. Does this mean that any acceptable social welfare orderings must belong to the class of generalized leximins that consider only tolerable inequality?

The answer to this question can be found in the re-examination of the invariance axioms with respect to scale transformations by Morreau and Weymark (2016).[7] Let us now consider the following two social welfare orderings. The first one evaluates income distributions in terms of U.S. dollars (USD), and the second one evaluates income distributions in terms of Japanese yen (JPY).

Example 1. Suppose the scale unit and the reference point (origin) are set to (\$1, \$0). Then, there exist a weight vector $w_{[N]} = (w_{[1]}, \ldots, w_{[n]})$ with $w_{[1]} \geq \cdots \geq w_{[n]}$ and a concave function g, for all income distributions $u_N = (u_{[1]}, \ldots, u_{[n]})$ and $v_N = (v_{[1]}, \ldots, v_{[n]})$ in USD with $u_{[1]} \leq \cdots \leq u_{[n]}$ and $v_{[1]} \leq \cdots \leq v_{[n]}$,

$$u_N \succcurlyeq v_N \leftrightarrow \sum_{i \in N} w_{[i]}\, g(u_{[i]}) \geq \sum_{i \in N} w_{[i]}\, g(v_{[i]}).$$

Example 2. Suppose the scale unit and the reference point (origin) are set to (¥1, ¥0) with the fixed nominal exchange rate as \$1 = ¥100. Then, there exist a weight vector $w_{[N]} = (w_{[1]}, \ldots, w_{[n]})$ with $w_{[1]} \geq \cdots \geq w_{[n]}$ and a concave function g which are defined in the above Example 1 such that, for all income distributions $u_N = (u_{[1]}, \ldots, u_{[n]})$ and $v_N = (v_{[1]}, \ldots, v_{[n]})$ in JPY with $u_{[1]} \leq \cdots \leq u_{[n]}$ and $v_{[1]} \leq \cdots \leq v_{[n]}$,

$$u_N \succcurlyeq v_N \leftrightarrow \sum_{i \in N} w_{[i]}\, g(0.01u_{[i]}) \geq \sum_{i \in N} w_{[i]}\, g(0.01v_{[i]}).$$

As Examples 1 and 2 clearly show, it is easy to construct a class of acceptable social welfare orderings that are nonlinear but invariant to any scale transformations, both in USD and JPY notations. However, if income distributions are judged without

[7] Morreau and Weymark (2016) is the first formal analysis to explicitly distinguish between a utility gap due to scale transformation and that of actual utility levels. They prove a neutrality theorem by using the scale-dependent axioms of the Pareto indifferent principle and the independence condition. Nebel (2021) also suggests that reformulating the concept of interpersonal comparisons of well-being correctly can make various social welfare orderings acceptable.

considering currency units, the Pareto principle makes a wrong judgment in which (100, 500) is strictly better than (1, 5), even though (100, 500) in JPY is equivalent to (1, 5) in USD in terms of their purchasing power. Since the traditional scale invariance conditions cannot distinguish *surface* well-being differences in *different* scale units and *actual* well-being differences in the *same* scale unit, they place severe restrictions on the functional form of social welfare orderings. In fact, this study shows that combining continuity and the Pareto principle with the traditional scale invariance conditions greatly limits the functional form of the social welfare ordering, yielding a weighted utilitarianism with variable weights.

In contrast, appropriately redefining invariance conditions and the scale-dependent axioms, we can obtain a broad class of efficient and equitable social welfare orderings which include non-linear functional forms. For example, a rank-dependent generalized utilitarianism is shown to be characterized by the desirable scale-dependent axioms (anonymity, the strong Pareto principle, and rank-separability). Furthermore, if some additional conditions are imposed, then a rank-weighted generalized utilitarianism and a rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering can be obtained. Then, the rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering is shown to violate Pigou-Dalton transfer equity in general and a kind of rank-weighted utilitarianism is shown to be the only one that can satisfy the equity condition. Also, we show that the rank-weighted generalized utilitarianism is a generalization of the Pareto egalitarian (Tungodden 2000; Tungodden and Vallentyne 2005), prioritarianism (Parfit 1991), and sufficientarianism (Crisp 2003; Bossert et al 2020).[8] It is worth noting that this general form of social welfare orderings can consider compassion for both relative inequality in well-being distributions and poverty level of well-being itself. In the sense that the rank-weighted generalized utilitarianism includes the leading social welfare orderings as a special form, this paper shows that the theoretical difference between key concepts in the analysis of distributional justice relies merely on a matter of the intensity of compassion for relative inequality and poverty.

The contributions of this study can be summarized as follows. First, the paper shows that the combination of traditional invariance conditions and standard axioms

[8] The Pareto egalitarianism is formally defined as a social welfare ordering implying a refinement of the maximin principle. This ordering is a special case of the generalized leximin. The prioritarianism is only a simple generalized utilitarianism. The definition of sufficientarianism is a bit complicated. Firstly, it compares subtotal well-being of the poor individuals below a poverty line. If the subtotal well-being is the same, then it secondly compares total well-being among whole individuals.

imposes severe constraints on the functional form of social welfare orderings. Second, reformulating the conventional framework appropriately, we characterize a general class of equitable and efficient social welfare orderings, rank-dependent general utilitarianisms that consider both relative inequality and poverty. Third, we show that the Pigou-Dalton transfer equity imposes a strong restriction on the functional forms of acceptable social welfare orderings and the only equitable rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering must be a simple form of rank-weighted utilitarianism. Fourth, the leading social welfare orderings proposed by ethical theories (the Pareto egalitarianism, prioritarianism, and sufficientarianism) are shown to be essentially the same since they are approximately included by the class of rank-weighted generalized utilitarianism. Fifth, these results are extended to the setting of social choice with variable populations.

The remainder of this paper is structured as follows. Section 2 explains the notation, definitions, and axioms in this study. Section 3 defines the traditional conditions of scale invariance and shows that the combination of the Pareto principle and continuity imposes constraints on the functional form of the social welfare ordering. Section 4 characterizes some acceptable social welfare orderings, such as the rank-dependent generalized utilitarianism, the rank-weighted generalized utilitarianism, and the rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering when the plausible axioms are redefined as scale-dependent forms. Furthermore, these general social welfare orderings are shown to imply the leading social welfare orderings in ethics. The last section summarizes this study and discuss the remaining issues.

2. Notations and Definitions

This section explains the notations and definitions in this paper. Let $\mathbb{R}$, $\mathbb{R}_{++}$, and $\mathbb{R}_{--}$be the sets of real numbers, positive real numbers, and negative real numbers, respectively. The fixed set of individuals is denoted by $N = \{1, \ldots, n\}$ with $n \geqq 3$. The set of all possible well-being vectors is denoted by $U = \mathbb{R}^N$. For all $u_N \in U$, $u_{[N]} = (u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ means a non-decreasing rearrangement of the well-being vector u_N, that is, $u_{[1]} \leq \cdots \leq u_{[n]}$. Let the set of ranks be $[N] = \{[1], [2], \ldots, [n]\}$. Following the traditional definition of Sen (1970), for an arbitrary set X, a binary relation defined on X is an *ordering* if and only if it satisfies completeness and transitivity.[9] A *social welfare ordering* $\succcurlyeq$ is defined on U. For all u_N, $v_N \in U$, $u_N \succcurlyeq v_N$ is interpreted that u_N is at least as socially good as v_N. Asymmetric and symmetric parts of $\succcurlyeq$ are given by $\succ$ and $\sim$, respectively.

Note that all social welfare orderings can be considered as *social ordering functions* in the sense that they always generate an ordering defined on the set of well-being profiles.[10] The next section will investigate so-called *cardinal full comparability*, which has been traditionally interpreted as each individual well-being is cardinal fully comparable.

Cardinal Full Comparability. $\forall\, u_N, v_N \in U, \forall\, a \in \mathbb{R}, \forall\, b \in \mathbb{R}_{++}, u_N \succcurlyeq v_N \Leftrightarrow (a+bu_i)_{i\in N} \succcurlyeq (a+bv_i)_{i\in N}$.

Next, let us define the standard axioms which seem to be plausible in the tradition of social choice theory. First, as an axiom of efficiency, the *strong Pareto principle* is defined as follows.

Strong Pareto Principle. $\forall\, u_N, v_N \in U$, if $u_N \geqq v_N$, then $u_N \succcurlyeq v_N$. Moreover, if $u_N > v_N$, then $u_N \succ v_N$.[11]

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[9] Completeness requires that for all x, y in X, $x \succcurlyeq y$ or $y \succcurlyeq x$. Transitivity requires that for all x, y, z in X, $(x \succcurlyeq y$ & $y \succcurlyeq z)$ implies $x \succcurlyeq z$.

[10] As a well-known neutrality theorem shows (Sen 1977), this paper implicitly assumes that a social ordering function satisfies *an independence condition* and *Pareto indifference*. See Morreau and Weymark (2016) for a neutrality theorem in the context of reformulated interpersonal comparability of well-being.

[11] The inequalities in vectors are defined as follows. For all u_N, $v_N \in U$, [$u_N \geqq v_N$ iff $u_i \geqq v_i$ for all i] and [$u_N > v_N$ iff $u_i \geqq v_i$ for all i and $u_j > v_j$ for some j].

The following *anonymity* condition requires a social welfare ordering to treat each individual well-being equally.

Anonymity. $\forall$ bijections π on N, $\forall u_N \in U$, $u_N \sim u_{\pi(N)}$.

The *continuity* says that both the upper contour set and the lower contour set of social welfare ordering should be closed.

Continuity. $\forall u_N \in U$, both $\{v_N \in U \mid v_N \succcurlyeq u_N\}$ and $\{v_N \in U \mid u_N \succcurlyeq v_N\}$ are closed.

The *rank-separability*, a weaker axiom of well-known *separability*,[12] requires a social welfare ordering to ignore well-being information about the same well-being in the same ranks between two profiles. Note that, as Ebert (1988a) shows, the rank-separability implies the anonymity in general.

Rank-Separability. $\forall u_{[N]}, v_{[N]}, u'_{[N]}, v'_{[N]} \in U$, if $\exists [M] \subseteq [N]$, $(\forall i \in [M]$, $u_{[i]} = u'_{[i]}$ & $v_{[i]} = v'_{[i]})$ and $(\forall j \in [N] \setminus [M]$, $u_{[j]} = v_{[j]}$ & $u'_{[j]} = v'_{[j]})$, then $u_{[N]} \succcurlyeq v_{[N]} \Leftrightarrow u'_{[N]} \succcurlyeq v'_{[N]}$.

Finally, consider the following *Pigou-Dalton transfer equity* as an axiom of equity. Suppose there is a well-being gap between two individuals in a well-being profile. Then, the Pigou-Dalton transfer equity states that the same amount of transfer from the rich to the poor will not at least reduce social welfare whenever the other well-being is fixed in the profile.[13]

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[12] Separability simply requires a social welfare ordering to ignore well-being information about indifferent individuals between two profiles. Obviously, this axiom is a stronger requirement than rank-separability. Note that separability implies rank-separability under the assumption of anonymity.

[13] Note that the Pigou-Dalton transfer equity makes sense only if cardinal interpersonal comparisons of well-being could be admissible. In other words, for any transfer of the same level of well-being from the rich to the poor to be meaningful, interpersonal comparability of well-being must be admitted not only for the ranking of the level of well-being, but also for the differences and ratios of well-being. There are, of course, many other equity requirements besides Pigou-Dalton transfer equity that have been discussed. One of the most important axioms is the Hammond equity, which is used to characterize the leximin rule. This equity axiom can be interpreted as a demand for extreme equality, or a demand for normal equity based on the concept of ordinal interpersonal comparisons of well-being. See Sakamoto (2020) for the theoretical relationship between equity axioms and interpersonal comparability of well-being following traditional invariance requirements.

Pigou-Dalton Transfer Equity. $\forall u_N, v_N \in U$, $\forall \varepsilon \in \mathbb{R}_{++}$, if $\exists i, j \in N$, $v_i - \varepsilon = u_i \geqq u_j = v_j + \varepsilon$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succcurlyeq v_N$.

These are the major axioms used in this paper. Using surprisingly few axioms, Section 3 shows a severe constraint on the functional form of social welfare orderings under the assumption of cardinal full comparability, and Section 4 examines a class of acceptable social welfare orderings.

3. Scale Invariance and Functional Forms of Social Welfare Orderings

This section shows that the combination of scale-invariance, continuity, and the Pareto principle in standard social choice theory imposes significant constraints on a functional form of a social welfare ordering. Indeed, the social welfare ordering is shown to be a weighted utilitarianism with *variable* weights that depend on well-being profiles and must be identical for any positive affine transformations of well-being. Moreover, we explain that there is no need to interpret the requirement of interpersonal comparability of well-being as the traditional axioms of scale invariance, and these invariance axioms can be appropriately reformulated by setting a unit and a reference point of the scale.

Study on how the combination of scale invariance and the Pareto principle affects a functional form of a social welfare function was initiated by the enlightening work of Sen (1970, Ch. 7*-9*). One of the early important contributions was from Roberts (1980a, Th. 4), showing that the combination of cardinal full interpersonal comparability and the weak Pareto principle imposes a restriction on a shape of the social welfare function which is represented by a linear combination of an average utility level and a homogeneous function of degree one with variables that are an individual utility level minus the average utility level.[14] Bossert and Weymark (2004) extended this result and showed that it can be represented by a weighted utilitarianism where its weight vectors are identical for all profiles through a positive affine transformation on the profile. This result is easily extended to a class of weighted utilitarianism with variable weights by using Euler's homogeneous function theorem.

Theorem 1. *Suppose that a social welfare ordering satisfies the axioms of cardinal full comparability, the strong Pareto, and continuity. Then, there exists a continuous and strictly increasing function* φ *that represents the social welfare ordering and satisfies followings:*

(i) $\varphi(u_N) = u$ *for all uniform profiles* $u_N = (u, \ldots, u)$ *and all* $u \in \mathbb{R}$*;*

(ii) $\varphi((a+bu_1, \ldots, a+bu_n)) = a+b\varphi(u_N)$ *for all* u_N*, all* $a \in \mathbb{R}$*, and all* $b \in \mathbb{R}_{++}$;

(iii) *if* φ *is differentiable at* u_N*, then it is represented by a weighted utilitarianism*

[14] Strictly speaking, it is necessary to require continuity. In the case without continuity, the strict inequalities generated by this functional form are sufficient for asymmetric factors of the social welfare ordering satisfying scale invariance and the weak Pareto.

with a weight vector that is proportional to $(\partial\varphi(u_N)/\partial u_1, \dots, \partial\varphi(u_N)/\partial u_n)$.

【Proof】By Blackorby et al. (2005, Th. 4. 1, p. 94), there exists a continuous and increasing function φ that represents a social welfare ordering and satisfies (i).

For any profile u_N, u_N is indifferent from a uniform profile $(\varphi(u_N), \dots, \varphi(u_N))$. Then, the cardinal full comparability implies that for all $a \in \mathbb{R}$ and $b \in \mathbb{R}_{++}$, a profile $(a+b\varphi(u_N), \dots, a+b\varphi(u_N))$ is indifferent from a profile $(a+bu_1, \dots, a+bu_n)$. Since φ represents the social welfare ordering, it follows $\varphi((a+bu_1, \dots, a+bu_n)) = \varphi((a+b\varphi(u_N), \dots, a+b\varphi(u_N)))$. By (i), we have $\varphi((a+bu_1, \dots, a+bu_n)) = a+b\varphi(u_N)$.

Finally, let φ be differentiable at u_N. Since φ is homogeneous of degree one due to (ii), Euler's homogeneous function theorem implies the following equation:

$$\varphi(u_N) = \sum_{i\in N} \frac{\partial\varphi(u_N)}{\partial u_i} u_i.$$

This obviously means that φ is represented by a weighted utilitarianism with a weight vector that is proportional to $(\partial\varphi(u_N)/\partial u_1, \dots, \partial\varphi(u_N)/\partial u_n)$.■

Note that the results in Theorem 1 also hold by requiring the *weak* Pareto principle instead of the strong Pareto principle. In fact, the same result of (i) in Theorem 1 holds for an efficiency requirement that is much weaker than the weak Pareto principle.[15] It is also easy to show that the same result of (ii) in Theorem 1 holds by replacing the invariance condition with the *ratio-scale full comparability*, which is weaker than the cardinal full comparability.[16] However, to avoid the complication of defining more axioms than necessary, this section considers the results of Theorem 1 by using only the axioms at hand.

When the separability is added to the axioms of Theorem 1, the weight vector is uniquely determined and its weights depend on individuals. Indeed, under the separability assumption, the continuous function φ that represents the social welfare ordering takes a form of total sum of functions φ_i that depends on individuals (that is, φ is additive with respect to individual well-being). In this case, the requirement of

[15] See Blackorby et al. (2005, Theorem 4. 1, p. 94). They introduce two weaker axioms of efficiency: *Pareto weak preference* and *minimal increasingness*.

[16] Even in the case of ratio-scale full comparability, the constraint on the functional form of the social welfare ordering is the same in the sense that it must become the weighted utilitarianism with variable weights. For example, a social welfare ordering based on Theil's entropy measure is shown to be ordinally equivalent to the functional form $\sum_{i\in N} u_i \log \bar{u}/u_i$, where $\bar{u}$ is the mean of u_N. This is clearly equivalent to the weighted utilitarianism with variable weights.

cardinal full comparability implies that φ_i is a linear function, and thus φ becomes a quasi-utilitarianism (Maskin 1978).[17] Even if we add the rank-separability, a weaker version of the separability, the weight vector is still uniquely determined, and its weights depend only on ranks. By a similar logic as in the case of separability, under the assumption of rank-separability, the continuous function φ is represented by a total sum of rank-dependent functions $\varphi_{[i]}$. Also, cardinal full comparability implies that $\varphi_{[i]}$ is a linear function, and φ must be a rank-weighted utilitarianism (Ebert 1988a).[18]

If the convexity of a social welfare ordering is added to the axioms in Theorem 1, then only a min-of-means social welfare ordering can survive (Gilboa and Schmeidler 1989; Ben-Porath et al. 1997). This social welfare ordering is represented by a weighted utilitarianism with variable weights, which is determined by minimizing the weighted sum in a unique convex and compact subset of weight vectors.[19] Note that in Theorem 1, there is no restriction on the set of weight vectors and a weight vector is not necessarily chosen to minimize the weighted utilitarianism.

As can be seen from the above discussion, Theorem 1 simply states that any Paretian and continuous social welfare orderings can be represented by weighted utilitarianisms with variable weight vectors that depend on well-being profiles whenever they satisfy scale invariance. In fact, the following social welfare orderings, that satisfy the Pareto principle, continuity, and scale invariance but not rank-separability, can be represented by a weighted utilitarianism with variable weights depending on the well-being profiles.

Example 3 (Ebert 1988a, Prop. 4). For all profiles u_N,

$$\varphi(u_N) = 1/n \left[\textstyle\sum_{i\in N} u_i + \sum_{i\in N:\, u_i < \bar{u}} (\bar{u} - u_i)\right],$$

[17] A quasi-utilitarianism can survive even in cases where there is no room for interpersonal comparisons of well-being and only *intrapersonal* comparisons of well-being are admissible. However, without any interpersonal comparisons of well-being, quasi-utilitarian judgments are interpreted as a generalization of dictatorship or oligarchy, in which social decisions are made by giving absolute priority to a specific individual or group.

[18] As clarified by Sakamoto (2020), the generalized leximin is a generalization of positional dictatorship and can be interpreted as a generalization of social welfare orderings that are lexicographic compositions of weighted sums of individual groups following rank-orders, from the simple leximin to the mild rank-weighted utilitarianism.

[19] Note that min-of-means social welfare orderings include both the quasi-utilitarianism and the rank-dependent utilitarianism. In fact, if the set of weight vectors is a singleton, the min-of-means social welfare ordering is equivalent to the quasi-utilitarianism. If the set of weight vectors W is symmetric (i.e., for all bijections π on N, w_N in W implies $w_{\pi(N)}$ in W), then it is equal to the rank-dependent utilitarianism.

where $\bar{u}$ is a mean of u_N. Suppose that $\hat{n}(u_N)$ is the number of individuals with well-being less than $\bar{u}$ at a profile u_N. Then, this social welfare ordering is a weighted utilitarianism with variable weights that are calculated to be proportional to a vector $(\hat{n}(u_N)/n, \ldots, \hat{n}(u_N)/n, 1+\hat{n}(u_N)/n, \ldots., 1+\hat{n}(u_N)/n)$ for each profile u_N. Note that this social welfare ordering obviously violates both rank-separability and convexity.

Example 4 (Roberts 1980a). For all profiles u_N,

$$\varphi(u_N) = 1/n\left[\sum_{i\in N} u_i - \sqrt{\sum_{i\in N}(u_i - \bar{u})^2}\,\right],$$

where $\bar{u}$ is a mean of u_N. This social welfare ordering balances a total sum and a standard deviation of an income distribution, and violates the rank-separability. Unlike the social welfare ordering in Example 3, it is shown to be convex.[20] Therefore, it can be represented as a min-of-means social welfare ordering that compares minimum weighted sums of well-being with respect to the unique convex and compact set of weight vectors, as shown in Gilboa and Schmeidler (1989).

Theorem 1 shows that simply requiring cardinal full comparability, the Pareto principle, and continuity restricts the functional form of the social welfare ordering to a weighted utilitarianism with variable weights, and eliminates possibilities of the other desirable social welfare orderings. Since the ordering is invariant for any positive affine transformations, a social decision between two income distributions ($1, $5, $10) and ($2, $3, $5) must be identical to the decision between (-$90, -$50, $0) and (-$80, -$70, -$50), and the decision between ($11 million, $15 million, $20 million) and ($12 million, $13 million, $15 million).[21] However, it is an extremely strong assumption that the decision on any two profiles must always be the same on the other profiles that are obtained from the original profiles through some positive affine transformations. It is not strange that some people wonder that the ordering need not be invariant for the transformed income distributions. The reason why this anomalous result occurs is that the social welfare ordering cannot distinguish a difference/ratio of well-being caused by

[20] The convexity of this social welfare ordering can be shown by giving the midpoint of any two well-being profiles and checking that its upper contour set is a midpoint convex set. The convexity of the upper contour set can be easily shown by using Cauchy-Bunyakovski-Schwarz inequality.

[21] The former income distributions are obtained by a transformation: 10 times the original income minus 100. The latter income distributions are obtained by a transformation: 1 million times the original income plus 10 million.

a scale transformation and that of actual well-being in the same scale.

For example, letting a nominal exchange rate be \$1=¥100, it makes sense to require that a ranking on two income distributions (\$1, \$5, \$10) and (\$2, \$3, \$5) should be identical to a ranking on two income distributions (¥100, ¥500, ¥1000) and (¥200, ¥300, ¥500). On the other hand, there seems to be little ethical basis for claiming that a decision on (\$1, \$5, \$10) and (\$2, \$3, \$5) should be the same as a decision on (\$100, \$500, \$1000) and (\$200, \$300, \$500). As Morreau and Weymark (2016) points out, we need to explore a class of social welfare orderings that are distribution-sensitive with respect to *actual* differences/ratios of well-being in the same scale but are insensitive with respect to *superficial* differences/ratios of well-being caused by scale transformations.

In order to properly deal with this problem due to scale invariance, consider a new framework proposed by Morreau and Weymark (2016). Let the set of scale parameters be Θ.[22] Then, for any scale parameter θ in Θ, all standard axioms in traditional social choice theory can be easily reformulated as requirements under the fixed scale θ. For example, the reformulated version of the Pareto principle requires that, for all two well-being profiles $u_N(\theta)$ and $v_N(\theta)$ under the same scale θ, if $u_N(\theta)$ is Pareto superior to $v_N(\theta)$, then $u_N(\theta)$ must be socially better than $v_N(\theta)$. This requirement makes it possible to yield different social judgments on profiles for different scale parameters, even if they look to be identical profiles numerically. Under the assumption that individual well-being is cardinal interpersonal comparable for the same scale parameter, computation of differences and ratios among individual well-being is meaningful, and various functional forms of social welfare ordering become possible, such as concave functions. To keep our notations and definitions simple, all axioms are reinterpreted as being imposed only on well-being profiles with the same scale parameter. This new interpretation allows us to find acceptable social welfare orderings that could take various functional forms other than the weighted utilitarianism.[23] The next section investigates a broad class of equitable and efficient social welfare orderings that is characterized by a reasonable system of *scale-dependent* axioms (anonymity, the Pareto

[22] A typical and useful example of scale parameters that may be helpful in economics would be a combination of a monetary unit and its origin. For example, the scale parameter of Japanese yen can be defined by setting one unit as 1 yen and a reference point as 0 yen in Japanese yen. As shown in the functional forms of Examples 1 and 2, if the scale transformation of well-being is properly defined, we can obtain a nonlinear functional form that is invariant to scale transformations and is distribution-sensitive.

[23] This reinterpretation is mathematically equivalent to as *numerical full comparability* (Blackorby et al. 2005, p. 114). Therefore, it is easy to predict that various functional forms can represent a class of acceptable social welfare orderings.

principle, Pigou-Dalton transfer equity, continuity, and rank-separability).

4. Axiomatic Characterizations of a Class of Rank-dependent Generalized Utilitarianisms

This section investigates a class of rank-dependent generalized utilitarianisms and characterizes its special forms, a rank-weighted generalized utilitarianism and a rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering. Moreover, we show that three leading social welfare orderings in the theory of distributive justice, the Pareto egalitarianisms (Tungodden 2000; Tungodden and Vallentyne 2005), prioritarianisms (Parfit 1991), and sufficientarianisms (Crisp 2003; Bossert et al. 2020), can be approximately represented by a rank-weighted generalized utilitarianism. As a result, the idea of social welfare that has been explored by the theory of distributive justice can be reduced to the simple problem about how much the society should show compassion for relative inequality and absolute poverty in the rank-weighted generalized utilitarianism. Finally, we introduce a stepwise form of the rank-weighted generalized utilitarianism satisfying the axiom of replication equivalence for a setting with variable populations.

First, let us define a rank-dependent generalized utilitarianism, and we then characterize it by a system of standard axioms in social choice theory. This social welfare ordering is merely a rank-dependent version of generalized utilitarianism. In fact, Ebert (1988a) characterizes it by using only simple standard axioms: anonymity, the strong Pareto principle, continuity, and rank-separability. Note that the only difference from the characterization result of a generalized utilitarianism is that separability is replaced by rank-separability in the system of axioms.

Definition. A social welfare ordering $\succcurlyeq^{RDGU}$ is a *rank-dependent generalized utilitarianism* if and only if for some continuous and strictly increasing function $g_{[i]}$, for all profiles u_N and v_N,

$$u_N \succcurlyeq^{RDGU} v_N \leftrightarrow \sum_{i\in N} g_{[i]}(u_{[i]}) \geq \sum_{i\in N} g_{[i]}(v_{[i]}).$$

This can include a various class of acceptable social welfare orderings. In fact, when $g_{[i]}(u_{[i]}) = f(u_{[i]}) - f(\tilde{u})$ holds (where $\tilde{u}$ is a given threshold and f is a given concave function), this ordering is a *critical-level generalized utilitarianism* (Blackorby and Donaldson 1984). Also, when $g_{[i]}(u_{[i]}) = \beta^{[i]}[f(u_{[i]}) - f(\tilde{u})]$ holds (where $\tilde{u}$ is a given threshold, f is a given concave function, and β is a given discount factor), it is a *rank-discounted critical-level generalized utilitarianism* (Zuber and Asheim 2012;

Asheim and Zuber 2014). Furthermore, this ordering includes a social welfare ordering that is distribution-sensitive when well-being level is sufficiently low, and is less distribution-sensitive when well-being level is fully high. For example, consider the following social welfare ordering.

Example 5. For some continuous, strictly increasing, concave, and semi-differentiable function g, for some weight vector $w_{[N]}$ with $w_{[1]} \geq \cdots \geq w_{[n]}$, for all profiles u_N and v_N,

$$u_N \succcurlyeq v_N \leftrightarrow \sum_{i \in N}(w_{[i]}\partial_{-}g(u_{[i]}) + g(u_{[i]})) \geq \sum_{i \in N}(w_{[i]}\partial_{-}g(v_{[i]}) + g(v_{[i]})),$$

where for all real numbers u, $\partial_{-}g(u)$ is a left derivative at u.

Since g is strictly increasing and concave in Example 5, a left derivative $\partial_{-}g(u)$ is always greater than zero and a decreasing function. This means the above function becomes less distributive-sensitive as the value of well-being increases. Thus, a rank-dependent generalized utilitarianism has an obvious advantage of being able to adjust its consideration for relative inequality according to well-being profiles.[24]

In the following theorems, we will characterize a class of acceptable social welfare orderings by the standard axioms in social choice theory, but as pointed out in the last paragraph of the previous section, we interpret them as scale-dependent axioms that only apply in the same scale.

Theorem 2. *A social welfare ordering satisfies the scale-dependent axioms of anonymity, the strong Pareto, continuity, and rank-separability if and only if it is a rank-dependent generalized utilitarianism. Moreover, Pigou-Dalton transfer equity is added to the axioms, then the functional form* $g_{[i]}$ *in the rank-dependent generalized utilitarianism satisfies the following inequality: For all u, u' with u > u', for any sufficiently small number* $\varepsilon > 0$, *for all ranks* $[i]$ *and* $[j]$ *with* $[i] > [j]$, $g_{[j]}(u' + \varepsilon) - g_{[j]}(u') \geq g_{[i]}(u + \varepsilon) - g_{[i]}(u)$.

[24] While a social welfare ordering that satisfies separability is easy to calculate, it has an obvious defect of ignoring all aspects of relative inequalities. As a result, it only shows compassion for absolute poverty levels. In contrast, a social welfare ordering that satisfies rank-separability can show compassion for both relative inequality and absolute poverty levels. If the society does not impose rank-separability on a social welfare ordering, then a class of acceptable social welfare orderings can consider overall deviances within well-being profiles, as shown in Example 4.

【Proof】By the characterization result in Ebert (1988a, Theorem 1), a social welfare ordering satisfies the axioms of anonymity, strong Pareto principle, continuity, and rank-separability if and only if it is a rank-dependent generalized utilitarianism.

Suppose now that the rank-dependent utilitarianism satisfies the Pigou-Dalton transfer equity. Consider two profile as follows: For all u, u' with $u > u'$, for any sufficiently small number $\varepsilon > 0$, for all ranks $[i]$ and $[j]$ with $[i] > [j]$

$$u_N = \left(u', \dots, \underbrace{u'}_{j\text{th rank}}, u, \dots, u, \underbrace{u+\varepsilon}_{i\text{th rank}}, \dots, u+\varepsilon \right),$$

$$v_N = \left(u', \dots, u', \underbrace{u'+\varepsilon}_{j\text{th rank}}, u, \dots, u, \underbrace{u}_{i\text{th rank}}, u+\varepsilon, \dots, u+\varepsilon \right).$$

By definitions of two profiles, v_N is obtained through a transfer from the ith rank individual to the jth rank individual at u_N. Since this social welfare ordering is represented by the rank-dependent generalized utilitarianism, the Pigou-Dalton transfer equity implies $\sum_{i\in N} g_{[i]}(v_{[i]}) \geq \sum_{i\in N} g_{[i]}(u_{[i]})$. Hence, it follows $g_{[j]}(u'+\varepsilon) - g_{[j]}(u') \geq g_{[i]}(u+\varepsilon) - g_{[i]}(u)$.■

The axiomatic characterization of rank-dependent generalized utilitarianism is a well-known and mathematically simple result (Ebert 1988a). Note, however, that the fact that imposing the Pigou-Dalton transfer equity on it greatly constrains the functional form has received little attention in social choice theory. If the rank-dependent function $g_{[i]}$ is differentiable, then requiring the Pigou-Dalton transfer equity implies $dg_{[j]}/du \geq dg_{[i]}/du$ for all $[i]$ and $[j]$ with $[i] < [j]$.[25] This is a major constraint on the curvature of the function. In fact, as we will show later in this section, the combination of the homotheticity and Pigou-Dalton transfer equity greatly restricts a class of acceptable social welfare orderings, yielding only the class of simple rank-weighted utilitarianisms.

As is well-known, if the separability is required to the axioms instead of rank-separability in Theorem 3, a social welfare ordering becomes a generalized utilitarianism (Blackorby et al. 2005, Th. 4. 7., p. 116). Although both the generalized utilitarianism and the rank-dependent generalized utilitarianism are easily characterized by these plausible and very simple axioms, their implications for welfare economics

[25] Note that the continuous function φ defined in Theorem 1 has the same property if it satisfies the Pigou-Dalton transfer equity.

seem to be significant. That is, if the society can agree that some comparisons of living standards among individuals are possible, then social welfare can be simply represented as a total sum of transformed individual well-being, showing compassion for relative inequality and poverty.[26] This finding is particularly helpful and practical in the context of assessing income or wealth distributions, and for evaluating QALYs (quality-adjusted life years) distributions for health economic evaluation. When evaluating the effects of public policies on social welfare, all we need is to specify the functional form $g_{[i]}$ following social compassion for relative inequality and those who are below the poverty line. Similarly, in the context of health economic evaluation, all we need is to specify the functional form $g_{[i]}$ following social norms for health inequality and those who are unhealthy or disabled.[27]

Let us now define and characterize a rank-weighted generalized utilitarianism, which is a special form of the rank-dependent generalized utilitarianism. This social welfare ordering can be characterized as a special form of the rank-dependent generalized utilitarianism in which its elasticity is constant with respect to well-being rank.

Definition. A social welfare ordering $\succcurlyeq^{RWGU}$ is a *rank-weighted generalized utilitarianism* if and only if for some continuous and strictly increasing function g, for some weight vector $w_{[N]}$ with $w_{[1]} \geq \cdots \geq w_{[n]}$, for all profiles u_N and v_N,

$$u_N \succcurlyeq^{RWGU} v_N \leftrightarrow \sum_{i \in N} w_{[i]}\, g(u_{[i]}) \geq \sum_{i \in N} w_{[i]}\, g(v_{[i]}).$$

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[26] It does not seem so strange to agree that even if exact comparisons of living standards are not possible, fuzzy but significant comparisons are possible. The claim that the society cannot compare the well-being of billionaires and the homeless according to the ascetic premise of ordinal interpersonal noncomparability of utilities may be true as a claim in the precision of science, but it is plainly harmful in the attempt of welfare economics to become an instrument for improving human society. Even if complete and rigorous comparisons of well-being are impossible, using surrogate indices such as income or asset levels for comparing standards of living, or developing a soft theory that allows for mathematical ambiguity, would provide a far more useful analysis of social welfare. See also Pigou's response to Lionel Robbins' criticism (Pigou 1920), Sen's analysis on interpersonal comparisons of utility (Sen 1970, Ch. 7*-9*), and Balinski and Laraki's insights on grade variables (Balinski and Laraki 2010).

[27] For an application of social welfare functions to the problem of health economic evaluation, see Cookson et al. (2021). In the literature, a rank-weighted utilitarianism or a generalized utilitarianism has been used in applied analysis, but a rank-dependent generalized utilitarianism is recommended to be used in the future. Note that depending on the shape of a function $g_{[i]}$, a calibration dilemma may arise (Nebel and Stefansson 2020).

The essence of this functional form is that, for all well-being levels u, the following relation holds:

$$g_{[1]}(u) : g_{[2]}(u) : \ldots : g_{[n]}(u) = w_{[1]} : w_{[2]} : \ldots : w_{[n]}.$$

In other words, for all ranks i and all well-being levels u, a marginal increment in the welfare level u ($= dg_{[i]}/du$) is proportional to the constant $w_{[i]}$, and the elasticity of the function $g_{[i]}$ ($= dg_{[i]}/du \times u/g_{[i]}(u)$) has the same value regardless of the rank. Therefore, if the above property is added to the axioms in Theorem 3, the rank-dependent generalized utilitarianism is equivalent to the rank-weighted generalized utilitarianism.

Theorem 3. *Suppose that a social welfare ordering satisfies the scale-dependent axioms of anonymity, the strong Pareto, continuity, rank-separability, and Pigou-Dalton transfer equity. Then, this social welfare ordering satisfies the above equation (1) if and only if it is a rank-weighted generalized utilitarianism.*

【Proof】

Rank-weighted generalized utilitarianism clearly satisfies these properties. Therefore, let us show only the sufficiency part.

Since a social welfare ordering satisfies anonymity, the strong Pareto principle, continuity, and rank separability, it is rank-dependent generalized utilitarianism by Theorem 2. Then, the formula (1) implies that there exists a function g such that: For all real numbers $u \in \mathbb{R}$,

$$g(u) = \frac{g_{[1]}(u)}{w_{[1]}} = \frac{g_{[2]}(u)}{w_{[2]}} = \cdots = \frac{g_{[n]}(u)}{w_{[n]}}.$$

This function is obviously well-defined, continuous, strictly increasing, and concave because of the properties of rank-dependent functions $g_{[i]}$. Thus, each rank-dependent function $g_{[i]}(u)$ can be denoted by $w_{[i]}\, g(u)$. This means that the rank-dependent generalized utilitarianism must be a rank-weighted generalized utilitarianism. ■

This simple social welfare ordering has more interesting properties than is apparent. As the following examples show, the three well-known social welfare orderings, the Paretian egalitarianism, prioritarianism, and sufficientarianism, which are proposed as acceptable candidates for social decisions considering the concept of distributive justice, are subclasses of rank-weighted generalized utilitarianisms in an *approximate way*.[28]

Example 6. A social welfare ordering $\succcurlyeq^{PE}$ is the *Paretian egalitarianism* if and only if for all profiles u_N and v_N,

$$u_{[1]} > v_{[1]} \rightarrow u_N \succ^{PE} v_N.$$

Suppose now that the highest weight $w_{[1]}$ is given by $(1\text{-}10^{-23}) \approx 1$ in the rank-weighted generalized utilitarianism. In this case, it seems reasonable to say that the rank-weighted generalized utilitarianism is approximately equal to the maximin ordering.[29]

As is clear from the definition, the Pareto egalitarianism is a *refinement* of the maximin social welfare ordering.[30] Tungodden (2000) shows that a social welfare ordering must be a refinement of the maximin whenever it satisfies the weak Pareto principle and the requirement of *perfect equality*.[31]

Example 7. A social welfare ordering $\succcurlyeq^{P}$ is the *prioritarianism* if and only if for some concave function g, for all profiles u_N and v_N,

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[28] The words "the *approximate way*" mean that the rank-weighted generalized utilitarianism can mimic the three social welfare orderings for any finite set of individual well-beings by properly adjusting the functional form and the weight vector. For example, although the leximin cannot be represented by a continuous real-valued function, the rank-weighted utilitarianism could be equivalent to the leximin by adjusting the weight vector for the finite set of well-beings. To define a class of social welfare orderings that include the Pareto egalitarianism, prioritarianism, and sufficientarianism in a strict sense, we must extend a class of rank-weighted generalized utilitarianisms in a somewhat more complicated way, just like the definition of the generalized leximin in Sakamoto (2020).

[29] Note that the rank-weighted utilitarianism can approximately include not only the maximin ordering but also the leximin ordering. For example, let us define a weight $w_{[i]}$ as $999*10^{-3[i]}$ for all ranks $[i]$ (=1, ..., n-1) and let $w_{[n]}$ be $10^{-3[i-1]}$. Since the weight vector gives an overwhelming weight for any higher rank, an ordering based on this weighted sum can be similar to the leximin ordering. In an analogous way, even the generalized leximin can also be approximately represented by the rank-weighted utilitarianism.

[30] A binary relation R is a refinement of a binary relation R', if the asymmetric part of R' is a subset of the asymmetric part of R.

[31] Perfect equality is defined as follows: for all two well-being profiles, if these profiles are Pareto incomparable, the one is a perfect equal well-being profile, and another is an unequal well-being profile, then the former should be strictly better than the latter.

$$u_N \succcurlyeq^P v_N \leftrightarrow \sum_{i\in N} g(u_i) \geq \sum_{i\in N} g(v_i).$$

As is clear from the definition, the prioritarianism is the same one as commonly called the generalized utilitarianism. Hence, if all rank-dependent weights are equal in the rank-weighted generalized utilitarianism, then it is equivalent to the prioritarianism.

Example 8. A social welfare ordering $\succcurlyeq^S$is the *sufficientarianism* if and only if, for some concave function *f*, and for all u_N and v_N,

$$u_N \succcurlyeq^S v_N \leftrightarrow \sum_{i\in N:\, u_i\leq\tilde{u}}[f(u_i)-f(\tilde{u})] > \sum_{i\in N:\, v_i\leq\tilde{u}}[f(v_i)-f(\tilde{u})]) \text{ or } \Big[\sum_{i\in N:\, u_i\leq\tilde{u}}[f(u_i)-f(\tilde{u})] = \sum_{i\in N:\, v_i\leq\tilde{u}}[f(v_i)-f(\tilde{u})] \;\&\; \sum_{i\in N:\, u_i>\tilde{u}}[f(u_i)-f(\tilde{u})] \geq \sum_{i\in N:\, v_i>\tilde{u}}[f(v_i)-f(\tilde{u})]\Big].$$

This formula of the sufficientarianism is defined by Bossert et al. (2020)[32], and it is equivalent to the definition of Adler (2019).[33] The sufficientarianism gives absolute priority to the poor who is at a well-being level below a poverty line. The rank-weighted generalized utilitarianism can be shown to be approximately equivalent to the sufficientarianism if each weight is equal among individuals and the functional form *g* is defined as follows[34]: for some concave function *f*, for some threshold $\tilde{u}$, and for all well-being profiles u_N,

$$g(u_i) = \begin{cases} 10^{1729}[f(u_i)-f(\tilde{u})], & if\ u_i \leq \tilde{u}, \\ 10^{-6174}[f(u_i)-f(\tilde{u})], & otherwise. \end{cases}$$

As shown in Examples 6-8 above, the Pareto egalitarianism, prioritarianism, and sufficientarianism are all just *approximate special cases* of the rank-weighted generalized utilitarianism. In this sense, the theory of distributive justice can seem to be

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32 Bossert et al. (2020) call this sufficientarianism a *critical-level sufficientarian social welfare ordering*.

33 The Adler's definition of sufficientarianism is as follows: there exist a concave function *f* and a threshold $\tilde{u}$ such that for all u_N and v_N, $u_N \succ^S v_N$ if and only if

(1) $\sum_{i\in N} f(\min\{u_i,\tilde{u}\}) > \sum_{i\in N} f(\min\{v_i,\tilde{u}\})$, or
(2) $\sum_{i\in N} f(\min\{u_i,\tilde{u}\}) = \sum_{i\in N} f(\min\{v_i,\tilde{u}\})$ & $\sum_{i\in N} f(\max\{u_i,\tilde{u}\}) > \sum_{i\in N} f(\max\{v_i,\tilde{u}\})$.

This definition is easily shown to be equivalent to that of Bossert et al. (2020).

34 Note that the functional form of the sufficientarianism has the similar property as Sen's *poverty focus*. If there are individuals who have a level of well-being below the poverty line, the sufficientarianism gives a lexicographic priority to those individuals. By appropriately defining the functional form g, the rank-weighted generalized utilitarianism could show compassion for only the poor, just like the sufficientarianism.

a simple problem about how to deal with considerations of relative inequality and poverty among various individuals. These results are summarized as the following theorem.

Theorem 4. *Let a range of individual well-being be the finite set. Then, for some weight vectors w and for some function g, one of the following three social welfare orderings: the Paretian egalitarianism, prioritarianism, and sufficientarianism, can be a special case of a rank-weighted generalized utilitarianism.*

Next, let us define the rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering (Blackorby and Donaldson 1982; Ebert 1988b), a subclass of the rank-weighted generalized utilitarianisms.

Definition. A social welfare ordering $\succcurlyeq^{RWABD}$ is a *rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering* if and only if for some parameters $\alpha, \gamma \in \mathbb{R}_{++}$, for some weight vector $(w_{[1]}, \dots, w_{[n]})$, for all profiles u_N and v_N,

$$u_N \succcurlyeq^{RWABD} v_N \leftrightarrow \sum_{i \in N:\, u_{[i]} \geq 0} w_{[i]}\, u_{[i]}^{\alpha} - \gamma \sum_{i \in N:\, u_{[i]} < 0} w_{[i]}\, (-u_{[i]})^{\alpha} \geq \sum_{i \in N:\, v_{[i]} \geq 0} w_{[i]}\, v_{[i]}^{\alpha} - \gamma \sum_{i \in N:\, v_{[i]} < 0} w_{[i]}\, (-v_{[i]})^{\alpha}.$$ [35]

It is easily shown that this social welfare ordering satisfies homotheticity, commonly known as a mathematically tractable property.

Definition. A social welfare ordering satisfies *homotheticity* if and only if $\forall u_N, v_N \in U, \forall \lambda \in \mathbb{R}_{++}, u_N \succcurlyeq v_N \leftrightarrow \lambda u_N \succcurlyeq \lambda v_N$.

Since the *ratio-scale full comparability* is mathematically equivalent to homotheticity, the following theorem follows immediately from a combination of well-known results of Blackorby and Donaldson (1982) and Ebert (1988b).[36]

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[35] This is the definition of the rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering on $\mathbb{R}^n$. Note that it becomes the rank-weighted *Nash* social welfare ordering when the domain is $\mathbb{R}_+^n$ and the parameter α goes to zero.

[36] Ebert (1988b) only proves the characterization theorem of rank-weighted *Atkinson*

Theorem 5. *A social welfare ordering satisfies the scale-dependent axioms of anonymity, the strong Pareto, continuity, rank-separability, and homotheticity if and only if it is a rank-weighted Atkinson-Blackorby-Donaldson social welfare ordering.*

At first glance, the rank-weighted Atkinson-Blackorby-Donaldson SWO seems to have quite good properties, but from the perspective of Pigou-Dalton transfer equity, it turns out it has a problem. In general, when the domain (the set of feasible well-being profiles) is a set of n-dimensional *positive* real numbers, α is interpreted as a parameter indicating inequality aversion. However, if the domain is a set of n-dimensional real numbers, the interpretation of this parameter invokes a conundrum. Indeed, the rank-weighted Atkinson-Blackorby-Donaldson SWO satisfies Pigou-Dalton transfer equity only if $\alpha = 1$ and $\gamma \geqq 1$ (*i.e.*, it must be a kind of simple rank-weighted utilitarianism).[37]

Given the following profiles, we can check that for any α except 1, the rank-weighted Atkinson-Blackorby-Donaldson SWO violates Pigou-Dalton transfer equity. Suppose now that α is in (0, 1) and $\gamma \geqq 1$ and the rank-weighted Atkinson-Blackorby-Donaldson SWO satisfies the Pigou-Dalton transfer equity. Consider two profiles as follows: For any sufficiently large negative number $u < 0$, for any sufficiently small number $\varepsilon > 0$, for all ranks $[i]$ and $[j]$ with $[i] > [j]$,

$$u_N = \left(u, \dots, \underbrace{u}_{j\text{th rank}}, 0, \dots, 0, \underbrace{1+\varepsilon}_{i\text{th rank}}, \dots, 1+\varepsilon \right),$$

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social welfare ordering that is defined on the sets of n-tuple *non-negative* real numbers. However, as Ebert (1988b) suggests, his result can be easily extended to an axiomatization of the rank-weighted *Atkinson-Blackorby-Donaldson* SWO with the domain of n-tuple real numbers by using the result in Blackorby and Donaldson (1982).

[37] Although the functional form of the rank-weighted Atkinson-Blackorby-Donaldson SWO has a major problem in defining social welfare, it may be an ideal way to represent individual preferences according to *prospect theory*. Let us now interpret the weights w and 0 as *subjective probability* and *reference point*, respectively. In this case, the functional form of the rank-weighted Atkinson-Blackorby-Donaldson SWO will be risk-averse in the domain of positive numbers and risk-loving in the domain of negative numbers. In this sense, this SWO may not be interesting in the context of social welfare, but it may be worth studying in the context of decision theory.

$$v_N = \left(u, \ldots, u, \underbrace{u+\varepsilon}_{j\text{th rank}}, 0, \ldots, 0, \underbrace{1}_{i\text{th rank}}, 1+\varepsilon, \ldots, 1+\varepsilon \right).$$

By definitions of two profiles, v_N can be obtained through a transfer from the ith rank individual to the jth rank individual at u_N. Since this social welfare ordering is represented by the rank-weighted Atkinson-Blackorby-Donaldson SWO and the Pigou-Dalton transfer equity implies $v_N \succcurlyeq u_N$, we have $w_{[i]}1^{\alpha} - \gamma w_{[j]}(-(u+\varepsilon))^{\alpha} \geq w_{[i]}(1+\varepsilon)^{\alpha} - \gamma w_{[j]}(-u)^{\alpha}$. If ε converges to zero, the equation implies $\alpha\gamma w_{[j]}(-u)^{\alpha-1} \geq \alpha w_{[i]}1^{\alpha-1}$.[38] This is equivalent to $\gamma w_{[j]}/w_{[i]} \geq (-u)^{1-\alpha}$. Since $\gamma w_{[j]}/w_{[i]}$ is constant, we can find a sufficiently large negative number u such that $\gamma w_{[j]}/w_{[i]} < (-u)^{1-\alpha}$. However, this obviously contradicts the Pigou-Dalton transfer principle. Note that even in cases where α is greater than 1 or γ is less than 1, it is easy to find profiles in which the rank-weighted Atkinson-Blackorby-Donaldson SWO violates the Pigou-Dalton transfer principle.

As illustrated in the above example, the only rank-weighted Atkinson-Blackorby-Donaldson SWO that satisfies the Pigou-Dalton transfer equity is a kind of rank-weighted utilitarianism, and this result can be summarized as the following new characterization theorem of the rank-weighted utilitarianism.

Theorem 6. *A social welfare ordering satisfies the scale-dependent axioms of anonymity, the strong Pareto, Pigou-Dalton transfer equity, continuity, rank-separability, and homotheticity if and only if it is a kind of rank-weighted utilitarianisms $\succcurlyeq^{RWU}$ as follows: For some parameter $\gamma \geq 1$, for some weight vector $w_{[N]}$ with $w_{[1]} \geq \cdots \geq w_{[n]}$, for all profiles u_N and v_N,*

$$u_N \succcurlyeq^{RWU} v_N \leftrightarrow \sum_{i\in N:\, u_{[i]}\geq 0} w_{[i]}u_{[i]} + \gamma \sum_{i\in N:\, u_{[i]}<0} w_{[i]}\, u_{[i]} \geq \sum_{i\in N:\, u_{[i]}\geq 0} w_{[i]}u_{[i]} + \gamma \sum_{i\in N:\, u_{[i]}<0} w_{[i]}\, u_{[i]}.$$

The homotheticity might be excellent in terms of mathematical tractability, but has great difficulties in terms of distributive justice. In fact, homotheticity makes it impossible to make flexible equity judgments based on levels of income impossible by requiring that a ranking of any two income distributions be invariant for any positive ratio-scale transformation. For example, consider a social decision problem between

[38] Note that, from the result of Theorem 2, it is possible to directly compare the first-order derivatives of the rank-dependent functions between two profiles.

two annual income distributions x_N = (\$10,000, \$50,000, \$150,000) and y_N = (\$20,000, \$30,000, \$50,000). Since everyone greatly sympathizes with the fact that an individual 1 is so poor to survive at x_N and the poverty is improved at y_N, it looks natural that the society strictly prefers y_N to x_N. However, what about the decision between the original distributions multiplied by 10, w_N = (\$100,000, \$500,000, \$1,500,000) and z_N = (\$200,000, \$300,000, \$500,000). If everyone enjoys a materially affluent life at both w_N and z_N, it would not be strange for the society to judge w_N to be better than z_N due to its large total income. The combination of the homotheticity and Pigou-Dalton transfer equity yields only simple rank-weighted utilitarianisms in a class of rank-dependent generalized utilitarianisms, eliminating even the possibility of weighted utilitarianisms with variable weights in Theorem 1. In this sense, the combination of these properties is a much stronger requirement than we might imagine.[39]

Finally, let us consider the problem of social choice with variable populations. Sakamoto and Mori (2020) show that a class of equitable and efficient SWOs that satisfy the replication equivalence can be represented by the stepwise form. A similar result can be obtained in this paper.[40] In fact, if we require the replication equivalence to the axioms of Theorem 3, a stepwise form of the rank-weighted generalized utilitarianism can immediately be obtained from the result of Sakamoto and Mori (2020). In order to define and characterize the stepwise form of social welfare orderings, let us introduce a step function based on a generalized utilitarianism and a weight function.

Definition. Given a continuous, strictly increasing, and concave function g, a function $g(u)_{[N]}: [0, 1] \to \mathbb{R}$ is called a rank-dependent generalized step function on u_N if and only if for all u_N, for all t in [0, 1], $u_{[N]}(t) = g(u_{[i]})$ whenever t in $[\,[i-1]/n, [i]/n\,)$.

Definition: A weight function w be defined on the closed interval [0, 1], where $w(t) \geqq$

[39] Separability and rank-separability are also mathematically tractable axioms. Note that both axioms impose severe restrictions on the functional form of social welfare orderings because they require the functional form to be additive with respect to each individual or rank. Without the rank-separability, we can obtain a flexible class of homothetic and equitable SWOs, as Example 4 illustrates.

[40] The stepwise forms of social welfare orderings allow us to provide more interesting comparisons of income distributions across different countries than the traditional GDP per capita method. In fact, Sakamoto and Mori (2020) propose the *k-quantile mean comparison method*, which compares each average income of each quintile from the bottom to the top. This method can be easily applied to the stepwise form of the rank-weighted generalized utilitarianism.

$w(t') \geqq 0$ for all t, t' with $t < t'$ and $\int_0^1 w(t)dt = 1$.

Using the above definitions, let us now define a stepwise rank-weighted generalized utilitarianism as follows.

Definition. A social welfare ordering $\succcurlyeq^{SRWGU}$ is a *stepwise rank-weighted generalized utilitarianism* if and only if for some continuous, strictly increasing, and concave function g, for some weight function w, for all profiles u_N and v_M,

$$u_N \succcurlyeq^{SRWGU} v_M \leftrightarrow \int_0^1 w(t)g(u)_{[N]}(t)dt \geqq \int_0^1 w(t)g(v)_{[M]}(t)dt.$$

The stepwise form of a SWO is characterized by the following *replication equivalence*, in addition to other plausible axioms. Replication equivalence requires that social welfare does not change for any k-replica of well-being profile because the social state replicated k times and the original social state have the same proportion of individuals with the same well-being level. To define replication equivalence, let $k * u_N$ be a k-replica of well-being profile u_N (i.e., $k * u_N = (\underbrace{u_N, \dots, u_N}_{k\ times})$).

Replication Equivalence: $\forall k \in \mathbb{N}, \forall u_N \in U, u_N \sim k * u_N$.

Combining replication equivalence and the axioms characterizing rank-weighted generalized utilitarianism, we can show necessary and sufficient conditions for the stepwise rank-weighted generalized utilitarianism by using the proof of Theorem 1 in Sakamoto and Mori (2020).

Theorem 7. *For all natural numbers n, suppose that a social welfare ordering satisfies the scale-dependent axioms of anonymity, strong Pareto, continuity, rank-separability, Pigou-Dalton transfer equity, and replication equivalence. Also, let it satisfy the property (1) in Theorem 3 for all n. Then, it must be a stepwise rank-weighted generalized utilitarianism. Moreover, the stepwise rank-weighted generalized utilitarianism satisfies all axioms the above.*

Whether stepwise forms of social welfare orderings can be regarded as acceptable solutions (commonly used name, *Parfit's theory X*) to social choice problems with variable populations needs further discussion.[41] As is clear from the axioms of the characterization theorem, this social welfare ordering has several advantages, while it cannot help facing what we call the *sadistic conclusion*.[42] The author does not consider the sadistic conclusion to be so pessimistic, and thinks that social welfare reflects merely relative desirability of social situations and there is no major problem with the stepwise forms of social welfare orderings. Another notable point is that there seems to be good reasons why the society should pay attention to non-welfarist information as well as welfarist information for social choice problems with variable populations.[43] Of course, different schools have different positions and they might define and analyze social welfare in various terms of desirability.[44] In this sense, what all theorists can

[41] The stepwise form of *number-dampened* generalized utilitarianism can be obtained if we require the replication invariance instead of replication equivalence, and strengthen the continuity to the extended continuity (Sakamoto and Mori 2020). Although the basic properties of efficiency and equity remain the same, the number-dampened version is positively responsive to population growth.

[42] Following Blackorby et al. (2005, p. 163), *the avoidance of the sadistic conclusion* is defined as follows:

$$\forall u_N, \forall v_M \in \cup_{N \subseteq \mathbb{N}} \mathbb{R}^N_{++}, \forall s_L \in \cup_{N \subseteq \mathbb{N}} \mathbb{R}^N_{--}, (u_N, v_M) \succcurlyeq (u_N, s_L).$$

That is, the sadistic conclusion means that the addition of a few individuals with negative utility is better than the addition of many individuals with positive utility. See Asheim and Zuber (2014) for various difficulties on the related issues.

[43] For example, consider the following two profiles that are numerically identical. The first one is obtained by terminating all the needy people with low utilities and leaving only those with high utilities. The second one is obtained by achieving the same utilities as the first one through sound economic development without killing anyone. These profiles are indifferent under the axioms of anonymity and the Pareto principle. However, it would be madness, no matter how modestly said, to claim that these profiles are equal. Here-in lies the apparent reason why non-welfarist information should be considered in the concept of social welfare. Furthermore, the author strongly believes that the non-welfarist information could play a main role, even in social choice problems with the same populations. Of course, when comparing well-being profiles without any unacceptable violations of human rights, there would be no need for non-welfarist judgments and there would be no problem with following the standard welfarist approach.

[44] See Sakamoto and Mori (2020) for the relationship between the stepwise forms of SWOs, the repugnant conclusion, and the sadistic conclusion. In general, the sadistic conclusion seems to invoke other ethical issues, such as how to set the threshold and how to deal with the slippery slope problem about the threshold. Moreover, since a class of equitable social welfare orderings is shown to face a trade-off between the repugnant conclusion and sadistic conclusion (Arrhenius 2000), the author thinks it is more

agree on is that social choice theory with variable populations should be said to still be a controversial field.

important to avoid the repugnant conclusion.

5. Concluding Remarks

This study shows that the combination of the strong Pareto principle and continuity under the traditional assumption of cardinal full comparability significantly restricts the functional form of the social welfare ordering and analyzes theoretical implications of appropriate reformulations of the invariance condition. Under the appropriate interpretation of scale invariance and interpersonal comparability of well-being, a plausible system of axioms needs to be reformulated in the scale-dependent manner. As a result, there is a broad class of social welfare orderings satisfying the axioms of scale-dependent efficiency and scale-dependent equity, and we can obtain a rank-dependent generalized utilitarianism as one of the suitable candidates for acceptable social welfare orderings. Furthermore, it is shown that a rank-weighted generalized utilitarianism, a special form of the rank-dependent generalized utilitarianism, includes three social welfare orderings of distributive justice that are well known in ethics (the Pareto egalitarianism, prioritarianism, and sufficientarianism). This suggests that the concept of social welfare which has been examined from various positions for a long time is related to a simple problem of *tolerable levels of relative inequality and poverty*. In the case of variable populations, we can illustrate that the rank-weighted generalized utilitarianism is easily represented by a stepwise form of social welfare ordering in Sakamoto and Mori (2020). Although this study analyzes a class of acceptable social welfare orderings under the strong assumption of cardinal interpersonal comparability of well-being, similar possibility results can be easily obtained under some mild requirements of interpersonal comparability of well-being. The dividing line between whether or not an acceptable social welfare ordering can be defined seems to lie in the degree of how far to admit interpersonal comparisons of well-being.[45] This study is only a small contribution toward the goal of constructing a practical and acceptable social welfare ordering, and further investigation is needed.

The remaining issues for further research can be clarified as follows.

First, the results of this study can be applied to various contexts of social

[45] With a lack of any interpersonal comparisons of well-being, social welfare orderings that satisfy the plausible axioms are either a dictatorship that exclusively reflects one individual preference relation (Arrow 1951; 1963) or a quasi-utilitarianism that prioritizes some preferences of certain groups of individuals over those of others (Maskin 1978). Note that the quasi-utilitarianism is obtained by admitting the cardinal *intrapersonal* comparison of well-being and by requiring continuity (Pivato 2015). See Sakamoto (2020) for theoretical relationships between the traditional invariance conditions and acceptable social welfare orderings that satisfy plausible axioms.

choice theory. For example, its application to the social choice problem under risky or uncertain situations would not be too difficult. In fact, we can easily define both the ex-ante and ex-post types of social welfare ordering.[46] Of course, it is inarguable that we still need more insights and investigations into the problem of risk and uncertainty. On the other hand, intergenerational equity issues may be difficult to extend our results to apply them because of various impossibilities such as future predictability and computability of acceptable social welfare orderings.

Second, our results require the continuity of a social welfare ordering for analytical simplification.[47] Without the continuity condition, a class of acceptable social welfare orderings will be similar to the generalized leximin in Sakamoto (2020). However, the mathematical notation such as the functional form $g_{[i]}$ will become unnecessarily complicated in the definition of the generalized leximin version of rank-dependent generalized utilitarianisms. Hence, this paper focuses on the result with the requirement of continuity, which makes the analysis clear.

Third, the question of how to apply scales and how much ambiguity to allow in the interpersonal comparability of well-being will be an important issue in practice. According to Stevens' classical classification (Stevens 1946), there are several types of scales: nominal, ordinal, interval, and ratio. The ratio scale is a suitable measure of income that is essentially an indication of purchasing power, while the ordinal and interval scales are suitable for evaluation of information on health and housing that is

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[46] Let the set of states be $\mathcal{S}$. For all states s, for all individuals i, π^s and u_i^s are interpreted as a probability of s and i' ex-post utility under s, respectively. Then, the *ex-ante* type of a social welfare ordering is defined as follows:

$$\sum_{i\in N} w_{[i]} g\left(\sum_{s\in\mathcal{S}} \pi^s u_i^s\right) \geq \sum_{i\in N} w_{[i]} g\left(\sum_{s\in\mathcal{S}} \pi^s v_i^s\right).$$

The *ex-post* type of a social welfare ordering is defined as follows:

$$\sum_{s\in\mathcal{S}} \pi^s \sum_{i\in N} w_{[i]} g\left(u_{[i]}^s\right) \geq \sum_{s\in\mathcal{S}} \pi^s \sum_{i\in N} w_{[i]} g\left(v_{[i]}^s\right).$$

The ex-ante SWO can be interpreted as the rank-weighted generalized utilitarianism based on *individual expected utilities*. The ex-post SWO can be interpreted as the *expected value* of the rank-weighted generalized utilitarianism based on *individual ex-post utilities*. The author does not have any convincing reasons for adopting either type of social welfare ordering. For recent elegant surveys in this field, see Fleurbaey (2018) and Mongin and Pivato (2016).

[47] As is well known, a continuous ordering can be represented by a continuous real-valued function (Debreu 1959).

used in multidimensional poverty indices.[48] However, as is well known, a poverty line on a certain component, which serves as a reference point in computation of the multi-dimensional poverty index, is often an ambiguous concept. Considering the problem of measurement errors, a comparison of living standards that allows for some degrees of ambiguity and deviation seems to be able to provide a more fruitful analysis rather than an exact comparison that automatically and simply aggregates the actual data that contains any kinds of human error. The methodological issue on how to synthesize and aggregate different measures with various scales is extremely complicated. In order to construct a consistent and acceptable theory that is robust to ambiguity and measurement errors, the application of some notable elements of social choice theory may be useful, such as Sen's incomplete social ranking (Sen 1970, Ch. 7*), Balinski and Laraki's grade variables (Balinski and Laraki 2010), fuzzy social choice, and Ng's perception scales (Ng 1975; Argenziano and Gilboa 2021).[49]

Fourth, although this study respects the anonymity condition as one of the plausible axioms, anonymity is only an axiom that should be applied to individuals who are regarded as being in the same situation under the same condition. As the capability approach aptly points out (Sen 1985), individuals who seem to face the same opportunities should be treated in a different manner when their abilities differ. For example, a physically challenged person and a person without any disabilities do not have the same opportunities in schooling and employment without any reasonable accommodation for disabilities. The simple demand of anonymity often gives rise to the major problem of equating discriminated groups and powerful groups in the context of evaluation of income distributions.[50] In order to capture various aspects of poverty

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48 The global MPI approach proposed by Alkire et al. (2018) uses information on health (nutrition and child mortality), education (years of schooling and school attendance), and living standards (cooking fuel, sanitation, drinking water, electricity, housing, and assets) for cross-national comparisons. The OECD's Better Life Index uses a variety of scales, including housing, income, jobs, community, education, environment, civic engagement, health, life satisfaction, safety, and work-life balance.

49 We must be very careful that theories focusing on ambiguity often bring new problems. For example, fuzzy logic yields a counter-intuitive relationship between the laws of excluded middle and non-contradiction. If we tried to overcome such a situation with the paraconsistent logic, we might suffer a great loss not only in logic but also in welfare economics.

50 Note that the concept of *reasonable accommodation* raises a new conundrum: to what extent should we consider it as reasonable? Of course, treating individuals with the same abilities in the same way is defended from the concept of horizontal equity. A limited restriction of anonymity to groups with the same abilities should be considered as an ethical requirement.

depending on the context of individual abilities and social barriers, it is necessary to explore new theories that make the functional form dependent on individual abilities and social barriers (a social welfare ordering that takes into account the capability approach).[51]

Finally, our results may seem to be trite in the sense that a social welfare ordering satisfying the plausible axioms is boiled down to a mediocre one that simply takes into account *tolerable inequality* and *tolerable poverty*.[52] However, beyond

[51] Without any interpersonal comparability of well-being, the dominance condition and the individual preference condition are incompatible with each other in the context of social evaluation on the set of resource allocations (*the indexing dilemma*). If we try to construct a social welfare ordering based solely on individual preferences under the assumption of interpersonal noncomparability of well-being, it yields a violation of the dominance condition. Therefore, welfare economics as an *instrument for the bettering of human life* needs to pay great attention not to become so fixated on the objectivity of interpersonal comparisons of utilities that greatly narrows the possibilities, leaving the evils of inequality and poverty, such as the terrible uncertainty overshadowing many families of the poor and the injurious luxury of some wealthy families, ignored. See Weymark (2017) for a survey on the indexing dilemma and the related difficulties.

[52] In the traditional Japanese view of nature worship and religion, and in the Confucian and Buddhist teachings that contributed to its formation and transformation, there exists the virtue of moderation and a sense of respect for the harmony of all life and nature (a sense that the human being is just another kind of animal and that all living things deserve a certain amount of respect). Therefore, the author has a sense that a society should avoid any extreme situations that could disrupt the balance of the biosphere and the sustainability of all living things, no matter the reason for the poverty or environmental destruction. This view may be very different from Western moral philosophy, which often tries to systematize various value judgments following fundamental ideal principles, with great emphasis on logical consistency and deduction. In the typical Asian value systems, although there are logical ambiguities and leaps that existentialism and a person who likes polemics badly imitates, and distorted ideologies that are often used to justify patriarchy, authority, class, and dictatorship, there are also notable insights into the virtue of harmony and moderation, which Aristotle focused on, compassion, which Hume analyzed as a kind of calm passion metamorphosed by the process of reason mind, and the need to help the weak and cooperate from a humanitarian perspective, which is also found in monotheism. In this sense, the author cannot help but feel happy that the combination of plausible axioms can yield a class of social welfare orderings, which tries to find suitable balance between poverty and inequality, that seems to embody both Western and Eastern virtues and cultural traditions. As Sen (1970; 2009) argues, there exists no normative principle that has absolute superiority over other principles in all situations, and all axioms in social choice theory should be interpreted as kinds of guidelines. Therefore, the author believes that the bright future of humans can be shaped by our efforts towards the best interpretation of justice and goodness in our time through democratic deliberation in an open society.

potentially competing ethical positions, the fact that the only way to define a social welfare ordering characterized by relatively uncontroversial axioms is in the functional form showing compassion for relative inequality and poverty levels, can be interpreted as a first step toward a consensus on the theory of distributive justice. The Pareto egalitarianism, prioritarianism, and sufficientarianism can be defined as special cases of the rank-weighted generalized utilitarianism. All we need to do now is to examine tolerable balance with respect to relative inequality and absolute poverty. The answer to this question is obviously beyond the scope of traditional economic analysis[53], but the author is strongly confirmed that meaningful and significant observations could be obtained through interdisciplinary investigations of economics, ethics, psychology, anthropology, archaeology, sociology, comparative religion studies, ethology, physiology, molecular biology, neuroscience, and computer science.[54]

[53] In order to answer this question, we should examine the problem of the transformation of normative values and behaviors based on mutual communication and cooperation with others. The assumption that individual preferences or values never change, and that social decision-making is based only on a profile of the fixed preferences and values, is different from observed facts that individual preferences and values are flexible through education and persuasion. Of course, the assumption that a social choice function always assigns rational results for all preference profiles is very attractive, and it is an appropriate and powerful setting for some contexts of analysis. However, in the context of determining social parameters that show compassion for inequality and poverty, it would be more important to consider a set of values that have been corrected by a rational consensus based on deliberation and sharing of scientific evidence on social problems, rather than the bias-ridden subjective views that people initially have.

[54] Note that any analysis based on experimental results is directly related to the classical problem of Hume's law regarding the division between "*is*" and "*ought to be*." For example, an experimental study on the income tax policy in the United States reports that a majority believes that the top tax rate should be between 20 and 40 percent, lower than the actual rate (Ballard-Rosa et al. 2017). Does this mean that the current inequality and poverty in the US is normatively justified? It seems that there are at least two necessary conditions for a decision on an issue to be normative: the decision-maker must have sufficient knowledge and information about the issue (the assumption of *perfect information*); the decision must be made from the third-party perspective that is different from the stakeholders in order to make it a fair decision (the assumption of the *impartial observer*). In this sense, if people are not making decisions based on correct information about the current poverty and inequality in the U.S. (e. g., inequality of job and educational opportunity, inequality of political influence, the whole picture of international tax avoidance and the hidden wealth, and the mechanism of poverty), then setting the aversion parameters of inequality and poverty by limited results in naive experiments would be a major flaw that should not be ignored.

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