# Choice, Opportunities, and Procedures: Collected Papers of Kotaro Suzumura

## Part I Rational Choice and Revealed Preference

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# Introduction

Kotaro Suzumura's work are mostly in the area of welfare economics and social choice theory. His work in the area of theoretical industrial organization, as well as his policy-oriented work on industrial policy, competition policy, and development policy, are derivatives from his work on welfare economics and social choice theory. This collection of his selected papers gathers his representative contributions in the major area of research, which are classified in seven parts.

The first part focuses on the axiomatic characterization of the concept of rational choice as purposive action. It was Lionel Robbins who first crystallized this classical notion of rationality, which Paul Samuelson elaborated into the celebrated edifice of revealed preference theory. Capitalizing on the contributions by Paul Samuelson, Hendrik Houthakker, Kenneth Arrow, Marcel Richter, Bengt Hansson, and Amartya Sen, Suzumura contributed to the axiomatic characterization of rational choice on the general domain. As an auxiliary step, Suzumura generalized Szpilrajn's classical extension theorem on binary relations in terms of his newly introduced concept of consistency. Consistency of a binary relation requires any preference cycle to involve indifference only. As shown by Suzumura, consistency is necessary and sufficient for the existence of an ordering extension of a binary relation. This novel concept of consistency and the generalized extension theorem are playing a basic role in this and many other contexts of choice and preference. This part contains Suzumura's basic contributions along these lines.

The second part focuses on the logical conflict between equity and efficiency in several distinct contexts. Generalizing the early contributions by Serge Kolm, Duncan Foley, and Hal Varian, which may be traced back even further to John Hicks and Jan Tinbergen, Suzumura contributed to build a bridge between the theory of fairness and the theory of social choice, thereby enriching both theories and clarifying their logical relationships. This assertion is substantiated by the first two papers in this part. There is another and even more classical concept of equity, which was introduced by Henry Sidgwick in the context of treating different generations equitably. The last two papers in this part are Suzumura's recent work on the possibility of ordering infinite utility streams on the basis of intergenerational equity and intertemporal efficiency.

The third part focuses on the Arrovian impossibility theorems in social choice theory. Arrow's original formulation of the problem of social choice was in terms of the social welfare function, the maximization of which subject to the feasibility constraints was construed to be the task of the social decision-maker. An early work of Suzumura in this arena was to see how Arrow's impossibility theorem fares if we get rid of the social welfare function and focus directly on the social choice per se. One of his more recent work in this arena asked how the Arrovian impossibility theorem fares if we treat an explicitly economic environment and weaken Arrow's axiom of independence of irrelevant alternatives by allowing richer information about individual preferences. This part also contains a paper in which the crucial concept of coherence was first introduced, and another paper on the logical relations between the compensation principles à la Kaldor, Hicks, Scitovsky and Samuelson, on the one hand, and the Bergson-Samuelson social welfare function, on the other.

The fourth part focuses on the logical coherence between social welfare and individual rights. It was Amartya Sen who posed the problem of compatibility of these two essential values in social choice, which he crystallized into the justly famous *impossibility of a Paretian liberal*. Suzumura's contribution in this arena is two-hold. In the first place, he could identify several escape routes from Sen's impasse, keeping Sen's formulation of individual rights in terms of individual decisiveness in social choice intact. In the second place, he came to the important recognition that Sen's original articulation of libertarian rights is incompatible with the classical concept of freedom of choice à la John Stuart Mill, and contributed to develop an alternative game-form articulation of individual rights. This part contains some of his representative contributions along these lines.

The fifth part focuses on the welfare effect of increasing competitiveness and interfirm collaboration. Contrary to the widespread and classical belief in competition as an efficient and decentralized mechanism for allocating resources, Suzumura proved what came to be called the *excess entry theorem* to the effect that the free-entry number of firms in the Cournot oligopoly market is socially excessive vis-à-vis the first-best number of firms as well as the second-best number of firms. This and related work motivated him to dig much deeper into the relationship between welfare, competition and collaboration. This part contains his major contributions in this area of research.

The sixth part focuses on the intrinsic, rather than instrumental, value of opportunities for making choice and procedures for decision-making. The inquiry along this line led Suzumura to go beyond consequentialism, which had remained almost unchallenged in the literature. He could obtain an axiomatic characterization of consequentialism and non-consequentialism, as well as some clarifications of the effects of going beyond consequentialism on such standard result as Arrow's general impossibility theorem in the theory of social choice.

The seventh part focuses on the analytical history of welfare economics. Although the nomenclature of welfare economics should be attributed to Arthur Pigou for his celebrated classic, *The Economics of Welfare*, the history of welfare economics could be traced back at least as far as to Adam Smith and Jeremy Bentham, and possibly further beyond Adam Smith, under the classical nomenclature of moral philosophy. Suzumura's work in this arena crystallize some of the crucial steps in the historical evolution of welfare economics, paying due attention to the informational basis of social welfare judgements, and the intrinsic value of social decision-making procedures and opportunities to choose. This part also contains two interviews with the great pioneers in the development of welfare economics and social choice theory, viz., Paul Samuelson and Kenneth Arrow.

# Chapter 1
# Rational Choice and Revealed Preference*

## 1 Introduction

According to the currently dominant view, the choice behaviour of an agent is construed to be rational if there exists a preference relation $R$ such that, for every set $S$ of available states, the choice therefrom is the set of "$R$-optimal" points in $S$.[1] There are at least two alternative definitions of $R$-optimality—$R$-maximality and $R$-greatestness. On the one hand, an $x$ in $S$ is said to be $R$-maximal in $S$ if there exists no $y$ in $S$ which is strictly preferred to $x$ in terms of $R$. On the other, an $x$ in $S$ is said to be $R$-greatest in $S$ if, for all $y$ in $S$, $x$ is at least as preferable as $y$ in terms of $R$. The former viewpoint can claim its relevance in view of the prevalent adoption of the concept of Pareto-efficiency in the theory of resource allocation processes.[2] The latter standpoint is deeply rooted in the well-developed theories of the integrability problem, revealed preference and social choice. The difference between these two definitions of rational choice is basically as follows. Any two states in a choice function which is $R$-maximal rational are either $R$-indifferent or $R$-incomparable, while any two states in a choice function which is $R$-greatest rational are $R$-indifferent.

A condition for rational choice has been put forward in terms of the $R$-greatestness interpretation of optimality (Hansson [4] and Richter [9, 10]). In this chapter, a condition for rational choice in its $R$-maximality interpretation will be presented. The condition in question will, in a certain sense, synthesize both concepts of rationality, because it can be seen that the rational choice in terms of $R$-maximality is rational in terms of $R$-greatestness as well. The role of various axioms of revealed preference and congruence in the theory of rational choice will also be clarified.

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[1] Arrow's seminal works [1, 2] are the main sources of the current theory of rational choice. Notable contributions in this field include Hansson [4], Richter [9, 10] and Sen [13], among others. See also Herzberger [5], Jamison and Lau [7] and Wilson [16]. In his recent paper [8], Plott axiomatized the concept of path-independent choice which is related to, but distinct from, the concept of rational choice. See Suzumura [15].

[2] See also Herzberger [5, pp.196-199], who calls an agent a *liberal maximizer* [*resp. stringent maximizer*], if he chooses $R$-maximal points [*resp.* $R$-greatest points] from the points available and favours the liberal maximizer as a model of rational agent.

In this kind of analysis, special care should be taken with the domain of the choice function. It was Arrow [1] who first suggested that "the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets". This line of enquiry was recently completed by Sen [13].[3] It is true, as was persuasively discussed by Sen [13, Section 6], that there is no convincing reason for our *restricting* the domain of the choice function to the class of convex polyhedras representing budget sets in the commodity space. At the same time, however, it should be admitted that there exists no specific reason for our *extending* the domain so as to include *all* finite sets. This being the case, no restriction whatsoever will be placed on the domain of the choice function in this chapter except that it should be a non-empty family of non-empty sets.

In Section 2, our conceptual framework will be presented. The main results will be stated in Section 3, the proofs thereof being given in Section 4. In Section 5 we will present some examples which will negate the converse of our theorems. Finally, Section 6 will be devoted to comparing our results with the Arrow-Sen theory, on the one hand, and the Richter-Hansson theory, on the other.

# 2 Definitions

Let $X$ be the basic set of all alternatives and let $K$ stand for the non-empty family of non-empty subsets of $X$. A suggested interpretation is that each and every $S \in K$ is the set of available alternatives which could possibly be presented to the agent. For the sake of brevity, a series of formal definitions will be given below.

*Definition* 1 (Preference Relation): A *preference relation* is a binary relation $R$ on $X$, that is to say, a subset of $X \times X$. If $(x, y) \in R$, we say that $x$ is at least as preferable as $y$. A *strict preference relation* associated with $R$ is a binary relation

$$P_R = \{(x, y) \in X \times X | (x, y) \in R \ \& \ (y, x) \notin R\}.$$

An *indifference relation* associated with $R$ is a binary relation

$$I_R = \{(x, y) \in X \times X | (x, y) \in R \ \& \ (y, x) \in R\}.$$

$R$ is said to be

(a) *complete* iff $\{(x, y), (y, x)\} \cap R \neq \emptyset$ for all $x, y \in X$,
(b) *acyclic* iff $(x, x) \notin T(P_R)$ for all $x \in X$,[4]
(c) *transitive* iff $[(x, y) \in R \ \& \ (y, z) \in R \Rightarrow (x, z) \in R]$ for all $x, y, z \in X$, and

[3] As was carefully noted by Sen [13, p.312], the Arrow-Sen theory works well even if the domain includes all pairs and all triples, but not all finite sets.

[4] For any binary relation $Q$ on $X$, $T(Q)$ stands for the *transitive closure* of $Q$: $T(Q) = \{(x, y) \in X \times X | (x, y) \in Q$ or $[(x, z^1), (z^k, z^{k+1}), (z^n, y) \in Q$ $(k = 1, \ldots, n-1)$ for some $\{z^1, \ldots, z^n\} \subset X]\}$. If $R$ is acyclic, there exists no strict preference cycle.

(d) an *ordering* iff it is complete as well as transitive.

*Definition* 2 (Maximal-Point Set and Greatest-Point Set): Let $R$ and $S$ be, respectively, a preference relation and an arbitrary subset of $X$. The subsets $M(S, R)$ and $G(S, R)$ of $S$, to be called the *R-maximal-point set* and the *R-greatest-point set* of $S$, respectively, are defined by

$$M(S, R) = \{x \in X | x \in S \;\&\; (y, x) \notin P_R \text{ for all } y \in S\}$$

and

$$G(S, R) = \{x \in X | x \in S \;\&\; (x, y) \in R \text{ for all } y \in S\}.$$

*Remark* 1. (i) For any $S \subset X$ and $R \subset X \times X, G(S, R) \subset M(S, R)$, and (ii) if $R$ is complete, $G(S, R) = M(S, R)$.[5] For any $x, y \in G(S, R)$, we have $(x, y) \in I_R$, while for any $x, y \in M(S, R)$, we have either $(x, y) \in I_R$ or $[(x, y) \notin R \;\&\; (y, x) \notin R]$.

*Definition* 3 (Choice Function): A *choice function* is a function $C$ on $K$ such that $C(S)$ is a non-empty subset of $S$ for all $S \in K$.

The intended interpretation is that, for any set $S$ of available alternatives, the subset $C(S)$ thereof represents the set of alternatives which are chosen from $S$. Associated with the given choice function $C$ on $K$, we can define various concepts of revealed preference.

*Definition* 4 (Revealed Preference Relations): Two preference relations $R^*$ and $R_*$ on $X$ such that

(a)$(x, y) \in R^*$ iff $[x \in C(S) \;\&\; y \in S]$ for some $S \in K$, and
(b)$(x, y) \in R_*$ iff $[x \notin S$ or $x \in C(S)$ or $y \notin C(S)]$ for all $S \in K$

are called the *revealed preference relations*.

In words, $x$ is said to be revealed $R^*$-preferred to $y$ if $x$ is chosen when $y$ is also available, and $x$ is said to be revealed $R_*$-preferred to $y$ if there exists no choice situation in which $y$ is chosen and $x$ is available but rejected. In order to connect these revealed preference relations to the revealed preference axioms, we introduce the following auxiliary concept.

*Definition* 5 ($C$-connectedness): A sequence of sets $(S_1, \ldots, S_n)$ in $K$ is said to be $C$-connected iff $S_k \cap C(S_{k+1}) \neq \emptyset$ for all $k \in \{1, \ldots n-1\}$ and $S_n \cap C(S_1) \neq \emptyset$.

It turns out that for our present purpose the following rather abstract formulation of the revealed preference axioms is the most convenient.

*Definition* 6 (Revealed Preference Axioms): A choice function $C$ on $K$ is said to satisfy

[5]For the proof of this well-known result, see Herzberger [5, Proposition P1] and Sen [12, Chapter 1*].

(a) *Weak Axiom of Revealed Preference* — (WARP), for short — iff for any $C$-connected pair $(S_1, S_2)$ in $K$, $S_1 \cap C(S_2) = C(S_1) \cap S_2$ holds,

(b) *Strong Axiom of Revealed Preference* — (SARP), for short — iff for any $C$-connected sequence $(S_1, \ldots, S_n)$ in $K$, $S_k \cap C(S_{k+1}) = C(S_k) \cap S_{k+1}$ for some $k \in \{1, \ldots, n-1\}$ holds, and

(c) *Hansson's Axiom of Revealed Preference* — (HARP), for short — iff for any $C$-connected sequence $(S_1, \ldots, S_n)$ in $K$, $S_k \cap C(S_{k+1}) = C(S_k) \cap S_{k+1}$ for all $k \in \{1, \ldots, n-1\}$.

At first sight, (WARP) and (SARP) in Definition 6, which are due originally to Hansson [4], might seem rather different from their traditional formulation, such as in Sen [13], but they are equivalent. In order to substantiate this claim, let us define another revealed preference relation $R^{**}$ by $(x, y) \in R^{**}$ iff $[x \in C(S)\ \&\ y \in S \backslash C(S)]$ for some $S \in K$. By definition, we have $R^{**} \subset R^*$. (In words, $x$ is said to be revealed $R^{**}$-preferred to $y$ if $x$ is chosen and $y$ is available but rejected.) In terms of $R^*$ and $R^{**}$, the common version of (WARP) is given by:

$$(x, y) \in R^{**} \Rightarrow (y, x) \notin R^*. \tag{1}$$

Similarly, the traditional formulation of (SARP) is given by:

$$(x, y) \in T(R^{**}) \Rightarrow (y, x) \notin R^*. \tag{2}$$

It will be shown in the Appendix that (1) and (2) are equivalent to (WARP) and (SARP), respectively, in Definition 6.

We are now in the position to introduce the concepts coined by Richter [9] and Sen [13].

*Definition* 7 (Congruence Axioms): A choice function $C$ on $K$ is said to satisfy

(a) *Weak Congruence Axiom* — (WCA), for short — iff for any $S \in K, (x, y) \in R^*$, $x \in S$ and $y \in C(S)$ imply $x \in C(S)$, and

(b) *Strong Congruence Axiom* — (SCA), for short — iff for any $S \in K, (x, y) \in T(R^*)$, $x \in S$ and $y \in C(S)$ imply $x \in C(S)$.

The central concepts of this paper are given by the following definition.

*Definition* 8 (Rational Choice Function): A choice function $C$ on $K$ is said to be

(a) *G-rational* iff there exists a preference relation $R$ such that $C(S) = G(S, R)$ for all $S \in K$, and

(b) *M-rational* iff there exists a preference relation $R$ such that $C(S) = M(S, R)$ for all $S \in K$.

A preference relation $R$ which rationalizes the choice function $C$ is called the *rationalization* of $C$.[6]

Intuitively, a choice function is said to be rational if we can interpret the stipulated choice behaviour as a kind of preference optimization. Two possible interpretations of this idea are formulated in Definition 8(a) and (b). It should be noted that the *M-rational choice function is G-rational but not vice versa*. This can be seen as follows. Let $C$ on $K$ be $M$-rational with the rationalization $R$. We define a binary relation $R'$ on $X$ by $[(x, y) \in R' \Leftrightarrow (y, x) \notin P_R]$ for all $x$ and $y$ in $X$. From Definition 2, we then have $M(S, R) = G(S, R')$ for all $S \in K$, so that $C$ is $G$-rational with the rationalization $R'$. In order to see that the $G$-rational choice function is not necessarily $M$-rational, let us consider an example where $X = \{x, y, z\}, K = \{S_1, S_2, S_3\}, S_1 = \{x, y\}, S_2 = \{x, z\}, S_3 = X, C(S_1) = S_1, C(S_2) = S_2$ and $C(S_3) = \{x\}$. This choice function is $G$-rational with the rationalization $R = \{(x, y), (y, x), (x, z), (z, x)\}$. Assume that this $C$ is $M$-rational with the rationalization $R'$. From $C(S_1) = S_1$ we obtain:

$$(x, y) \in I_{R'} \text{ or } [(x, y) \notin R' \ \& \ (y, x) \notin R'] \tag{3}$$

From $C(S_2) = S_2$ we obtain:

$$(x, z) \in I_{R'} \text{ or } [(x, z) \notin R' \ \& \ (z, x) \notin R'] \tag{4}$$

From $C(S_3) = \{x\}$ we obtain:

$$[(x, y) \in P_{R'} \text{ or } (z, y) \notin P_{R'}] \ \& \ [(x, z) \in P_{R'} \text{ or } (y, z) \in P_{R'}]. \tag{5}$$

From (3), (4) and (5) we obtain $(z, y), (y, z) \in P_{R'}$, which contradicts Definition 1. Thus the choice function in question is not $M$-rational.

In view of Definition 2 and the associated remark, it is clear that both concepts of rationality coincide if the rationalization is complete. If this complete rationalization satisfies the transitivity axiom as well, we say, following Richter [10], that the choice function is *regular-rational*.

Finally, let us introduce the concept of normality.

*Definition* 9 (Normal Choice Function): Let two functions $G^*$ and $M^*$ on $K$ be defined by $G^*(S) = G(S, R^*)$ and $M^*(S) = M(S, R^*)$ for all $S \in K$. A choice function $C$ on $K$ is said to be

(a) *G-normal* iff $C(S) = G^*(S)$ for all $S \in K$, and
(b) *M-normal* iff $C(S) = M^*(S)$ for all $S \in K$.

[6]It should be noted that (i) there exists an irrational choice and (ii) the rationalization of the rational choice function is not necessarily unique. The following examples will establish these points.

*Example* 1*. $X = \{x, y, z\}$, $K = \{S_1, S_2\}$, $S_1 = X$, $S_2 = \{x, y\}$, $C(S_1) = \{y\}$, and $C(S_2) = \{x\}$. It is easy to see that this choice function is neither $G$-rational nor $M$-rational.

*Example* 2*. $X = \{x, y, z\}$, $K = \{S_1, S_2\}$, $S_1 = \{x, y\}$, $S_2 = \{y, z\}$, $C(S_1) = \{x\}$ and $C(S_2) = \{y, z\}$. This choice function has two $G$-rationalizations

$$R_1 = \{(x, y), (y, z), (z, y)\}, R_2 = \{(x, y), (y, z), (z, y), (x, z), (z, x)\}$$

and two $M$-rationalizations $R'_1 = \{(x, y), (y, z), (z, y)\}, R'_2 = \{(x, y)\}$.

# 3 Theorems

We are now ready to investigate the structure of rational choice functions. At the outset, we set down the equivalence which holds between revealed preference axioms and congruence axioms.

**Theorem 1.** (i) (WARP), (WCA) *and the property* ($R^* \subset R_*$) *are mutually equivalent*. (ii) (HARP) *and* (SCA) *are equivalent*.

Our concept of $G$-normality is identical with Richter's $V$-axiom which he proposed as a necessary and sufficient condition for $G$-rationality [10, p.33]. The role of our $M$-rationality is made clear by the following.

**Theorem 2.** *An M-normal choice function is G-normal.*

In view of Richter's Theorem and our Theorem 2, it is important to find an economically meaningful condition which assures the $M$-normality of the choice function. This is where the revealed preference axioms come in.

**Theorem 3.** *A choice function satisfying* (WARP) *is M-normal.*

By combining Theorem 2 and Theorem 3 we can see the role played by (WARP) in the theory of rational choice.

**Theorem 4.** *A choice function satisfying* (WARP) *is M- as well as G-normal.*

We have seen that the concept of $M$-rationality and that of $G$-rationality coincide if the rationalization satisfies the axiom of completeness. This being the case, it is important to have the following:

**Theorem 5.** *An M-rational choice function is complete rational.*

Hansson [4] and Richter [9] showed that the necessary and sufficient condition for the regular-rationality of the choice function is (HARP) or, equivalently, (SCA). On the other hand, Theorem 4 shows us the relevance of (WARP) in the theory of rational choice. Why do we need (SARP)? Our answer is given by the following theorem.

**Theorem 6.** *A choice function satisfying* (SARP) *is acyclic and complete rational.*

These theorems will be proved in the next section.

# 4 Proofs

*Proof of Theorem* 1 (i)
Step 1 [(WARP) $\Rightarrow$ ($R^* \subset R_*$)]. If ($R^* \subset R_*$) does not hold, we have $(x, y) \in R^* \backslash R_*$ for some $x, y \in X$. Then there exist $S$, $S' \in K$ such that $x \in C(S), y \in S, x \in S', x \notin C(S')$ and $y \in C(S')$, so that we have $x \in S' \cap C(S), y \in C(S') \cap S$ and $x \notin C(S') \cap S$. Thus (WARP) does not hold. Hence (WARP) implies ($R^* \subset R_*$).

Step 2 [($R^* \subset R_*$) $\Rightarrow$ (WCA)]. Suppose ($R^* \subset R_*$) and let $(x, y) \in R^*, x \in S$ and $y \in C(S)$ for any $S \in K$. Then we have $(x, y) \in R_*, x \in S$ and $y \in C(S)$, which imply $x \in C(S)$ by virtue of Definition 4(b). Thus (WCA) holds.

Step 3 [(WCA) $\Rightarrow$ (WARP)]. Let $C$ satisfy (WCA). Let $(S_1, S_2)$ be a $C$-connected pair in $K$ and let $x$ and $y$ be taken arbitrarily from $S_1 \cap C(S_2)$ and $C(S_1) \cap S_2$, respectively. Because of $x \in C(S_2)$ and $y \in S_2$, we have $(x, y) \in R^*$ which, coupled with $x \in S_1$ and $y \in C(S_1)$, implies $x \in C(S_1)$ thanks to (WCA). Thus we have $S_1 \cap C(S_2) \subset C(S_1) \cap S_2$.

Similarly we can verify that $S_1 \cap C(S_2) \supset C(S_1) \cap S_2$. Thus (WARP) holds. $\|$

*Proof of Theorem* 1 (ii)
Step 1 [(HARP) $\rightarrow$ (SCA)]. Let $(x, y) \in T(R^*), x \in S$ and $y \in C(S)$ for an $S \in K$. Then either ($\alpha$) $(x, y) \in R^*$, or ($\beta$) there exist $\{z^1, \ldots, z^{n-1}\} \subset X$ and $\{S_1, \ldots, S_n\} \subset K$ such that $x \in C(S_1), z^k \in S_k \cap C(S_{k+1}) (k = 1, \ldots, n-1)$, and $y \in S_n$. In case ($\alpha$), we have $x \in C(S)$, since (HARP) implies (WARP), which is equivalent to (WCA). In case ($\beta$), taking $x \in S$ and $y \in C(S)$ into consideration, $(S, S_1, \ldots, S_n)$ is seen to be $C$-connected, so that we obtain $S \cap C(S_1) = C(S) \cap S_1$ by virtue of (HARP). Thus $x \in C(S)$. In any case, (SCA) holds.

Step 2 [(SCA) $\Rightarrow$ (HARP)]. Let a sequence $(S_1, \ldots, S_n)$ in $K$ be $C$-connected and let $z^k$ and $z^n$ be taken arbitrarily from $S_k \cap C(S_{k+1})$ and $S_n \cap C(S_1)$, respectively ($k = 1, \ldots, n-1$). It will be shown that $S_1 \cap C(S_2) = C(S_1) \cap S_2$. By definition, we have $z^1 \in S_1, (z^1, z^n) \in T(R^*)$ and $z^n = C(S_1)$, so that (SCA) entails $z^1 \in C(S_1)$. Noticing $z^1 \in C(S_2) \subset S_2$, we have $S_1 \cap C(S_2) \subset C(S_1) \cap S_2$. Next, let $z$ be an arbitrary point of $C(S_1) \cap S_2$. Then $z \in S_2, (z, z^1) \in R^* \subset T(R^*)$ and $z^1 \in C(S_2)$, so that we have $z \in C(S_2)$ thanks to (SCA). Thus we have $C(S_1) \cap S_2 \subset S_1 \cap C(S_2)$, yielding $S_1 \cap C(S_2) = C(S_1) \cap S_2$. In a similar way, we can show $C(S_k) \cap S_{k+1} = S_k \cap C(S_{k+1})$ for all $k \in \{1, \ldots, n-1\}$. Thus (HARP) is implied. $\|$

*Proof of Theorem* 2
Let us establish:

$$C(S) \subset G^*(S) \subset M^*(S) \quad \text{for all} \quad S \in K. \tag{6}$$

For any $S \in K$, let $x \in C(S)$. Then for all $y \in S, (x, y) \in R^*$, so that $x \in G^*(S)$. The other part follows from Remark 1. Thanks to (6) and Definition 9, the assertion of the theorem holds. $\|$

*Proof of Theorem* 3
In view of (6), we have only to show that (WARP) implies:

$$M^*(S) \subset C(S) \quad \text{for all} \quad S \in K \tag{7}$$

Let any $S \in K$ be fixed once and for all and let $x \in M^*(S)$. ($M^*(S) \neq \emptyset$ for all $S \in K$ because of (6) and Definition 3.) Then $x \in S$ and for all $y \in S$, either ($\alpha$) [$y \notin C(S')$ or $x \notin S'$] for all $S' \in K$, or ($\beta$) [$x \in C(S'')$ & $y \in S''$] for some $S'' \in K$. Let $y \in C(S)$. Then ($\alpha$) cannot hold for $S' = S$, so that ($\beta$) must hold for $y \in C(S)$. By virtue of (WARP), $x \in C(S)$ follows from $x \in S \cap C(S'')$ and $y \in C(S) \cap S''$, entailing (7). ‖

*Proof of Theorem* 4
By virtue of Theorem 3, (WARP) implies the $M$-normality of $C$, which implies its $G$-normality, thanks to Theorem 2. The assertion of the theorem then follows from Definitions 8 and 9. ‖

*Proof of Theorem* 5
By $M$-normality, we have $C(S) = M^*(S)$ for all $S \in K$. Let $R^0$ be defined on $X$ by:

$$[(x,y) \in R^0 \Leftrightarrow (y,x) \notin P_{R^*}] \quad \text{for all} \quad x, y \in X. \tag{8}$$

Then $R^0$ is a complete relation and, by its very definition, $M^*(S) = G(S, R^0)$ for all $S \in K$. Taking Remark 1 into consideration, the assertion of the theorem follows. ‖

*Proof of Theorem* 6
(SARP) implies (WARP), so that we have $C(S) = G(S, R^0) = M(S, R^0)$ for all $S \in K$ by virtue of Theorems 3 and 5, where $R^0$ is a complete relation defined by (8). Noticing $P_{R^*} = P_{R^0}$, we have only to show $(x,x) \notin T(P_{R^*})$ for all $x \in X$. Suppose, to the contrary, that there exists an $x \in X$ such that $(x,x) \in T(P_{R^*})$ does hold. Then there exist $\{z^1, \ldots, z^{n-1}\} \subset X$ and $\{S_1, \ldots, S_n\} \subset K$ satisfying $x \in C(S_1), z^k \in [S_k \backslash C(S_k)] \cap C(S_{k+1})$ ($k = 1, \ldots, n-2$) and $x \in S_n \backslash C(S_n)$. Then $(S_1, \ldots, S_n)$ is a $C$-connected sequence in $K$ but $S_k \cap C(S_{k+1}) \neq C(S_k) \cap S_{k+1}$ for all $k \in \{1, \ldots, n-1\}$, so that (SARP) does not hold. Thus under (SARP), $R^0$ must be acyclic. ‖

# 5 Counter-Examples

In this section, counter-examples are given for the converse of our Theorems. The set of all alternatives is always taken as $X = \{x, y, z\}$.

($a$) By Definition 6, (HARP) implies (SARP) which, in its turn, implies (WARP). The converse does not hold, as was shown by Hansson [4, Theorem 5 and Theorem 6]. By Definition 7, (SCA) implies (WCA). The converse does not hold as is shown by the following:

*Example* 1. $K = \{S_1, S_2, S_3\}, S_1 = \{x, y\}, S_2 = \{y, z\}, S_3 = \{x, z\}, C(S_1) = S_1, C(S_2) = S_2$ and $C(S_3) = \{x\}$. This $C$ has a revealed preference relation

$$R^* = \{(x,x), (y,y), (z,z), (x,y), (y,x), (y,z), (z,y), (x,z)\}.$$

(SCA) is not satisfied, because $(z, x) \in T(R^*), x \in C(S_3), z \in S_3$ but $z \notin C(S_3)$. On the other hand, (WCA) is easily seen to be satisfied. $\|$

($b$) The converse of Theorem 2 does not hold in general; that is to say, there exists a $G$-normal choice function which is not $M$-normal.

*Example* 2. $K = \{S_1, S_2\}, S_1 = X, S_2 = \{x, z\}, C(S_1) = \{z\}$, and $C(S_2) = \{x, z\}$. In this case, $R^* = \{(z, z), (z, x), (z, y), (x, z), (x, x)\}$, so that we have $G^*(S_1) = \{z\}$ and $G^*(S_2) = \{x, z\}$. Thus $C$ is $G$-normal. But $M^*(S_1) = \{x, z\} \neq C(S_1)$, so that $C$ is not $M$-normal. $\|$

($c$) The converse of Theorem 3 does not hold in general. An example of a choice function which is $M$-rational but does not satisfy (WARP) will do.

*Example* 3. $K = \{S_1, S_2\}, S_1 = X, S_2 = \{y, z\}, C(S_1) = \{x, y\}$ and $C(S_2) = S_2$. For this $C$, $R^* = \{(x, x), (y, y), (z, z), (x, y), (x, z), (y, x), (y, z), (z, y)\}$. This $C$ is $M$-normal as can be verified. (WARP) is, however, not satisfied, because $S_1 \cap C(S_2) = \{y, z\}$ and $C(S_1) \cap S_2 = \{y\}$. $\|$

($d$) The converse of Theorem 4 is not true in general, as is shown by the following example.

*Example* 4. $K = \{S_1, S_2\}, S_1 = X, S_2 = \{x, y\}$, $C(S_1) = \{x\}$ and $C(S_2) = S_2$. Corresponding to this $C$, we have $R^* = \{(x, x), (y, y), (x, y), (x, z), (y, x)\}$. Therefore, $M^*(S_1) = \{x, y\} \neq C(S_1)$. This $C$ is not $M$-normal, hence (WARP) does not hold, thanks to Theorem 3. But we have $C(S) = G(S, R_1) = M(S, R_2)$ for all $S \in K$, where $R_1 = \{(x, x), (y, y), (x, y), (y, x), (x, z)\}$ and $R_2 = \{(z, y), (x, z)\}$. $\|$

($e$) The converse of Theorem 5 is falsified by the following example.

*Example* 5. $K = \{S_1, S_2, S_3\}, S_1 = X, S_2 = \{x, y\}, S_3 = \{y, z\}, C(S_1) = \{y\}, C(S_2) = S_2$ and $C(S_3) = \{y\}$. For this $C$, $R^* = \{(x, x), (y, y), (x, y), (y, x), (y, z)\}$. This $C$ is not $M$-normal, because $M^*(S_1) = \{x, y\} \neq C(S_1)$. But

$$R = \{(x, y), (y, x), (y, z), (z, x), (x, x), (y, y), (z, z)\}$$

has the property $C(S) = G(S, R)$ for all $S \in K$ and $R$ is complete, so that $C$ is complete rational. $\|$

($f$) The converse of Theorem 6 is falsified by the following example.

*Example* 6. $K = \{S_1, S_2, S_3\}, S_1 = \{x, y\}, S_2 = \{y, z\}, S_3 = \{x, z\}, C(S_1) = \{x\}, C(S_2) = \{y\}$ and $C(S_3) = S_3$. As can be verified,

$$R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (z, x), (x, z)\}$$

is an acyclic and complete relation such that $C(S) = G(S, R)$ for all $S \in K$. (SARP) is, however, not satisfied, because $S_1 \cap C(S_2) = \{y\}, S_2 \cap C(S_3) = \{z\}, S_3 \cap C(S_1) = \{x\}, C(S_1) \cap S_2 = \emptyset$ and $C(S_2) \cap S_3 = \emptyset$. ‖

($g$) (WARP) is not strong enough to assure the acyclic and complete rationality of the choice function.

*Example* 7. $K = \{S_1, S_2, S_3\}, S_1 = \{x, y\}, S_2 = \{y, z\}, S_3 = \{x, z\}, C(S_1) = \{x\}, C(S_2) = \{y\}$ and $C(S_3) = \{z\}$. In this case, $R^* = \{(x, y), (y, z), (z, x)\}$. We are going to show that $R^* \subset R_*$. We have $(x, y) \in R_*$, because $x \in C(S_1), x \notin S_2$ and $y \notin C(S_3)$. Similarly $z \notin C(S_1), y \in C(S_2)$ and $y \notin S_3$ entail $(y, z) \in R_*$, while $z \notin S_1, x \notin C(S_2)$ and $z \in C(S_3)$ show $(z, x) \in R_*$. Thus, thanks to Theorem 1 (i), this choice function satisfies (WARP). The unique complete rationalization thereof is, however, a cyclic one

$$R^* = \{(x, y), (y, z), (z, x)\}. \quad \|$$

# 6 Concluding Remarks

In conclusion, let us compare our results with Arrow-Sen theory, on the one hand, and the Richter-Hansson theory, on the other. We are concerned with a choice function $C$ on the family $K$. The choice mechanism $C$ is said to be $G$-rational (*resp.* $M$-rational) if there exists a preference relation $R$ such that the choice set $C(S)$ can be identified with the set of all $R$-greatest points (*resp.* $R$-maximal points) in $S$ for any possible choice situation $S \in K$. Note here that the concepts of $G$-rationality and $M$-rationality have nothing to do with the transitivity of the rationalization $R$. If $R$ happens to satisfy the ordering axiom of completeness and transitivity, $C$ is said to be regular-rational. Arrow [1] and Sen [13] showed that:

($\alpha$) (SCA),(WCA),(SARP) *and* (WARP) *are mutually equivalent necessary and sufficient conditions for regular-rationality if $K$ includes all the pairs and all the triples taken from the basic set $X$.*

It seems to us that the cost paid for the neat result ($\alpha$) is rather high. Their assumption on the content of $K$ might not generally be admissible, hence depriving their result of its general applicability. On the other hand, Richter [9] and Hansson [4] made virtually no restrictions on the content of $K$ and established that:

($\beta$) (SCA) *and* (HARP) *are mutually equivalent necessary and sufficient conditions for regular-rationality.*

Later, Richter [10] extended his conceptual framework and established the following characterization of $G$-rational choice functions.

($\gamma$) *$G$-normality is a necessary and sufficient condition for $G$-rationality.*

[Insert **Figure 1** around here.]

In this paper, we have systematically examined the structure of $G$-rational and $M$-rational choice functions. The domain of the choice function is assumed simply to be a non-empty family of non-empty sets. Thus we may reasonably claim the general applicability of our results. The role of (SARP), (WARP), (WCA) and $M$-normality are clarified in this general setting.

Our results, together with ($\beta$) and ($\gamma$), are summarized in Figure 1. An arrow indicates implication, and cannot in general be reversed. The contrast with the Arrow-Sen result ($\alpha$) is clear.

# 7 Appendix

(a) *The Equivalence between* (1) *and* (WARP)
[(WARP) $\Rightarrow$ (1)] Let us assume that $(x, y) \in R^{**}$ and $(y, x) \in R^*$ for some $x$ and $y$ in $X$. Then there exist $S_1$ and $S_2$ in $K$ such that $x \in C(S_1)$, $y \in S_1 \backslash C(S_1)$, $x \in S_2$ and $y \in C(S_2)$. It follows that $(S_1, S_2)$ is a $C$-connected pair in $K$ but $y \notin C(S_1) \cap S_2$ and $y \in S_1 \cap C(S_2)$ so that (WARP) does not hold.

[(1) $\Rightarrow$ (WARP)] Let $(S_1, S_2)$ be a $C$-connected pair in $K$ such that $x \in S_1 \cap C(S_2)$ and $x \notin C(S_1) \cap S_2$ for some $x$ in $X$. Then $(y, x) \in R^{**}$ and $(x, y) \in R^*$ for any $y \in C(S_1) \cap S_2$, negating (1).

(b) *The Equivalence between* (2) *and* (SARP)
[(SARP) $\Rightarrow$ (2)] Suppose that $(x, y) \in T(R^{**})$ and $(y, x) \in R^*$ for some $x$ and $y$ in $X$. Then either $(x, y) \in R^{**}$ or $(x, z^1)$,$(z^1, z^2)$, $\ldots$, $(z^n, y) \in R^{**}$ for some $z^1, \ldots, z^n \in X$. In view of (a) and Definition 6, we have only to consider the latter case. In this case, there exist $S_1, \ldots, S_n$ and $S$ in $K$ such that $x \in C(S_1)$, $z^1 \in S_1 \backslash C(S_1)$, $z^1 \in C(S_2)$, $z_2 \in S_2 \backslash C(S_2), \ldots,$ $z^n \in C(S_{n+1})$, $y \in S_{n+1} \backslash C(S_{n+1})$, $y \in C(S)$ and $x \in S$. Then $(S_1, \ldots, S_{n+1}, S)$ is a $C$-connected sequence in $K$. But

$$z^1 \in [S_1 \cap C(S_2)] \backslash [C(S_1) \cap S_2], z^2 \in [S_2 \cap C(S_3)] \backslash [C(S_2) \cap S_3], \ldots,$$
$$y \in [S_{n+1} \cap C(S)] \backslash [C(S_{n+1}) \cap S],$$

so that (SARP) does not hold.

[(2)$\Rightarrow$(SARP)] Let $(S_1, \ldots, S_n)$ be any $C$-connected sequence in $K$. If

$$S_{n-1} \cap C(S_n) = C(S_{n-1}) \cap S_n,$$

there remains nothing to be proved. Suppose, then, that $S_{n-1} \cap C(S_n) \neq C(S_{n-1}) \cap S_n$. Firstly, suppose that there exists an $x \in X$ such that $x \in C(S_{n-1}) \cap C(S_n)$ and $x \notin S_{n-1} \cap C(S_n)$. In this case, for any $y \in S_{n-1} \cap C(S_n)$, we have $(x, y) \in R^*$ and $(y, x) \in R^{**}$ in contradiction to (2). Secondly, suppose that we have $x \notin C(S_{n-1}) \cap S_n$, $x \in S_{n-1} \cap C(S_n)$ and $y \in C(S_{n-1}) \cap S_n$ for some $x$ and $y$ in $X$. Here again we have $(x, y) \in R^*$ and

$(y, x) \in R^{**}$ in contradiction to (2). Finally, suppose that $C(S_{n-1}) \cap S_n = \emptyset$. $(S_1, \ldots, S_n)$ being a $C$-connected sequence in $K$, $x \in S_{n-1} \cap C(S_n)$ for some $x$ in $X$. If $x \in C(S_{n-1})$, we have $x \in C(S_{n-1}) \cap C(S_n) \subset C(S_{n-1}) \cap S_n$, a contradiction. Thus we obtain $x \in S_{n-1} \backslash C(S_{n-1})$. If $S_{n-2} \cap C(S_{n-1}) = C(S_{n-2}) \cap S_{n-1}$, there remains nothing to be proved. Otherwise, we repeat the above procedure to obtain a $z^{n-1} \in [S_{n-2} \backslash C(S_{n-2})] \cap C(S_{n-1})$. This algorithm leads us either to

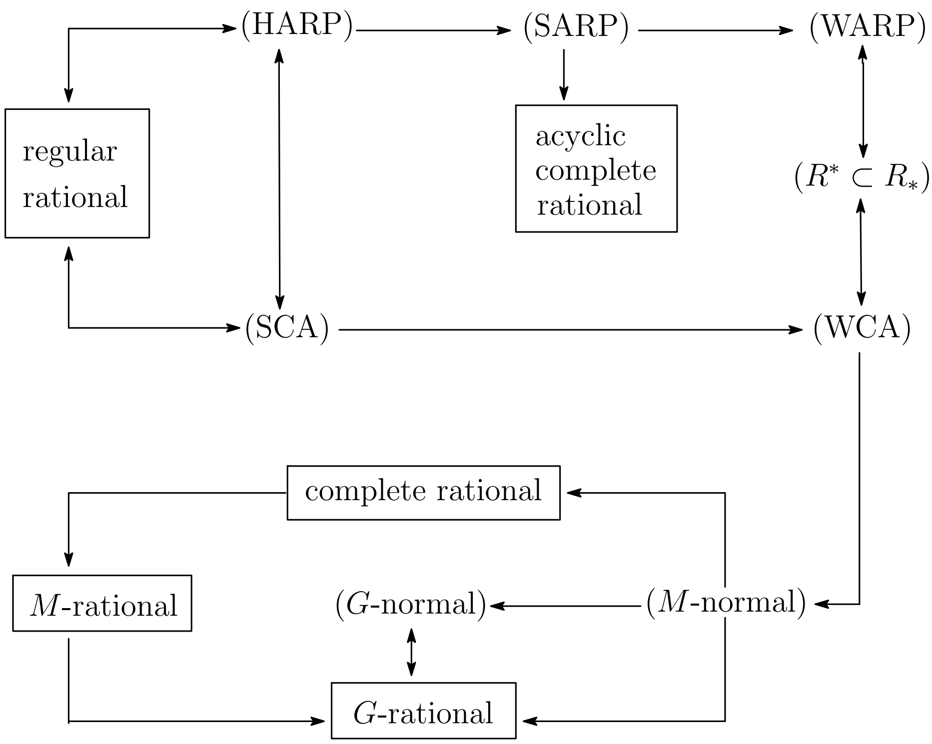
$$S_k \cap C(S_{k+1}) = C(S_k) \cap S_{k+1} \quad \text{for some} \quad k \in \{1, \ldots, n-1\} \qquad \ldots (1^*)$$

or to:

There exist $z^2, \ldots, z^{n-1}$ such that $z^2 \in [S_1 \backslash C(S_1)] \cap C(S_2)$,

$$z^3 \in [S_2 \backslash C(S_2)] \cap C(S_3), \ldots, z^{n-1} \in [S_{n-2} \backslash C(S_{n-2})] \cap C(S_{n-1}) \qquad \ldots (2^*)$$

In the latter case, take a $z^1 \in C(S_1) \cap S_n$. Then we obtain $(z^1, x) \in T(R^{**})$ and $(x, z^1) \in R^*$, in contradiction to (2). $\|$


(HARP)
(SARP)
(WARP)
regular rational
acyclic complete rational
$(R^* \subset R_*)$
(SCA)
(WCA)
complete rational
$M$-rational
($G$-normal)
($M$-normal)
$G$-rational

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# Chapter 2
# Houthakker's Axiom in the Theory of Rational Choice*

## 1 Introduction

In his celebrated classic paper [4], Houthakker strengthened Samuelson's weak axiom of revealed preference [7, 8] into what he called *semitransitivity*, and showed that the Lipschitz-continuous demand function of a competitive consumer satisfying his axiom does possess a generating utility function.[1] Uzawa [11], Arrow [1], and others extended the conceptual framework of revealed preference theory so as to make it applicable to a wider class of problems. Instead of confining our attention to a demand function of a competitive consumer, we are now concerned with a *choice function* over a family of nonempty subsets of a basic nonempty set. A natural question suggests itself: What property of a choice function guarantees the existence of a generating preference ordering (GPO)? In particular, does Houthakker's axiom, if suitably reformulated, qualify as such? It is this problem of the existence of a GPO that constitutes the problem of *rationalizability* of a choice function, which is a choice-functional counterpart of the integrability problem in demand theory.

In the literature we have two answers to this question, depending on the extent of the domain of a choice function. On the one hand, if the family over which a choice function is defined contains all finite subsets of the whole space, the weak axiom ensures the existence of a GPO and that the *strong axiom of revealed preference* (which was so named by Samuelson [8] and attributed to Houthakker) is equivalent to the weak axiom [1, 9]. On the other hand, if we do not impose any such additional assumption on the domain, the strong axiom is necessary but not sufficient for the existence of a GPO [3, 10].[2] Richter [5] and Hansson [3] proposed in this general setting a necessary and

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[1]Further results on this problem are found in papers collected in [2]. See, especially, Uzawa [2, Chap. 1] and Hurwicz-Richter [2, Chap. 3].

[2]This statement is true for one version of the strong axiom, which is used by Arrow [1], Hansson [3], Sen [9] and Suzumura [10]. More about this in the final section.

sufficient condition for the existence of a GPO, which was called the *congruence axiom* by Richter.

The purpose of this paper is to show that Houthakker's semitransitivity axiom, if suitably formalized in the choice-functional context, is in fact necessary and sufficient for the existence of a GPO. Therefore it follows that, contrary to the prevailing interpretation, the strong axiom is not a legitimate formalization of Houthakker's axiom which is *equivalent* to the congruence axiom. In other words, Houthakker's axiom, unlike the strong axiom, provides us with the precise restriction on a choice function for the rationalizability thereof, just as it provided us with the precise restriction on a Lipschitz-continuous demand function for the integrability thereof.

# 2 Rationalizability

**2.1.** Let $X$ be a nonempty set which stands for a fixed universe of alternatives. We assume that there is a well-specified family $K$ of nonempty subsets of $X$. The pair $(X, K)$ will be called a *choice space*. A *choice function* on a choice space $(X, K)$ is a function $C$ defined on $K$ which assigns a non-empty subset (*choice set*) $C(S)$ of $S$ to each $S \in K$.

A *preference relation* $R$ on $X$ is a binary relation on $X$, namely a subset of a Cartesian product $X \times X$. Associated with a given preference relation $R$, an infinite sequence of binary relations $\{R^{(\tau)}\}_{\tau=1}^{\infty}$ is defined by $R^{(1)} = R, R^{(\tau)} = \{(x, y) \in X \times X | (x, z) \in R^{(\tau-1)}$ & $(z, y) \in R$ for some $z \in X\} (\tau \geq 2)$. The *transitive closure* of $R$ is then defined by $T(R) = \cup_{\tau=1}^{\infty} R^{(\tau)}$.[3] The *strict preference relation* $P_R$ corresponding to a preference relation $R$ is an asymmetric component of $R$:

$$P_R = \{(x, y) \in X \times X | (x, y) \in R \ \& \ (y, x) \notin R\}. \tag{1}$$

We say that a preference relation $R$ is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$, *acyclic* if $(x, x) \notin T(P_R)$, and *complete* if either $(x, y) \in R$ or $(y, x) \in R$ for all $x$ and $y$ in $X$. $R$ is said to be an *ordering* if it is transitive and complete. For every $S \in K$, we define

$$G(S, R) = \{x \in X | x \in S \ \& \ (x, y) \in R \text{ for all } y \in S\}, \tag{2}$$

which is the set of all $R$-greatest points in $S$.

**2.2.** A preference relation $R$ on $X$ is said to rationalize a choice function $C$ on $(X, K)$ if we have

$$C(S) = G(S, R) \quad \text{for every} \quad S \in K. \tag{3}$$

A choice function $C$ is said to be *rational* if there exists a preference relation $R$ which rationalizes $C$. ($R$ is then called a *rationalization* of $C$.) If a choice function $C$ is rational

[3]It is easy to see that $T$ satisfies the axiom of *closure operations*: $(\alpha)$ $R \subset T(R)$ for every $R$, $(\beta)$ $R \subset R'$ implies $T(R) \subset T(R')$ for every $R$ and $R'$, $(\gamma)$ $T[T(R)] = T(R)$ for every $R$, and $(\delta)$ $T(\emptyset) = \emptyset$.

with an acyclic rationalization, we say that $C$ is *acyclic rational*. Similarly, if $C$ is rational with an ordering rationalization, we say that $C$ is *full rational*.

**2.3.** Let $C$ be a choice function on $(X, K)$ which is fixed once and for all. Two revealed preference relations $R^*$ and $R^{**}$ are induced from $C$ as follows. We define a binary relation $R^*$ on $X$ by

$$R^* = \{(x, y) \in X \times X | x \in C(S) \ \& \ y \in S \text{ for some } S \in K\} \tag{4}$$

and, when $(x, y) \in R^*$, we say that $x$ is *revealed $R^*$-preferred to $y$*. Similarly, we define

$$R^{**} = \{(x, y) \in X \times X | x \in C(S) \ \& \ y \in S \backslash C(S) \text{ for some } S \in K\} \tag{5}$$

and, when $(x, y) \in R^{**}$, we say that $x$ is *revealed $R^{**}$-preferred to $y$*. It is easy to see that $R^*$ and $R^{**}$ are related by the following relation:

$$P_{R^*} \subset P_{R^{**}} \subset R^{**} \subset R^*. \tag{6}$$

We have only to show that $P_{R^*} \subset P_{R^{**}}$, the remaining inclusions in (6) being obvious by definition. If $(x, y) \in P_{R^*}$, then we have

$$x \in C(S) \ \& \ y \in S \qquad \text{for some} \qquad S \in K, \tag{7}$$

and

$$y \notin C(S') \qquad \text{or} \qquad x \notin S' \qquad \text{for all} \qquad S' \in K. \tag{8}$$

If we apply (8) for $S' = S$, this (coupled with (7)) yields

$$x \in C(S) \ \& \ y \in S \backslash C(S) \qquad \text{for some} \qquad S \in K, \tag{9}$$

while (8) implies

$$y \notin C(S') \quad \text{or} \quad x \notin S' \quad \text{or} \quad x \in C(S') \quad \text{for all} \quad S' \in K. \tag{10}$$

It follows from (9) and (10) that $(x, y) \in P_{R^{**}}$.

We now turn from our revealed preference relations to revealed preference axioms. A finite sequence $\{x^1, x^2, \ldots, x^n\} (n \geq 2)$ in $X$ is called an *$H$-cycle of order $n$* if we have $(x^1, x^2) \in R^{**}, (x^\tau, x^{\tau+1}) \in R^*$ $(\tau = 2, \ldots, n-1)$ and $(x^n, x^1) \in R^*$. Similarly, a finite sequence $\{x^1, x^2, \ldots, x^n\} (n \geq 2)$ in $X$ is called an *SH-cycle of order $n$* if we have $(x^1, x^2) \in R^*, (x^\tau, x^{\tau+1}) \in R^{**}$ $(\tau = 2, \ldots, n-1)$ and $(x^n, x^1) \in R^{**}$. In view of (6) it is clear that *an SH cycle of some order* is *an $H$ cycle of the same order*. This being the case, the exclusion of an $H$ cycle of any order excludes, *a fortiori*, the existence of an $SH$ cycle of any order. We now introduce the following two revealed preference axioms.

Houthakker's Revealed Preference Axiom. *There exists no H-cycle of any order*.

Strong Axiom of Revealed Preference. *There exists no SH-cycle of any order*.

Clearly, Houthakker's axiom is stronger than the strong axiom. We have shown in [10] that the strong axiom is necessary but not sufficient for full rationality and that it is sufficient but not necessary for acyclic rationality. In the final section we will argue that the above-stated Houthakker's axiom is a proper choice-functional counterpart of Houthakker's semitransitivity in demand theory.

**2.4.** We are now ready to put forward our theorem.

Rationalizability Theorem. *A choice function $C$ is full rational if and only if it satisfies Houthakker's axiom of revealed preference.*

*Proof of Necessity.* If $C$ is full rational with an ordering rationalization $R$, then we have (3). Suppose that there exists a sequence $\{x^1, x^2, \ldots, x^n\}(n \geq 2)$ such that $(x^1, x^2) \in R^{**}$ and $(x^\tau, x^{\tau+1}) \in R^*$ $(\tau = 1, 2, \ldots, n-1)$. Then there exists a sequence $\{S^1, S^2, \ldots, S^{n-1}\}$ in $K$ such that $x^1 \in C(S^1)$, $x^2 \in S^1 \backslash C(S^1), x^\tau \in C(S^\tau)$, and $x^{\tau+1} \in S^\tau$ $(\tau = 2, \ldots, n-1)$. Since $C$ is full rational we then have $(x^1, x^2) \in P_R$ and $(x^\tau, x^{\tau+1}) \in R$ $(\tau = 2, \ldots, n-1)$, which entails $(x^1, x^n) \in P_R$, thanks to the transitivity of $R$. But this result excludes the possibility that $(x^n, x^1) \in R^*$, so that there exist no $H$-cylcle of any order.

*Proof of Sufficiency.* Let a *diagonal* $\Delta$ be defined by

$$\Delta = \{(x, x) \in X \times X | x \in X\}, \tag{11}$$

and define a binary relation $Q$ by

$$Q = \Delta \cup T(R^*). \tag{12}$$

It is easy to see that $Q$ is transitive and *reflexive*: $(x, x) \in Q$ for all $x$ in $X$. Thanks to a corollary of Szpilrajn's theorem [3, Lemma 3] there exists an ordering $R$ which subsumes $Q$; namely, there exists an ordering $R$ such that

$$Q \subset R, \tag{13}$$

and

$$P_Q \subset P_R. \tag{14}$$

We are going to show that this $R$ in fact satisfies

$$R^* \subset R, \tag{15}$$

and

$$P_{R^*} \subset P_R. \tag{16}$$

The former is obvious in view of $R^* \subset T(R^*)$, (12), and (13). To prove the latter we have only to show that $P_{R^*} \subset P_Q$, thanks to (14). Assume $(x, y) \in P_{R^*}$, which means $(x, y) \in R^*$ and $(y, x) \notin R^*$. From $(x, y) \in R^*$ it follows that $(x, y) \in Q$. It only remains to be shown that $(y, x) \notin Q$. Assume, therefore, that $(y, x) \in Q$. Clearly, $(y, x) \notin \Delta$, else we cannot have $(x, y) \in P_{R^*}$. It follows that $(y, x) \in T(R^*)$, which, in combination with $(x, y) \in P_{R^*} \subset R^{**}$, implies the existence of an $H$-cycle of some order, a contradiction. Therefore (15) and (16) are valid.

Let an $S \in K$ be chosen and let $x \in C(S)$. Then $(x, y) \in R^*$ for all $y \in S$. In view of (15) we then have $x \in G(S, R)$. It follows that

$$C(S) \subset G(S, R). \tag{17}$$

Next let $x \in S \backslash C(S)$ and take $y \in C(S)$, so that $(y, x) \in R^{**}$. If we have $(x, y) \in R^*$ it turns out that $\{x, y\}$ is an $H$-cycle of order 2, a contradiction. Therefore we must have $(x, y) \notin R^*$, which, in view of $(y, x) \in R^{**} \subset R^*$, implies $(y, x) \in P_{R^*}$. Thanks to (16) we then have $(y, x) \in P_R$, entailing $x \in S \backslash G(S, R)$. Therefore we obtain

$$G(S, R) \subset C(S). \tag{18}$$

As (17) and (18) are valid for any $S \in K$ we have shown that $C$ is full rational with an ordering rationalization $R$. This completes the proof.

# 3 Comments on the Literature

It only remains to make some comments on the existing literature.

(i) Houthakker [4, pp.162-163] introduced his semitransitivity axiom in terms of a demand function $h$ on the family of competitive budgets. Let $\Omega, p$, and $m$ be the commodity space, a competitive price vector, and an income. Then $h$ is a function on the family of all budget sets

$$B(p, m) = \{x \in \Omega | px \leq m\}. \tag{19}$$

We consider a sequence $\{x^\tau\}_{\tau=1}^{T}$ in $\Omega$ satisfying

$$x^\tau = h(p^\tau, p^\tau x^\tau) \qquad \text{for every} \qquad \tau \in \{1, 2, \ldots, T\}, \tag{20}$$

$$x^{\tau+1} \in B(p^\tau, p^\tau x^\tau) \qquad \text{for every} \qquad \tau \in \{1, 2, \ldots, T-1\}, \tag{21}$$

and

$$x^{\tau+1} \neq h(p^\tau, p^\tau x^\tau) \quad \text{for } \textit{at least one} \quad \tau \in \{1, 2, \ldots, T-1\}. \tag{22}$$

Houthakker's semitransitivity then requires that $x^1 \notin B(p^T, p^T x^T)$. It will be noticed that what we called Houthakker's axiom of revealed preference in Section 2.3 is a natural reformulation of this requirement in terms of a choice function $C$.

(ii) What Samuelson [8, pp. 370-371] called the strong axiom is the same requirement as Houthakker's, save for the replacement of (22) by

$$x^{\tau+1} \neq h(p^\tau, p^\tau x^\tau) \quad \text{for } \textit{every} \quad \tau \in \{1, 2, \ldots, T-1\}. \tag{23}$$

Our strong axiom of revealed preference in Section 2.3 is, it will be noticed, a natural extension of Samuelson's axiom in the context of a choice function.

(iii) We have shown in [10] that Hansson's strong axiom of revealed preference [3] is, despite its apparent difference, equivalent to that of ours in Section 2.3.

(iv) Richter [5, p.637] argued that "the revealed preference notions employed in [the Weak Axiom of Samuelson and the Strong Axiom of Houthakker] are relevant only to the special case of competitive consumers, so that axioms also have meaning only in that limited context." However, these axioms can be and have been generalized beyond the narrow confinement of competitive consumers. Besides, Richter himself defined in [6] the weak and the strong axioms for a *single-valued* choice function. There is no reason, furthermore, that we should not consider these axioms in terms of a set-valued choice function.

In conclusion, it is hoped that our result will help to clarify the central role played by Houthakker's axiom in the whole spectrum of revealed preference theory.

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# Chapter 3
# Consistent Rationalizability*

## 1 Introduction

Rationalizability is an important issue in the analysis of economic decisions. It provides a means to test theories of choice, including — but not limited to — traditional consumer demand theory. The central question to be addressed is as follows: are the observed choices of an economic agent compatible with our standard theories of choice as being motivated by optimizing behaviour? More precisely, can we find a preference relation with suitably defined properties that generates the observed choices as the choice of greatest or maximal elements according to this relation? This question has its origin in the theory of consumer demand but has since been explored in more general contexts, including both individual and collective choice. By formulating necessary and sufficient conditions for the existence of a rationalizing relation, testable restrictions on observable choice behaviour implied by the various theories are established.

Samuelson [12] began his seminal paper on revealed preference theory with a remark that "[f]rom its very beginning the theory of consumer's choice has marched steadily towards greater generality, sloughing off at successive stages unnecessarily restrictive conditions" (Samuelson [12, p.61]). Even after Samuelson [12; 13, Chapter V; 14; 15] laid the foundations of "the theory of consumer's behaviour freed from any vestigial traces of the utility concept" (Samuelson [12, p.71]), the exercise of Ockham's Razor persisted within revealed preference theory. Capitalizing on Georgescu-Roegen's [5, p.125; 15, p.222] observation that the intuitive justification of the axioms of revealed preference theory has nothing to do with the special form of budget sets, but instead, is based on the implicit consideration of choices from two-element sets. Arrow [2] expanded the analysis of rational choice and revealed preference beyond consumer choice problems. He pointed out that "the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened

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to include all finite sets" (Arrow, [2, p.122]). Sen [16, p.312] defended Arrow's domain assumption by posing two important questions: "why assume the axioms [of revealed preference] to be true only for 'budget sets' and not for others?" and "[a]re there reasons to expect that some of the rationality axioms will tend to be satisfied in choices over 'budget sets' but not for other choices?"

While it is certainly desirable to liberate revealed preference theory from the narrow confinement of budget sets, the admission of all finite subsets of the universal sets into the domain of a choice function may well be unsuitable for many applications. In this context, two important groups of contributions stand out. In the first place, Richter [10; 11], Hansson [7] and Suzumura [17; 19; 20, Chapter 2] developed the theory of rational choice and revealed preference for choice functions with general non-empty domains which do not impose any extraneous restrictions whatsoever on the class of feasible sets. In the second place, Sen [16] showed that Arrow's results (as well as others with similar features) do not hinge on the full power of the assumption that *all* finite sets are included in the domain of a choice function — it suffices if the domain contains all two-element and three-element sets.

It was in view of this current state of the art that Bossert, Sprumont and Suzumura [3] examined two crucial types of general domain in an analysis of several open questions in the theory of rational choice. The first is the general domain à la Richter, Hansson and Suzumura, and the second is the class of *base domains* which include all singletons and all two-element subsets of the universal set. The status of the general domain seems to be impeccable, as the theory developed on this domain is relevant in whatever choice situations we may care to specify. The base domains also seem to be on safe ground, as the concept of rational choice as maximizing choice is intrinsically connected with pairwise comparisons: singletons can be viewed as pairs with identical components, whereas two-element sets represent pairs of distinct alternatives. As Arrow [1, p.16] put it, "one of the consequences of the assumptions of rational choice is that the choice in any environment can be determined by a knowledge of the choices in two-element environments."

In this chapter, we focus on the rationalizability of choice functions by means of *consistent* relations. The concept of consistency was first introduced by Suzumura [18], and it is a weakening of transitivity requiring that any preference cycle should involve indifference only. As was shown by Suzumura [18; 20, Chapter 1], consistency is necessary and sufficient for the existence of an ordering extension of a binary relation. For that reason, consistency is a central property for the analysis of rational choice as well: in order to obtain a rationalizing relation that is an ordering, an extension procedure is, in general, required in order to ensure that the rationalization is complete. Violations of transitivity are quite likely to be observed in practical choice situations. For instance, Luce's [9] well-known coffee-sugar example provides a plausible argument against assuming that indifference is always transitive: the inability of a decision-maker to perceive 'small' differences in alternatives is bound to lead to intransitivities. As this example illustrates, transitivity frequently is too strong an assumption to impose in the context of individual choice. In collective choice problems, it is even more evident that the plausibility of transitivity can be questioned. On the other hand, it is difficult to interpret observed choices as 'rational' if they do not possess any coherence property. Because of Suzumura's [18] result, consistency can be considered a weakening of transitivity that is minimal in the sense that it cannot be weakened further without abandoning all hope of finding a rationalizing ordering extension.

To further underline the importance of consistency, note that this property is precisely what is required to prevent the problem of a 'money pump.' If consistency is violated, there exists a preference cycle with at least one strict preference. In this case, the agent under consideration is willing to trade an alternative $x_0$ for another alternative $x_1$ (where 'willingness to trade' is to be interpreted as being at least as well-off after the trade as before), $x_1$ for an alternative $x_2$ and so on until we reach an alternative $x_K$ such that the agent *strictly prefers* getting back $x_0$ to retaining possession of $x_K$. Thus, at the end of a chain of exchanges, the agent is willing to pay a positive amount in order to get back the alternative it had in its possession in the first place — a classical example of a money pump.

We examine consistent rationalizability under two domain assumptions. The first is, again, the general domain assumption where no restrictions whatsoever are imposed, and the second weakens the base domain hypothesis: we merely require the domain to contain all two-element sets but not necessarily all singletons, and we refer to those domains as *binary domains.* Thus, our results are applicable in a wide range of choice problems. Unlike many contributions to the theory of rational choice, we do not have to assume that triples are part of the domain. Especially the first domain assumption — the general domain — is highly relevant because it can accommodate any choice situation that arises in the analysis of both individual and collective choice. For instance, our results are applicable in traditional demand theory but in more general environments as well.

Depending on the additional properties that can be imposed on rationalizations (reflexivity and completeness), different notions of consistent rationalizability can be defined. We characterize all but one of those notions in the general case, and all of them in the case of binary domains. It is worth noting that we obtain full characterization results on binary domains (in particular, on domains that do not have to contain any triples), even though consistency imposes a restriction on possible cycles of any length.

In Section 2, the notation and our basic definitions are presented, along with some preliminary lemmas. Section 3 develops the theory of consistent rationalizability on general domains, whereas Section 4 expounds the corresponding theory on binary domains. Some concluding remarks are collected in Section 5.

# 2 Preliminaries

The set of positive (resp. non-negative) integers is denoted by $\mathbb{N}$ (resp. $\mathbb{N}_0$). For a set $S$, $|S|$ is the cardinality of $S$. Let $X$ be a universal non-empty set of alternatives. $\mathcal{X}$ is the power set of $X$ excluding the empty set. A choice function is a mapping $C\colon \Sigma \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \Sigma$, where $\Sigma \subseteq \mathcal{X}$ with $\Sigma \neq \emptyset$ is the domain of $C$. Note that $C$ maps $\Sigma$ into the set of all *non-empty* subsets of $X$. Thus, using Richter's [11] terminology, the choice function $C$ is assumed to be *decisive.* Let $C(\Sigma)$ denote the image of $\Sigma$ under $C$, that is, $C(\Sigma) = \cup_{S \in \Sigma} C(S)$. In addition to arbitrary non-empty domains, to be called *general domains,* we consider *binary domains* which are domains $\Sigma \subseteq \mathcal{X}$ such that $\{S \in \mathcal{X} \mid |S| = 2\} \subseteq \Sigma$.

Let $R \subseteq X \times X$ be a (binary) relation on $X$. The asymmetric factor $P(R)$ of $R$ is given by $(x, y) \in P(R)$ if and only if $(x, y) \in R$ and $(y, x) \notin R$ for all $x, y \in X$. The symmetric factor $I(R)$ of $R$ is defined by $(x, y) \in I(R)$ if and only if $(x, y) \in R$ and $(y, x) \in R$ for all $x, y \in X$. The non-comparable factor $N(R)$ of $R$ is given by $(x, y) \in N(R)$ if and only if $(x, y) \notin R$ and $(y, x) \notin R$ for all $x, y \in X$.

A relation $R \subseteq X \times X$ is (i) *reflexive* if, for all $x \in X$, $(x, x) \in R$; (ii) *complete* if, for all $x, y \in X$ such that $x \neq y$, $(x, y) \in R$ or $(y, x) \in R$; (iii) *transitive* if, for all $x, y, z \in X$, $[(x, y) \in R$ and $(y, z) \in R]$ implies $(x, z) \in R$; (iv) *consistent* if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$; (v) *$P$-acyclical* if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$.

The *transitive closure* of $R \subseteq X \times X$ is denoted by $\overline{R}$, that is, for all $x, y \in X$, $(x, y) \in \overline{R}$ if there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and $x^K = y$. Clearly, $\overline{R}$ is transitive and, because we can set $K = 1$, it follows that $R \subseteq \overline{R}$. For future reference, we state the following well-known result the proof of which is straightforward and thus omitted (see Suzumura [20, pp.11–12]).

**Lemma 1** *Let $R$ and $Q$ be binary relations on $X$. If $R \subseteq Q$, then $\overline{R} \subseteq \overline{Q}$.*

The *direct revealed preference relation* $R_C \subseteq X \times X$ of a choice function $C$ with an arbitrary domain $\Sigma$ is defined as follows. For all $x, y \in X$, $(x, y) \in R_C$ if there exists $S \in \Sigma$ such that $x \in C(S)$ and $y \in S$. The *(indirect) revealed preference relation* of $C$ is the transitive closure $\overline{R_C}$ of the direct revealed preference relation $R_C$.

For $S \in \Sigma$ and a relation $R \subseteq X \times X$, the set of $R$-greatest elements in $S$ is $\{x \in S \mid (x, y) \in R$ for all $y \in S\}$, and the set of $R$-maximal elements in $S$ is $\{x \in S \mid (y, x) \notin P(R)$ for all $y \in S\}$. A choice function $C$ is greatest-element rationalizable if there exists a relation $R$ on $X$, to be called a *G-rationalization,* such that $C(S)$ is equal to the set of $R$-greatest elements in $S$ for all $S \in \Sigma$. $C$ is maximal-element rationalizable if there exists a relation $R$ on $X$, to be called an *M-rationalization,* such that $C(S)$ is equal to the set of $R$-maximal elements in $S$ for all $S \in \Sigma$. We use the term *rationalization* in general discussions where it is not specified whether greatest-element rationalizability or maximal-element rationalizability is considered.

If a rationalization is required to be reflexive and complete, the notions of greatest-element rationalizability and that of maximal-element rationalizability coincide. Without these properties, however, this is not necessarily the case. Greatest-element rationalizability is based on the idea of chosen alternative weakly dominating all alternatives in the feasible set under consideration, whereas maximal-element rationalizability requires chosen elements not to be strictly dominated by any other feasible alternative. Specific examples illustrating the differences between those two concepts will be discussed later.

Depending on the properties that we might want to impose on a rationalization, different notions of rationalizability can be defined. For simplicity of presentation, we use the following notation. **G** (resp. **RG**; **CG**; **RCG**) stands for greatest-element rationalizability by means of a consistent (resp. reflexive and consistent; complete and consistent; reflexive, complete and consistent) G-rationalization. Analogously, **M** (resp. **RM**; **CM**; **RCM**) is maximal-element rationalizability by means of a consistent (resp. reflexive and consistent; complete and consistent; reflexive, complete and consistent) M-rationalization. Note that we do not identify consistency explicitly in these acronyms even though it is assumed to be satisfied by the rationalization in question. This is because consistency is required in all of the theorems presented in this paper, so that the use of another piece of notation would be redundant and likely increase the complexity of our exposition. However, note that the two lemmas stated below do not require consistency. In particular, the implication of part (i) of Lemma 3 does not apply to rationalizability by a consistent relation; see also Theorem 1.

We conclude this section with two further preliminary results. We first present the following lemma, the first part of which is due to Samuelson [12; 14]; see also Richter [11]. It states that the direct revealed preference relation must be contained in any G-rationalization and, moreover, that if an alternative $x$ is directly revealed preferred to an alternative $y$, then $y$ cannot be strictly preferred to $x$ by any M-rationalization.

**Lemma 2** *(i) If $R$ is a G-rationalization of $C$, then $R_C \subseteq R$.*
*(ii) If $R$ is an M-rationalization of $C$, then $R_C \subseteq R \cup N(R)$.*

**Proof.** (i) Suppose that $R$ is a G-rationalization of $C$ and $x, y \in X$ are such that $(x, y) \in R_C$. By definition of $R_C$, there exists $S \in \Sigma$ such that $x \in C(S)$ and $y \in S$. Because $R$ is a G-rationalization of $C$, we obtain $(x, y) \in R$.

(ii) Suppose $R$ is an M-rationalization of $C$ and $x, y \in X$ are such that $(x, y) \in R_C$. By way of contradiction, suppose $(x, y) \notin R \cup N(R)$. Therefore, $(y, x) \in P(R)$. Because $R$ maximal-element rationalizes $C$, this implies $x \notin C(S)$ for all $S \in \Sigma$ such that $y \in S$. But this contradicts the hypothesis $(x, y) \in R_C$. ■

Our final preliminary observation concerns the relationship between maximal-element rationalizability and greatest-element rationalizability when no further restrictions are imposed on a rationalization. This applies, in particular, when consistency is not imposed. Moreover, an axiom that is necessary for either form of rationalizability is presented. This requirement is referred to as the V-axiom in Richter [11]; we call it direct-revelation coherence in order to have a systematic terminology throughout this chapter.

**Direct-Revelation Coherence:** For all $S \in \Sigma$, for all $x \in S$, if $(x, y) \in R_C$ for all $y \in S$, then $x \in C(S)$.

Suzumura [17] establishes that, in the absence of any requirements on a rationalization, maximal-element rationalizability implies greatest-element rationalizability. Furthermore, Richter [11] shows that direct-revelation coherence is necessary for greatest-element rationalizability by an arbitrary G-rationalization on an arbitrary domain. We summarize these observations in the following lemma. For completeness, we provide a proof.

**Lemma 3** *(i) If $C$ is maximal-element rationalizable, then $C$ is greatest-element rationalizable.*

*(ii) If $C$ is greatest-element rationalizable, then $C$ satisfies direct-revelation coherence.*

**Proof.** (i) Suppose $R$ is an M-rationalization of $C$. It is straightforward to verify that $R' = \{(x, y) \mid (y, x) \notin P(R)\}$ is a G-rationalization of $C$.

(ii) Suppose $R$ is a G-rationalization of $C$, and let $S \in \Sigma$ and $x \in S$ be such that $(x, y) \in R_C$ for all $y \in S$. By part (i) of Lemma 2, $(x, y) \in R$ for all $y \in S$. Because $R$ is a G-rationalization of $C$, this implies $x \in C(S)$. ■

There are alternative notions of rationality such as that of Kim and Richter [8] who proposed the concept of *motivated choice:* $C$ is a motivated choice if there exist a relation $R$ on $X$, which is to be called a *motivation* of $C$, such that

$$C(S) = \{x \in S \mid (y, x) \notin R \text{ for all } y \in S\}$$

for all $S \in \Sigma$. This property is implied by maximal-element rationalizability but the converse implication is not true. Moreover, $C$ is a motivated choice if and only if $C$ is greatest-element rationalizable. Indeed, $R$ greatest-element rationalizes $C$ if and only if its dual $R^d$, which is defined by

$$(x, y) \in R^d \Leftrightarrow (y, x) \notin R$$

for all $x, y \in X$, is a motivation of $C$.

Richter [11] shows that direct-revelation coherence is not only necessary but also sufficient for greatest-element rationalizability on an arbitrary domain, without any further restrictions imposed on the G-rationalization. Moreover, the axiom is necessary and sufficient for greatest-element rationalizability by a reflexive (but otherwise unrestricted) rationalization on an arbitrary domain. The requirement remains, of course, necessary for greatest-element rationalizability if we restrict attention to binary domains. As shown below, if we add consistency as a requirement on a rationalization, direct-revelation coherence by itself is sufficient for neither greatest-element rationalizability nor for maximal-element rationalizability, even on binary domains.

# 3 General Domains

In this section, we impose no restrictions on the domain $\Sigma$. We begin our analysis by providing a full description of the logical relationships between the different notions of rationalizability that can be defined, given our consistency assumption imposed on a rationalization. The possible definitions of rationalizability that can be obtained depend on whether reflexivity or completeness are added to consistency. Furthermore, a distinction between greatest-element rationalizability and maximal-element rationalizability is made. For convenience, a diagrammatic representation is employed: all axioms that are depicted within the same box are equivalent, and an arrow pointing from one box $b$ to another box $b'$ indicates that the axioms in $b$ imply those in $b'$, and no further implications are true without additional assumptions regarding the domain of $C$.

**Theorem 1** *Suppose* $\Sigma$ *is a general domain. Then*

$$\begin{array}{ccc} & \boxed{\textbf{RCG}, \textbf{CG}, \textbf{RCM}, \textbf{CM}} & \\ \swarrow & & \searrow \\ \boxed{\textbf{RG}, \textbf{G}} & & \boxed{\textbf{RM}, \textbf{M}} \end{array}$$

**Proof.** We proceed as follows. In Step 1, we prove the equivalence of all axioms that appear in the same box. In Step 2, we show that all implications depicted in the theorem statement are valid. In Step 3, we provide examples demonstrating that no further implications are true in general.

**Step 1.** For each of the three boxes, we show that all axioms listed in the box are equivalent.

**1.a.** We first prove the equivalence of the axioms in the top box.

Clearly, **RCG** implies **CG** and **RCM** implies **CM**. Moreover, if a relation $R$ is reflexive and complete, it follows that the set of $R$-greatest elements in $S$ is equal to the set of $R$-maximal elements in $S$ for any $S \in \Sigma$. Therefore, **RCG** and **RCM** are equivalent.

To see that **CM** implies **RCM**, suppose $R$ is a consistent and complete M-rationalization of $C$. Let

$$R' = R \cup \{(x, x) \mid x \in X\}.$$

Clearly, $R'$ is reflexive. $R'$ is consistent and complete because $R$ is. That $R'$ is an M-rationalization of $C$ follows immediately from the observation that $R$ is.

To complete Step 1.a of the proof, it is sufficient to show that **CG** implies **RCG**. Suppose $R$ is a consistent and complete G-rationalization of $C$. Let

$$R' = [R \cup \Delta \cup \{(y, x) \mid x \notin C(\Sigma) \ \& \ y \in C(\Sigma)\}] \setminus \{(x, y) \mid x \notin C(\Sigma) \ \& \ y \in C(\Sigma)\},$$

where $\Delta = \{(x, x) | x \in X\}$. Clearly, $R'$ is reflexive by definition.

To show that $R'$ is complete, let $x, y \in X$ be such that $x \neq y$ and $(x, y) \notin R'$. By definition of $R'$, this implies

$$(x, y) \notin R \text{ and } [x \notin C(\Sigma) \text{ or } y \in C(\Sigma)]$$

or

$$x \notin C(\Sigma) \text{ and } y \in C(\Sigma).$$

If the former applies, the completeness of $R$ implies $(y, x) \in R$ and, by definition of $R'$, we obtain $(y, x) \in R'$. If the latter is true, $(y, x) \in R'$ follows immediately from the definition of $R'$.

Next, we show that $R'$ is consistent. Let $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ be such that $(x^{k-1}, x^k) \in R'$ for all $k \in \{1, \ldots, K\}$. Clearly, we can, without loss of generality, assume that $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. We distinguish two cases.

(i) $x^0 \notin C(\Sigma)$. In this case, it follows that $x^1 \notin C(\Sigma)$; otherwise we would have $(x^1, x^0) \in P(R')$ by definition of $R'$, contradicting our hypothesis. Successively applying this argument to all $k \in \{1, \ldots, K\}$, we obtain $x^k \notin C(\Sigma)$ for all $k \in \{1, \ldots, K\}$. By definition of $R'$, this implies $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. By the consistency of $R$, we must have $(x^K, x^0) \notin P(R)$. Because $x^K \notin C(\Sigma)$, this implies, according to the definition of $R'$, $(x^K, x^0) \notin P(R')$.

(ii) $x^0 \in C(\Sigma)$. If $x^K \notin C(\Sigma)$, $(x^K, x^0) \notin P(R')$ follows immediately from the definition of $R'$. If $x^K \in C(\Sigma)$, it follows that $x^{K-1} \in C(\Sigma)$; otherwise we would have $(x^{K-1}, x^K) \notin R'$ by definition of $R'$, contradicting our hypothesis. Successively applying this argument to all $k \in \{1, \ldots, K\}$, we obtain $x^k \in C(\Sigma)$ for all $k \in \{1, \ldots, K\}$. By definition of $R'$, this implies $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. By the consistency of $R$, we must have $(x^K, x^0) \notin P(R)$. Because $x^0 \in C(\Sigma)$, this implies, according to the definition of $R'$, $(x^K, x^0) \notin P(R')$.

Finally, we show that $R'$ is a G-rationalization of $C$. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $(x, y) \in R'$ for all $y \in S$. If $|S| = 1$, $x \in C(S)$ follows immediately because $C(S)$ is non-empty. If $|S| \geq 2$, we obtain $x \in C(\Sigma)$. Because $R$ is a G-rationalization of $C$, this implies $(x, x) \in R$. By definition of $R'$, $(x, z) \in R$ for all $z \in C(S)$. Therefore, $(x, z) \in R$ for all $z \in C(S) \cup \{x\}$. Suppose, by way of contradiction, that $x \notin C(S)$. Because $R$ is a G-rationalization of $C$, it follows that there exists $y \in S \setminus (C(S) \cup \{x\})$ such that $(x, y) \notin R$. The completeness of $R$ implies $(y, x) \in P(R)$. Let $z \in C(S)$. It follows that $(z, y) \in R$ because $R$ is a G-rationalization of $C$ and, as established earlier, $(x, z) \in R$. This contradicts the consistency of $R$.

To prove the converse implication, suppose $x \in C(S)$. Because $R$ is a G-rationalization of $C$, we have $(x, y) \in R$ for all $y \in S$. In particular, this implies $(x, x) \in R$ and, according to the definition of $R'$, we obtain $(x, y) \in R'$ for all $y \in S$.

**1.b.** The proof that **RM** and **M** are equivalent is analogous to the proof of the equivalence of **RCM** and **CM** in Step 1.a.

**1.c.** Clearly, **RG** implies **G**. Conversely, suppose $R$ is a consistent G-rationalization of $C$. Let

$$R' = (R \cup \mathit{\Delta}) \setminus \{(x, y) \mid x \notin C(\Sigma) \ \&\ x \neq y\}.$$

Clearly, $R'$ is reflexive.

Next, we prove that $R'$ is consistent. Let $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ be such that $(x^{k-1}, x^k) \in R'$ for all $k \in \{1, \ldots, K\}$. Again, we can assume that $x^{k-1} \neq x^k$ for all

$k \in \{1, \ldots, K\}$. By definition of $R'$, $x^0 \in C(\Sigma)$. The rest of the proof follows as in part (ii) of the consistency of the relation $R'$ in Step 1.a.

It remains to be shown that $R'$ is a G-rationalization of $C$. Let $S \in \Sigma$ and $x \in S$.

First, suppose $(x, y) \in R'$ for all $y \in S$. By definition of $R'$, $(x, y) \in R$ for all $y \in S \setminus \{x\}$. Analogously to the corresponding argument in Step 1.a, the assumption $x \notin C(S)$ implies the existence of $y \in S \setminus (C(S) \cup \{x\})$ such that $(x, y) \notin R$, a contradiction.

Finally, suppose $x \in C(S)$. This implies $(x, y) \in R$ for all $y \in S$ because $R$ is a G-rationalization of $C$. Furthermore, because $C(S) \subseteq C(\Sigma)$, we have $x \in C(\Sigma)$. By definition of $R'$, this implies $(x, y) \in R'$ for all $y \in S$.

**Step 2.** The implications corresponding to the arrows in the theorem statement are straightforward.

**Step 3.** Given Steps 1 and 2, to prove that no further implications are valid, it is sufficient to provide examples showing that (a) **M** does not imply **G**; and (b) **G** does not imply **M**. Note that this independence of **M** and **G** in the presence of consistency does not contradict part (i) of Lemma 3—consistency is not required in the lemma.

**3.a. M** does not imply **G**.

**Example 1** Let $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. For future reference, note that $\Sigma$ is a binary domain. Define the choice function $C$ by letting $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x, z\}$ and $C(\{y, z\}) = \{y\}$. This choice function is maximal-element rationalizable by the consistent (and reflexive) rationalization

$$R = \{(x, x), (y, y), (y, z), (z, z)\}.$$

Suppose $C$ is greatest-element rationalizable by a consistent rationalization $R'$. Because $C(\Sigma) = X$, greatest-element rationalizability implies that $R'$ is reflexive. Therefore, because $y \in C(\{y, z\})$ and $z \notin C(\{y, z\})$, we must have $(y, z) \in R'$ and $(z, y) \notin R'$. Therefore, $(y, z) \in P(R')$. Because $R'$ is a G-rationalization of $C$, $z \in C(\{x, z\})$ implies $(z, x) \in R'$ and $x \in C(\{x, y\})$ implies $(x, y) \in R'$. This yields a contradiction to the assumption that $R'$ is consistent.

**3.b.** To prove that **G** does not imply **M**, we employ an example due to Suzumura [17, pp.151–152].

**Example 2** Let $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{x, z\}, \{x, y, z\}\}$, and define $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x, z\}$ and $C(\{x, y, z\}) = \{x\}$. This choice function is greatest-element rationalizable by the consistent (and reflexive) rationalization

$$R = \{(x, x), (x, y), (x, z), (y, x), (y, y), (z, x), (z, z)\}.$$

Suppose $R'$ is an M-rationalization of $C$. Because $z \in C(\{x, z\})$, maximal-element rationalizability implies $(x, z) \notin P(R')$ and, consequently, $z \notin C(\{x, y, z\})$ implies

$(y, z) \in P(R')$. Analogously, $y \in C(\{x, y\})$ implies, together with maximal-element rationalizability, $(x, y) \notin P(R')$ and, consequently, $y \notin C(\{x, y, z\})$ implies $(z, y) \in P(R')$. But this contradicts the above observation that we must have $(y, z) \in P(R')$. Note that consistency (or any other property) of $R'$ is not invoked in the above argument. Moreover, $R$ is reflexive. Thus, **RG** does not even imply maximal-element rationalizability by an arbitrary rationalization. ■

We now provide characterizations of two of the three notions of rationalizability identified in the above theorem. The first is a straightforward consequence of Richter's [10] result and the observation that consistency is equivalent to transitivity in the presence of reflexivity and completeness. Richter [10] shows that the *congruence* axiom is necessary and sufficient for greatest-element rationalizability by a transitive, reflexive and complete rationalization. Congruence is defined as follows.

**Congruence:** For all $x, y \in X$, for all $S \in \Sigma$, if $(x, y) \in \overline{R_C}$, $y \in C(S)$ and $x \in S$, then $x \in C(S)$.

We obtain the following:

**Theorem 2** *$C$ satisfies* **RCG** *if and only if $C$ satisfies congruence.*

**Proof.** As is straightforward to verify, a relation is consistent, reflexive and complete if and only if it is transitive, reflexive and complete. The result now follows immediately from the equivalence of congruence and greatest-element rationalizability by a transitive, reflexive and complete rationalization established by Richter [10]. ■

In order to characterize **G** (and, therefore, **RG**; see Theorem 1), we employ the *consistent closure* of the direct revealed preference relation $R_C$. The consistent closure of a relation $R$ is analogous to the transitive closure: the idea is to add all pairs to the relation $R$ that must be in a G-rationalizing relation due to the requirement that the rationalization be consistent. Define the consistent closure $R^*$ of $R$ by

$$R^* = R \cup \{(x, y) \mid (x, y) \in \overline{R} \;\&\; (y, x) \in R\}.$$

Clearly, $R \subseteq R^*$. To illustrate the definition of the consistent closure and its relationship to the transitive closure, consider the following examples.

**Example 3** Let $X = \{x, y, z\}$ and $R = \{(x, x), (x, y), (y, y), (y, z), (z, x), (z, z)\}$. We obtain $R^* = \overline{R} = X \times X$.

**Example 4** Let $X = \{x, y, z\}$ and $R = \{(x, y), (y, z)\}$. We have $R^* = R$ and $\overline{R} = \{(x, y), (y, z), (x, z)\}$.

In Example 3, the consistent closure coincides with the transitive closure, whereas in Example 3, the consistent closure is a strict subset of the transitive closure. More generally, $R^*$ is always a subset of $\overline{R}$; see Lemma 4 below. Moreover, the lemma establishes an important property of $R^*$: just as $\overline{R}$ is the smallest transitive relation containing $R$, $R^*$ is the smallest consistent relation containing $R$.

**Lemma 4** *Let $R$ be a binary relation on $X$.*
*(i) $R^* \subseteq \overline{R}$.*
*(ii) $R^*$ is the smallest consistent relation containing $R$.*

**Proof.** (i) Suppose that $(x, y) \in R^*$. By definition,

$$(x, y) \in R \text{ or } [(x, y) \in \overline{R} \ \& \ (y, x) \in R].$$

If $(x, y) \in R$, $(x, y) \in \overline{R}$ follows because $R \subseteq \overline{R}$. If $[(x, y) \in \overline{R} \ \& \ (y, x) \in R]$, $(x, y) \in \overline{R}$ is implied trivially.

(ii) We first prove that $R^*$ is consistent. Suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in R^*$ for all $k \in \{1, \ldots, K\}$. We show that $(x^K, x^0) \notin P(R^*)$. By part (i) of the lemma, $(x^{k-1}, x^k) \in \overline{R}$ for all $k \in \{1, \ldots, K\}$, and the transitivity of $\overline{R}$ implies

$$(x^0, x^K) \in \overline{R}. \tag{1}$$

If $(x^K, x^0) \notin R^*$, we immediately obtain $(x^K, x^0) \notin P(R^*)$ and we are done. Now suppose that $(x^K, x^0) \in R^*$. By definition of $R^*$, we must have

$$(x^K, x^0) \in R \text{ or } [(x^K, x^0) \in \overline{R} \ \& \ (x^0, x^K) \in R].$$

If $(x^K, x^0) \in R$, (1) and the definition of $R^*$ together imply $(x^0, x^K) \in R^*$ and, thus, $(x^K, x^0) \notin P(R^*)$. If $(x^K, x^0) \in \overline{R}$ and $(x^0, x^K) \in R$, $(x^0, x^K) \in R^*$ follows because $R \subseteq R^*$. Again, this implies $(x^K, x^0) \notin P(R^*)$ and the proof that $R^*$ is consistent is complete.

To show that $R^*$ is the smallest consistent relation containing $R$, suppose that $Q$ is an arbitrary consistent relation containing $R$. To complete the proof, we establish that $R^* \subseteq Q$. Suppose that $(x, y) \in R^*$. By definition of $R^*$,

$$(x, y) \in R \text{ or } [(x, y) \in \overline{R} \ \& \ (y, x) \in R].$$

If $(x, y) \in R$, $(x, y) \in Q$ follows because $R$ is contained in $Q$ by assumption. If $(x, y) \in \overline{R}$ and $(y, x) \in R$, Lemma 1 and the assumption $R \subseteq Q$ together imply that $(x, y) \in \overline{Q}$ and $(y, x) \in Q$. If $(x, y) \notin Q$, we obtain $(y, x) \in P(Q)$ in view of $(y, x) \in Q$. Since $(x, y) \in \overline{Q}$, this contradicts the consistency of $Q$. Therefore, we must have $(x, y) \in Q$. ■

Analogously to Lemma 2 (i), we obtain the following:

**Lemma 5** *If $R$ is a consistent G-rationalization of $C$, then $R_C^* \subseteq R$.*

**Proof.** Suppose that $R$ is a consistent G-rationalization of $C$ and $(x, y) \in R_C^*$. By definition,

$$(x, y) \in R_C \text{ or } [(x, y) \in \overline{R_C} \ \& \ (y, x) \in R_C].$$

If $(x, y) \in R_C$, Lemma 2 implies $(x, y) \in R$. If $(x, y) \in \overline{R_C}$ and $(y, x) \in R_C$, Lemmas 2 and 1 together imply $(x, y) \in \overline{R}$ and $(y, x) \in R$. If $(x, y) \notin R$, it follows that $(y, x) \in P(R)$ in view of $(y, x) \in R$. Because $(x, y) \in \overline{R}$, this contradicts the consistency of $R$. Therefore, $(x, y) \in R$. ■

Part (ii) of Lemma 2 does not generalize in an analogous fashion. To see this, consider again Example 1. We have $R_C^* = X \times X$ and, therefore, $(z, y) \in R_C^*$. But $(y, z) \in P(R)$ according to the consistent M-rationalization $R$ defined in the example, and it follows that $R_C^* \nsubseteq R \cup N(R)$.

The following axiom is a strengthening of direct-revelation coherence which we call consistent-closure coherence. It is obtained by replacing $R_C$ with its consistent closure $R_C^*$ in the definition of direct-revelation coherence.

**Consistent-Closure Coherence:** For all $S \in \Sigma$, for all $x \in S$, if $(x, y) \in R_C^*$ for all $y \in S$, then $x \in C(S)$.

We now obtain the following:

**Theorem 3** *$C$ satisfies* **G** *if and only if $C$ satisfies consistent-closure coherence.*

**Proof.** To prove the only-if part of the theorem, suppose $R$ is a consistent G-rationalization of $C$ and let $S \in \Sigma$ and $x \in S$ be such that $(x, y) \in R_C^*$ for all $y \in S$. By Lemma 5, $(x, y) \in R$ for all $y \in S$. Thus, because $R$ is a G-rationalization of $C$, $x \in C(S)$.

Now suppose $C$ satisfies consistent-closure coherence. We complete the proof by showing that $R_C^*$ is a consistent G-rationalization of $C$. That $R_C^*$ is consistent follows from Lemma 4. To prove that $R_C^*$ is a G-rationalization of $C$, suppose first that $S \in \Sigma$ and $x \in S$. Suppose $(x, y) \in R_C^*$ for all $y \in S$. Consistent-closure coherence implies $x \in C(S)$. Conversely, suppose $x \in C(S)$. By definition, this implies $(x, y) \in R_C$ for all $y \in S$ and, because $R_C \subseteq R_C^*$, we obtain $(x, y) \in R_C^*$ for all $y \in S$. ■

# 4 Binary Domains

We now turn to the special case of binary domains. These domains are of interest because they represent a natural weakening of some domains studied in the earlier literature on rational choice. In particular, the binary-domain assumption is implied by the requirement that $\Sigma$ contains all non-empty and finite subsets of $X$, by the assumption that the domain contains all pairs and all triples and by the requirement that $\Sigma$ is a base domain. Moreover, binary domains occur naturally in applications such as tournaments where a pairwise comparison of all agents is performed; consider, for example, a round-robin tournament.

In the case of binary domains, the presence of all two-element sets in $\Sigma$ guarantees that every G-rationalization must be complete and, as a consequence, all rationality requirements involving greatest-element rationalizability and consistency become equivalent. In contrast, maximal-element rationalizability by a consistent and complete rationalization remains a stronger requirement than maximal-element rationalizability by a consistent and reflexive rationalization. These observations are summarized in the following theorem.

**Theorem 4** *Suppose $\Sigma$ is a binary domain. Then*

$$\boxed{\textbf{RCG}, \textbf{CG}, \textbf{RCM}, \textbf{CM}, \textbf{RG}, \textbf{G}}$$
$$\downarrow$$
$$\boxed{\textbf{RM}, \textbf{M}}$$

**Proof.** We divide the proof into the same three steps as in Theorem 1.

**Step 1.** We prove the equivalence of the axioms for each of the two boxes.

**1.a.** Using Theorem 1, the equivalence of the axioms in the top box follows from the observation that any consistent G-rationalization of $C$ must be complete, given that $\Sigma$ is binary.

**1.b.** This part is already proven in Theorem 1.

**Step 2.** Again, the implication corresponding to the arrow in the theorem statement is straightforward.

**Step 3.** To prove that the reverse implication is not valid, Example 1 can be employed. ■

As shown in Theorem 4, there are only two different versions of rationalizability for binary domains. Consequently, we can restrict attention to the rationalizability axioms **G** and **M** in this case, keeping in mind that, by Theorem 4, all other rationalizability requirements involving consistent rationalizations are covered as well. Although there are some analogies between the results in this section and some of the theorems established in Bossert, Sprumont and Suzumura [3], our characterizations are novel because, unlike the earlier paper, they employ consistency and they apply to binary domains rather than base domains.

First, we show that **G** (and all other axioms that are equivalent to it according to Theorem 4) is characterized by the following weak congruence axiom (see Bossert, Sprumont and Suzumura, [3]).

**Weak Congruence:** For all $x, y, z \in X$, for all $S \in \Sigma$, if $(x, y) \in R_C$, $(y, z) \in R_C$, $x \in S$ and $z \in C(S)$, then $x \in C(S)$.

In contrast to congruence, weak congruence applies not to chains of direct revealed preference of an arbitrary length, but merely to chains involving three elements. For binary domains, weak congruence is necessary and sufficient for all forms of greatest-element rationalizability involving a consistent G-rationalization.

**Theorem 5** *Suppose $\Sigma$ is a binary domain. $C$ satisfies* **G** *if and only if $C$ satisfies weak congruence.*

**Proof.** By Theorem 4, **G** is equivalent to **RCG** given that $\Sigma$ is a binary domain. Moreover, as mentioned earlier, consistency is equivalent to transitivity in the presence of reflexivity and completeness. Theorem 3 in Bossert, Sprumont and Suzumura [3] states that greatest-element rationalizability by a reflexive, complete and transitive relation is equivalent to weak congruence, provided that $\Sigma$ is a binary domain. The result follows immediately as a consequence of this observation. ■

Finally, we establish that direct-revelation coherence and P-acyclicity of $R_C$ together are necessary and sufficient for **M** (and **RM**) on a binary domain. This result is analogous to the characterization of greatest-element rationalizability by a P-acyclical, reflexive and complete rationalization on base domains (domains that contain all singletons in addition to all two-element sets) in Bossert, Sprumont and Suzumura ([3, Theorem 5]).

**Theorem 6** *Suppose $\Sigma$ is a binary domain. $C$ satisfies* **M** *if and only if $C$ satisfies direct-revelation coherence and $R_C$ is P-acyclical.*

**Proof.**

**Step 1.** We first show that **M** implies that $R_C$ is P-acyclical (that direct-revelation coherence is implied follows from Lemma 3). Suppose $R$ is a consistent M-rationalization of $C$. By way of contradiction, suppose $R_C$ is not P-acyclical. Then there exist $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ such that $(x^{k-1}, x^k) \in P(R_C)$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R_C)$. Because $\Sigma$ is a binary domain, $\{x^{k-1}, x^k\} \in \Sigma$ for all $k \in \{1, \ldots, K\}$ and $\{x^0, x^K\} \in \Sigma$. By definition of $R_C$, it follows that $x^k \notin C(\{x^{k-1}, x^k\})$ for all $k \in \{1, \ldots, K\}$ and $x^0 \notin C(\{x^0, x^K\})$. Because $R$ is an M-rationalization of $C$, it follows that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R)$, contradicting the consistency of $R$.

**Step 2.** We show that direct-revelation coherence and the P-acyclicity of $R_C$ together imply **M**. Define $R = P(R_C)$. Clearly, $P(R) = R = P(R_C)$ and, consequently, $R$ is consistent because $R_C$ is P-acyclical.

It remains to be shown that $R$ is an M-rationalization of $C$. Let $S \in \Sigma$ and $x \in S$. Suppose first that $x$ is $R$-maximal in $S$, that is, $(y, x) \notin P(R)$ for all $y \in S$. If $S = \{x\}$, $x \in C(S)$ follows from the non-emptiness of $C(S)$. Now suppose $S \neq \{x\}$, and let $y \in S \setminus \{x\}$. Because $\Sigma$ is a binary domain, $\{x, y\} \in \Sigma$. If $x \in C(\{x, y\})$, we obtain $(x, y) \in R_C$ by definition. If $x \notin C(\{x, y\})$, it follows that $(y, x) \in R_C$ and, because $(y, x) \notin P(R) = P(R_C)$ by assumption, we again obtain $(x, y) \in R_C$. By direct-revelation coherence, it follows that $x \in C(S)$.

Now suppose $x \in C(S)$. This implies $(x, y) \in R_C$ for all $y \in S$ and, therefore, $(y, x) \notin P(R_C) = P(R)$ for all $y \in S$. Therefore, $x$ is $R$-maximal in $S$. ■

# 5 Concluding Remarks

The only notion of consistent rationalizability that is not characterized in this chapter is maximal-element rationalizability by means of a consistent (and reflexive) rationalization on a general domain. The reason why it is difficult to obtain necessary and sufficient conditions in that case is the existential nature of the requirements for maximal-element rationalizability. It is immediately apparent that the revealed preference relation must be respected by any greatest-element rationalization, whereas this is not the case for maximal-element rationalizability (see Lemma 3). In order to exclude an element from a set of chosen alternatives according to maximal-element rationalizability, it merely is required that there exists (at least) *one* element in that set which is strictly preferred to the alternative to be excluded. The problem of identifying necessary and sufficient conditions for that kind of rationalizability is closely related to the problem of determining the *dimension* of a quasi-ordering; see, for example, Dushnik and Miller [4]. Because this is an area that is still quite unsettled, it is not too surprising that characterizations of maximal-element rationalizability on general domains are difficult to obtain. To the best of our knowledge, this is a feature that is shared by *all* notions of maximal-element rationalizability that are not equivalent to one of the notions of greatest-element rationalizability on general domains: we are not aware of any characterization results for maximal-element rationalizability on general domains unless the notion of maximal-element rationalizability employed happens to coincide with one of the notions of greatest-element rationalizability. Thus, there are important open questions to be addressed in future work in this area of research.

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# Chapter 4
# Rationalizability of Choice Functions on General Domains Without Full Transitivity*

## 1 Introduction

The intuitive conception of rational choice as optimizing behaviour, irrespective of the nature of the objective to be optimized (be it by a single agent or by a group of individuals), has been studied extensively in the literature. Beginning with the revealed preference theory of consumer demand on competitive markets, which is due to Samuelson [11; 12, Chapter V; 13; 14] and Houthakker [6], the early phase of the theory of rational choice was devoted to the analysis of choices from budget sets only.

Uzawa [22] and Arrow [1] freed this theory from this exclusive concern by introducing the general concept of a choice function defined on the domain of all subsets of a universal set of alternatives. Following this avenue, Sen [16; 17], Schwartz [15], Bandyopadhyay and Sengupta [2], and many others succeeded in characterizing optimizing choice corresponding to fine demarcations in the degree of consistency of the objective to be maximized. Most notably, the theory of rational choice on such full domains was greatly simplified by the equivalence results between several revealed preference axioms, for example, the weak axiom of revealed preference and the strong axiom of revealed preference, whose subtle difference had been regarded as lying at the heart of the integrability problem for

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a competitive consumer. However, this simplification was obtained at a price which some may think is much too high. Instead of assuming that the domain of a choice function consists solely of the set of budget sets, it is assumed that "the domain includes all finite subsets of [the universal set of alternatives for choice] whether or not it includes any other subset" (Sen [16; 17, p.47]). It deserves emphasis that "it is not necessary that even all finite sets be included in the domain. All the results and proofs would continue to hold even if the domain includes all pairs and triples but not all finite sets" (Sen, [16; 17, pp.48-49]).

Whatever stance one may want to take *vis-à-vis* Sen's argument in favor of his domain assumption, it is interesting to see what we can make of the concept of a rational choice function irrespective of which assumption we care to specify on its domain, thereby focusing directly on what the logic of rational choice — and nothing else — entails in general. A crucial step along this line was taken by Richter [9, 10], Hansson [4] and Suzumura [19; 20; 21, Chapter 2] who assumed the domain of a choice function to be an arbitrary family of non-empty subsets of an arbitrary non-empty universal set of alternatives without any algebraic or topological structure. These authors succeeded in axiomatizing the concept of a fully rational choice function, that is, a choice function resulting from the optimization of an underlying transitive preference ordering. Yet, "cold winds blow through unstructured sets" (Howard [7, p.xvii]), and there remains a large gap between the theory of rational choice functions with the Arrow-Sen domain and that with the Richter-Hansson domain. In more concrete terms, the Richter-Hansson approach has not yet delivered an axiomatization of rational choice functions where the underlying preference relation is not fully transitive but possesses weaker properties such as quasi-transitivity or acyclicity. Such weakenings of transitivity are particularly relevant in the context of *social* choice. The purpose of this chapter is to narrow down this gap along two lines.

In the first place, we focus on choice functions defined on what we call base domains, which contain all singletons and pairs of alternatives included in some universal set. On these domains, we provide axiomatizations of choice functions rationalized by preference relations that are not fully transitive. In addition, a new characterization of transitive rational choice is provided for those domains. The concept of a rational choice as an optimizing choice is binary in nature in that the choice from any (possibly very large) set is to be accounted for in terms of a binary relation. In this sense, base domains seem to be the most natural domains to work with in the theory of rational choice. Triples need not be included in a base domain even though consistency properties involving three or more alternatives (namely, quasi-transitivity and acyclicity) are imposed, a feature which distinguishes our approach from the Arrow-Sen framework.

In the second place, we develop new necessary conditions for choice functions defined on arbitrary domains to be rationalized by preference relations that are merely quasi-transitive or acyclical. Furthermore, in the acyclical case, we present a new sufficient condition, and we prove that it is weaker than a set of sufficient conditions in the earlier literature.

Within the context of the consistency properties of transitivity, quasi-transitivity, and acyclicity, we also explore the implications of all possible notions of rational choice as an optimizing choice both on arbitrary domains and on base domains. Furthermore, we analyze both maximal-element rationalizability and greatest-element rationalizability (see Sen [18], for example).

The remainder of this chapter is organized as follows. Section 2 introduces the concept of rationalizability, along with some preliminary observations. Logical relationships are examined in Section 3. Section 4 contains our characterization results on base domains. In Section 5, we present sufficient conditions and necessary conditions on arbitrary domains. Section 6 concludes.

## 2 Rationalizable Choice Functions

The set of positive (resp. non-negative) integers is denoted by $\mathbb{N}$ (resp. $\mathbb{N}_0$). For a set $S$, $|S|$ is the cardinality of $S$. Let $X$ be a universal non-empty set of alternatives. $\mathcal{X}$ is the power set of $X$ excluding the empty set. A choice function is a mapping $C\colon \Sigma \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \Sigma$, where $\Sigma \subseteq \mathcal{X}$ with $\Sigma \neq \emptyset$ is the domain of $C$. In addition to arbitrary non-empty domains, we consider *binary domains* which are domains $\Sigma \subseteq \mathcal{X}$ such that $\{S \in \mathcal{X} \mid |S| = 2\} \subseteq \Sigma$, and *base domains* which are domains $\Sigma \subseteq \mathcal{X}$ such that $\{S \in \mathcal{X} \mid |S| = 1 \text{ or } |S| = 2\} \subseteq \Sigma$.

Let $R \subseteq X \times X$ be a binary relation on $X$. The asymmetric factor $P(R)$ of $R$ is given by $(x, y) \in P(R)$ if and only if $(x, y) \in R$ and $(y, x) \notin R$ for all $x, y \in X$.

A relation $R \subseteq X \times X$ is (i) reflexive if, for all $x \in X$, $(x, x) \in R$; (ii) complete if, for all $x, y \in X$ such that $x \neq y$, $(x, y) \in R$ or $(y, x) \in R$; (iii) transitive if, for all $x, y, z \in X$, $[(x, y) \in R$ and $(y, z) \in R]$ implies $(x, z) \in R$; (iv) quasi-transitive if $P(R)$ is transitive; (v) acyclical if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$; (vi) asymmetric if, for all $x, y \in X$, $(x, y) \in R$ implies $(y, x) \notin R$.

The transitive closure of $R \subseteq X \times X$ is denoted by $\overline{R}$, that is, for all $x, y \in X$,

$(x, y) \in \overline{R}$ if there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x^0 = x$, $x^K = y$ and $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Clearly, $\overline{R}$ is transitive and, because we can set $K = 1$, it follows that $R \subseteq \overline{R}$.

The *direct revealed preference relation* $R_C \subseteq X \times X$ of a choice function $C$ with an arbitrary domain $\Sigma$ is defined as follows. For all $x, y \in X$, $(x, y) \in R_C$ if there exists $S \in \Sigma$ such that $x \in C(S)$ and $y \in S$. The *(indirect) revealed preference relation* of $C$ is the transitive closure $\overline{R_C}$ of the direct revealed preference relation $R_C$. If $\Sigma$ is a base domain, the *base relation* $B_C \subseteq X \times X$ of $C$ is defined by letting, for all $x, y \in X$, $(x, y) \in B_C$ if $x \in C(\{x, y\})$.

For $S \in \Sigma$ and a relation $R \subseteq X \times X$, the set of $R$-greatest elements in $S$ is $G(S, R) = \{x \in S \mid (x, y) \in R \text{ for all } y \in S\}$, and the set of $R$-maximal elements in $S$ is $M(S, R) = \{x \in S \mid (y, x) \notin P(R) \text{ for all } y \in S\}$. A choice function $C$ is *greatest-element rationalizable* if there exists a relation $R$ on $X$ such that $C(S) = G(S, R)$ for all $S \in \Sigma$. $C$ is *maximal-element rationalizable* if there exists a relation $R$ on $X$ such that $C(S) = M(S, R)$ for all $S \in \Sigma$.

Ours is the standard definition of maximal-element rationalizability that is used in the traditional literature on rational (social and individual) choice. We are aware that there are interesting alternatives such as that of Kim and Richter [8] who proposed the concept of *motivated choice:* $C$ is a motivated choice if there exists a relation $R$ on $X$, which is to be called a *motivation* of $C$, such that

$$C(S) = \{x \in S \mid (y, x) \notin R \text{ for all } y \in S\}$$

for all $S \in \Sigma$. This property is implied by maximal-element rationalizability but the converse implication is not true. Moreover, $C$ is a motivated choice if and only if $C$ is greatest-element rationalizable. Indeed, $R$ greatest-element rationalizes $C$ if and only if its dual $R^d$, defined by

$$(x, y) \in R^d \Leftrightarrow (y, x) \notin R$$

for all $x, y \in X$, is a motivation of $C$. Because our version of maximal-element rationalizability has been employed extensively in the literature and because we think it has strong intuitive appeal, we use it in this chapter. In fact, the analysis of the differences that emerge between the two traditional versions of rationalizability once full transitivity is no longer required is one of the major purposes of this chapter.

Depending on the properties that we might want to impose on a rationalizing relation, different notions of rationalizability can be defined. In particular, our focus is on transitivity, quasi-transitivity, and acyclicity. We refer to those properties as consistency conditions, and we let T,Q,A,R, and C stand for transitivity, quasi-transitivity, acyclicity, reflexivity, and completeness, respectively. Each notion of rationalizability is identified by a list of properties assumed to be satisfied by the rationalizing relation, followed by the type of rationalizability (greatest-element or maximal-element rationalizability). For example, QC-G means greatest-element rationalizability by a quasi-transitive and complete relation, ARC-M is maximal-element rationalizability by an acyclical, reflexive, and complete relation, etc.

We conclude this section with some preliminary results. We first present the following lemma, due to Samuelson [11; 13]; see also Richter [10]. It states that the direct revealed preference relation must be contained in any greatest-element rationalizing relation.

**Lemma 1 (Samuelson [11; 13])** *If $R$ greatest-element rationalizes $C$, then $R_C \subseteq R$.*

If $R$ is transitive and greatest-element rationalizes $C$, it also follows that the strict preference relation corresponding to $R_C$ must be contained in the strict preference relation of $R$, that is, $P(R_C) \subseteq P(R)$ (see Bossert [3]). On an arbitrary domain, this result is no longer true if transitivity is weakened to quasi-transitivity.

**Example 1** *Let $X = \{x, y\}$, $\Sigma = \{\{x, y\}\}$, and $C(\{x, y\}) = \{x\}$. The relation $R$ defined by*

$$R = \{(x, x), (x, y), (y, x)\}$$

*is quasi-transitive and greatest-element rationalizes $C$ but we have*

$$P(R_C) = \{(x, y)\} \not\subseteq \emptyset = P(R).$$

Even if a greatest-element rationalizing relation $R$ is reflexive and complete, $P(R_C)$ need not be contained in $P(R)$.

**Example 2** *Let $X = \{x, y, z, w\}$, $\Sigma = \{\{x, z\}, \{x, y, w\}\}$, and define $C(\{x, z\}) = \{z\}$ and $C(\{x, y, w\}) = \{x, w\}$. The relation $R$ given by*

$$R = (X \times X) \setminus \{(x, z), (y, w)\}$$

*is quasi-transitive, reflexive, and complete and greatest-element rationalizes $C$. We have $(x, y) \in P(R_C)$ and $(x, y) \notin P(R)$ and, hence, $P(R_C) \not\subseteq P(R)$.*

The implication discussed above does hold on a base domain even if no consistency requirement such as transitivity, quasi-transitivity or acyclicity is imposed.

**Lemma 2** *Suppose $\Sigma$ is a base domain. If $R$ greatest-element rationalizes $C$, then $P(R_C) \subseteq P(R)$.*

**Proof.** Suppose $(x, y) \in P(R_C)$ for some $x, y \in X$. This implies $(x, y) \in R_C$ and, by Lemma 1, $(x, y) \in R$. By way of contradiction, suppose $(y, x) \in R$. Because $\Sigma$ is a base domain, $\{y\} \in \Sigma$. By the non-emptiness of $C(\{y\})$, $y \in C(\{y\})$. Hence, $(y, y) \in R_C$ and, using Lemma 1 again, $(y, y) \in R$. Because $\Sigma$ is a base domain, $\{x, y\} \in \Sigma$. Because $(y, x) \in R$, $(y, y) \in R$, and $R$ greatest-element rationalizes $C$, we must have $y \in C(\{x, y\})$ and hence $(y, x) \in R_C$. But this contradicts the assumption that $(x, y) \in P(R_C)$. ■

A final preliminary observation concerns an axiom that is necessary for greatest-element rationalizability even without any restrictions on a rationalizing relation. This requirement is referred to as the V-axiom in Richter [10]; we call it D-congruence (D for "direct revelation") in order to have a systematic terminology throughout this chapter.

**D-Congruence:** For all $S \in \Sigma$, for all $x \in S$, if $(x, y) \in R_C$ for all $y \in S$, then $x \in C(S)$.

We state Richter's [10] result that D-congruence is necessary for greatest-element rationalizability by an arbitrary relation on an arbitrary domain.

**Lemma 3 (Richter [10])** *If $C$ is greatest-element rationalizable, then $C$ satisfies D-congruence.*

Richter [10] shows that D-congruence is not only necessary but also sufficient for greatest-element rationalizability by an arbitrary binary relation on an arbitrary domain. Moreover, the axiom is necessary and sufficient for greatest-element rationalizability by a reflexive (but otherwise unrestricted) relation on an arbitrary domain. The requirement remains, of course, necessary for rationalizability if we restrict attention to base domains. However, if we add a consistency requirement such as transitivity, quasi-transitivity, or acyclicity, D-congruence by itself is not sufficient for rationalizability, even on base domains.

# 3 Logical Relationships

We provide a full description of the logical relationships between the different notions of rationalizability that can be defined in this setting. The possible definitions of rationalizability that can be obtained depend subtly on which consistency requirement is adopted (namely, transitivity, quasi-transitivity or acyclicity) and on whether reflexivity or completeness are added. Furthermore, a distinction between greatest-element rationalizability and maximal-element rationalizability is made. Note that, without any additional properties, maximal-element rationalizability implies greatest-element rationalizability but this implication is no longer valid if additional properties are imposed on a rationalization; see Theorem 1 below.

We state all logical relationships between the different notions of rationality analyzed in this chapter in two theorems — one for arbitrary domains and one for base domains. For convenience, a diagrammatic representation is employed: all axioms that are depicted within the same box are equivalent, and an arrow pointing from one box $b$ to another box $b'$ indicates that the axioms in $b$ imply those in $b'$, and the converse implication is not true. In addition, of course, all implications resulting from chains of arrows depicted in the diagram are valid.

**Theorem 1** *Suppose $\Sigma$ is a general domain. Then*

$$
\begin{tikzcd}
& \boxed{\textit{TRC-G, TC-G, TR-G, T-G, TRC-M, TC-M}} \arrow[d] & \\
& \boxed{\textit{TR-M, T-M, QRC-G, QRC-M, QC-M, QR-M, Q-M}} \arrow[dl] \arrow[dr] & \\
\boxed{\textit{ARC-G, ARC-M, AC-M, AR-M, A-M}} \arrow[d] & & \boxed{\textit{QC-G}} \arrow[d] \arrow[dll] \\
\boxed{\textit{AC-G}} \arrow[d] & & \boxed{\textit{QR-G}} \arrow[d] \\
\boxed{\textit{AR-G, A-G}} & & \boxed{\textit{Q-G}} \arrow[ll]
\end{tikzcd}
$$

**Proof.** We proceed as follows. In Step 1, we prove the equivalence of all axioms that appear in the same box. In Step 2, we show that all implications depicted in the theorem statement are valid. In Step 3, we demonstrate that no further implications are true other than those resulting from chains of implications established in Step 2.

**Step 1.** We prove the equivalence of the axioms for each of the four boxes containing more than one axiom.

**1.a.** We first prove the equivalence of the axioms in the top box.

That TRC-G, TC-G, TR-G, and T-G are equivalent follows directly from Richter's [9; 10] results.

Because maximal elements and greatest elements coincide for a reflexive and complete relation, it follows that TRC-G and TRC-M are equivalent.

Finally, we show that TC-M implies TRC-M. Suppose $R$ is a transitive and complete relation that maximal-element rationalizes $C$, that is, $C(S) = M(S, R)$ for all $S \in \Sigma$. Let

$$R' = R \cup \{(x, x) \mid x \in X\}.$$

It follows immediately that $R'$ is reflexive, complete, and transitive. Furthermore, $P(R') = P(R)$ and, therefore, $M(S, R') = M(S, R) = C(S)$ for all $S \in \Sigma$, which implies that $R'$ maximal-element rationalizes $C$.

**1.b.** Next, we prove that the axioms in the second box from the top are equivalent.

That TR-M and T-M are equivalent can be shown using the same construction as in the proof of the equivalence of TRC-M and TC-M.

Clearly, QRC-G and QRC-M are equivalent because greatest and maximal elements coincide for a reflexive and complete relation.

Next, we show that Q-M implies QRC-M. Suppose $R$ is a quasi-transitive relation that maximal-element rationalizes $C$. Define $R'$ by

$$R' = \{(x, y) \in X \times X \mid (y, x) \notin P(R)\}. \tag{1}$$

Regardless of the properties possessed by $R$, $R'$ is always reflexive and complete and, furthermore, $P(R') = P(R)$ and hence

$$G(S, R') = M(S, R') = M(S, R) \text{ for all } S \in \Sigma. \tag{2}$$

Since $R$ is quasi-transitive and $P(R') = P(R)$, $R'$ is quasi-transitive as well.

That $R'$ maximal-element rationalizes $C$ follows immediately from (2) and the assumption that $R$ maximal-element rationalizes $C$.

To complete this part of the proof, it is sufficient to establish the equivalence of T-M and QRC-G.

First, we show that T-M implies QRC-G. Suppose $C$ is maximal-element rationalizable by a transitive relation $R$. Define the relation $R'$ as in (1). As in the argument proving the previous implication, $R'$ is quasi-transitive, reflexive, and complete, and (2) implies that $R'$ greatest-element rationalizes $C$.

To prove that QRC-G implies T-M, suppose $R$ is a quasi-transitive, reflexive, and complete relation that greatest-element rationalizes $C$. Define

$$R' = P(R).$$

$R'$ is transitive because $R$ is quasi-transitive. Furthermore, we have $P(R') = P(R)$ and hence

$$M(S, R') = M(S, R) = G(S, R) \text{ for all } S \in \Sigma, \tag{3}$$

where the second equality follows from reflexivity and completeness of $R$. Since $R$ greatest-element rationalizes $C$ it follows from (3) that $R'$ maximal-element rationalizes $C$.

**1.c.** We prove that the axioms ARC-G and all axioms involving maximal-element rationalizability by an acyclical relation are equivalent.

Again, the equivalence of ARC-G and ARC-M follows immediately because the greatest and maximal elements of a reflexive and complete relation coincide.

Finally, we show that A-M implies ARC-M. Suppose $R$ is an acyclical relation that maximal-element rationalizes $C$. Define $R'$ as in (1). Again, it is clear that $R'$ is reflexive and complete. Since $R$ is acyclical and $P(R') = P(R)$, $R'$ is acyclical as well. The argument showing that $R'$ maximal-element rationalizes $C$ is identical to the one used in 1.b.

**1.d.** To complete the first part of the proof, it remains to be shown that A-G implies AR-G. Suppose $R$ is acyclical and greatest-element rationalizes $C$. Define

$$R' = (R \cup \{(x, x) \mid x \in X\}) \setminus \{(x, y) \in X \times X \mid (x, x) \notin R \text{ and } y \neq x\}. \tag{4}$$

Clearly, $R'$ is reflexive. Furthermore, by definition of $R'$, we have

$$[(x, x) \notin R \Rightarrow (x, y) \notin R'] \text{ for all } x \in X, \text{ for all } y \in X \setminus \{x\}. \tag{5}$$

Now suppose $R'$ is not acyclical. Then there exist $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ such that $(x^{k-1}, x^k) \in P(R')$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R')$. Clearly, we can, without loss of generality, assume that the $x^k$ are pairwise distinct. By (5), $(x^{k-1}, x^{k-1}) \in R$ for all $k \in \{1, \ldots, K+1\}$. But this implies that we have $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R)$ by definition of $R'$, contradicting the acyclicity of $R$.

We now prove that $R'$ greatest-element rationalizes $C$. For future reference, note that the argument used in the proof does not depend on any of the properties of $R$ other than the observation that it greatest-element rationalizes $C$. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. Because $R$ greatest-element rationalizes $C$, we have $(x, y) \in R$ for all $y \in S$ which, in particular, implies $(x, x) \in R$. Therefore, by (4), $(x, y) \in R'$ for all $y \in S$ and hence $x \in G(S, R')$.

Now suppose $x \in G(S, R')$. Therefore, $(x, y) \in R'$ for all $y \in S$. If $S = \{x\}$, $x \in C(S)$ follows from the non-emptiness of $C(S)$. If there exists $y \in S$ such that $y \neq x$, (5) implies $(x, x) \in R$. Therefore, because $(x, y) \in R'$ implies $(x, y) \in R$ for all $y \in S$ and $R$ greatest-element rationalizes $C$, we immediately obtain $x \in C(S)$.

**Step 2.** The only non-trivial implication is that QC-G implies QR-G. Suppose $R$ is quasi-transitive and complete and greatest-element rationalizes $C$. Define the (reflexive) relation $R'$ as in (4). Next, we prove that $R'$ is quasi-transitive. Suppose $(x, y) \in P(R')$ and $(y, z) \in P(R')$. By (5), $(x, x) \in R$ and $(y, y) \in R$. Suppose $(x, y) \notin P(R)$. Because $R$ is complete, we have $(y, x) \in R$. Because $(y, y) \in R$, it follows that $(y, x) \in R'$ by definition of $R'$, contradicting $(x, y) \in P(R')$. Therefore, $(x, y) \in P(R)$.

We now distinguish two cases.

**Case a.** $(z, z) \in R$. Analogously to the above proof demonstrating that $(x, y) \in P(R)$, we obtain $(y, z) \in P(R)$ in this case. Because $R$ is quasi-transitive, it follows that $(x, z) \in P(R)$. Because $(x, x) \in R$ and $(x, z) \in P(R)$, we must have $(x, z) \in R'$ by definition of $R'$. Furthermore, $(z, x) \notin R$ implies $(z, x) \notin R'$ by definition of $R'$ and, consequently, we obtain $(x, z) \in P(R')$.

**Case b.** $(z, z) \notin R$. By (5), we obtain $(z, x) \notin R'$. Suppose $(x, z) \notin R'$. Because $(x, x) \in R$, this implies $(x, z) \notin R$ by definition of $R'$ and hence $(z, x) \in P(R)$ by the completeness of $R$. Because $R$ is quasi-transitive, we obtain $(z, y) \in P(R)$ and hence $(y, z) \notin R$. Because $(z, z) \notin R$, the definition of $R'$ implies $(y, z) \notin R'$, contradicting $(y, z) \in P(R')$. Therefore, $(x, z) \in R'$ and, because $(z, x) \notin R'$, it follows that $(x, z) \in P(R')$.

That $R'$ greatest-element rationalizes $C$ can be shown using the same proof as in 1.d.

**Step 3.** To prove that no further implications other than those resulting from Step 2 are valid, it is sufficient to provide examples showing that (a) QRC-G does not imply T-G; (b) QC-G does not imply ARC-G; (c) ARC-G does not imply Q-G; (d) QR-G does not imply AC-G; and (e) Q-G does not imply QR-G.

**3.a.** QRC-G does not imply T-G.

**Example 3** *Let* $X = \{x, y, z\}$ *and* $\Sigma = \mathcal{X} \setminus \{\{x, y, z\}\}$. *Define the choice function* $C$ *by letting* $C(\{t\}) = \{t\}$ *for all* $t \in X$, $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{z\}$, *and* $C(\{y, z\}) = \{y, z\}$. *This choice function is greatest-element rationalizable by the quasi-transitive, reflexive, and complete relation*

$$R = (X \times X) \setminus \{(x, z)\}.$$

*Suppose* $C$ *is greatest-element rationalizable by a transitive relation* $R'$. *Because* $x \in C(\{x, y\})$, *we must have* $(x, x) \in R'$ *and* $(x, y) \in R'$. *Analogously,* $y \in C(\{y, z\})$ *implies*

*$(y,z) \in R'$. By the transitivity of $R'$, it follows that $(x,z) \in R'$ and, together with $(x,x) \in R'$, $x \in C(\{x,z\})$, contradicting the definition of $C$.*

**3.b.** QC-G does not imply ARC-G.

**Example 4** *Let $X = \{x,y,z,w\}$ and $\Sigma = \{\{x,w\},\{y,z\},\{y,w\},\{x,y,w\},\{y,z,w\}\}$, and define $C(\{x,w\}) = \{w\}$, $C(\{y,z\}) = \{y\}$, $C(\{y,w\}) = \{y,w\}$, $C(\{x,y,w\}) = \{w\}$, and $C(\{y,z,w\}) = \{y\}$. This choice function is greatest-element rationalized by the quasi-transitive and complete relation $R$ given by*

$$\{(x,y),(x,z),(x,w),(y,y),(y,z),(y,w),(z,x),(z,y),(z,w),(w,x),(w,y),(w,w)\}.$$

*Suppose $R'$ is acyclical, reflexive, and complete and greatest-element rationalizes $C$.*

*Because $C(\{y,z\}) = \{y\}$ and $R'$ is reflexive, we obtain $(y,z) \in P(R')$. Analogously, because $C(\{x,w\}) = \{w\}$ and $R'$ is reflexive, we must have $(w,x) \in P(R')$.*

*Because $y \in C(\{y,w\})$ and $y \notin C(\{x,y,w\})$, we must have $(y,x) \notin R'$ and, because $R'$ is complete, it follows that $(x,y) \in P(R')$. Analogously, because $w \in C(\{y,w\})$ and $w \notin C(\{y,z,w\})$, we must have $(w,z) \notin R'$ and, because $R'$ is complete, it follows that $(z,w) \in P(R')$.*

*Therefore, we have established that $(x,y) \in P(R')$, $(y,z) \in P(R')$, $(z,w) \in P(R')$, and $(w,x) \in P(R')$, contradicting the acyclicity of $R'$.*

**3.c.** ARC-G does not imply Q-G.

**Example 5** *Let $X = \{x,y,z\}$ and $\Sigma = \mathcal{X} \setminus \{\{x,y,z\}\}$. Define the choice function $C$ by letting $C(\{t\}) = \{t\}$ for all $t \in X$, $C(\{x,y\}) = \{x\}$, $C(\{x,z\}) = \{x,z\}$, and $C(\{y,z\}) = \{y\}$. This choice function is greatest-element rationalizable by the acyclical, reflexive, and complete relation*

$$R = \{(x,x),(x,y),(x,z),(y,y),(y,z),(z,x),(z,z)\}.$$

*Suppose $C$ is greatest-element rationalizable by a quasi-transitive relation $R'$. Because $y \in C(\{y,z\})$, we have $(y,y) \in R'$. Therefore, $y \notin C(\{x,y\})$ implies $(x,y) \in P(R')$. Analogously, $z \in C(\{x,z\})$ implies $(z,z) \in R'$ and, therefore, $z \notin C(\{y,z\})$ implies $(y,z) \in P(R')$. Because $R'$ is quasi-transitive, it follows that $(x,z) \in P(R')$ and hence $(z,x) \notin R'$. Because $R'$ greatest-element rationalizes $C$, this implies $z \notin C(\{x,z\})$, contradicting the definition of $C$.*

**3.d.** QR-G does not imply AC-G.

**Example 6** *Let* $X = \{x, y, z\}$ *and* $\Sigma = \{\{x, y\}, \{x, z\}, \{x, y, z\}\}$. *Define the choice function* $C$ *by letting* $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x, z\}$, *and* $C(\{x, y, z\}) = \{x\}$. $C$ *is greatest-element rationalizable by the quasi-transitive and reflexive relation*

$$R = \{(x, x), (x, y), (x, z), (y, x), (y, y), (z, x), (z, z)\},$$

*but it cannot be greatest-element rationalized by a complete relation. By way of contradiction, suppose* $R'$ *is such a relation. By completeness, we must have*

$$(y, z) \in R' \tag{6}$$

*or*

$$(z, y) \in R'. \tag{7}$$

*Suppose (6) is true. Because* $R'$ *greatest-element rationalizes* $C$ *and* $y \in C(\{x, y\})$, *it follows that* $(y, x) \in R'$ *and* $(y, y) \in R'$. *Together with (6) and the greatest-element rationalizability of* $C$ *by* $R'$, *we obtain* $y \in C(\{x, y, z\})$, *contradicting the definition of* $C$.

*Now suppose (7) is true. Because* $R'$ *greatest-element rationalizes* $C$ *and* $z \in C(\{x, z\})$, *it follows that* $(z, x) \in R'$ *and* $(z, z) \in R'$. *Together with (7) and the greatest-element rationalizability of* $C$ *by* $R'$, *we obtain* $z \in C(\{x, y, z\})$, *contradicting the definition of* $C$.

**3.e.** Q-G does not imply QR-G.

**Example 7** *Let* $X = \{x, y, z, w\}$ *and* $\Sigma = \{\{x, y\}, \{y, z\}, \{z, w\}, \{x, z, w\}\}$, *and define the choice function* $C$ *by letting* $C(\{x, y\}) = \{y\}$, $C(\{y, z\}) = \{z\}$, $C(\{z, w\}) = \{z, w\}$, *and* $C(\{x, z, w\}) = \{w\}$. *This choice function is greatest-element rationalized by the quasi-transitive relation* $R$ *given by*

$$\{(x, y), (y, x), (y, y), (z, y), (z, z), (z, w), (w, x), (w, z), (w, w)\}.$$

*Suppose* $R'$ *is quasi-transitive and reflexive and greatest-element rationalizes* $C$. *By reflexivity,* $(x, x) \in R'$ *and, because* $x \notin C(\{x, y\})$, *we must have* $(y, x) \in P(R')$. *Because* $y \in C(\{x, y\})$, *it follows that* $(y, y) \in R'$ *and, hence,* $y \notin C(\{y, z\})$ *implies* $(z, y) \in P(R')$. *By quasi-transitivity, we obtain* $(z, x) \in P(R')$.

*Because* $z \in C(\{z, w\})$, *it follows that* $(z, z) \in R'$ *and* $(z, w) \in R'$. *Together with* $(z, x) \in P(R')$ *and the assumption that* $R'$ *greatest-element rationalizes* $C$, *we obtain* $z \in C(\{x, z, w\})$, *which contradicts the definition of* $C$.

■

Regarding the implications presented in the above theorem, it is worth pointing out some surprising differences between those notions of rationalizability encompassing transitivity and those that merely require quasi-transitivity or acyclicity. Most strikingly, as soon as we weaken full transitivity to quasi-transitivity, not even reflexivity is implied as a property of a greatest-element rationalizing relation. On the other hand, all notions of maximal-element rationalizability coincide if merely quasi-transitivity rather than full transitivity is required.

The results regarding the logical relationships between our rationalizability axioms simplify dramatically when base domains are considered. The presence of all one-element and two-element sets in $\Sigma$ guarantees that every greatest-element rationalizing relation must be reflexive and complete and, as a consequence, all rationality requirements involving greatest-element rationalizability with a given consistency requirement become equivalent. All implications of Theorem 1 are preserved and, other than those just mentioned, there are no additional ones. Those demanding transitivity are stronger than those where merely quasi-transitivity is required which, in turn, imply (but are not implied by) all axioms where the rationalizing relation is acyclical. These observations are summarized in the following theorem.

**Theorem 2** *Suppose $\Sigma$ is a base domain. Then*

$$\boxed{\textit{TRC-G, TC-G, TR-G, T-G, TRC-M, TC-M}}$$

$$\downarrow$$

$$\boxed{\textit{TR-M, T-M, QRC-G, QC-G, QR-G, Q-G, QRC-M, QC-M, QR-M, Q-M}}$$

$$\downarrow$$

$$\boxed{\textit{ARC-G, AC-G, AR-G, A-G, ARC-M, AC-M, AR-M, A-M}}$$

The implications and equivalences of the theorem follow immediately from Theorem 1 and the assumption that $\Sigma$ is a base domain. Furthermore, Examples 3 and 5 can be employed to demonstrate that the implications between boxes are strict. Thus, no formal proof is required.

As shown in Theorem 2, there are only three different versions of rationalizability for base domains. As a consequence, we can restrict attention to the rationalizability axioms

TRC-G, QRC-G, and ARC-G in this case, keeping in mind that, by Theorem 2, all other rationalizability requirements discussed in this chapter are covered as well by our results.

Note that, in the case of transitive greatest-element rationalizability, all definitions of rationalizability are equivalent even if $\Sigma$ only contains all two-element sets but not necessarily the singletons; this is a consequence of the observation that if $R$ is a transitive (and complete) relation greatest-element rationalizing $C$, it is always possible to find a reflexive and transitive (and complete) relation that contains $R$ and rationalizes $C$ as well; see Richter [9; 10]. Therefore, the equivalence of all axioms involving a transitive greatest-element rationalization can be established for binary domains as well. We do not state the corresponding result formally as a separate theorem because our focus is on quasi-transitive and acyclical rationalizability in this chapter.

# 4 Characterizations for Binary and Base Domains

If we restrict attention to base domains (that is, domains $\Sigma$ that contain all one-element and two-element sets), the analysis of quasi-transitive and acyclical rationalizability is significantly less complex than in the case of an arbitrary domain, and we obtain "clean" characterization results. In addition, we formulate a new characterization result regarding transitive rationalizability for binary domains. The full power of a base domain is not required in the transitive case because reflexivity can always be added as a property of a rationalizing relation as long as transitivity is satisfied.

Note that the assumption of having a base domain differs in an important aspect from the assumption used by Sen [16], which stipulates that not only the sets of cardinality two, but also those of cardinality three are in $\Sigma$. (Sen did not require the singletons to be in the domain due to the observation that, for transitive greatest-element rationalizability, reflexivity can always be added as a property of a greatest-element rationalizing relation — see the discussion at the end of the previous section.) It is interesting to note that, in order to obtain useful and applicable results for quasi-transitive and acyclical rationalizing relations, those sets of cardinality three are not required in the domain, even though these consistency properties impose restrictions on three or more alternatives which may be distinct. It is not sufficient to assume that we have a binary domain (that is, a domain containing all two-element sets). The singleton sets are needed if transitivity is weakened to quasi-transitivity or acyclicity because, without full transitivity, reflexivity of a rationalizing relation can no longer be guaranteed. Base domains have also been used by Herzberger [5] but he did not pursue the same questions we address in this chapter.

## 4.1 Transitive Rationalizability

In the case of binary domains, we obtain a new characterization of TRC-G that employs a weaker axiom than Richter's [9] congruence axiom to be defined in Section 5. This axiom — which we call T-congruence — is defined as follows.

**T-Congruence:** For all $x, y, z \in X$, for all $S \in \Sigma$, if $(x, y) \in R_C$, $(y, z) \in R_C$, $x \in S$ and $z \in C(S)$, then $x \in C(S)$.

Note that, in contrast to congruence, T-congruence does not apply to chains of direct revealed preference of an arbitrary length but merely to chains involving three elements. For binary domains, T-congruence is necessary and sufficient for TRC-G. Of course, T-congruence is necessary for greatest-element rationalizability by a transitive relation on an arbitrary domain but it is not sufficient unless specific domain assumptions are made.

**Example 8** *Let $X = \{x, y, z, w\}$, $\Sigma = \{\{x, y\}, \{y, z\}, \{z, w\}, \{x, w\}\}$, and define $C$ by $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\})$, $C(\{z, w\}) = \{z\}$, and $C(\{x, w\}) = \{w\}$. This choice function satisfies T-congruence but it is not greatest-element rationalizable by a transitive relation. To see this, suppose $R$ is transitive and greatest-element rationalizes $C$. The definition of $C$ yields $(x, y) \in R_C$, $(y, z) \in R_C$, $(z, w) \in R_C$, and $(x, x) \in R_C$. By Lemma 1, $(x, y) \in R$, $(y, z) \in R$, $(z, w) \in R$, and $(x, x) \in R$. Because $R$ is transitive, we must have $(x, w) \in R$. By definition of greatest-element rationalizability, it follows that $x \in C(\{x, w\})$, a contradiction.*

It is an interesting observation that binary domains are sufficient to obtain results of that nature involving transitivity, even though those domains do not necessarily contain all triples. This is in contrast to Sen's [16] results which crucially depend on having all triples available in the domain. We obtain

**Theorem 3** *Suppose $\Sigma$ is a binary domain. $C$ satisfies TRC-G if and only if $C$ satisfies T-congruence.*

**Proof.** Let $\Sigma$ be a binary domain. This implies that $R_C$ is complete—see the proof of Theorem 2.

**Step 1.** That TRC-G implies T-congruence follows immediately from Richter's [9] result and the observation that T-congruence is weaker than his congruence axiom.

**Step 2.** We show that T-congruence implies TRC-G. Let

$$R = R_C \cup \{(x, x) \mid x \in X\}.$$

Clearly, $R$ is reflexive by definition, and $R$ is complete because $R_C$ is complete. Next, we prove that $R$ is transitive. Suppose $(x, y) \in R$ and $(y, z) \in R$ for some $x, y, z \in X$. If $x = z$, $(x, z) \in R$ follows from the reflexivity of $R$. If $x \neq z$, it follows that $\{x, z\} \in \Sigma$ because $\Sigma$ is a binary domain. By T-congruence, $x \in C(\{x, z\})$ and hence $(x, z) \in R_C$ which, by Lemma 1, implies $(x, z) \in R$.

Finally, we show that $R$ greatest-element rationalizes $C$. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x, y) \in R_C$ for all $y \in S$ and hence $(x, y) \in R$ for all $y \in S$. Hence, $x \in G(S, R)$.

Now suppose $x \in G(S, R)$, that is, $(x, y) \in R$ for all $y \in S$. If $S = \{x\}$, we have $C(S) = \{x\}$ because $C(S)$ is non-empty and hence $(x, x) \in R_C$. If there exists $y \in S$ such that $y \neq x$, it follows that $(x, y) \in R_C$ and, by definition of $R_C$, $x$ must be chosen for some feasible set in $\Sigma$. Thus, again, $(x, x) \in R_C$. Therefore, $(x, y) \in R_C$ for all $y \in S$. Let $z \in C(S)$. This implies $(z, z) \in R_C$. Because $z \in S$, $(x, z) \in R_C$. Letting $y = z$ in the definition of T-congruence, the axiom implies $x \in C(S)$. ■

Clearly, if $\Sigma$ is a base domain rather than merely a binary domain, $R_C$ is reflexive and can be used as the rationalizing relation in the above theorem. A corollary of Theorem 3 is that, on a base domain, T-congruence and congruence are equivalent. In general, this need not be the case: congruence always implies T-congruence but the reverse implication is not true on all domains.

## 4.2 Quasi-Transitive Rationalizability

To obtain a set of necessary and sufficient conditions for QRC-G in the case of a base domain, we add the following Q-congruence axiom to the D-congruence axiom introduced in Section 2.

**Q-Congruence:** For all $x, y, z \in X$, for all $S \in \Sigma$, if $(x, y) \in P(R_C)$, $(y, z) \in P(R_C)$ and $x \in S$, then $z \notin C(S)$.

Together with D-congruence, Q-congruence guarantees that the direct revealed preference relation $R_C$ is quasi-transitive. Note, again, that we do not need to impose a restriction regarding chains of (strict) revealed preferences of arbitrary length. We obtain

**Theorem 4** *Suppose $\Sigma$ is a base domain. $C$ satisfies QRC-G if and only if $C$ satisfies D-congruence and Q-congruence.*

**Proof.** Let $\Sigma$ be a base domain. Therefore, $R_C$ is reflexive and complete — see the proof of Theorem 2.

**Step 1.** We first show that QRC-G implies Q-congruence (that D-congruence is implied follows from Lemma 3). Suppose $R$ is a quasi-transitive relation that greatest-element rationalizes $C$. Let $x, y, z \in X$ and $S \in \Sigma$ be such that $(x, y) \in P(R_C)$, $(y, z) \in P(R_C)$, and $x \in S$. By Lemma 2, $(x, y) \in P(R)$ and $(y, z) \in P(R)$ and, because $R$ is quasi-transitive, $(x, z) \in P(R)$. This implies $(z, x) \notin R$ and because $R$ greatest-element rationalizes $C$, we have $z \notin C(S)$.

**Step 2.** We show that D-congruence and Q-congruence together imply QRC-G. First, we prove that $R_C$ is quasi-transitive. Suppose $(x, y) \in P(R_C)$ and $(y, z) \in P(R_C)$ for some $x, y, z \in X$. Because $\Sigma$ is a base domain, $\{x, z\} \in \Sigma$. By Q-congruence, $z \notin C(\{x, z\})$ and hence $x \in C(\{x, z\})$ which implies $(x, z) \in R_C$. Since $R_C$ is reflexive, $(z, z) \in R_C$. If $(z, x) \in R_C$, D-congruence implies $z \in C(\{x, z\})$, a contradiction. Therefore, $(x, z) \in P(R_C)$.

The rest of the proof proceeds as in Richter [10] by showing that $R_C$ greatest-element rationalizes $C$, given D-congruence. Let $S \in \Sigma$ and $x \in S$. Suppose $x \in C(S)$. This implies $(x, y) \in R_C$ for all $y \in S$ and hence $x \in G(S, R_C)$. Now suppose $x \in G(S, R_C)$, that is, $(x, y) \in R_C$ for all $y \in S$. By D-congruence, $x \in C(S)$. ■

D-congruence and Q-congruence are independent on base domains, as shown by means of the following examples.

**Example 9** *Let $X = \{x, y, z\}$ and $\Sigma = \mathcal{X} \setminus \{\{x, y, z\}\}$, and define $C(\{t\}) = \{t\}$ for all $t \in X$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. This choice function satisfies D-congruence but violates Q-congruence.*

**Example 10** *Let $X = \{x, y, z\}$ and $\Sigma = \mathcal{X}$, and define $C(\{t\}) = \{t\}$ for all $t \in X$, $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y, z\}$, $C(\{x, z\}) = \{x, z\}$, and $C(\{x, y, z\}) = \{y, z\}$. This choice function satisfies Q-congruence but violates D-congruence.*

Q-congruence is a weaker axiom than the quasi-transitivity of $R_C$; it is only in conjunction with D-congruence that it implies that the revealed preference relation is quasi-transitive. Strengthening Q-congruence to the quasi-transitivity of $R_C$ does not allow us to drop D-congruence in the above characterization. Note that the above example showing that Q-congruence does not imply D-congruence is such that $R_C$ is quasi-transitive, and recall that D-congruence is necessary for greatest-element rationalizability on any domain (Lemma 3).

Given that we employ a base domain, it is natural to ask whether the base relation $B_C$ could be used in place of the revealed preference relation $R_C$ in the formulation of D-congruence and Q-congruence. This is not the case.

**Example 11** *Let* $X = \{x, y, x\}$ *and* $\Sigma = \mathcal{X}$*, and define* $C(\{t\}) = \{t\}$ *for all* $t \in X$*,* $C(\{x, y\}) = \{y\}$*,* $C(\{x, z\}) = \{z\}$*,* $C(\{y, z\}) = \{y, z\}$*, and* $C(\{x, y, z\}) = \{x, y, z\}$*. This choice function satisfies the modifications of D-congruence and Q-congruence where* $R_C$ *is replaced with* $B_C$ *but it does not satisfy D-congruence (and, thus, fails to be greatest-element rationalizable by any binary relation). Note that replacing* $R_C$ *with* $B_C$ *leads to a weakening of D-congruence but to a strengthening of Q-congruence.*

## 4.3 Acyclical Rationalizability

If quasi-transitivity is weakened to acyclicity, it seems natural to replace Q-congruence by the following A-congruence axiom in order to obtain a characterization of the respective rationalizability property on a base domain.

**A-Congruence:** For all $x, y \in X$, for all $S \in \Sigma$, if $(x, y) \in \overline{P(R_C)}$, $x \in S$ and $y \in C(S)$, then $x \in C(S)$.

It is indeed the case that D-congruence and A-congruence together are necessary and sufficient for ARC-G on base domains. However, A-congruence by itself is stronger than the acyclicity of $R_C$ and, thus, a stronger characterization result can be obtained by employing acyclicity instead of A-congruence.

**Theorem 5** *Suppose* $\Sigma$ *is a base domain.* $C$ *satisfies ARC-G if and only if* $C$ *satisfies D-congruence and* $R_C$ *is acyclical.*

**Proof.** Let $\Sigma$ be a base domain. Again, it follows that $R_C$ is reflexive and complete.

**Step 1.** We first show that ARC-G implies that $R_C$ is acyclical (again, that D-congruence is implied follows from Lemma 3).

Suppose $R$ is an acyclical, reflexive, and complete relation that greatest-element rationalizes $C$. Let $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ be such that $(x^{k-1}, x^k) \in P(R_C)$ for all $k \in \{1, \ldots, K\}$. By Lemma 2, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. Because $R$ is acyclical, we have $(x^K, x^0) \notin P(R)$ and, since $R$ is reflexive and complete, $(x^0, x^K) \in R$. Because $R$ is reflexive, $(x^K, x^K) \in R$. Because $\Sigma$ is a base domain, $\{x^0, x^K\} \in \Sigma$. Because $R$ greatest-element rationalizes $C$, $x^0 \in C(\{x^0, x^K\})$ and hence $(x^0, x^K) \in R_C$, which implies $(x^K, x^0) \notin P(R_C)$.

**Step 2.** D-congruence and the acyclicity of $R_C$ together imply ARC-G because D-congruence implies that $R_C$ greatest-element rationalizes $C$, as was shown in the last paragraph of the proof of Theorem 4. ■

That D-congruence and the acyclicity of $R_C$ are independent is shown by Examples 9 and 10. Analogously, D-congruence cannot be replaced with an axiom that merely applies to the base relation $B_C$ instead of $R_C$.

# 5 Conditions for Arbitrary Domains

In this section, we examine greatest-element rationalizability and maximal-element rationalizability on completely arbitrary domains under various assumptions regarding the properties of a rationalizing relation. Because Richter's [9] and Suzumura's [20; 21, p. 48] results characterizing TRC-G for arbitrary domains are well-known, we only discuss the quasi-transitive and asymmetric cases.

## 5.1 Quasi-Transitive Rationalizability

If we move from a base domain to an arbitrary domain, the conjunction of Q-congruence and D-congruence ceases to be sufficient for QRC-G, as can be seen from Example 4. Moreover, Q-congruence is not a necessary condition for QRC-G either, as demonstrated by the following example.

**Example 12** *Let* $X = \{x, y, z, u, v, w\}$, $\Sigma = \{\{x, y, u\}, \{x, z, w\}, \{y, z, v\}\}$, *and define* $C(\{x, y, u\}) = \{x, u\}$, $C(\{x, z, w\}) = \{z, w\}$, *and* $C(\{y, z, v\}) = \{y, v\}$. *This choice function is greatest-element rationalizable by the quasi-transitive, reflexive, and complete relation* $R$ *given by*

$$\{(x, t) \mid t \in X \setminus \{w\}\} \cup \{(y, t) \mid t \in X \setminus \{u\}\} \cup \{(z, t) \mid t \in X \setminus \{v\}\}$$
$$\cup \{(u, t) \mid t \in X\} \cup \{(v, t) \mid t \in X\} \cup \{(w, t) \mid t \in X\}.$$

*Since* $(x, y) \in P(R_C)$ *and* $(y, z) \in P(R_C)$, *Q-congruence requires* $z \notin C(\{x, z, w\})$, *contradicting the definition of* $C$.

The formulation of necessary and sufficient conditions for greatest-element or maximal-element rationalizability by a quasi-transitive relation is a complex task. Suzumura [21, p. 50] shows that the strong axiom of revealed preference is a sufficient (but not a necessary) condition for QRC-G. Now we present a condition that is necessary (but not sufficient, even if combined with D-congruence) for QRC-G.

The axiom we employ involves a recursive construction. The idea is to identify circumstances that force a strict preference between two elements of $X$ and impose a condition ensuring that this forced strict preference is transitive, as required by the quasi-transitivity of a rationalizing relation.

Suppose $C$ is greatest-element rationalizable by a quasi-transitive, reflexive, and complete relation $R$. Consider a feasible set $S \in \Sigma$ and distinct elements $x, y \in S$ such that $y$ is not chosen in $S$ but $y$ is directly revealed preferred to all $z \in S \setminus \{x, y\}$. By Lemma 1, $(y, z) \in R$ for all $z \in S \setminus \{x, y\}$ and, together with the reflexivity and completeness of $R$, $y \in S \setminus C(S)$ requires that $x$ be declared strictly preferred to $y$ according to $R$ and, by quasi-transitivity, all chains of strict preference thus established must be respected as well. Moreover, once it is implied that $x$ is declared strictly preferred to $y$ according to the above argument (or, more generally, according to the transitive closure of the relation thus obtained), this strict preference may have further implications: there may exist another set $T \in \Sigma$ such that $x, y \in T$, $x$ is declared preferred to all $z \in T \setminus \{x, y, w\}$ for some $w \in S \setminus \{x, y\}$, and $x$ is not chosen in $T$. In that case, we must declare a strict preference for $w$ over $x$ according to $R$. This procedure can be repeated recursively, and we now present a formal definition of this recursion, followed by a necessary condition for QRC-G based on this recursive construction. This recursion is analogous to the one employed in Bossert [3].

Define the relation $F_C^0$ on $X$ as follows. For all $x, y \in X$, $(x, y) \in F_C^0$ if there exists $S \in \Sigma$ such that $x \in S \setminus \{y\}$, $y \in S \setminus C(S)$, and $(y, z) \in R_C$ for all $z \in S \setminus \{x, y\}$.

Now let $i \in \mathbb{N}$. Let $J_C^{i-1} = R_C \cup \overline{F_C^{i-1}}$, and define the relation $F_C^i$ on $X$ as follows. For all $x, y \in X$, $(x, y) \in F_C^i$ if

$$[(x, y) \in \overline{F_C^{i-1}}] \quad \text{or} \quad [\text{there exists } S \in \Sigma \text{ such that } x \in S \setminus \{y\}, y \in S \setminus C(S),$$
$$\text{and } (y, z) \in J_C^{i-1} \text{ for all } z \in S \setminus \{x, y\}].$$

Finally, let $J_C^\infty = \cup_{i \in \mathbb{N}_0} J_C^i$ and $F_C^\infty = \cup_{i \in \mathbb{N}_0} \overline{F_C^i}$. The following axiom turns out to be necessary for QRC-G.

**Recursive Q-Congruence:** For all $x, y \in X$, if $(x, y) \in F_C^\infty$, then $(y, x) \notin J_C^\infty$.

We obtain

**Theorem 6** *If $C$ satisfies QRC-G, then $C$ satisfies recursive Q-congruence. The converse implication is not true.*

**Proof.** Suppose $R$ is a quasi-transitive, reflexive, and complete relation that greatest-element rationalizes $C$. We first prove that

$$F_C^\infty \subseteq P(R). \tag{8}$$

Clearly, by definition of $F_C^\infty$, it is sufficient to prove that $\overline{F_C^i} \subseteq P(R)$ for all $i \in \mathbb{N}_0$. We proceed by induction.

**Step 1.** $i = 0$. We first show that $F_C^0 \subseteq P(R)$. Suppose $(x, y) \in F_C^0$ for some $x, y \in X$. By definition, there exists $S \in \Sigma$ such that $(y, z) \in R_C$ for all $z \in S \setminus \{x, y\}$, $x \in S \setminus \{y\}$, and $y \in S \setminus C(S)$. By Lemma 1, $(y, z) \in R$ for all $z \in S \setminus \{x, y\}$ and, because $R$ is reflexive, $(y, y) \in R$. Because $y \in S \setminus C(S)$ and $R$ greatest-element rationalizes $C$, we must have $(y, x) \notin R$. Hence, because $R$ is complete, $(x, y) \in P(R)$.

Because $R$ is quasi-transitive (that is, $P(R)$ is transitive) and $\overline{F_C^0}$ is the transitive closure of $F_C^0$, it follows that $\overline{F_C^0} \subseteq P(R)$.

**Step 2.** Let $i \in \mathbb{N}$ and suppose $\overline{F_C^j} \subseteq P(R)$ for all $j \in \{0, \ldots, i-1\}$. Let $(x, y) \in F_C^i$ for some $x, y \in X$. By definition, there are two cases.

**2.a.** $(x, y) \in \overline{F_C^{i-1}}$. In this case, $(x, y) \in P(R)$ follows from the induction hypothesis.

**2.b.** There exists $S \in \Sigma$ such that $(y, z) \in J_C^{i-1}$ for all $z \in S \setminus \{x, y\}$, $x \in S \setminus \{y\}$, and $y \in S \setminus C(S)$. Let $z \in S \setminus \{x, y\}$. By definition of $J_C^{i-1}$ we have $(y, z) \in R_C$ or $(y, z) \in \overline{F_C^{i-1}}$. If $(y, z) \in R_C$, Lemma 1 implies $(y, z) \in R$. If $(y, z) \in \overline{F_C^{i-1}}$, $(y, z) \in R$ follows from the induction hypothesis. Therefore, $(y, z) \in R$ for all $z \in S \setminus \{x, y\}$ and, using the same argument as in Step 1, we obtain $F_C^i \subseteq P(R)$ and, by the quasi-transitivity of $R$, $\overline{F_C^i} \subseteq P(R)$. This completes the proof of (8).

Next, we prove that

$$J_C^\infty \subseteq R. \tag{9}$$

Again, it is sufficient to prove that $J_C^i \subseteq R$ for all $i \in \mathbb{N}_0$. Let $(x, y) \in J_C^i$ for some $x, y \in X$. By definition, $(x, y) \in R_C$ or $(x, y) \in \overline{F_C^i}$. If $(x, y) \in R_C$, $(x, y) \in R$ follows from Lemma 1. If $(x, y) \in \overline{F_C^i}$, $(x, y) \in R$ follows from the proof of (8). Therefore, (9) is true.

To complete the proof that QRC-G implies recursive Q-congruence, we proceed by contradiction. Suppose recursive Q-congruence is violated. Then there exists $x, y \in X$ such that $(x, y) \in F_C^\infty$ and $(y, x) \in J_C^\infty$. By (8) and (9), we have $(x, y) \in P(R)$ and $(y, x) \in R$, a contradiction.

To see that the converse implication is not true, consider the following example.

**Example 13** *Let $X = \{x, y, z, u, v, w\}$ and $\Sigma = \{\{x, y, u\}, \{y, z, v\}, \{x, z, w\}\}$ and define $C$ by letting $C(\{x, y, u\}) = \{x\}$, $C(\{y, z, v\}) = \{y\}$, and $C(\{x, z, w\}) = \{z\}$. It is straightforward to check that $F_C^0 = \emptyset$. It follows that $J_C^0 = R_C$, $J_C^i = J_C^0 = R_C$ for all $i \in \mathbb{N}$, and $F_C^\infty = F_C^i = F_C^0 = \emptyset$ for all $i \in \mathbb{N}$. Thus, recursive Q-congruence is trivially satisfied. Because we will use this example in the following subsection as well, we show that*

*$C$ cannot be greatest-element rationalized by an acyclical, reflexive, and complete relation (and, thus, it cannot be greatest-element rationalized by a quasi-transitive, reflexive, and complete relation). Suppose, by way of contradiction, that $R$ is acyclical, reflexive, and complete and greatest-element rationalizes $C$. Because $y, u \in \{x, y, u\} \setminus C(\{x, y, u\})$ and $R$ is reflexive and complete, we must have*

$$\textbf{1.a.}\ (x, y) \in P(R) \ \textit{or}\ \textbf{1.b.}\ [(x, u) \in P(R) \ \textit{and}\ (u, y) \in P(R)].$$

*Analogously, because $z, v \in \{y, z, v\} \setminus C(\{y, z, v\})$ and $x, w \in \{x, z, w\} \setminus C(\{x, z, w\})$, we have*

$$\textbf{2.a.}\ (y, z) \in P(R) \ \textit{or}\ \textbf{2.b.}\ [(y, v) \in P(R) \ \textit{and}\ (v, z) \in P(R)]$$

*and*

$$\textbf{3.a.}\ (z, x) \in P(R) \ \textit{or}\ \textbf{3.b.}\ [(z, w) \in P(R) \ \textit{and}\ (w, x) \in P(R)].$$

*If 1.a, 2.a, and 3.a are true, we immediately obtain a contradiction to the acyclicity of $R$.*

*If 1.a, 2.a, and 3.b are true, we have $(x, y) \in P(R)$, $(y, z) \in P(R)$, $(z, w) \in P(R)$, and $(w, x) \in P(R)$, contradicting the acyclicity of $R$. Because of the symmetric role played by $x, y$, and $z$, analogous contradictions are obtained whenever statements i.a, j.a, and k.b are true for any distinct values of $i, j, k \in \{1, 2, 3\}$.*

*If 1.a, 2.b, and 3.b are true, we obtain $(x, y) \in P(R)$, $(y, v) \in P(R)$, $(v, z) \in P(R)$, $(z, w) \in P(R)$, and $(w, x) \in P(R)$, again a violation of acyclicity. Using the symmetric role of $x, y$, and $z$ again, analogous contradictions are obtained whenever statements i.a, j.b, and k.b are true for any distinct values of $i, j, k \in \{1, 2, 3\}$.*

*Finally, if 1.b, 2.b, and 2.c are true, we obtain $(x, u) \in P(R)$, $(u, y) \in P(R)$, $(y, v) \in P(R)$, $(v, z) \in P(R)$, $(z, w) \in P(R)$, and $(w, x) \in P(R)$, and acyclicity is violated again.*

■

We noted in Lemma 3 that D-congruence is a necessary condition for greatest-element rationalizability by any relation. This raises the question whether recursive Q-congruence implies D-congruence. To see that this is indeed the case, suppose D-congruence is violated. Then there exist $S \in \Sigma$ and $x \in S$ such that $(x, y) \in R_C$ for all $y \in S$ and $x \notin C(S)$. By definition, this implies $(y, x) \in F_C^0 \subseteq F_C^\infty$ for all $y \in S \setminus \{x\}$. Because $C(S) \subseteq S$ is non-empty and $x \in S \setminus C(S)$, $S \setminus \{x\}$ is non-empty. Consider any $y \in S \setminus \{x\}$. Because $(x, y) \in R_C$ we have $(x, y) \in J_C^\infty$. Therefore, we obtain $(y, x) \in F_C^\infty$ and $(x, y) \in J_C^\infty$, contradicting recursive Q-congruence.

Furthermore, recursive Q-congruence and Q-congruence are independent. The choice function in Example 12 satisfies recursive Q-congruence (by Theorem 6; note that it satisfies QRC-G) but violates Q-congruence. Conversely, the choice function in Example 4 satisfies Q-congruence but violates recursive Q-congruence. To see this, note first that $R_C = \{(y,y),(y,z),(y,w),(w,x),(w,y),(w,w)\}$. In view of the definition of $C$, it follows that $F_C^0 = \{(x,y),(y,z),(z,w),(w,x)\}$, and the transitive closure of $F_C^0$ is therefore given by $\overline{F_C^0} = X \times X$. It follows that $F_C^\infty = J_C^\infty = X \times X$, a contradiction to recursive Q-congruence.

Because no particular assumptions are formulated regarding the domain of the choice function, it seems that conditions that are both necessary and sufficient cannot be formulated without invoking existential clauses. Moreover, contrary to the transitive case, quasi-transitivity of a greatest-element rationalizing relation does not imply that the asymmetric factor of the revealed preference relation must be contained in this rationalizing relation—see the discussion regarding Lemma 2 in Section 2. These observations appear to be an important part of the reason why there does not exist much literature on the subject of quasi-transitive rational choice on arbitrary domains. Rather than constructing the relation $F_C^\infty$ one pair of alternatives at a time, a tighter necessary condition would be to establish the existence of an alternative $x \in S$ such that $x$ can be declared better than $y$ if $y$ is feasible but not chosen in $S$. This kind of condition involves an existential clause, and conditions of that nature are difficult to verify in practice and, therefore, are of limited interest.

## 5.2 Acyclical Rationalizability

A-congruence (and, thus, the acyclicity of $R_C$) fails to be sufficient for ARC-G even in the presence of D-congruence, as Example 4 shows. Moreover, the acyclicity of $R_C$ (and, thus, A-congruence) is not necessary for ARC-G in the case of a general domain. This is established by Example 12; note that, in Example 12, we have $(x,y) \in P(R_C)$, $(y,z) \in P(R_C)$, and $(z,x) \in P(R_C)$.

As is the case for quasi-transitivity, the formulation of necessary and sufficient conditions for acyclical rationalizability appears to necessitate the use of axioms involving existential clauses. We provide a discussion analogous to the one for quasi-transitivity to illustrate the issues involved. First, we present a new sufficient condition for ARC-G. To do so, we employ the relation $E_C$ introduced in Suzumura [19]. It is defined as follows. For all $x,y \in X$, $(x,y) \in E_C$ if there exists $S \in \Sigma$ such that $x \in C(S)$ and $y \in S \setminus C(S)$.

**Strong A-Congruence:** For all $x, y \in X$, for all $S \in \Sigma$, if $(x, y) \in \overline{E_C} \cup R_C$, $x \in S$ and $y \in C(S)$, then $x \in C(S)$.

We obtain

**Theorem 7** *If $C$ satisfies strong A-congruence, then $C$ satisfies ARC-G. The converse implication is not true.*

**Proof.** Suppose $C$ satisfies strong A-congruence. First, we prove that $\overline{E_C}$ is asymmetric. Suppose, by way of contradiction, that there exist $x, y \in X$ such that $(x, y) \in \overline{E_C}$ and $(y, x) \in \overline{E_C}$. Therefore, there exist $K, K' \in \mathbb{N}$, $x^0, \ldots, x^K \in X$, and $z^0, \ldots, z^{K'} \in X$ such that $x^0 = z^{K'} = x$, $x^K = z^0 = y$, $(x^{k-1}, x^k) \in E_C$ for all $k \in \{1, \ldots, K\}$, and $(z^{k-1}, z^k) \in E_C$ for all $k \in \{1, \ldots, K'\}$. Hence, $(x, z^{K'-1}) \in \overline{E_C}$ and $(z^{K'-1}, x) \in E_C$. By definition of $E_C$, there exists $S \in \Sigma$ such that $z^{K'-1} \in C(S)$ and $x \in S \setminus C(S)$, contradicting strong A-congruence.

Now define

$$R = \{(x, y) \in X \times X \mid (y, x) \notin \overline{E_C}\} \cup R_C.$$

Clearly, $R$ is reflexive and complete by the asymmetry of $\overline{E_C}$. To prove that $R$ is acyclical, we first derive the asymmetric factor of $R$. By definition, $(x, y) \in P(R)$ if

$$[(y, x) \notin \overline{E_C} \text{ or } (x, y) \in R_C] \text{ and } [(x, y) \in \overline{E_C} \text{ and } (y, x) \notin R_C]$$

which is equivalent to

$$[(y, x) \notin \overline{E_C} \text{ and } (x, y) \in \overline{E_C} \text{ and } (y, x) \notin R_C] \text{ or}$$
$$[(x, y) \in R_C \text{ and } (x, y) \in \overline{E_C} \text{ and } (y, x) \notin R_C].$$

Using the asymmetry of $\overline{E_C}$, this is equivalent to

$$[(x, y) \in \overline{E_C} \text{ and } (y, x) \notin R_C] \text{ or } [(x, y) \in \overline{E_C} \text{ and } (x, y) \in P(R_C)]$$

or, equivalently,

$$[(x, y) \in \overline{E_C} \text{ and } (y, x) \notin R_C]. \tag{10}$$

Now we establish the acyclicity of $R$. Suppose there exist $K \in \mathbb{N}\setminus\{1\}$ and $x^0, \ldots, x^K \in X$ such that $x^0 = x$, $x^K = y$, $(x^{k-1}, x^k) \in \overline{E_C}$ and $(x^k, x^{k-1}) \notin R_C$ for all $k \in \{1, \ldots, K\}$. Because $\overline{E_C}$ is transitive, we have $(x, y) \in \overline{E_C}$ and, by the asymmetry of $\overline{E_C}$, $(y, x) \notin \overline{E_C}$. By (10), this implies $(y, x) \notin P(R)$.

Finally, we show that $R$ greatest-element rationalizes $C$. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$. This implies $(x, y) \in R_C$ for all $y \in S$ and hence $(x, y) \in R$ for all $y \in S$, and we have $x \in G(S, R)$.

Now suppose $x \in G(S, R)$. This implies

$$[(y, x) \notin \overline{E_C} \text{ or } (x, y) \in R_C] \text{ for all } y \in S. \tag{11}$$

Let $z \in C(S) \subseteq S$. This implies $(z, x) \in R_C$. Because $z \in S$, (11) implies

$$(z, x) \notin \overline{E_C} \tag{12}$$

or

$$(x, z) \in R_C. \tag{13}$$

If (12) is true, we must have $x \in C(S)$ because otherwise $(z, x) \in E_C$ by definition and hence $(z, x) \in \overline{E_C}$.

If (13) is true, $x \in C(S)$ follows from strong A-congruence.

To show that strong A-congruence is not implied by ARC-G, note that Example 12 can be employed here as well: QRC-G (and, thus, ARC-G) is satisfied but we have $(x, z) \in \overline{E_C}$ and $z \in C(\{x, z, w\})$ and $x \in \{x, z, w\} \setminus C(\{x, z, w\})$, contradicting strong A-congruence. ■

Strong A-congruence is a tighter sufficient condition for ARC-G than the set of sufficient conditions established in Suzumura [21, p. 51]. The conditions used in this earlier contribution are the following.

**Weak Axiom of Revealed Preference:** For all $x, y \in X$, for all $S \in \Sigma$, if $(x, y) \in E_C$ and $x \in S$, then $y \notin C(S)$.

**No $E_C$-Cycles:** For all $x, y \in X$, for all $S \in \Sigma$, if $(x, y) \in \overline{E_C}$, $x \in S$ and $y \in C(S)$, then $x \in C(S)$.

We obtain

**Theorem 8** *If $C$ satisfies the weak axiom of revealed preference and no $E_C$-cycles, then $C$ satisfies strong A-congruence. The converse implication is not true.*

**Proof.** Suppose strong A-congruence is violated. Then there exist $x, y \in X$ such that $(x, y) \in \overline{E_C} \cup R_C$ and $(y, x) \in E_C$. If the weak axiom of revealed preference is satisfied, it follows that $(x, y) \notin R_C$. Therefore, we must have $(x, y) \in \overline{E_C}$, and we obtain an $E_C$-cycle.

To see that the converse implication is not true, consider the following example.

**Example 14** *Let* $X = \{x, y, z\}$ *and* $\Sigma = \{\{x, y\}, \{x, y, z\}\}$ *and define* $C$ *by letting* $C(\{x, y\}) = \{x, y\}$ *and* $C(\{x, y, z\}) = \{x\}$*. It is straightforward to verify that* $C$ *satisfies strong A-congruence but violates the weak axiom of revealed preference.*

■

To obtain a necessary condition for ARC-G, we employ a recursive construction as in the previous subsection. Let $H_C^0 = F_C^0$ and, for $i \in \mathbb{N}$, let $L_C^{i-1} = R_C \cup \overline{H_C^{i-1}}$ and define the relation $H_C^i$ as follows. For all $x, y \in X$, $(x, y) \in H_C^i$ if

$$[(x, y) \in H_C^{i-1}] \quad \text{or} \quad \begin{array}{l}[\text{there exists } S \in \Sigma \text{ such that } x \in S \setminus \{y\}, y \in S \setminus C(S), \\ \text{and } (y, z) \in L_C^{i-1} \text{ for all } z \in S \setminus \{x, y\}].\end{array}$$

Furthermore, let $L_C^\infty = \cup_{i \in \mathbb{N}_0} L_C^i$ and $H_C^\infty = \cup_{i \in \mathbb{N}_0} H_C^i$.

Analogously to recursive Q-congruence, the following condition is necessary for ARC-G.

**Recursive A-Congruence:** For all $x, y \in X$, if $(x, y) \in H_C^\infty$, then $(y, x) \notin L_C^\infty$.

We now obtain

**Theorem 9** *If* $C$ *satisfies ARC-G, then* $C$ *satisfies recursive A-congruence. The converse implication is not true.*

**Proof.** Suppose $R$ is an acyclical, reflexive, and complete relation that greatest-element rationalizes $C$. To demonstrate that A-congruence is implied, we first prove

$$H_C^\infty \subseteq P(R). \tag{14}$$

Again, by definition of $H_C^\infty$, it is sufficient to prove that $H_C^i \subseteq P(R)$ for all $i \in \mathbb{N}_0$. We proceed by induction.

**Step 1.** $i = 0$. Because $H_C^0 = F_C^0$, Step 1 of the proof of Theorem 6 can be employed to conclude that $H_C^0 \subseteq P(R)$.

**Step 2.** Let $i \in \mathbb{N}$ and suppose $H_C^j \subseteq P(R)$ for all $j \in \{0, \ldots, i-1\}$. Let $(x, y) \in H_C^i$ for some $x, y \in X$. By definition, there are two cases.

**2.a.** $(x, y) \in H_C^{i-1}$. In this case, $(x, y) \in P(R)$ follows from the induction hypothesis.

**2.b.** There exists $S \in \Sigma$ such that $(y, z) \in L_C^{i-1}$ for all $z \in S \setminus \{x, y\}$, $x \in S \setminus \{y\}$, and $y \in S \setminus C(S)$. Let $z \in S \setminus \{x, y\}$. By definition of $L_C^{i-1}$ we have $(y, z) \in R_C$ or $(y, z) \in \overline{H_C^{i-1}}$. If $(y, z) \in R_C$, Lemma 1 implies $(y, z) \in R$. If $(y, z) \in \overline{H_C^{i-1}}$, there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x^0 = y$, $x^K = z$ and $(x^{k-1}, x^k) \in H_C^{i-1}$ for all

$k \in \{1, \ldots, K\}$. By the induction hypothesis, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$, and the acyclicity and the completeness of $R$ together imply $(y, z) \in R$. Therefore, $(y, z) \in R$ for all $z \in S \setminus \{x, y\}$ and, using the same argument as in Step 1 of the proof of Theorem 6, we obtain $H^i_C \subseteq P(R)$. This completes the proof of (14).

Furthermore, we have

$$L^{\infty}_C \subseteq R; \tag{15}$$

the proof of this claim is analogous to the proof of (9).

To complete the proof that recursive A-congruence is satisfied, the same argument as in the proof of Theorem 6 can be employed, where $F^{\infty}_C$ and $J^{\infty}_C$ are replaced with $H^{\infty}_C$ and $L^{\infty}_C$, and (8) and (9) are replaced with (14) and (15).

To see that the converse implication is not true, note that the choice function in Example 13 satisfies recursive A-congruence (trivially because $H^{\infty}_C = \emptyset$) but it cannot be greatest-element rationalized by an acyclical relation. ■

Recursive A-congruence implies D-congruence; the proof is analogous to the proof establishing that recursive Q-congruence implies D-congruence.

Furthermore, recursive A-congruence does not imply the acyclicity of $R_C$ (and, thus, fails to imply A-congruence). This is shown by Example 13. Conversely, recursive A-congruence is not implied by A-congruence (and, thus, it is not implied by the acyclicity of $R_C$; Example 4 establishes this claim.

# 6 Concluding Remarks

We conclude this chapter with a brief discussion of some open problems. As mentioned in the text, it seems very difficult to obtain useful necessary and sufficient conditions for QRC-G or for ARC-G on general domains. Given that very indirect implications of preference maximization have to be taken into consideration, the nature of the problem suggests that existential clauses have to be invoked in order to arrive at full characterizations. See also Bossert [3] for analogous difficulties in a different framework.

If the formulation of clean necessary and sufficient conditions for QRC-G and for ARC-G turns out to be too complex a task, the following more modest objective might be an issue to be addressed in future work. Note that the strong axiom of revealed preference, a sufficient condition for QRC-G, is not implied by TRC-G and, analogously, strong A-congruence is not implied by QRC-G. Likewise, recursive Q-congruence does not imply ARC-G. One direction in which the results of this chapter could be extended is to find a condition that is intermediate in strength between TRC-G and QRC-G, and a condition that is implied by QRC-G and implies ARC-G in order to obtain tighter bounds on possible characterizations.

We conclude by remarking that the strong axiom of revealed preference cannot be weakened to strong A-congruence to get a tighter sufficient condition for QRC-G (this is established by Example 5). Similarly, recursive A-congruence cannot be strengthened to recursive Q-congruence to get a tighter necessary condition for ARC-G (again, this is established by Example 5).

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