Welfarist-Consequentialism, Similarity of Attitudes, and Arrow's General Impossibility Theorem*

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Abstract

Two features of Arrow's social choice theory are critically scrutinized. The first feature is the *welfarist-consequentialism*, which not only bases social judgements about right or wrong actions on the assessment of their consequences, but also assesses consequences in terms of people's welfare and nothing else. The second feature is a *similarity of people's attitudes* towards social outcomes as a possible resolvent of the Arrow impossibility theorem. Two extended frameworks, one *consequentialist* and the other *non-consequentialist*, are developed. Both frameworks are shown to admit some interesting resolutions of Arrow's general impossibility theorem, which are rather sharply contrasting with Arrow's own perspective.

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1 Introduction

The purpose of this paper is to re-examine Arrow's *general impossibility theorem* with special reference to the following two basic features of his celebrated *Social Choice and Individual Values*. The first feature is the *welfarist-consequentialism*, which claims that the social judgements on right or wrong actions should be based on the assessment of their consequential states of affairs, where the assessment of consequences is conducted exclusively in terms of people's welfare, their preference satisfaction, or people getting what they want.[1,2] Not only is Arrow's own analysis based squarely on the welfarist-consequentialism in this sense, but also this basic feature permeates through the entire edifice of contemporary social choice theory.[3] The second feature is the perception that "the possibility of social welfare judgements rests upon a similarity of attitudes toward social alternatives [Arrow (1963, p.69)]." To substantiate this claim analytically, Arrow (1963, p.81) showed that "it is possible to construct suitable social welfare functions if we feel entitled to say in advance that the tastes of individuals fall within certain prescribed realms of similarity." It goes without saying that a large portion of the subsequent developments in social choice theory is devoted to the exploration of Arrow's important insight to this effect.[4]

To gauge the extent to which Arrow's impossibility theorem and the resolution thereof hinge on these two basic features of his framework, this paper develops two extended frameworks in which individuals are supposed to express their preferences not only about consequential outcomes, but also about opportunity sets from which outcomes are chosen. Two such frameworks are identified below: a *consequentialist* framework and a *non-consequentialist* framework. It is shown that the counterpart of Arrow's impossibility theorem still holds in the consequentialist framework if the society is composed exclusively of individuals who show similar attitudes toward social alternatives, whereas a resolution of Arrow's impossibility theorem can be found if there is a diversity of attitudes among individuals. Thus, in the consequentialist conceptual framework, it is in fact a *dissimilarity* rather than a *similarity* among individuals that serves as a *deux ex machina* vis-à-vis Arrow's general impossibility theorem. In contrast with this verdict on the consequentialist framework, an interesting resolution of Arrow's general impossibility theorem exists in the non-consequentialist framework, which may work even in the *homogeneous* society where all individuals exhibit a similarity of attitudes toward outcomes and opportunities.

The structure of this paper is as follows. In Section 2, we lay the foundation of our analysis by introducing some basic notation and definitions. Section 3 is devoted to examining how Arrow's general impossibility theorem fares in the consequentialist framework, whereas Section 4 conducts the corresponding analysis in the non-consequentialist framework. Section 5 describes the way how these results can be generalized, and Section 6 concludes this paper with several observations.

2 The Basic Notation and Definitions

Let X be the set of all conventionally defined social states, which are mutually exclusive and jointly exhaustive. It is assumed that X satisfies $3 \leq \#X < \infty$. The elements of X are denoted by $x, y, z, \cdots$, and they are called *outcomes.* K denotes the set of all non-empty subsets of X. The elements of K are denoted by $A, B, C, \cdots$, and they are called *opportunity sets.* Let $X \times K$ be the Cartesian product of X and K. The elements of $X \times K$ are denoted by $(x, A), (y, B), (z, C), \cdots$, and they are called *extended alternatives.* The intended meaning of $(x, A) \in X \times K$ is that the outcome x is chosen from the opportunity set A, but this interpretation will be vacuous if $x \notin A$. Thus, let $\Omega \subseteq X \times K$ be such that $x \in A$ whenever $(x, A) \in \Omega$.

Let $N = \{1, 2, \cdots, n\}$ be the set of all individuals in the society, where $2 \leq n = \#N < \infty$. Each individual $i \in N$ is assumed to have an extended preference ordering R_i over Ω, which is *reflexive, complete* and *transitive.* For any $(x, A), (y, B) \in \Omega$, $(x, A) R_i (y, B)$ is meant to imply that i feels at least as good when choosing x from A as when choosing y from B. The asymmetric part and the symmetric part of R_i are denoted by $P(R_i)$ and $I(R_i)$, respectively, which denote the strict preference relation and the indifference relation of $i \in N$.

Let $\wp$ be the set of all logically possible orderings over Ω. Then a *profile* $\mathbf{R} = (R_1, R_2, \cdots, R_n)$ of extended individual preference orderings, one extended ordering for each individual, is an element of $\wp^n$. An *extended social welfare function* (ESWF) is a function f which maps each and every profile in some subset D_f of $\wp^n$ into $\wp$. When $R = f(\mathbf{R})$ holds for some $\mathbf{R} \in D_f$, $I(R)$ and $P(R)$ stand, respectively, for the social indifference relation and the social strict preference relation corresponding to R. Given an ESWF f, the problem of social choice we envisage in this paper can be phrased as follows. Suppose that a profile $\mathbf{R} \in D_f$ and a set $S \subseteq X$ of feasible social alternatives are given. Then the best social choice from S can be identified to be an $x^* \in S$ such that $(x^*, S)R(x, S)$ holds for all $x \in S$, where $R = f(\mathbf{R})$. To make this interpretation natural as well as sensible, we assume that each and every $x \in X$ denote a *public* alternative such as a list of public goods to be provided in the society, or the description of a candidate in the public election.

2.1 Domain Restriction

In order to make our problem both analytically tractable and interesting, we assume that each individual's extended preference ordering R_i $(i \in N)$, which defines an admissible profile $\mathbf{R} = (R_1, R_2, \cdots, R_n) \in D_f$, satisfies the following two conditions:

Independence (IND): For all $(x, A), (y, B) \in \Omega$ and all $z \in X - A \cup B$,

$$(x, A)R_i(y, B) \Leftrightarrow (x, A \cup \{z\})R_i(y, B \cup \{z\}).$$

Simple Indifference (SI): For all $x \in X$ and all $y, z \in X - \{x\}$,

$$(x, \{x, y\})I(R_i)(x, \{x, z\}).$$

IND corresponds to the standard independence axiom used in the literature (see, for example, Pattanaik and Xu (1990)). It requires that, for all opportunity sets $A, B \in K$, if an alternative $z \in X$ is not in both A and B, then the preference ranking over $(x, A \cup \{z\})$ and $(y, B \cup \{z\})$ corresponds to the preference ranking over (x, A) and (y, B). **SI** requires that choosing x from the two simple cases consisting of two alternatives each is regarded as indifferent no matter what alternative is added to x.

We may as well assume the following condition:

Monotonicity (MON): For all $(x, A), (x, B) \in \Omega$,

$$B \subseteq A \Rightarrow (x, A)R_i(x, B).$$

MON makes an explicit use of information about the opportunity aspect of choice situations. It requires that choosing an outcome x from the opportunity set A is at least as good as choosing the same x from the opportunity set B which is a subset of A. In the present context, this axiom seems very reasonable.

The following result summarizes the implication of the above three conditions.

Lemma 1. If R_i satisfies (IND), (SI) and (MON), then for all $(x, A), (x, B) \in \Omega$, $|A| \geq |B| \Rightarrow (x, A)R_i(x, B)$.

Proof. Let R_i satisfy (IND), (SI), and (MON). Let $(x, A), (x, B) \in \Omega$ be such that $|A| \geq |B|$.

If $|A| = |B| = 1$, then $A = B = \{x\}$. By reflexivity of R_i, it follows immediately that $(x, A)I(R_i)(x, B)$. If $|A| = |B| = 2$, then by (SI), $(x, A)I(R_i)(x, B)$ follows from (SI) directly. Thus, we have proved the following:

(2.1) for all $(x, A), (x, B) \in \Omega$, if $|A| = |B| \leq 2$, then $(x, A)I(R_i)(x, B)$.

To prove that $|A| = |B| = m > 2 \Rightarrow (x, A)I(R_i)(x, B)$, we use the induction method. Suppose

(2.2) for all $(x, S), (x, T) \in \Omega$ such that $|S| = |T| < m$, $(x, S)I(R_i)(x, T)$.

If there exists $y \in A \cap B$ such that $y \neq x$, then, from (2.2), $(x, A - \{y\})I(R_i)(x, B - \{y\})$. By (IND), $(x, A)I(R_i)(x, B)$ follows immediately. If $A \cap B = \{x\}$, then consider $C = (A - \{a\}) \cup \{b\}$ where $a \in A - \{x\}$ and $b \in B - \{x\}$. From the previous argument, clearly, $(x, A)I(R_i)(x, C)$ and $(x, B)I(R_i)(x, C)$. Thus, by the transitivity of R_i, $(x, A)I(R_i)(x, B)$.

From (2.1) and (2.2), noting the finiteness of X, we have

(2.3) for all $(x, A), (x, B) \in \Omega$, if $|A| = |B|$, then $(x, A)I(R_i)(x, B)$.

Consider now that $|A| > |B|$. If $|B| = 1$, that is, $|B| = \{x\}$, by (MON), $(x, A)R_i(x, B)$ follows immediately. Similarly, if $A = X$, then, by (MON), $(x, A) = (x, X)R_i(x, B)$. Let $|X| > |A| > |B| > 1$. Clearly, in this case, there exists $C \in K$ such that $|C \cup B| = |A|$. From (2.3), $(x, C \cup B)I(R_i)(x, A)$. By (MON), $(x, C \cup B)R_i(x, B)$. Hence, $(x, A)R_i(x, B)$ follows from the transitivity of R_i immediately. ■

Thus, these simple conditions impose a mild restriction on each individual's extended preference ordering to the effect that *each individual is not averse to richer opportunities, viz., a larger opportunity set does not do any harm to him.*[5,6]

Throughout Sections 3 and 4, we assume that each and every profile $\mathbf{R}$ $= (R_1, R_2, \cdots, R_n) \in D_f$ is such that R_i satisfies the conditions **IND, SI** and **MON** for all $i \in N$.

2.2 Arrovian Conditions in the Extended Framework

In addition to the domain restriction on D_f introduced above, we introduce three conditions on f, which are slight modifications of Arrow's own conditions in Arrow (1963). They are well known, and require no further explanation.

Strong Pareto Principle (SP): For all $(x, A), (y, B) \in \Omega$, and for all $\mathbf{R} = (R_1, R_2, \cdots, R_n) \in D_f$, if $(x, A)P(R_i)(y, B)$ holds for all $i \in N$, then we have $(x, A)P(R)(y, B)$, and if $(x, A)I(R_i)(y, B)$ holds for all $i \in N$, then we have $(x, A)I(R)(y, B)$, where $R = f(\mathbf{R})$.

Independence of Irrelevant Alternatives (IIA): For all $\mathbf{R^1} = (R_1^1, R_2^1, \cdots, R_n^1)$, $\mathbf{R^2} = (R_1^2, R_2^2, \cdots, R_n^2) \in D_f$, and for all $(x, A), (y, B) \in \Omega$, if $[(x, A)R_i^1(y, B) \Leftrightarrow (x, A)R_i^2(y, B)]$ holds for all $i \in N$, then $[(x, A)R^1(y, B) \Leftrightarrow (x, A)R^2(y, B)]$ holds, where $R^1 = f(\mathbf{R^1})$ and $R^2 = f(\mathbf{R^2})$.

Non-Dictatorship (ND): There exists no $i \in N$ such that $P(R_i) = P(R)$ holds for all $\mathbf{R} = (R_1, R_2, \cdots, R_n) \in D_f$, where $R = f(\mathbf{R})$.

3 Arrovian Impossibility Theorems in the Consequentialist Framework

In this section, we discuss Arrovian impossibility theorems in the framework which is broader than welfarist-consequentialism, yet lies within consequentialism. Following Suzumura and Xu (1999), let us identify two types of an individual whom we have a reason to call a *consequentialist*:

Extreme Consequentialist: An individual $i \in N$ is said to be an *extreme consequentialist* if, for all $(x, A), (x, B) \in \Omega$, it is true that $(x, A)I(R_i)(x, B)$.

Strong Consequentialist: An individual i is said to be a *strong consequentialist* if, for all $(x, A), (y, B) \in \Omega$,
(a) if $(x, \{x\})I(R_i)(y, \{y\})$, then $\#A \geq \#B \Leftrightarrow (x, A)R_i(y, B)$; and
(b) if $(x, \{x\})P(R_i)(y, \{y\})$, then $(x, A)P(R_i)(y, B)$.

Thus, an extreme consequentialist ranks two extended alternatives (x, A) and (y, B) simply in terms of their outcomes x and y, giving no relevance to the opportunity sets A and B from which x and y are chosen. In contrast, a strong consequentialist ranks two alternatives (x, A) and (y, B) in complete accordance with their outcomes x and y only if he has a strict preference between choosing x from the singleton set $\{x\}$, and choosing y from the singleton set $\{y\}$. If he is indifferent between choosing x from the singleton set $\{x\}$, and choosing y from the singleton set $\{y\}$, his preference ranking between (x, A) and (y, B) is in accordance with the cardinality comparison between A and B. It is to this limited extent that a strong consequentialist reveals his preference for opportunity, thereby exhibiting his impure consequentialist side of attitude.

As the following lemma shows, for an extreme as well as a strong consequentialist, imposing the conditions **IND, SI** and **MON** does not in fact restrict his/her preferences at all.

Lemma 2. An extreme as well as a strong consequentialist's extended preference orderings must always satisfy the three conditions **IND, SI** and **MON**.

Proof. It is easy to check that both an extreme and a strong consequentialist's extended preference orderings satisfy **SI** and **MON**. We now prove that **IND** is satisfied by an extreme and a strong consequentialist's extended preference orderings.

An Extreme Consequentialist: Let i be an extreme consequentialist and R_i be his extended preference ordering. Let $(x, A), (y, B) \in \Omega$ and $z \in X - A \cup B$ be such that $(x, A)R_i(y, B)$ holds. By definition of an extreme consequentialist, we must have $(x, A)I(R_i)(x, A \cup \{z\})$ and $(y, B)I(R_i)(y, B \cup \{z\})$. Then transitivity of R_i implies $(x, A \cup \{z\})R_i(y, B \cup \{z\})$. The converse implication may be similarly verified. Therefore, **IND** holds for an extreme consequentialist's extended preference orderings.

A Strong Consequentialist: Let j be a strong consequentialist and R_j be his extended preference ordering. Let $(x, A), (y, B) \in \Omega$ and $z \in X - A \cup B$ be such that $(x, A)R_j(y, B)$ holds. We distinguish three cases that exhaust all possibilities: (a) $(x, A)I(R_j)(y, B)$; (b) $x = y$ and $(x, A)P(R_j)(y, B)$; and (c) $x \neq y$ and $(x, A)P(R_j)(y, B)$. In case (a), according to the definition of a strong consequentilist, it must be that $(x, \{x\})I(R_j)(y, \{y\})$ and $\#A = \#B$. Then, by the definition of strong consequentialism, it follows that $(x, A \cup \{z\})I(R_j)(y, B \cup \{z\})$. In case (b), since $x = y$, by the definition of a strong consequentialist, we must have $\#A > \#B$. Clearly, $\#(A \cup \{z\}) > \#(B \cup \{z\})$. Hence, $(x, A \cup \{z\})P(R_j)(y, B \cup \{z\})$ follows from the definition of strong consequentialism. Finally, in case (c), we must have that

$(x,\{x\})P(R_j)(y,\{y\})$. Then, $(x,A\cup\{z\})P(R_j)(y,B\cup\{z\})$ follows immediately from the definition of a strong consequentialist. The converse may be similarly verified. Hence, a strong consequentialist's extended preference orderings satisfy **IND**. ■

Let us now introduce three domain restrictions on f by specifying some appropriate subsets of D_f. In the first place, let $D_f(E)$ be the set of all profiles in D_f such that all individuals are extreme consequentialists. In the second place, let $D_f(E\cup S)$ be the set of all profiles in D_f such that at least one individual is an extreme consequentialist and at least one individual is a strong consequentialist.[7] Finally, let $D_f(S)$ be the set of all profiles in D_f such that all individuals are strong consequentialists.

We are now ready to present our results, beginning with the consequentialist framework. The first result is a simple restatement of Arrow's general impossibility theorem save for the restriction on the domain of the extended social welfare function and a slight strengthening of the Pareto principle.

Theorem 1. Suppose that all individuals are extreme consequentialists. Then, there exists no extended social welfare function f with the domain $D_f(E)$ which satisfies **SP, IIA** and **ND**.
Proof. Suppose that there exists an ESWF f on $D_f(E)$ which satisfies **SP** as well as **IIA**. Since all individuals are extreme consequentialists,

(3.1) $\forall\ i\in N: (x,A)R_i(y,B)\Leftrightarrow (x,X)R_i(y,X)$

holds for all $(x,A),(y,B)\in\Omega$ and for all $\mathbf{R}=(R_1,R_2,\cdots,R_n)\in D_f(P)$. Note that the conditions **IND, SI** and **MON** impose no restriction whatsoever on the profile $\mathbf{R}=(R_1,R_2,\cdots,R_n)$ even when, for each and every $i\in N$, R_i is restricted on $\Omega_X:=\{(x,X)\in X\times K|x\in X\}$. Note also that **SP** and **IIA** imposed on f imply that the same conditions must be satisfied on the restricted space Ω_X. By virtue of the Arrow impossibility theorem, therefore, there exists a dictator, say $d\in N$, for f on the restricted space Ω_X. That is, for all $\mathbf{R}\in D_f(E)$ and all $(x,X),(y,X)\in\Omega_X$, $(x,X)P(R_d)(y,X)\Rightarrow(x,X)P(R)(y,X)$. We now show that for all $(x,A),(y,B)\in\Omega$, $(x,A)P(R_d)(y,B)\Rightarrow(x,A)P(R)(y,B)$, viz., d is a dictator for f on the full space Ω. Note that, since d is an extreme consequentialist, we must have that $(x,A)P(R_d)(y,B)$ iff $(x,X)P(R_d)(y,X)$. Since all individuals are extreme consequentialists, it must be true that $(x,A)I(R_i)(x,X)$ and $(y,B)I(R_i)(y,X)$ for all $i\in N$. Therefore, by **SP**,

$(x, A)I(R)(x, X)$ and $(y, B)I(R)(y, X)$. By virture of the transitivity of R, it then follows that $(x, X)P(R)(y, X) \Rightarrow (x, A)P(R)(y, B)$. That is, we have shown that $(x, A)P(R_d)(y, B) \Rightarrow (x, A)P(R)(y, B)$. In other words, d is a dictator for f on the full space Ω. Therefore, there exists no ESWF that satisfies **SP, IIA** and **ND**. ■

Thus, the similarity of attitudes among individuals in the sense that all individuals are extreme consequentialists brings back an essentially Arrovian impossibility result. The message of this theorem can be strengthened by proving the next theorem which asserts that the impossibility result disappears if an extreme consequentialist and a strong consequentialist coexist in the society.

Theorem 2. Suppose that there exist at least one extreme consequentialist and at least one strong consequentialist in the society. Then, there exists an extended social welfare function f with the domain $D_f(E \cup S)$ which satisfies **SP, IIA** and **ND**.

Proof. Let $e \in N$ be an extreme consequentialist and $s \in N$ be a strong consequentialist. Note that

(3.2) $\forall\ (x, A), (x, B) \in \Omega : (x, A)I(R_e)(x, B),$

(3.3) $\forall\ (x, A), (y, B) \in \Omega :$ if $(x, \{x\})I(R_s)(y, \{y\})$, then $(x, A)R_s(y, B) \Leftrightarrow \#A \geq \#B,$

and

(3.4) $\forall\ (x, A), (y, B) \in \Omega :$ if $(x, \{x\})P(R_s)(y, \{y\})$, then $(x, A)P(R_s)(y, B)$

hold. Now consider the following ESWF: For all $(x, A), (y, B) \in \Omega$, if $(x, \{x\})P(R_s)$ $(y, \{y\})$, then $(x, A)R(y, B)$ if and only if $(x, A)R_s(y, B)$; if $(x, \{x\})I(R_s)(y, \{y\})$, then $(x, A)R(y, B)$ if and only if $(x, A)R_e(y, B)$, where $R = f(\mathbf{R})$. By construction, this ESWF satisfies **SP, IIA** and **ND**. The binary relation R generated by this ESWF is clearly reflexive and complete. We now show that R is transitive. Let $(x, A), (y, B)$ and $(z, C) \in \Omega$ be such that $(x, A)R(y, B)$ and $(y, B)R(z, C)$. Note that since $(x, A)R(y, B)$, by the ESWF constructed above, we cannot have $(y, \{y\})P(R_s)(x, \{x\})$. Then, by the completeness of R_s, there are only two cases to be distinguished and separately addressed: (a) $(x, \{x\})I(R_s)(y, \{y\})$; and (b) $(x, \{x\})P(R_s)(y, \{y\})$.

Case (a): In this case, we must have $(x, A)R_e(y, B)$. If $(y, \{y\})I(R_s)(z, \{z\})$, then from $(y, B)R(z, C)$, it follows that $(y, B)R_e(z, C)$. Then, the transitivity of R_e implies $(x, A)R_e(z, C)$. By the transitivity of R_s, $(x, \{x\})I(R_s)(z, \{z\})$. Therefore, $(x, A)R(z, C)$ iff $(x, A)R_e(z, C)$. Hence, $(x, A)R(z, C)$ follows from $(x, A)R_e(z, C)$. If $(y, \{y\})P(R_s)(z, \{z\})$, then, by the transitivity of R_s, it follows that $(x, \{x\})P(R_s)$ $(z, \{z\})$. Therefore, $(x, A)R(z, C)$ iff $(x, A)R_s(z, C)$. s being a strong consequentialist, given that $(x, \{x\})P(R_s)(z, \{z\})$, we must have $(x, A)P(R_s)(z, C)$. Therefore, $(x, A)P(R)(z, C)$. Hence, $(x, A)R(z, C)$ holds. Note that, given $(y, B)R(z, C)$, we cannot have $(z, \{z\})P(R_s)(y, \{y\})$. Therefore, the transitivity of R holds in case (a).

Case (b): In this case, we must have $(x, A)P(R_s)(y, B)$, hence $(x, A)P(R)(y, B)$. Since $(y, B)R(z, C)$, we must then have $(y, \{y\})R_s(z, \{z\})$. By the transitivity of R_s, it follows that $(x, \{x\})P(R_s)(z, \{z\})$. Thus, $(x, A)P(R_s)(z, C)$ follows from s being a strong consequentialist. By construction, in this case, $(x, A)R(z, C)$ iff $(x, A)R_s(z, C)$. Hence, $(x, A)P(R)(z, C)$. Therefore, the transitivity of R holds in case (b).

Combining cases (a) and (b), the transitivity of R is proved. ■

Consistently with the messages of Theorem 1 and Theorem 2, we can prove the following:

Theorem 3. Suppose that all individuals are strong consequentialists. Then, there exists no extended social welfare function f with the domain $D_f(S)$ which satisfies **SP, IIA** and **ND**.

Proof. Since all individuals are strong consequentialists, we have the following for all $i \in N$: For all $(x, A), (y, B) \in \Omega$, if $(x, \{x\})I(R_i)(y, \{y\})$, then $\#A \geq \#B \Leftrightarrow (x, A)R_i(y, B)$, whereas if $(x, \{x\})P(R_i)(y, \{y\})$, then $(x, A)R_i(y, B) \Leftrightarrow (x, X)R_i(y, X)$. Suppose that an ESWF f satisfies **SP** and **IIA**, and consider all triples $(x, A), (y, B)$ and $(z, C) \in \Omega$ such that x, y and z are all distinct. Since all individuals are strong consequentialists and f has the domain $D_f(S)$, there exists no restriction on each individual's strict extended preference orderings over $\{(x, A), (y, B), (z, C)\}$. Thus, there is a dictator over the triple $\{(x, A), (y, B), (z, C)\}$. Note that the triple $\{(x, A), (y, B), (z, C)\}$, where $A \neq X$, coincides with the triple $\{(x, X), (y, B), (z, C)\}$ over the pair $\{(y, B), (z, C)\}$. Hence the dictator over the triple $\{(x, A), (y, B), (z, C)\}$ must in fact be independent of the set $A \in K$. The same argument can be applied to $B \in K$ as well as $C \in K$. Hence, for *all* triples $\{(x, A), (y, B), (z, C)\}$, we must have a single dictator. Call him $d \in N$ and consider a triple $(x, A), (y, B)$ and $(z, C) \in \Omega$ such that x, y and z are all distinct. Consider any $(x, A^*) \in \Omega$, where $A \neq A^*$. If $A^* \subset A$, all individuals being strong consequentialists, **SP** implies that $(x, A)P(R)(x, A^*)$, where $R = f(\mathbf{R})$. Similarly, if $A \subset A^*$, all individuals being strong consequentialists, **SP** implies that $(x, A^*)P(R)(x, A)$, where $R = f(\mathbf{R})$. If neither A is a subset of A^*, nor A^* is a subset of A, all individuals being strong consequentialists, we must have $(x, A)R_i(x, A^*)$ iff $\#A \geq \#A^*$. Then, **SP** implies that $(x, A)R(x, A^*)$ iff $\#A \geq \#A^*$, where $R = f(\mathbf{R})$. Hence, d is a dictator over Ω. Therefore, there exists no ESWF satisfying **SP, IIA** and **ND**. ■

The message of these simple results seems very clear. Within the consequentialist framework, if all individuals are either extreme consequentialists or strong consequentialists, we have Arrovian impossibility results. As the society becomes diverse by having at least one extreme consequentialist and at least one strong consequentialist simultaneously, however, it is possible to design an extended social welfare function that satisfies the Arrow conditions. Thus, *it is the diversity of the society, or the heterogeneity of population in the society, that plays a crucial role in resolving the Arrow impossibility theorem within the consequentialist framework.*

4 Arrovian Impossibility Theorem in the Non-Consequentialist Framework

Let us now turn to the examination of Arrow's impossibility theorem in a non-consequentialist framework. Our first task is to clarify what precisely we mean by non-consequentialism. Following Suzumura and Xu (1999), let us define an individual to be a non-consequentialist as follows:[8]

Non-Consequentialist: An individual $i \in N$ is said to be a *non-consequentialist* if, for all $(x, A), (y, B) \in \Omega$, (a) $\#A > \#B \Rightarrow (x, A)P(R_i)(y, B)$; and (b) $\#A = \#B \Rightarrow [(x, A)R_i(y, B) \Leftrightarrow (x, \{x\})R_i(y, \{y\})]$.

Thus, a non-consequentialist is a person whose preference ranking over two extended alternatives $(x, A), (y, B) \in \Omega$ are such that, whenever the opportunity set A contains more alternatives than the opportunity set B, (x, A) is ranked higher than (y, B). It is only when A and B contain the same number of alternatives that (x, A) and (y, B) are ranked exactly the same as $(x, \{x\})$ and $(y, \{y\})$. In this sense, a non-consequentialist is in sharp contrast with both an extreme consequentialist and a strong consequentialist.

The following lemma shows that, for a non-consequentialist, imposing the conditions **IND, SI** and **MON** introduced in Section 2 does not in fact restrict his/her preferences at all.

Lemma 3. A non-consequentialist's extended preference orderings must satisfy the three conditions **IND, SI** and **MON**.

Proof. It can be checked easily that a non-consequentialist's extended preference orderings satisfy **SI** and **MON**. We now show that **IND** is also satisfied by a non-consequentialist's extended preference orderings.

Let i be a non-consequentialist. Let $(x, A), (y, B) \in \Omega$ and $z \in X - A \cup B$. Suppose $(x, A)R_i(y, B)$. There are two cases to consider: (a) $\#A > \#B$ and (b) $\#A = \#B$. In case (a), clearly, $(x, A)P(R_i)(y, B)$ and $\#(A \cup \{z\}) > \#(B \cup \{z\})$. Hence, $(x, A \cup \{z\})P(R_i)(y, B \cup \{z\})$ follows from i being a non-consequentialist. In case (b), we have $(x, A)R_i(y, B)$ iff $(x, \{x\})R_i(y, \{y\})$ and $\#(A \cup \{z\}) = \#(B \cup \{z\})$. Then, $(x, A \cup \{z\})R_i(y, B \cup \{z\})$ iff $(x, \{x\})R_i(y, \{y\})$ follows from i being a non-consequentialist. Noting that R_i is complete, **IND** is therefore satisfied by R_i. ■

Let $D_f(N)$ be the domain of an extended social welfare function f such that there exists at least one person, say $n^* \in N$, who is a non-consequentialist *uniformly* for all profiles $\mathbf{R} = (R_1, R_2, \cdots, R_n) \in D_f(N)$. Such a person will be called a uniform non-consequentialist over $D_f(N)$.

Theorem 4. Suppose that there exists at least one person who is a uniform non-consequentialist over $D_f(N)$. Then, there exists an extended social welfare function f with the domain $D_f(N)$ that satisfies **SP, IIA** and **ND**.

Proof. Let $n^* \in N$ be a uniform non-consequentialist over $D_f(N)$. Then, for all $\mathbf{R} = (R_1, R_2, \cdots, R_n) \in D_f(N)$ and all $(x, A), (y, B) \in \Omega$, $\#A > \#B$ implies $(x, A)P(R_{n^*})(y, B)$. Consider now the following ESWF f: For all $(x, A), (y, B) \in \Omega$,

if $\#A > \#B$, then $(x, A)P(R)(y, B)$;
if $\#A = \#B = 1$, then $(x, \{x\})R(y, \{y\})$ iff $(x, \{x\})R_1(y, \{y\})$;
if $\#A = \#B = 2$, then $(x, A)R(y, B)$ iff $(x, A)R_2(y, B)$;
$\vdots$

if $A = B = X$, then $(x, A)R(y, A)$ iff $(x, A)R_k(y, B)$, where $k = \min\{\#N, \#X\}$,

where $R = f(\mathbf{R})$. It is easy to verify that this f satisfies **SP, IIA** and **ND**.[9] It is also clear that R generated by this ESWF is reflexive and complete. We now show that R is transitive as well. Let $(x, A), (y, B), (z, C) \in \Omega$ be such that $(x, A)R(y, B)$ and $(y, B)R(z, C)$. Then, clearly, $\#A \geq \#B$ and $\#B \geq \#C$. If $\#A > \#B$ or $\#B > \#C$, then $\#A > \#C$. By the constructed ESWF, $(x, A)P(R)(z, C)$ follows easily. Thus, the transitivity of R holds for this case. Now, suppose $\#A = \#B = \#C$. Note that in this case, for all $(a, G), (b, H) \in \Omega$ such that $\#G = \#H = \#A$, $(a, G)R(b, H)$ iff $(a, G)R_k(b, H)$ where $k \in N$ and $k = \min\{\#N, \#A\}$. Therefore, the transitivity of R follows from the transitivity of R_k. The above two cases exhaust all the possibilities. Therefore, R is transitive. ■

It is worthwhile to emphasize that, unlike the extreme consequentialist or the strong consequentialist, a non-consequentialist is able to guarantee the existence of an Arrovian extended social welfare function *by himself*, and his ability is not nullified even in the homogeneous society where all individuals are non-consequentialists.

5 Generalizations

Although our analysis so far invoked the simple cardinality measure of the richness of opportunities, which is often criticized for its naivety, some of our results can go far beyond this special measure. To see this, let Θ be a complete orderings over K such that, for all $A, B \in K$, $A\Theta B$ holds if and only if A contains no less opportunity than B. $P(\Theta)$ and $I(\Theta)$ stand, respectively, for the asymmetric part and the symmetric part of Θ. The set X of all conventionally defined social states can be partitioned by $I(\Theta)$. Let $\mathcal{K}_\Theta$ denote the family of equivalence classes in accordance with Θ. For each $A \in K$, let $E_\Theta(A) \in \mathcal{K}_\Theta$ be the equivalence class determined by A. We can then define a linear ordering Θ^* on $\mathcal{K}_\Theta$ by

(5.1) For all $E_\Theta(A), E_\Theta(B) \in \mathcal{K}_\Theta$, $E_\Theta(A)\Theta^* E_\Theta(B) \Leftrightarrow A\Theta B$.

In what follows, we assume that $(\Theta, \mathcal{K}_\Theta)$ satisfies the following two basic requirements:

Assumption U: The richness measure of opportunities, Θ, is unanimously held by all individuals in the society.

Assumption R: There exist at least two equivalence classes in $\mathcal{K}_\Theta$.

The definitions of consequentialists and non-consequentialists now read as follows:

Extreme Consequentialist: An individual $i \in N$ is said to be an *extreme consequentialist* if, for all $(x, A), (x, B) \in \Omega$, it is true that $(x, A)I(R_i)(x, B)$.

Strong Consequentialist: An individual $i \in N$ is said to be a *strong consequentialist* if, for all $(x, A), (y, B) \in \Omega$,

(a) if $(x, \{x\})I(R_i)(y, \{y\})$, then $A\Theta B \Leftrightarrow (x, A)R_i(y, B)$; and

(b) if $(x, \{x\})P(R_i)(y, \{y\})$, then $(x, A)P(R_i)(y, B)$.

Non-Consequentialist: An individual $i \in N$ is said to be a *non-consequentialist* if, for all $(x, A), (y, B) \in \Omega$,

(a) $AP(\Theta)B \Rightarrow (x, A)P(R_i)(y, B)$; and

(b) $AI(\Theta)B \Rightarrow [(x, A)R_i(y, B) \Leftrightarrow (x, \{x\})R_i(y, \{y\})]$.

We can now generalize our results in Sections 3 and 4 for the framework discussed in this section as follows. The proofs of these results are similar to those of Theorems 1, 2, 3 and 4, and we may safely omit them.

Theorem 5. Suppose that all individuals are extreme consequentialists. Then, there exists no extended social welfare function f with the domain $D_f(E)$ which satisfies **SP, IIA** and **ND**.

Theorem 6. Suppose that there exists at least one extreme consequentialist and at least one strong consequentialist in the society. Then, there exists an extended social welfare function f with the domain $D_f(E \cup S)$ which satisfies **SP, IIA** and **ND**.

Theorem 7. Suppose that all individuals are strong consequentialists. Then, there exists no extended social welfare function f with the domain $D_f(S)$ which satisfies **SP, IIA** and **ND**.

Theorem 8. Suppose that there exists at least one person who is a uniform non-consequentialist over $D_f(N)$. Then, there exists an extended social welfare function

f with the domain $D_f(N)$ which satisfies **SP, IIA** and **ND**.

Thus, our basic results in this paper do not in fact hinge on somewhat controversial cardinality measure of the richness of opportunities.

6 Concluding Remarks

This paper developed two extended analytical frameworks of social choice theory in order to check how and to what extent Arrow's general impossibility theorem hinges on his basic assumption of welfarist-consequentialism. Another motivation of our analysis was to see whether or not Arrow's observation that "the possibility of social welfare judgments rests upon a similarity of attitudes toward social alternatives" could be substantiated in the arena which is wider than welfarist-consequentialism.

The starting point of our analysis was an extended individual preference ordering defined over the pairs of social states and opportunity sets to which these social states belong.[10] It seems to us that people are prepared to say that choosing an alternative x from an opportunity set A is at least as good as choosing an alternative y from an opportunity set B. Negating the possibility of expressing such an extended preference ordering altogether is tantamount to saying that *there is no intrinsic value in the act of choice as such,* since we are then not in the position to say that choosing x from A, which includes x among others, is better than choosing x from $\{x\}$, which in fact means no effective choice at all. The concept of extended preference orderings enabled us to formulate a wider conceptual framework for analyzing social choice, and we could identify two such frameworks: the *consequentialist framework* and the *non-consequentialist framework.* The former is concerned with a society in which at least one consequentialist, either extreme or strong, is residing, whereas the latter is concerned with a society in which at least one non-consequentialist is residing.

Within the consequentialist framework, it was shown that the Arrovian impossibility theorem strenuously comes back if all individuals are either extreme consequentialists or strong consequentialists, whereas a more diverse society resided simultaneously by at least one extreme consequentialist and at least one strong consequentialist admits the existence of an Arrovian extended social welfare function. In this sense, it is the *diversity* rather than *similarity* of individual attitudes towards social alternatives in the society that helps resolve the Arrow impossibility theorem within the consequentialist framework. The logical fate of the non-consequentialist society is rather different. Indeed, within the non-consequentialist framework, it was possible to guarantee the existence of an Arrovian extended social welfare function as long as there exists at least one non-consequentialist in the society, and this ability is not nullified even if the society is *homogeneous* so that all individuals are non-consequentialists. Although these results are first established by using a naive cardinality measure of the richness of opportunities, their validity does not hinge on this arguably controversial measure.

It is hoped that our results, though simple, would be suggestive enough to motivate further exploration of the wider conceptual frameworks of social choice theory.

End Notes

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[1] See Sen (1979a; 1979b; 1995), and Sen and Williams (1982) for more detailed account of welfarist-consequentialism and criticisms thereof. See also Arrow (1987), Hammond (1987), and Suzumura (1996; 1999; 2000).

[2] Welfarist-consequentialism has been under intensive scrutiny and harsh criticism in recent years by, e.g., Bossert, Pattanaik and Xu (1994), Pattanaik and Xu (1990; 1998), Sen (1985; 1995; 1996), and Suzumura (1996; 1999; 2000). It is often argued that individuals are ready to express preferences not only over outcomes, but also over opportunity sets from which outcomes are chosen. It is not infrequently suggested that such an extended preference should be duly taken into consideration in the analysis of how socially right or wrong actions should be determined. However, most of the preceding attempts to respond to these arguments and suggestions are concerned with the *ranking of opportunity sets* in terms of freedom of choice and/or overall well-being of individuals. To the best of our knowledge, the question as to how individuals' extended preference orderings over outcomes and opportunities should be aggregated into the social preference ordering has been left unexplored in this literature.

[3] It is true that Sen's (1970, Chapter 6*; 1976; 1979a; 1979b; 1995) well-known criticism against the welfaristic foundations of normative economics and social choice theory, which capitalizes on his *impossibility of a Paretian liberal*, sharply brought the importance of non-welfaristic features of consequences to the fore. However, it still keeps us within the broad territory of consequentialism. See, also, Suzumura (1978; 1996).

[4] See, among others, Black (1958), Kuga and Nagatani (1974), and Sen (1970, Chapter

10*; 1986).

[5] It may be argued that the measurement of opportunity in terms of the cardinality of the opportunity set is naive, and one should take such information as similarities among outcomes into consideration. This can be done as in Pattanaik and Xu (2000) using the minimum of the cardinalities of informationally equivalent classes rather than the cardinality of the opportunity set *per se*. It is for the purpose of keeping our framework as simple as possible that we are using in this paper the cardinality approach in measuring opportunity. See, however, Section 5 below.

[6] Recollect that we are neglecting decision-making cost and other factors that make a larger opportunity set a liability rather than a credit. For some arguments which may cast reasonable doubts on the universal worth of having a larger opportunity set rather than a smaller one, the interested readers are referred to Dworkin (1982).

[7] It is assumed that an extreme consequentialist (resp. a strong consequentialist) for a profile $\mathbf{R} \in D_f(E \cup S)$ remains to be an extreme consequentialist (resp. a strong consequentialist) *uniformly* for all profiles in $D_f(E \cup S)$.

[8] In the terminology coined by Suzumura and Xu (1999), a non-consequentialist defined here is called a *strong non-consequentialist*. Since this is the only category of non-consequentialism which is relevant in the present context, we have simplified our circumlocution by avoinding the adjective "strong".

[9] It may be worthwhile to note that the extended social welfare function constructed in this proof has some nice features. When $\#X + 1 \geq n$, every individual can dictate one "layer" of the extended social alternatives. Indeed, we assign the non-consequentialist to dictate over (x, A) and (y, B) in Ω such that $\#A \neq \#B$. For each of all other individuals, we assign him to dictate over (x, A) and (y, B) in Ω such that $\#A = \#B$ coincides with his "index". When $\#X + 1 < n$, the number of "layers" is less than sufficient to assign each individual a "layer" for him to dictate. In this case, however, we can divide the population into two groups, Group 1 and Group 2. Group 2, which consists of $(\#X + 1)$ individuals including the non-consequentialist, will dictate over specific extended social alternatives assigned to him. The remainder of individuals form Group 1, which consists of $(n - \#X - 1)$ individuals, decide which individual in Group 2 should dictate which "layer" of alternatives, allowing

the non-consequentialist to dictate over (x, A) and (y, B) with $\#A \neq \#B$. In this fashion, all individuals are assigned to actively participate in the process of social decision-making.

[10] As far as we are aware, Gravel (1994; 1998) was the first who analyzed the concept of extended preference orderings. However, he assumed the existence of two individual preference orderings, viz. the preference ordering r on the set of options X, on the one hand, and the extended preference ordering R on the Cartesian product of X and K, where K is the set of opportunity sets from which the individual chooses. His analysis was focussed on the possible conflict between these preference orderings, and no role whatsoever is played in his analysis by the concept of consequentialism and non-consequentialism. See also Sen (1997) for an implicit approach of using extended preference orderings in social choice theory.

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