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Beauty Contests and Fat Tails in Financial Markets

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Abstract

This paper demonstrates that fat-tailed distributions of trade volume and stock returns emerge in a simultaneous-move herding model of rational traders who infer other traders' private information on the value of assets by observing aggregate actions. Without parametric assumptions on the private information, I analytically show that the traders' aggregate actions follow a power-law distribution with exponential truncation. Numerical simulations show that the model is able to generate the fat-tailed distributions of returns as observed empirically. I argue that the learning among a large number of traders leads to a criticality condition for the power-law clustering of actions.

Keywords: Herd behavior, trade volume, stock return, fat tail, power law

JEL classification code: G14

1 Introduction

Since Mandelbrot [27] and Fama [14], it has been well established that the short-term stock returns exhibit a fat-tailed, leptokurtic distribution. Jansen and de Vries [20], for example, estimated the exponent of the power-law tail to be in the range 3 to 5, which warrants a finite variance and yet deviates greatly from the normal distribution in the fourth moment. This anomaly in the tail and kurtosis has been considered as a reason for the excess volatility of stock returns.

Efforts to explain this anomaly have been ongoing for long. A traditional economic explanation for the excess volatility of the volumes and returns relies on the traders' rational herd behavior. In a situation where a trader's private information on the asset value is partially revealed by her transaction, the trader's action can cause an avalanche of similar actions by the other traders. This idea of a chain reaction through the revelation of private information has been extensively studied in the literature of herd behavior, informational cascade, and information aggregation. However, there have been few attempts to explain the fat tail in this framework. This paper shows that the chain reaction of information revelation leads to the fat-tail distributions of the traders' aggregate actions and asset returns.

I consider a model of a large number of informed traders who receive imperfect private information on the true value of an asset. The traders simultaneously choose whether to buy one unit of the asset or not to buy at all. I consider a rational expectations equilibrium in which each trader submits her demand schedule conditional on price. The trader's rational choice is based on her private information as well as the information revealed by the other traders' actions through the equilibrium price. The price is set by an auctioneer who aggregates the informed traders' demand and matches the demand with the supply schedule submitted by uninformed traders. The equilibrium is a mapping from the space of private information of all traders to that of the aggregate actions. The larger the aggregate buying action, the more is the traders' subjective belief on the state of a high asset value, and the more likely each trader is to buy the asset. Hence, the traders' strategy exhibits complementarity, and their actions are positively correlated.

I derive the probability distribution of the equilibrium aggregate action and show that it decays as a power function with exponential truncation. The speed of the exponential truncation is determined by the strength of the strategic complementarity among traders. This analytical result is obtained by a new method that utilizes a fictitious stochastic tatonnement process to characterize the aggregate actions. The power-law distribution implies a large kurtosis. Thus, the power-law result illustrates that a significant magnitude of aggregate risk exists even when the uncertainty in the transaction volume solely stems from the idiosyncratic private information drawn by a large number of traders.

I extend the model dynamically in which the traders receive private information repeatedly. Suppose that the initial belief started far below the threshold belief. Then, traders buy only if they receive extremely good news. As private information is accumulated over time, however, the average belief increases toward a threshold at which some traders start buying regardless of the other traders' inactions. Such traders' buying actions trigger the other traders' buying, which results in a herd that follows a power-law distribution. This process implies that a large amount of private information tends to be revealed at once around the point of time when the average belief reaches some threshold. Thus, even though the subjective belief converges to the true value of an asset in the long run, the price process toward the true value can deviate significantly from the smooth path that would occur if the private information is fully revealed in each period. A sizable portion of the total price adjustment toward the true value is accounted for by the rare events of synchronized actions of a large number of traders.

The model can account for the non-normal, fat-tailed distribution of daily stock returns. The private information partially revealed by informed traders in the equilibrium transaction is reflected in the resulting shift in the price. I define the difference in the logarithm of price as the stock return. The impact of the transaction volume on the return is determined by the supply function of uninformed traders (or "liquidity suppliers"), which is calibrated as in Gabaix, Gopikrishnan, Plerou, and Stanley [15]. Each realization of the private information profile results in an equilibrium return, and thus, the returns distribution is obtained by Monte Carlo simulations of information draws. The simulated distribution is shown to resemble well the daily returns distribution observed in the Tokyo Stock Exchange.

As a herding model, our model is analogous to the Keynes' beauty contest. Each trader recognizes that the other traders have private information that is as valuable as her own. When each trader tries to match with the behavior of an average trader, the resulting equilibrium exhibits fragility due to perfect strategic complementarity. In addition, if the trader's action is discrete, the equilibrium becomes locally unique and allows quantitative characterization of the fluctuations due to the randomness in private information. This paper formalizes the idea of perfect strategic complementarity among the traders with private information, and shows that a power-law distribution of the aggregate actions emerges naturally in this setup.

An extensive array of literature addresses the issue of imitative behavior in financial markets. The models of herd behavior and informational cascade by Scharfstein and Stein [32], Banerjee [4], and Bikhchandani, Hirshleifer, and Welch [5] have been applied to financial market crashes by Lee [24] and Chari and Kehoe [10] among others. While the benchmark herd behavior model provides a robust intuition for rational herding, it typically exhibits an all-or-nothing herding due to its particular information structure implied by sequential trading. Some modification is due in order to apply the intuition of herd behavior to stochastic fluctuations. Gul and Lundholm [16], for example, have demonstrated an emergence of stochastic herding by endogenizing the traders' choice of waiting time. I extend this line of research by employing a simple simultaneous-move model of traders. This approach is related to Caplin and Leahy [9] who argue that the aggregate revelation of dispersed information in the market tends to occur suddenly as the last straw that breaks the camel's back. Stretching their analogy, this paper claims that, when the camel's back breaks, the rupture size is distributed according to a power law. Another underlying theme of this paper is the aggregation of private information (Vives [37]) or idiosyncratic shocks (Jovanovic [21]; Durlauf [13]). This paper shows that the aggregation of private information in the market leads to a non-trivial, structured fluctuation that is characterized by a power law.

The technical analysis I employ is linked to the field of critical phenomena in statistical physics. Recently, a number of statistical physicists investigated the empirical fluctuations of financial markets.[1] Some papers in this literature reproduce the empirical power laws by introducing the methodology used for critical phenomena to the herd behavior models (Bak, Paczuski, and Shubik [2]; Cont and Bouchaud [11]; Stauffer and Sornette [36]). Two questions have been raised for the models of critical phenomena. One is that they lack the model of traders' purposeful behavior and rational learning, which hinders the integration of their methodology to the existing body of financial

[1] A survey of these attempts is provided in Bouchaud and Potters [7] and Mantegna and Stanley [28].

economics. The other is a fundamental question as to why at all the market has to exhibit criticality. The power-law fluctuation occurs typically only at the critical value of a parameter that governs the connectivity of the networked traders. Gabaix et al. [15] address these questions by incorporating the trader's optimal behavior and by relating the power laws for the volumes and returns to Zipf's law for the size distribution of firms. This paper proposes an alternative by showing that the market necessarily converges to the critical point as a result of the purposeful behavior of individual traders who gain information from each other.

The remainder of the paper is organized as follows. In Section 2, a simple static model is presented. Section 3 analytically derives the power-law distribution and provides an intuition for the mechanism behind the fat tail. The model is also extended dynamically, and the power-law distribution is shown to occur at the state in which the heterogeneous beliefs of traders evolve. Section 4 shows by numerical simulations that the equilibrium volumes follow a power law and that the equilibrium returns distribution matches its empirical counterpart. Section 5 discusses the role of discrete actions and symmetric information structure in the model. Section 6 concludes.

2 Model

2.1 Model and equilibrium

In this section, I consider the simplest case in which each trader receives private information just once. I consider a financial market with N informed traders, where N is a large finite number. The informed traders do not know the true state of the economy, but each of them receives private information x_i that correlates with the state. The signal is private, meaning that trader i does not observe the information received by the other traders. The economy can take one of two possible states, H and L. The asset is worth 1 in state H and 0 in state L. The private information x_i is drawn independently across i from a known distribution F in state H and from G in state L.

A share of the asset is assumed to be indivisible, and the unit of transaction is normalized by the number of traders, $1/N$. Traders choose either to buy it or not. The situation where traders choose to sell or not can be symmetrically analyzed. The buying action of trader i is denoted by $a_i = 1$ and the non-buying action is denoted by $a_i = 0$. The informed trader's choice is conditional on the price of asset, p. The demand function of the informed traders is denoted by $a_i = d(p, x_i)$. The aggregate demand function is defined as $D(p) = \sum_{i=1}^{N} d(p, x_i)/N$.

In addition to the informed traders, there are uninformed traders who act as liquidity suppliers of the asset. I assume that the uninformed traders have an aggregate supply function $S(p)$, which is upward sloping and satisfies $S(p_0) = 0$. Thus, uninformed traders are contrarians who sell when the price is high and buy when the price is low. The equilibrium price is determined so that it clears the market: $D(p) = S(p)$. I define a sequence of price points p_k, $k = 1, 2, \ldots, N$, at which price the demand from k informed traders is met by suppliers: $S(p_k) = k/N$.

The transaction is implemented by an auctioneer. The auctioneer receives the demand and supply schedules D and S from the informed and uninformed traders, and clears the market by setting price p_m so that $D(p_m) = S(p_m)$.[2] Given p_m, the posterior belief of informed trader i for state H to occur is denoted by $b_{i,1}$. Informed

[2]This implementation of a rational expectations equilibrium by the submission of demand schedules follows Bru and Vives [8]. Without the information aggregation by the auctioneer, the model becomes similar to that of Minehart and Scotchmer [30], who showed that the traders cannot agree to disagree in a rational expectations equilibrium, i.e., the equilibrium may not exist, or if it exists, it is a herding equilibrium where all the traders choose the same action.

traders are risk-neutral and maximize their subjective expected payoff. The expected payoff of a trader is 0 when $a_i = 0$ regardless of the belief, whereas it is equal to $b_{i,1} - p_1$ when $a_i = 1$. Thus, trader i buys the asset if and only if $b_{i,1} \geq p$.

For each realization of x, a rational expectations equilibrium consists of price p_m, allocation a, demand schedule d, and posterior belief b_1, such that (i) for any p, $d(p, x_i)$ maximizes trader i's expected payoff evaluated at the posterior belief $b_{i,1}$ for any i, (ii) $b_{i,1}$ is consistent with the realized private information x_i and p_m for any i, and (iii) the auctioneer clears the market as $S(p_m) = \sum_{i=1}^{N} a_i$, and delivers the orders $a_i = d(p_m, x_i)$.

2.2 Information structure and optimal strategy

I impose a standard assumption that private information has the monotone likelihood ratio property (MLRP). I define an odds function $\delta(x_i) = g(x_i)/f(x_i)$, where f and g are derivatives of F and G, respectively. MLRP requires δ to be monotone. Without loss of generality, I assume that δ is strictly decreasing. Namely, a larger x_i implies a larger likelihood of H. I also assume that the prior belief for H to occur is common across i at b_0. This assumption is imposed for the sake of simplicity, and is relaxed in Section 3.2, where the belief is allowed to evolve heterogeneously over periods.

Using a likelihood ratio $\theta_{i,1} \equiv (1 - b_{i,1})/b_{i,1}$, the optimality condition for a buying action $b_{i,1} \geq p$ is equivalently expressed as $\theta_{i,1} \leq 1/p - 1$. Thanks to MLRP, the optimal demand schedule of trader j follows a threshold rule:

$$a_j = \begin{cases} 1 & \text{if } x_j \geq \bar{x}(m), \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\bar{x}(m)$ denotes the value of private information at which trader j is indifferent between buying and not buying given p_m.

Under the threshold rule, the likelihood ratios revealed by an inaction ($a_j = 0$) and by a buying action ($a_j = 1$) are derived as follows, respectively:

$$A(\bar{x}) \equiv \frac{\Pr(x_i < \bar{x} \mid L)}{\Pr(x_i < \bar{x} \mid H)} = \frac{G(\bar{x})}{F(\bar{x})}, \tag{2}$$

$$B(\bar{x}) \equiv \frac{\Pr(x_j \geq \bar{x} \mid L)}{\Pr(x_j \geq \bar{x} \mid H)} = \frac{1 - G(\bar{x})}{1 - F(\bar{x})}. \tag{3}$$

As in Smith and Sørensen [34], MLRP implies that, for any value of $\bar{x}$ in the interior of the support of F and G,

$$A(\bar{x}) > \delta(\bar{x}) > B(\bar{x}) > 0, \tag{4}$$

and that $A(\bar{x})$ and $B(\bar{x})$ are strictly decreasing in $\bar{x}$:

$$\frac{dA}{d\bar{x}} = \frac{g(\bar{x})}{F(\bar{x})} - \frac{G(\bar{x})f(\bar{x})}{F(\bar{x})^2} = \frac{f(\bar{x})}{F(\bar{x})}\left(\delta(\bar{x}) - A(\bar{x})\right) < 0, \tag{5}$$

$$\frac{dB}{d\bar{x}} = -\frac{g(\bar{x})}{1 - F(\bar{x})} + \frac{(1 - G(\bar{x}))f(\bar{x})}{(1 - F(\bar{x}))^2} = \frac{f(\bar{x})}{1 - F(\bar{x})}\left(B(\bar{x}) - \delta(\bar{x})\right) < 0. \tag{6}$$

A buying trader infers from the equilibrium price p_m that there are $m - 1$ informed traders buying and $N - m$ not buying. For a non-buying trader, the expected payoff is zero regardless of the trader's posterior belief. The posterior likelihood for a buying trader can be determined using the threshold rule above. Consider a buying trader at price p_1. If this bid is struck by the auctioneer, it implies that the other $N - 1$ informed traders do not bid at p_1. Thus, their actions (inactions) reveal the likelihood $A(\bar{x}(1))^{N-1}$. The posterior likelihood in this case is $\theta_{i,1} = A(\bar{x}(1))^{N-1}\delta(x_i)\theta_0$. Thus, the threshold is determined by

$$1/p_1 - 1 = A(\bar{x}(1))^{N-1}\delta(\bar{x}(1))\theta_0. \tag{7}$$

Similarly, a trader buying at p_k knows that, if the bid is executed, there are $k - 1$ traders bidding at p_k and $N - k$ traders not buying at p_k. Then, the threshold $\bar{x}(k)$ is obtained by solving

$$1/p_k - 1 = A(\bar{x}(k))^{N-k} B(\bar{x}(k))^{k-1} \delta(\bar{x}(k))\theta_0. \tag{8}$$

Given the threshold behavior above, I obtain the aggregate demand $D(p_k)$ as the number of informed traders with $x_i \geq \bar{x}(k)$ for $k = 1, 2, \ldots, N$. I set $D(p_0) = D(p_1)$, since p_0 and p_1 convey the same information to an informed trader that there is no other buying trader, while the purchasing cost p_0 is lower than p_1. Then, I obtain the following.

Proposition 1 *There exists $\bar{N}$ such that for any $N > \bar{N}$, there exists an equilibrium outcome m for each realization of x.*

By taking the log-difference of (8), I obtain

$$\log \frac{A(\bar{x}(k))}{B(\bar{x}(k))} + \log \frac{1/p_{k+1} - 1}{1/p_k - 1} = (N-k-1) \log \frac{A(\bar{x}(k+1))}{A(\bar{x}(k))} + k \log \frac{B(\bar{x}(k+1))}{B(\bar{x}(k))} + \log \frac{\delta(\bar{x}(k+1))}{\delta(\bar{x}(k))}. \tag{9}$$

Note that $\inf(\log A - \log B)$ is strictly positive and independent of N, while $\log(1/p_{k+1} - 1) - \log(1/p_k - 1)$ is of order $1/N$. Hence, the left-hand side is strictly positive for a sufficiently large N. The right-hand side is strictly positive only if $\bar{x}(k+1) < \bar{x}(k)$, since $A' < 0$, $B' < 0$, and $\delta' < 0$. Then, $\bar{x}(k)$ is strictly decreasing in k. Define $\mathcal{S} = \{0, 1, 2, \ldots, N\}$ as a set of possible equilibrium outcome m, and define an aggregate reaction function $\Gamma : \mathcal{S} \mapsto \mathcal{S}$ for each realization of $x = (x_1, x_2, \ldots, x_N)$ such that $\Gamma(m) = D(p_m)$. Since $\Gamma(m)$ is the number of traders with $x_i \geq \bar{x}(m)$ for $m = 1, 2, \ldots, N$, and since $\Gamma(0) = \Gamma(1)$, a decreasing $\bar{x}$ implies that Γ is non-decreasing in m for any realization of x. Since Γ is a non-decreasing mapping of a finite discrete set $\mathcal{S}$ onto itself, there exists a non-empty closed set of fixed points of Γ by Tarski's fixed point theorem. $\square$

The threshold strategy (1) allows multiple equilibria for each realization of x. Here, I focus on the case where the auctioneer selects the minimum number of buying traders, m^*, among possible equilibria for each x. The equilibrium selection maps each realization of x to m^*. Thus, m^* is a random variable if viewed unconditionally on x, and its probability distribution is determined by the probability measure of x and the equilibrium selection mapping. By this selection assumption, I exclude the fluctuations that arise purely from informational coordination such as in sunspot equilibria. Thus, under this selection, I can show that even the minimum shift in price $\log p_{m^*} - \log p_0$ exhibits large fluctuations. I further characterize the probability distribution of m^* and p_{m^*} in the following sections.

3 Analytical results

3.1 Derivation of the power law

In this section, I analytically derive the power-law distribution of the minimum aggregate action m^* defined in the model. I propose a method to characterize the distribution of m^* by using a fictitious tatonnement process. In so doing, I clarify the condition for the power law of m^* and provide an economic interpretation for the mechanism that generates the power law. I assume that the true state of the economy is H throughout the paper. The case of L can be analyzed similarly.

The minimum equilibrium m^* is known to be reached by the best response dynamics of traders, which Vives [39] called an informational tatonnement. The best response dynamics requires the traders to know only the "aggregate" information, m in this case. Cooper [12] argued that the parsimonious informational requirement of the best response dynamics makes it a desirable equilibrium selection algorithm when there exist multiple equilibria and when it is difficult for traders to coordinate their actions.

I start by showing that the informational tatonnement converges to m^*. I continue to use the notation developed for Proposition 1: I use $\mathcal{S}$ for the set of m and Γ for the aggregate reaction function on m.

Proposition 2 *Consider an informational tatonnement process m_u, where $m_0 = 0$ and $m_u = \Gamma(m_{u-1})$ for $u = 1, 2, \ldots, T$, where the stopping time T is the smallest u such that $m_u - m_{u-1} = 0$. Then, m_u converges to the minimum equilibrium m^* for each realization of x. Moreover, the threshold decreases over the informational tatonnement process: $\bar{x}(m_{u+1}) < \bar{x}(m_u)$ for any realization of x.*

Proof: Applying Vives [38], it is directly shown that this tatonnement always reaches a fixed point m_T of Γ, since Γ is increasing, $\mathcal{S}$ is finite, and $m_0 = 0$ is the minimum in $\mathcal{S}$. Further, m_T must coincide with the minimum fixed point m^* for the following reason. Suppose that there exists another fixed point m that is strictly smaller than m_T. Then I can pick $u < T$ such that $m_u < m < m_{u+1}$. Applying the non-decreasing function Γ, I obtain $\Gamma(m_u) \leq \Gamma(m)$. Then, $m_{u+1} \leq m$. This contradicts $m < m_{u+1}$. $\square$

Since the informational tatonnement starts from $m_0 = 0$ and converges to m^*, I can express m^* as the sum of increments in the tatonnement process. Moreover, the informational tatonnement can be regarded as a stochastic process, once it is viewed unconditionally on the private information x. Thus, m^* can be expressed as the sum of a stochastic process that starts from and converges to zero. I use this tatonnement as an algorithm to compute the minimum aggregate action m^*. The idea of characterizing an equilibrium outcome by a stochastic process is similar in spirit to Kirman [23].

By showing that the threshold $\bar{x}$ decreases over the stochastic process, Proposition 2 establishes that there exists a non-trivial chance of a chain reaction during the tatonnement. A trader who chooses to buy in u will continue to choose to buy in $u + 1$, since the threshold is lowered. A trader who does not buy in u might choose to buy in $u+1$. The conditional probability of a non-buying trader switching to buying in response to $m_u - m_{u-1}$ is defined as follows:

$$q_u \equiv \int_{\bar{x}_u}^{\bar{x}_{u-1}} f(x)dx/F(\bar{x}_{u-1}), \quad u = 1, 2, \ldots, N, \tag{10}$$

where $\bar{x}_u$ is a shorthand for $\bar{x}(m_u)$. q_u is always non-negative because of the decreasing threshold. Thus, $m_{u+1} - m_u$, the number of traders who buy in $u+1$ for the first time, conditional on the tatonnement history up to u, follows a binomial distribution with population parameter $N - m_u$ and probability parameter q_u. The distribution of m_1 follows a binomial distribution with population N and probability $q_0 \equiv 1 - F(\bar{x}_0)$. This completely defines the stochastic tatonnement process, as summarized in the following proposition.

Proposition 3 *Consider a stochastic process $m_u - m_{u-1}$, $u = 1, 2, \ldots, T$, where $m_0 = 0$. Suppose that $m_{u+1} - m_u$ conditional on $m_u - m_{u-1}$ follows a binomial distribution with population $N - m_u$ and probability q_u, which is determined by (10) and $\bar{x}_u = \bar{x}(m_u/N)$. Further, suppose that m_1 follows a binomial distribution with population N and probability q_0. Then, the minimum equilibrium number of buying traders m^* follows the same distribution as m_T, the cumulative sum of the process.*

Proposition 3 establishes that the minimum equilibrium m^* is equal to the sum of a binomial process. A binomial distribution permits Poisson approximation when the population is "large" and the probability is "small." The approximation holds in our tatonnement if the probability q_u is of order $1/N$. I show that this is the case.

Proposition 4 *As $N \to \infty$, the binomial process $m_{u+1} - m_u$ asymptotically follows a branching process with a state-dependent Poisson random variable with mean*

$$\phi_u = \frac{A(\bar{x}_{u-1})}{\delta(\bar{x}_{u-1})} \frac{\log A(\bar{x}_{u-1}) - \log B(\bar{x}_{u-1})}{A(\bar{x}_{u-1})/B(\bar{x}_{u-1}) - 1}. \tag{11}$$

Moreover, $\phi_u \to 1$ *when* $\sup_x(G(x) - F(x)) \to 0$.

Proof: Equation (9) implies that $\bar{x}(k) - \bar{x}(k+1)$ is of order $1/N$. From Equation (10), I obtain that $q_u = (f(\bar{x}_{u-1})/F(\bar{x}_{u-1}))(\bar{x}(m_{u-1}) - \bar{x}(m_u)) + O(1/N^2)$. Thus, q_u is also of order $1/N$. The asymptotic mean of the binomial variable $m_{u+1} - m_u$ conditional on $m_u - m_{u-1} = 1$ is derived as follows:

$$\begin{aligned}
\phi_u &\equiv \operatorname*{plim}_{N\to\infty} q_u|_{m_u - m_{u-1}=1}(N - m_u) \\
&= \operatorname*{plim}_{N\to\infty} \frac{f(\bar{x}_{u-1})}{F(\bar{x}_{u-1})} \frac{\log A(\bar{x}_{u-1}) - \log B(\bar{x}_{u-1})}{\frac{N-m_{u-1}-1}{N}\frac{A'(\bar{x}_{u-1})}{A(\bar{x}_{u-1})} + \frac{m_{u-1}}{N}\frac{B'(\bar{x}_{u-1})}{B(\bar{x}_{u-1})}} \frac{-(N-m_u)}{N} && (12) \\
&= \operatorname*{plim}_{N\to\infty} \frac{\log A(\bar{x}_{u-1}) - \log B(\bar{x}_{u-1})}{\frac{N-m_{u-1}}{N}\left(1 - \frac{\delta(\bar{x}_{u-1})}{A(\bar{x}_{u-1})}\right) + \frac{m_{u-1}}{N}\frac{F(\bar{x}_{u-1})}{1-F(\bar{x}_{u-1})}\left(\frac{\delta(\bar{x}_{u-1})}{B(\bar{x}_{u-1})} - 1\right)} \frac{N-m_u}{N}, && (13)
\end{aligned}$$

where I used (9) and the fact that the difference of $\log p_k$ is of order $1/N$ for the second equation and (5) and (6) for the third equation. Note that m_u/N converges to $1 - F(\bar{x}_{u-1})$ with probability 1 for a fixed threshold $\bar{x}_{u-1}$ as $N \to \infty$ by the strong law of large numbers. Then, $(m_u/(N - m_u))(F(\bar{x}_{u-1})/(1 - F(\bar{x}_{u-1})))$ converges to 1 with probability 1. Applying this to (13), I obtain the expression (11). Using that $\bar{x}(m_u) - \bar{x}(m_{u-1})$ is of order $1/N$, I obtain that $q_u(N - m_u) \to \phi_u(m_u - m_{u-1})$ for $N \to \infty$. Hence, $m_{u+1} - m_u$ asymptotically follows a Poisson distribution with mean $\phi_u(m_u - m_{u-1})$, which is equivalently a $(m_u - m_{u-1})$-times convolution of a Poisson distribution with mean ϕ_u. Thus, the binomial process asymptotically follows a branching process in which each parent bears a random number of children that follows a Poisson distribution with mean ϕ_u.

When the distribution G is taken closer to F, $A(\bar{x})/B(\bar{x}) \to 1$ holds since $A/B = (1/F - 1)/(1/G - 1)$. As I take $A(\bar{x})/B(\bar{x}) \to 1$, the first fraction in the right-hand side of (11) converges to 1 because of $A > \delta > B$, and the second fraction also converges to 1 by l'Hospital's rule. Thus, $\phi_u \to 1$. □

A branching process is a stochastic integer process of population in which each individual ("parent") in a generation bears a random number of "children" in the next generation. Proposition 4 shows that the number of newly buying traders in each step $u+1$ asymptotically follows a Poisson distribution with mean $\phi_u(m_u - m_{u-1})$. Since m_u is an integer and since Poisson distribution is infinitely divisible, the tatonnement process asymptotically follows the branching process in which each newly buying trader in u induces a random number of other traders to buy in $u+1$ according to a Poisson distribution with mean ϕ_u.

Proposition 4 shows that ϕ_u does not depend on N for a given level of threshold $\bar{x}$. This means that q_u, the probability of a non-buying trader switching to buying in u, is of order $1/N$. This also implies that the decrease in the threshold $\bar{x}$ in each step is of order $1/N$. This property is important for the tatonnement process to generate a non-degenerate distribution of the total number of buying traders m^*. If q_u is of order less than $1/N$, then m^* converges to zero as $N \to \infty$. If q_u is of order greater than $1/N$, the process explodes to infinity with probability one as $N \to \infty$. Only when q_u is of order $1/N$, m^* exhibits non-degenerate stochastic fluctuations.

Now, I derive the distribution of m^* as the cumulative sum of the informational tatonnement. Proposition 4 shows that the tatonnement can be approximated by a branching process with state-dependent Poisson mean, and that the Poisson mean is close to 1 when the informativeness of the private information is vanishingly small. When the Poisson mean is constant, I obtain the following distribution for the total size of the branching process.

Proposition 5 *Consider a branching process $m_u - m_{u-1}$ in which $m_0 = 0$, m_1 follows a Poisson distribution with mean μ, and $m_{u+1} - m_u$ conditional on $m_u - m_{u-1} = 1$ follows a Poisson distribution with mean ϕ. Then, the sum of the branching process*

m_T has the following distribution:

$$\Pr(m_T = m) = \frac{\mu e^{-(\phi m+\mu)}}{m!}(\phi m + \mu)^{m-1} \tag{14}$$

$$\propto e^{-(\phi-1-\log\phi)m} m^{-1.5}, \tag{15}$$

where the second line holds asymptotically as $m \to \infty$.

Proof: The sum m_T conditional on $m_1 = 1$ follows a closed form distribution known as Borel-Tanner distribution in the queuing theory [22]. Equation (14) is derived by mixing the Borel-Tanner distribution and the Poisson distribution for m_1, and (15) is obtained by applying Stirling's formula [31]. □

Proposition 5 states that the sum of the Poisson branching process follows a power-law distribution with exponent 0.5 with exponential truncation (the exponent is defined for a cumulative distribution). Since our informational tatonnement does not generally have a constant branching mean during the process for a finite N, Proposition 5 applies to m^* asymptotically only when $N \to \infty$ and $\sup(G - F) \to 0$. I provide an explanation on the analytical property of this asymptotic case, while the deviation from this asymptotic result for a finite economy is examined by numerical simulations in Section 4.

The tail distribution as in (15) is known to be obtained not only for the Poisson branching process but also for any branching process (see Harris [17] and Sornette [35] for the robustness of this result for generalized branching processes). For the case $\phi > 1$, there is a non-zero probability for m^* to be infinite. This is because the branching process $m_{u+1} - m_u$ does not reach 0 with a non-zero probability if $\phi > 1$, whereas it reaches 0 in a finite step with probability one if $\phi \leq 1$.

The exponential tail holds for a large finite m^* either for the subcritical case $\phi < 1$ or the supercritical case $\phi > 1$ (Harris [17]). The speed of exponential truncation is determined by $|\phi - 1 - \log \phi|$. The speed of the exponential decay slows down as ϕ becomes close to 1, and disappears when $\phi = 1$. At this critical level $\phi = 1$, the branching process becomes a martingale, and the distribution (14) has a power-law tail. The power exponent (in the cumulative form) is 0.5, which is less than one, implying that the mean of m^* diverges to infinity if $\phi = 1$. The exponent 0.5 is closely related to the same exponent that appears for the distribution of the first return time of a martingale. The diverging mean can also be shown as follows. The branching process has a recursive characteristic, in which the total number of offsprings originating from a child has the same probability distribution as the total number of offsprings originating from the parent of the child. Let $H(s)$ denote the probability generating function of the total number of offsprings generated by one parent, and let $J(s)$ denote the probability generating function of the number of children each parent bears. Then, the relation $H(s) = sJ(H(s))$ must hold, where $J(H(s))$ is the probability generating function for the offsprings originating from all children of the parent, and s is multiplied to $J(H(s))$ because $H(s)$ counts the parent in itself. By taking a derivative and evaluating it at $s = 1$, I obtain $H'(1) = 1 + J'(1)H'(1)$, where $J'(1) = 1$ if the mean number of children per parent is one. Hence, $H'(1)$ does not have a finite solution if $J'(1) = 1$, implying that the total population does not have a finite mean.

What is the economic intuition for such a large fluctuation of the minimum aggregate actions m^*? The key to the fluctuation is that each informed trader responds to the *average* behavior of the other informed traders. This can be seen from the threshold condition (8) for a fixed fraction of buying traders $\alpha = m/N$ at the limit of N: $(1 - \alpha) \log A(\bar{x}) + \alpha \log B(\bar{x}) = 0$. When a trader buys, the other traders adjust their beliefs not only due to the observed action but also due to the observed inactions of the other traders. The buying action decreases the likelihood ratio of the other traders by a factor of B/A, and thus, decreases their threshold $\bar{x}$. The inactions of these traders under the revised threshold partially reveal their private information in turn in favor of not buying. As a result of these two forces, the threshold is shifted so that the impact on the public information caused by the triggering buying action is canceled out by the disbelief subsequently revealed by the inaction of bear traders. Since there are N traders in the market, the impact of one buying action on the threshold is of order $1/N$. This implies that the probability of a trader induced to buy by the lowered threshold is of order $1/N$. Since there are $N-m$ traders who can be induced to buy potentially, the branching mean of the newly buying traders ϕ becomes of order N^0. This is a natural order of magnitude for ϕ in the case of the symmetric information structure as I assume here, because the traders have no reason to imagine that the information revealed by an action should weigh more or less than that revealed by an inaction. Hence, it is a robust feature of herd behavior models that the chain-reaction mechanism operates near the criticality when the size of traders in terms of informational weights is not extremely diversified.

3.2 Dynamic extension

The results in the static model were derived under the assumption of homogeneous prior belief. The results hold even if the prior belief is heterogeneous. A particularly interesting case is when the belief evolves over time as private information is drawn repeatedly. In this case, even though I maintain the assumption that the prior belief in the initial period is homogeneous, the belief in the subsequent periods will be heterogeneous due to the past private information. In this sequence of static equilibria, I show that the informational tatonnement continues to be characterized as before.

The dynamic extension not only relaxes the assumption of common prior belief but also ensures that the propagation effect shown in the static model is triggered at some point of time. The limiting behavior of q_0 when $N \to \infty$ is ambiguous in the static model. This leaves a possibility that the chain reaction is practically never triggered for a large N if $q_0 N \to 0$. This is because the traders will rarely react to the private information at the beginning of the tatonnement where no other traders reveal their information, if the prior belief is very low.

It turns out in the dynamic model that the traders eventually learn the true state as they accumulate private information. This implies that regardless of the level of the initial prior belief or N, the belief increases to the level at which traders start buying even though the other traders are not buying. This triggers the propagation of buying actions. This dynamics is similar to the self-organized criticality of Bak et al. [3] with respect to the traders' average belief converging to the state at which the size distribution of herd behavior is characterized by a power law.

3.2.1 Heterogeneous belief

I dynamically extend the basic model as follows. Each trader i draws private information $x_{i,t}$ repeatedly over periods for $t = 1, 2, \ldots$. The private information is identically and independently distributed across traders and periods. I consider the same asset as before that is worth 1 in H and 0 in L. Traders are given an opportunity to buy this asset regardless of their past actions. Noise traders provide the supply function that has the same elasticity as in the static model but a different intercept $S(p_{t-1}) = 0$. The intercept p_{t-1} reflects the equilibrium price in the previous period, as it incorporates the information revealed to the public in that period. Informed traders submit their demand schedule to an auctioneer who clears the market as in the static model. To maximize the expected payoff of the transaction in t, trader i buys the asset in t

if $b_{i,t} \geq p_t$ and does not buy otherwise. There is no dynamic aspect involved in the traders' decision other than updating the belief.

Informed traders observe their private information history $x_i^t = (x_{i,1}, x_{i,2}, \dots, x_{i,t})$. I study a sequence of static equilibria (p_t, a_t, b_t), $t = 1, 2, \dots$, such that the action $a_{i,t}$ maximizes trader i's expected period payoff under the subjective belief $b_{i,t}$ (defined for the state H as before), which is consistent with the trader's observation.

The prior belief at the initial period is common, $b_{i,0} = b_0$, but the belief is allowed to evolve stochastically as the traders draw information repeatedly. Thus, the belief in each period $t > 0$ is heterogeneous across traders with a particular structure wherein the heterogeneity stems only from the distribution functions F and G that are ordered by MLRP.

Given an action profile history a^{t-1}, all traders are divided into 2^{t-1} groups according to their action history a_i^{t-1}. Let $n_{k,t}$ denote the number of traders in the k-th group for $k = 1, 2, \dots, 2^{t-1}$ (hence, $\sum_{k=1}^{2^{t-1}} n_{k,t} = N$), and $m_{k,t}$ denote the number of buying traders in the same group. Let $X_{k,s}$ for $s < t$ denote the domain of $x_{i,s}$ that is consistent with a_i^s under the threshold strategy supposed in Proposition 6 for trader i who belongs to group k. The likelihood ratios revealed by an action history of a non-buying trader i and a buying trader j in group k are written as follows:

$$A_{k,t} = \frac{\int_{X_{k,1}} \cdots \int_{X_{k,t-1}} G(\bar{x}_t(P_{k,t}, x_i^{t-1}))dG(x_{i,t-1}) \cdots dG(x_{i,1})}{\int_{X_{k,1}} \cdots \int_{X_{k,t-1}} F(\bar{x}_t(P_{k,t}, x_i^{t-1}))dF(x_{i,t-1}) \cdots dF(x_{i,1})}, \quad (16)$$

$$B_{k,t} = \frac{\int_{X_{k,1}} \cdots \int_{X_{k,t-1}} (1 - G(\bar{x}_t(P_{k,t}, x_j^{t-1})))dG(x_{j,t-1}) \cdots dG(x_{j,1})}{\int_{X_{k,1}} \cdots \int_{X_{k,t-1}} (1 - F(\bar{x}_t(P_{k,t}, x_j^{t-1})))dF(x_{j,t-1}) \cdots dF(x_{j,1})}. \quad (17)$$

In the static model, informed traders submit the demand schedule conditional on p_m, where the conditioning on p_m is equivalent to conditioning on m. In the dynamically extended model, I assume that informed traders submit their demand schedules conditional on the vector of the number of buying traders in each group, $(m_{k,t})_k$. Let $P_{k,t}$ denote the information inferred by trader i in group k. $P_{k,t}$ is written as

$$P_{k,t} = \left(\prod_h A_{h,t}^{n_{h,t}-m_{h,t}} B_{h,t}^{m_{h,t}} \right) / B_{k,t}. \tag{18}$$

I show that the equilibrium threshold strategy still exists in this setup.

Proposition 6 *For each realization of x^t, there exists an equilibrium outcome $(m_{k,t})$ and thresholds $\bar{x}_t$ such that the action profile a_t satisfies the optimal threshold rule:*

$$a_{i,t} = \begin{cases} 1 & \text{if } x_{i,t} \geq \bar{x}_t(P_{k,t}, x_i^{t-1}), \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The proof is deferred to Appendix A. The tatonnement process is characterized by a mixture of binomial distributions, which asymptotically follows a Poisson distribution. The threshold decreases over steps during the tatonnement within each node of history, and hence, we have a well-defined tatonnement process where a chain reaction of buying actions is possible.

3.2.2 Self-organized criticality

In this dynamic extension, traders accumulate private information that is independent across periods. Thus, through Bayesian learning by observing private information and aggregate actions, traders eventually learn the true state almost surely.

Proposition 7 *The subjective belief $b_{i,t}$ converges to 1 as $t \to \infty$ almost surely.*

Proof: The likelihood ratio for the private information history, $\prod_{\tau=1}^{t} \delta(x_{i,\tau})$, converges to zero as in Billingsley [6]. The proof is outlined as follows. The likelihood ratio $\theta_{i,t}$ follows a martingale in the probability measure of the private information under the true state: $E(\theta_{i,t} \mid \theta_{i,t-1}, H) = \theta_{i,t-1}$. Further, the likelihood ratio is bounded from below at zero by construction. Then, the martingale convergence theorem asserts that the likelihood ratio converges in distribution to a random variable. Moreover, the probability measures represented by the distributions F^T and G^T for a sequence of private information $(x_{i,1}, x_{i,2}, \ldots, x_{i,T})$ are mutually singular when $T \to \infty$, since $x_{i,t}$ is independent across t. Then, $\prod_{\tau=1}^{t} \delta_{i,\tau}$ converges to zero.

Hence, $\theta_{i,t}$ converges to zero if $P_{k,t}$ remains finite for $t \to \infty$. $P_{k,t}$ is finite for a finite $\bar{x}_t$ when N is finite. When $\bar{x}_t$ tends to a positive infinity, $P_{k,t}$ decreases to a finite value since $A_{k,t}$ and $B_{k,t}$ are decreasing in $\bar{x}_t$ and positive. When $\bar{x}_t$ tends to a negative infinity, all traders eventually choose to buy. Hence, $\prod_h A_{h,t}^{n_{h,t}-m_{h,t}}$ tends to one, and $P_{k,t}$ tends to $\prod_h B_{h,t}^{m_{h,t}}/B_{k,t}$. I showed that $B_{k,t} < (1/p_t - 1)/(\theta_0 P_{k,t})$ in the proof of Proposition 6. If $P_{k,t}$ tends to a positive infinity as $\bar{x}_t$ tends to a negative infinity, then, this inequality contradicts the fact that $P_{k,t}$ tends to $\Pi_h B_{h,t}^{m_{h,t}}/B_{k,t}$ for any finite N. Thus, $P_{k,t}$ is finite as $t \to \infty$. Hence, $\theta_{i,t}$ is dominated by private information as $t \to \infty$ and converges to zero, and $b_{i,t}$ converges to 1 almost surely. $\square$

Proposition 7 means that the belief converges to the true state eventually. This is a natural consequence of the fact that traders have infinitely precise information in the long run as they accumulate their own private information repeatedly. The convergence of belief implies that there is no possibility for herd behavior in the long run in the narrow sense that we have an infinite sequence of traders taking actions on the basis of a wrong belief or of traders completely neglecting their private information.

The convergence of belief to the true state H means that all traders will buy eventually. This implies that some traders start buying even without any other trader buying at some point of the process toward convergence. Such a buying action triggers the chain reaction of buying. Thus, the converging belief assures that the triggering actions eventually occur and almost surely cause the fat-tailed aggregate actions. The logic is analogous to the self-organized criticality proposed by Bak et al. [3]. In Bak's sand-pile model, the distribution of avalanche size depends on a slowly-varying variable (the slope of the sand pile), and the dynamics of the slope variable has a global sink exactly at the critical point at which the avalanche size exhibits a power-law distribution. In our model, the average belief corresponds to the slope in the sand-pile model. The chain reaction is rarely triggered when the average belief is far below the threshold. As private information accumulates, the average belief increases toward the threshold. This ensures that the triggering buying action will occur eventually.

4 Numerical results on volumes and returns distributions

In Section 3.1, I derived the power-law distribution analytically without specifying F and G, but only asymptotically when N is taken to infinity and F is taken to G. In this section, I show by numerical simulations that the probability distribution of the minimum equilibrium aggregate action m^* follows a dampened power law when N is finite for a parametrized distribution of private information. In the simulations, I set the number of informed traders N as finite but relatively large: 500 and 1000. F and G are specified as normal distributions with mean 1 and 0, respectively, and with standard deviation σ that is relatively large: 25 and 50. The large standard deviation relative to the difference in mean captures the situation where the information value of the news is small. The common prior belief is set at $b_0 = 0.5$, reflecting the situation where traders put equal probabilities on the two states, and the initial price level fully reflects the belief as $p_0 = 0.5$.

It is extensively discussed in the literature as to how informed traders' trades are incorporated in prices. For example, the no-trade theorem by Milgrom and Stokey [29] argued that no stable relation between the price and trades is necessarily predicted if the price reflects the publicly available information instantly. Smith [33] showed that a trader's timing of trade will not be affected by public information either, since the price movement that reflects the public information will cancel out the effect of the public information on the trader's belief. Considering this difficulty, Avery and Zemsky [1] proposed that another dimension of uncertainty is needed for herd behavior models to deal with stock price fluctuations. In this paper, I follow Gabaix et al. [15] who provide a micro-foundation for the square-root specification of a price impact function with a Barra model of uninformed traders that have a mean-variance preference and zero bargaining power against informed traders. Namely, I specify the supply schedule by uninformed traders as $S(p) = p_0((m-1)/N)^\gamma$ with $\gamma = 0.5$. This square-root specification falls within the empirically estimated range of the price impact (Lillo et al. [25]) and is used in the estimation of the price impact function (Hasbrouck and Seppi [18]).

A profile of private information x is randomly drawn 100,000 times, and m^* is computed for each draw. Figure 1 plots the inverted cumulative distribution of m^* for different values of σ and N. The inverted distribution $\Pr_>(m^*)$ is cumulated from above, and is thus 0 at $m^* = N$ and 1 at $m^* = 0$. The distribution is plotted in log-log scale, and thus, a linear line indicates a power law $\Pr_>(m^*) \propto m^{*-\xi}$, where the slope of the linear line ξ is called the exponent of the power law. The simulated distributions appear linear for smaller values of m, and decay fast when m is close to N. This conforms to the model prediction that m^* follows a dampened power-law distribution. The asymptotic analysis also predicted $\xi = 0.5$. As shown in the left panel of Figure 1,

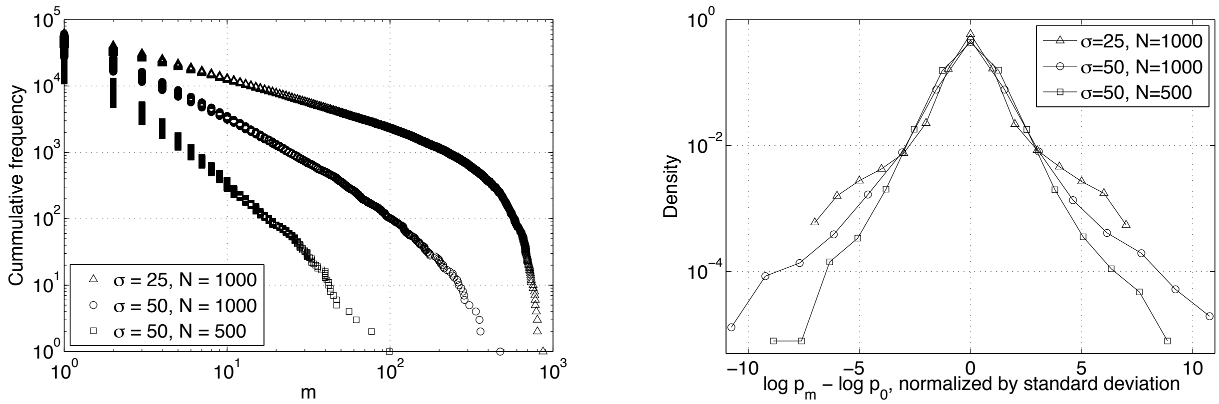

Figure 1: Left: Inverted cumulative distributions of the minimum equilibrium number of buying traders m^*. Right: Distributions of price shifts $\log p_m - \log p_0$

I observe ξ around 0.5 when $\sigma = 25$ and $N = 1000$, but it takes larger values for other parameter sets. This might result from the fact that exponential truncation occurs at a small value of m in these cases, or that the state-dependence of ϕ is strong enough to cause a large deviation from the predicted exponent ξ.[3]

Our model also determines price p_m for each equilibrium number of buying traders m. I interpret the shifts in log price caused by the equilibrium transactions, $\log p_m - \log p_0$, as stock returns, and plot the distributions of the simulated returns in the right panel of Figure 1. The density is logarithmically scaled, and thus, a linear decline indicates an exponential distribution. Notice that the returns are normalized by standard deviations, and the normalized returns span a wide range from -10 to 10. Thus, the plots well indicate that the simulated returns distributions exhibit the pattern of fat tails with exponential truncation.

[3]Sornette [35] shows that the power-law exponent increases by 1 when the parameter ϕ travels across the criticality.

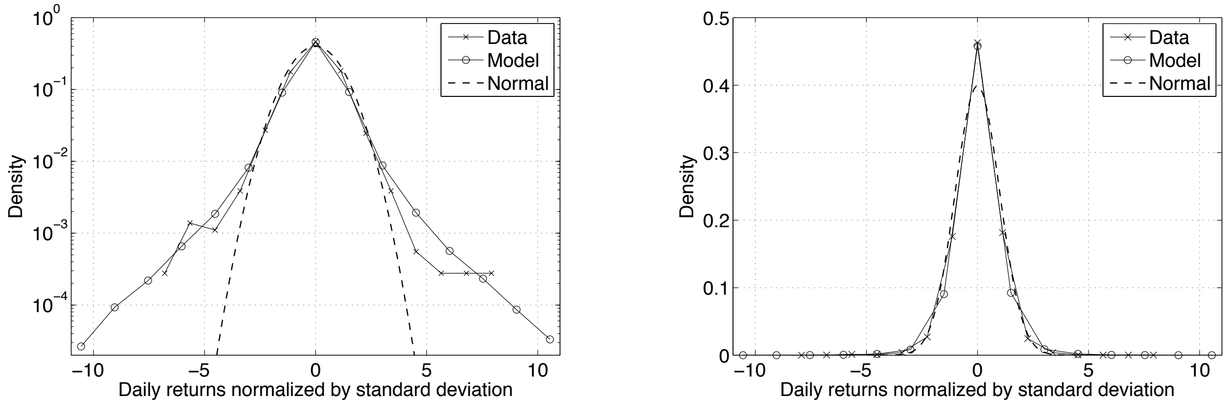

Figure 2: Distributions of TOPIX daily returns, simulated returns $\log p_m - \log p_0$, and a standard normal distribution (Left: semi-log scale, Right: linear scale)

The simulated distribution of returns is compared to the empirical distribution in Figure 2. The returns distributions are plotted in semi-log scale in the left panel and in linear scale in the right panel. The daily returns data is generated using the TOPIX stock index in the Tokyo Stock Exchange during 1998-2010. I define the daily return as the log difference from the opening price to the closing price, rather than the return in a business day, in order to homogenize the time horizon of each observed return.

The simulated distribution is generated under the parameter set $N = 1000$ and $\sigma = 48.5$. The standard deviation of the information, σ, is set so that the density estimate at returns 0 matches with the empirical distribution, as shown in the right panel of Figure 2. The other parameters are set as in the previous simulations. I observe that the tail distribution (especially the left tail) of the empirical returns is well replicated by the simulated tail in the left panel.

The herding behavior in our model results in the slow decay of the probability distribution of the fraction of buying traders m/N. This contrasts well to the case where traders act independently on their own private information. In this case, their action a_i is independent across i, and thus, the central limit theorem predicts that $m/\sqrt{N}$ asymptotically follows a normal distribution. Simulations also show that the variance of m^*/N does not decline as N increases. The simulated variances are 8%, 9%, and 11% for the cases of N = 500, 1000, and 2000, respectively. This contrasts again with the case of no strategic complementarity, where the variance should decline linearly in N according to the law of large numbers.

A power-law tail implies that the distribution belongs to the domain of the attraction of a stable law, i.e., the sum of the random variables distributed according to a power law also follows the power law with the same exponent. Thus, if the herd size of traders follows a power law independently in each period, the cumulated size of the herds over periods should also follow a power law with the same exponent. If the power law is truncated exponentially, the central limit theorem comes into effect in the accumulation, and thus, the cumulated herd size would converge to a normal distribution as the time horizon increases. Hence, the distribution of herd size will exhibit a transition from a fat tail to a normal tail as the time horizon of the return increases. This is indeed empirically observed in the stock returns (Mantegna and Stanley [28]).

5 Discussions

5.1 Variation in market microstructure

In the benchmark model above, I characterize the distribution of the smallest equilibrium m^* under the assumption that the informed traders do not know this selection rule when multiple equilibria exist. This implies that the dampened power law characterizes a portion of fluctuations that arise from the local propagation of information among traders in the situation where the actual equilibrium fluctuations may also involve global shifts in equilibrium due to the sunspot-like coordination. In this section, I provide an alternative model in which informed traders know that the auctioneer selects the smallest m^*. This modification has subtle effects on equilibrium, and therefore, I can no longer characterize the dynamic case where the information is drawn repeatedly. However, the analysis for the static model still holds as shown below.

When traders know the selection rule, they not only know the number of buying and non-buying traders at p_m but also know that there is no equilibrium at a price below p_m. Thus, they infer that there are at least $k+1$ traders buying at p_k for any $k < m$. This is equivalent to the fact that there are two traders with information greater than $\bar{x}(1)$ and one trader with $x_i \geq \bar{x}(k)$ for each $k = 2, 3, \dots, m-1$. The threshold condition is modified as

$$\begin{aligned} 1/p_1 - 1 &= A(\bar{x}(1))^{N-1}\delta(\bar{x}(1))\theta_0, & (20) \\ 1/p_2 - 1 &= A(\bar{x}(2))^{N-2}B(\bar{x}(1))\delta(\bar{x}(2))\theta_0, & (21) \\ 1/p_m - 1 &= A(\bar{x}(m))^{N-m}B(\bar{x}(1))\Pi_{k=1}^{m-2}B(\bar{x}(k))\delta(\bar{x}(m))\theta_0,\ m = 3, 4, \dots, N. & (22) \end{aligned}$$

I can show that $\bar{x}(m)$ is decreasing in m, and thus, the fictitious tatonnement follows a stochastic process as earlies. The simulated distributions of aggregate actions and returns are shown in Figure 3. The distribution of m^* shows a faster exponential truncation for the case of $N = 1000$, compared to the case of the benchmark model. For $N = 500$, there is a small 0.2% probability for an explosive equilibrium $m^* = N$ in which all the informed traders buy. The distributions of returns are similar to the benchmark model except for the occurrence of an explosive equilibrium for the case of $N = 500$. The difference from the benchmark model arises from an altered ϕ_u, the mean number of buying traders induced by a buying trader in tatonnement step

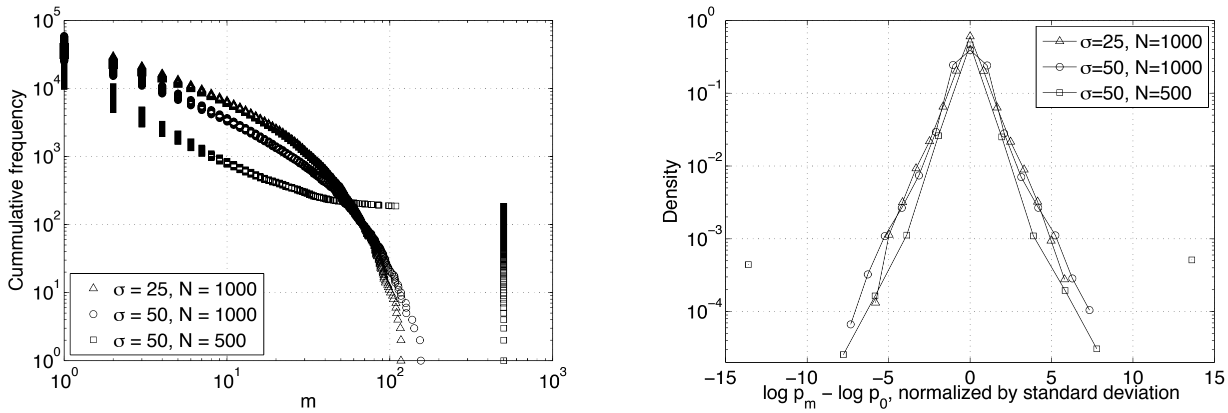

Figure 3: Simulated distributions of m^* (left) and $\log p_m - \log p_0$ (right) when informed traders know the auctioneer's selection rule of m^*.

u. When the traders know that there cannot be a smaller equilibrium, they can infer more precise information from the traders who receive a very high signal x_i. Thus, the threshold $\bar{x}(k)$ decreases in k faster than in the benchmark model. This results in a different fluctuation pattern in m (the left panel), but the distribution of returns (the right panel) seems to be less affected.

5.2 Discrete actions

The discreteness of the action space plays an important role in the rational herd behavior model. In our model, traders can only choose either to buy or not. Suppose instead that each trader chooses an action from a continuous action space and that the action corresponds to the private information one-to-one. Suppose that the private information is drawn from an exponential distribution, for example, and that the state space is the possible mean of the distribution, $\{\lambda, \mu\}$, where $\lambda > \mu$ and the true state is λ. Then, the likelihood ratio of observing an average information profile $\langle x \rangle$

is $(\lambda/\mu)^N e^{(1/\lambda - 1/\mu)N\langle x \rangle}$. Since $\langle x \rangle \to \lambda$ and $(\lambda/\mu)e^{(1-\lambda/\mu)} < 1$, the likelihood ratio converges to 0 as N increases. Thus, for a large N, one round of information draw is sufficient for the market to learn the true state. This contrasts with our result that the amount of information revealed in equilibrium varies greatly depending on the realization of the information profile x.

The aggregate effect of the binary choice shown in this model has an implication on the effectiveness of the Tobin tax scheme. The tax levied on the transactions of the assets held for short term raises the transaction costs, and thus, can suppress speculative trades. A byproduct of the increased transaction costs and the decreased trades is the inhibition of the revelation of private information. This model suggests that the inhibition may result in a larger aggregate fluctuation. An increase in transaction costs will decrease the frequency of transaction and increase the volume per transaction for each trader. In the situation where the information inferences among traders give rise to the aggregate fluctuations of volumes and returns, the magnitude of fluctuations can be suppressed by inducing the traders to trade more frequently. In the limiting case when the traders trade continuously, the stochastic fluctuations of the herd size disappear as discussed above in the case of continuous action space. This mechanism corresponds to the empirical finding by Hau [19] that the volatility of stock prices is increased by an increase in transaction costs.

The effect of the transaction costs on the aggregate fluctuations can be illustrated as follows. Suppose that traders need to trade a certain amount of assets in each year. Suppose that traders can divide the total amount into monthly transactions if the transaction cost is low, whereas they can afford only one transaction a year if the transaction cost is high. If traders tend to herd in transactions due to the revelation of private information associated with the transactions, I observe a small monthly herding when the transaction cost is low, whereas I observe a large yearly herding when the transaction cost is high. If the herd involves all the traders at once, and if the total volume traded per year is fixed at X, the second moment of daily volume is $12(X/12)^2/365$ in the case of monthly herdings while it is $X^2/365$ in yearly herdings. Thus, while the Tobin tax does shift the occurrence of herding from the high frequency domain to low, it can result in an increase in the daily volatility of trades.

The present model assumes a discrete state space in addition to the binary action. Even in a continuous state space, however, the structure of the information revelation will not be altered. This is because traders still form a threshold behavior, and thus, the revealed private information is lumped into two groups, below or above the threshold. The essential environment for the threshold rule, which is the necessary ingredient for the stochastic chain reaction, is that the action space is coarser than the state space so that the private information is not fully revealed by actions.

The binary state space, or the "either-or" uncertainty, arises typically when two alternative interpretations emerge among traders for a set of observations on the true value of assets. Consider the period of a prominent rise in stock prices, for example. It imposes an either-or uncertainty if the traders are divided into two camps as to whether the price rise is justifiable by fundamentals or it is a bubble. The type of stochastic herding shown in our model is plausible in this situation. After a long period of the accumulation of own private information without much revelation of the private information to the market, the average belief moves toward the level at which some traders start to trade on their private information alone even though no other traders trade. Such traders trigger the other traders to follow, and this herd size obeys the fat-tailed distribution. Thus, in the convergence process toward the true value of the asset, a bulk of adjustments is made by the tail instances of the stochastic herd. At the tail event, a number of traders become simultaneously convinced by one interpretation through the process of information revelation in which one trader's conviction exerts a positive influence toward the conviction of another.

This paper mainly aims at explaining the fat-tailed fluctuations observed in the high-frequency domain for which the empirical evidence is most supportive. However, some empirical studies such as Jansen and de Vries [20] and Longin [26] suggest that the largest crashes and booms in the history can be understood as an extreme event within the same power-law tail, instead of being outliers. Authors such as Lee [24] and Chari and Kehoe [10] pursue to apply the logic of herd behavior to rare large events such as market crashes. Our model is consistent with their view that market crashes are caused by the same mechanism that causes price fluctuations in normal times. Of course, the uncertainty on asset fundamentals varies in terms of its impact on prices, and the underlying uncertainties for the historic crashes are probably more potent than the uncertainties that drive daily fluctuations. However, for both crashes and daily fluctuations, the traders exchange their information through the same market mechanism. Thus, low-frequency events and high-frequency events may well share the same mechanism through which the traders' views crystallize collectively.

5.3 Information structure

In an illuminating paper, Gabaix et al. [15] provide an explanation for the power laws in stock volumes and returns by focusing on trader heterogeneity. They start from the observation that trader size follows a power law, and argue that the power law transmits to the transaction volumes and price movements even if the traders' actions are independent. I take a different approach to explain the fat-tailed distributions by the interactions of traders who receive private information. This paper shows that the power-law distribution can emerge even if the traders are symmetric in size.

This paper employs a model of rational herding that provides an economic foundation for the traders' apparent imitative behaviors. It has been pointed out in the literature of critical phenomena that the imitative behaviors can lead to fat-tailed fluctuations, if the imitation occurs stochastically around a particular "critical" probability somehow. This paper provides an economic reason as to why the imitative behavior occurs at the critical probability. The basic property that warrants the imitative traders model to generate a non-degenerate, non-explosive fluctuation is that the impact of an average trader's revealed information is $1/N$ in terms of the likelihood of another trader's action. This property naturally arises when the private information of any trader is as worthy as another trader's, and an action by a trader is as informative as an inaction by another.

It has been suggested that the information structure is a crucial factor that determines the nature of the distribution of the aggregate fluctuations. An important example is the standard herd behavior model with sequential trading such as Banerjee's [4]. In such as case, a trader can only observe the actions of traders who have taken actions earlier. Then, the first trader exerts overwhelming influence on the subsequent traders, resulting in complete herding in which all traders take the same action. The standard herding model is extended to the models of critical phenomena in the market with networked traders such as Cont and Bouchaud [11] and Stauffer and Sornette [36]. They explain the power-law distributions of returns with heterogeneous or localized reference groups of traders generated by an exogenous random process. In contrast, this paper shows that stochastic herding with a power law distribution emerges in the symmetric structure of information inference among traders. The model can be extended to incorporate the heterogeneous information structure that modifies the threshold (8). The analytical method to characterize the aggregate fluctuations by the fictitious tatonnement remains valid, and the resulting distribution will be affected by the heterogeneity in the traders' reference groups.

6 Conclusion

This paper analyzed aggregate fluctuations that arise from the information inference among traders in financial markets. In a class of herd behavior models in which each trader infers other traders' private information only by observing their actions, I found that the number of traders who take the same action at equilibrium can exhibit a large variation. The size of the synchronized actions follows a power-law distribution with exponential truncation. The model prediction was fitted to the empirical fat-tail distribution of stock returns. The parameters that determine the power-law distribution and its exponential truncation were identified by a new analytical method that utilizes a fictitious tatonnement process. I also showed that such chain reactions are eventually triggered almost surely in the situation where private information is drawn repeatedly over time. This implies that the model features a self-organized criticality: traders' belief converges to the point at which the fluctuations of the aggregate actions follow a power law.

The power-law distribution of aggregate actions emerges when the information structure of traders is symmetric. Every trader receives private information of the same magnitude of informativeness on the true value of an asset. Thus, an action by a trader is as informative as an inaction by another. When information is revealed by a trader's buying action, the inaction of the other traders reveals their private information in favor of not buying. This counter-revelation is facilitated by a change in the threshold private information of order $1/N$, which leads to the criticality condition for the aggregate fluctuations. Thus, the information inference model provides an economic foundation for the models of critical phenomena in the market with interacting traders. Finally, the model implies that an increase in transaction costs raises the lumpiness of discrete actions, and thus, increases aggregate volatility.

Appendix

A Proof of Proposition 6

I define the threshold function $\bar{x}_t(P_{k,t}, x_i^{t-1})$ at which trader i is indifferent between buying and not buying. It is implicitly determined by

$$\frac{1}{p_t} - 1 = P_{k,t}\theta_0\delta(\bar{x}_t)\prod_{\tau=1}^{t-1}\delta(x_{i,\tau}). \tag{23}$$

It follows that $\delta(\bar{x}_t(P_{k,t}, x_i^{t-1}))\prod_{\tau=1}^{t-1}\delta(x_{i,\tau})$ is equal to $(1/p_t - 1)/(\theta_0 P_{k,t})$, and thus, is constant across i in group k. Then, $A_{k,t} > (1/p_t - 1)/(\theta_0 P_{k,t}) > B_{k,t}$ can be shown as follows. The numerator of $A_{k,t}$ is expanded as

$$\int_{X_{k,1}} \cdots \int_{X_{k,t-1}} G(\bar{x}_t(P_{k,t}, x_i^{t-1}))\delta(x_{i,t-1})dF(x_{i,t-1}) \cdots \delta(x_{i,1})dF(x_{i,1}) \tag{24}$$

$$> \int_{X_{k,1}} \cdots \int_{X_{k,t-1}} F(\bar{x}_t(P_{k,t}, x_i^{t-1}))\delta(\bar{x}_t(P_{k,t}, x_i^{t-1}))\Pi_{\tau=1}^{t-1}\left(\delta(x_{i,\tau})dF(x_{i,\tau})\right) \tag{25}$$

$$= \frac{1/p_t - 1}{\theta_0 P_{k,t}} \int_{X_{k,1}} \cdots \int_{X_{k,t-1}} F(\bar{x}_t(P_{k,t}, x_i^{t-1}))dF(x_{i,t-1}) \cdots dF(x_{i,1}). \tag{26}$$

The integral in (26) is equal to the denominator of $A_{k,t}$, and thus, $A_{k,t} > (1/p_t - 1)/(\theta_0 P_{k,t})$ holds. Similarly, I obtain $B_{k,t} < (1/p_t - 1)/(\theta_0 P_{k,t})$.

Suppose that m_t increases due to an increase in $m_{k,t}$. $P_{k,t}$ decreases by $A_{k,t} > B_{k,t}$ and (18). Then, $\bar{x}_t$ needs to adjust in order to satisfy (23). By using $A_{k,t} > (1/p_t - 1)/(\theta_0 P_{k,t}) > B_{k,t}$, I get that the logarithms of $A_{k,t}$ and $B_{k,t}$ are decreasing in $\bar{x}_t$ as in the static model. Further, $\delta(\bar{x}_t)$ is decreasing in $\bar{x}_t$. Thus, $\bar{x}_t$ in (23) decreases in response to the increase in m_t. The decreasing $\bar{x}_t$ entails a non-decreasing reaction function of m_{t+1} defined for each realization of x^t. Hence, the existence of an equilibrium is established by Tarski's fixed point theorem. This completes the proof.

The informational tatonnement process is characterized by a set of binomial distributions with probability $q_{k,u}$ and population n_k. When m_k and $\bar{x}_u - \bar{x}_{u+1}$ are small and n_k is large, the binomial allows a Poisson approximation. Thus, the sum of traders who buy in step u is approximated by a Poisson with mean $\sum_k n_k q_{k,u}$. Hence, the tatonnement approximately follows a Poisson branching process.

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