A Mediator Approach to Mechanism Design with Limited Commitment*

Niccolò Lomys† Takuro Yamashita‡

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Abstract

We study mechanism design with limited commitment. In each period, a principal offers a "spot" contract to a privately informed agent without committing to future contracts. In contrast to the classical model with a fixed information structure, we allow for all admissible information structures. We represent the information structure as a fictitious mediator and re-interpret the model as a mechanism design problem for the committed mediator. We construct examples to explain why new equilibrium outcomes can arise when considering general information structures. Next, we apply our approach to durable-good monopoly. In the seller-optimal mechanism, trade dynamics and welfare substantially differ from those in the classical model: the seller offers a discount to the high-valuation buyer in the initial period, followed by the high surplus-extracting price until an endogenous deadline, when the buyer's information is revealed without noise. The Coase conjecture fails. We also discuss unmediated implementation of the seller-optimal outcome.

Keywords: Mechanism Design; Limited Commitment; Information Design; Communication Equilibrium; Durable-Good Monopoly; Coase Conjecture.

JEL Classification: C7; D4; D8.

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†Toulouse School of Economics, University of Toulouse Capitole; niccolo.lomys@tse-fr.eu.

‡Toulouse School of Economics, University of Toulouse Capitole; takuro.yamashita@tse-fr.eu.

1 Introduction

We study information structures in mechanism design problems with limited commitment where, in each period, a principal offers a "spot" contract to a privately informed agent without committing to future contracts. Durable-good monopoly is a classical application and is also central in this paper: time is discrete, the horizon is infinite, and, in each period, a seller (he) who owns a good and a buyer (she) may trade. If they do, the game ends; otherwise, the game continues in the next period. The buyer's willingness to pay for the good is binary, is her private information, and does not vary over time. With full commitment, the seller's optimal long-term mechanism takes the form of a perfectly rigid posted price (see, for example, Baron and Besanko, 1984). That is, the seller sets a price for the good at the beginning of the interaction, and the buyer buys in the initial period (resp., never buys) if her willingness to pay for the good is above (resp., below) that price. Accordingly, trade is inefficient, and both parties earn some ex ante surplus. With limited commitment, the celebrated *Coase conjecture* (Coase, 1972) argues that the seller's rent is diluted relative to the full-commitment case. In particular, in the limiting case of perfect patience, trade occurs at a price equal to the lowest buyer's valuation and with no delay; thus, trade is efficient. The Coase conjecture is formally shown by Stokey (1981), Fudenberg, Levine, and Tirole (1985), and Gul, Sonnenschein, and Wilson (1986) for the case in which the seller can only post prices.

Some recent contributions observe that the standard model of mechanism design with limited commitment (such as that in the above papers) imposes an implicit assumption on the players' information structure. Thus, they study how the conclusions change under alternative, but a specific class of, informational assumptions. For example, Doval and Skreta (2022) consider the case in which, in each period, the principal can garble the information input by the agent (namely, her type report) in that period. In a durable-good monopoly example with two periods, they show that it is not an equilibrium for the seller to post a price in each period; thus, new equilibrium outcomes arise. When the time horizon is infinite, however, Doval and Skreta (2021) show that the seller again finds it optimal to post prices even if the contracts as in Doval and Skreta (2022) are available; thus, the Coase conjecture survives. Brzustowski, Georgiadis-Harris, and Szentes

(2021) consider a class of long-term contracts, subject to the constraint that the seller does not want to revise the offered contract in the future, reflecting the limited commitment nature of the problem. Their long-term contract determines the entire sequence of allocations as a function of the buyer's initial type report, but without revealing any information to the seller other than that revealed through each period's trading outcome. In particular, the buyer's initial report can affect allocations in the future, simultaneously keeping the seller in the dark. With these "smart contracts", the Coase conjecture now fails: even in the limiting case of perfect patience, the seller can guarantee himself a non-vanishing expected revenue.[1]

As such, the recent literature points out that the information structure plays a crucial role in the predictions of mechanism design with limited commitment and, in particular, in the durable-good monopoly setting. Although each of the above papers considers some specific classes of information structures and studies their implications, a more systematic study of the entire class of information structures seems important. For example, whereas Brzustowski et al. (2021) show that keeping the buyer's type report undisclosed to a limited-committed seller can be beneficial for the seller's ex ante expected revenue, is this the only way in which the seller deviates from the Coase conjecture? What is the optimal information structure from the seller's (or the buyer's or the society's) viewpoint? What about efficiency properties? Our goal is precisely this systematic analysis of information structures in mechanism design with limited commitment.

We begin by formalizing a general model of mechanism design with limited commitment in which any information structure that satisfies the following conditions is *admissible*. First, a player's private information in the game remains private whenever the player would like to do so. In the durable-good monopoly application, without this restriction, the seller's optimal information structure would trivially be the one in which he *knows* the buyer's willingness to pay. Second, each period's realized allocation is publicly observed in that period. In the durable-good monopoly application, observability of whether trade happens and, if so, at which price seems a natural assumption. Third, the contract chosen by

[1]The above discussion is about the "gap case", in which the Coase conjecture holds in all equilibria of the model with price-posting sellers. In the "no-gap case," the Coase conjecture fails even with posted prices if buyers use non-stationary strategies (Ausubel and Deneckere, 1989).

the principal in each period is publicly observed in that period. This assumption is natural in many applications and also reflects the idea of limited commitment.

Our analysis allows for a rich class of admissible information structures. For example, period-by-period noisy disclosure as in Doval and Skreta (2022, 2021) and/or delayed disclosure as in Brzustowski et al. (2021) are possible, but so are also more sophisticated history-dependent information structures.

After introducing our model, we construct three examples that explain why the set of equilibrium outcomes can expand once we allow for general information structures. Our examples identify three representative reasons for why this can be the case: delayed disclosure of stored information; punishing a principal's future self who deviates to a non-equilibrium contract by deleting any planned disclosure of stored information; punishing a principal's current or future self who deviates to a non-equilibrium contract by selecting a suboptimal continuation equilibrium.[2]

We identify equilibrium outcomes under all admissible information structures building on the notion of *(sequential) communication equilibrium* of Forges (1986), Myerson (1986), and Sugaya and Wolitzky (2021) (hereafter, FMSW). They consider mediated communication in extensive-form games and establish various versions of *revelation principle* depending on the equilibrium concept of interest. Roughly, these results state that any equilibrium outcome given any information structure is attainable as an equilibrium outcome with the *canonical* information structure (and viceversa). In the canonical information structure, in each period: (i) a *(fictitious) mediator* (he) privately asks for each player's information and then privately recommends each player's action for the period; and (ii) players truthfully report their private information to the mediator and obey his recommendation. In a sense, we apply the (sequential) communication equilibrium notion to the setting in which one of the players is a limited-committed principal.

However, precisely that the principal is one of the players is the main challenge we face. A naive application of the revelation principle in FMSW would imply that, in the canonical information structure, the mediator recommends one of the feasible contracts to the principal, and the principal prefers obeying to deviating to any other contract. This statement, however, is not very useful in characterizing possible outcomes because the set of feasible contracts is too large to be tractably

[2]Of course, additional reasons may be possible.

handled. First, the revelation principle does not apply to the principal's contract space.[3] Potentially, the contract the mediator should recommend to the principal in each period might be a very complex indirect contract. Second, the set of obedience constraints could be large for the same reason, and there is not much guidance as to which (indirect) contracts would be relevant for potential deviation.[4]

To circumvent these challenges, we propose an indirect approach for the durable-good monopoly application. First, we consider an auxiliary game in which the mediator—not the seller—proposes an allocation in each period after he privately asks for each player's information at the beginning of that period. This outcome-based approach avoids the complications related to the mediator's contract recommendations to the seller as the set of allocations is much more well-structured than the unrestricted set of contracts.

Second, we identify necessary conditions for outcomes to form an equilibrium of the game. More specifically, in the durable-good monopoly application, we focus only on the seller's deviations to the constant contact that ends the game by allocating the good to all buyers types at a price equal to the lowest buyer's valuation. To the extent that all the other obedience constraints are ignored, this problem is a relaxation of the original problem and is much simpler to solve; we interpret it as a form of *relaxed revelation principle* (Theorem 1).

Third, we identify sufficient conditions for the outcomes satisfying the relaxed problem to be equilibrium outcomes of the original game. We consider another auxiliary problem where the seller's deviation necessarily triggers reversion to the Coasean outcome from that point on. Theorem 2 shows that this provides a lower bound on the seller-optimal equilibrium outcome. Theorems 7 and 8 establish what we call the *mediated implementation* result: the seller-optimal outcome in this lower-bound problem is indeed attainable in the original problem. In Section 4.4, we also consider an alternative notion of implementation, *unmediated implementation*, where the seller—not the mediator—directly controls information as in Doval and Skreta (2022, 2021)—crucially, however, not only to his next-period self but also to his later selves—or as in Brzustowski et al. (2021). We establish when mediated and unmediated implementation coincide.

[3]See Laffont and Tirole (1988) and Bester and Strausz (2000, 2001, 2007).

[4]To the best of our knowledge, this challenge is novel in the literature.

Our characterization of the seller-optimal mechanism in the durable-good monopoly application, summarized by Theorems 3–6, uncovers the following results of economic substance. First, the optimal information structure specifies when the buyer's private information is publicly revealed; until then, no information—except whether a trade has occurred or not—is revealed. [5] Second, in the seller's optimal mechanism, trade occurs with delay and is inefficient, even in the limiting case of perfect patience, as opposed to what the Coase conjecture affirms.

Intuitively, delayed (but precise) information disclosure increases the seller's bargaining power. Relative to the case in which no such information arrives, the seller's incentive to offer a more aggressive price is stronger because even if the buyer does not buy at that aggressive price, full extraction is still possible once the time comes. Furthermore, once the seller becomes aggressive in some period, he can also be more aggressive in the previous period as the buyer has less continuation payoff conditional on no trading. In this sense, the aggressiveness of each seller's self is a "strategic complement" to each other. Indeed, except for the initial period, the seller continues to offer a price equal to the highest buyer's valuation in every period until the time of revelation. This price pattern is completely different from the classical pattern of decreasing posted prices: our case may rather be interpreted as an initial "fire sale" followed by the rigid high price.

Related Literature. Our paper contributes to the literature on mechanism design with limited commitment. The failure of the revelation principle is well-known since the seminal contributions of Laffont and Tirole (1988) and Bester and Strausz (2001), further developed in Bester and Strausz (2000) and Bester and Strausz (2007). Bester and Strausz (2007) are the first to propose the mediator approach based on Forges (1986) and Myerson (1986) to mechanism design problems with limited commitment. In Bester and Strausz (2007), the game has two periods, and the mechanism design is only in the initial period; in the second period, the principal just selects an action from a given set. Therefore, the main challenge we face in our long-horizon setting—namely, how to handle potentially large sets of contracts for the principal, both on-path and off-path—does not arise.

[5] That the seller benefits from delayed information disclosure is consistent with Brzustowski et al. (2021). However, our result shows that the seller can do better than in Brzustowski et al. (2021)—even in the limiting case of perfect patience—and, moreover, that is the best possible.

This challenge also makes our problem a non-trivial application of the (sequential) communication equilibrium notion of Forges (1986), Myerson (1986), and Sugaya and Wolitzky (2021). As we briefly discuss in the conclusion, we believe that our approach can be useful in other potential applications of (sequential) communication equilibrium in which some players' action space is complicated.

Several papers study durable-good monopoly (or, equivalently, bargaining with one-sided incomplete information) as a representative application of mechanism design with limited commitment.[6] Classically, the literature of durable-good monopoly implicitly assumes that the seller's action is simply a price offer. The (limited-committed) mechanism-design perspective adds by allowing for more general contracts (Skreta, 2006; Doval and Skreta, 2022, 2021; Brzustowski et al., 2021). In contrast to these papers, which consider specific classes of contracts and/or information structures, we propose a more systematic treatment of them.

Our paper also relates to the literature that checks the robustness of the Coase conjecture or documents its failure. For instance, Feinberg and Skrzypacz (2005) show that higher-order uncertainty can generate delay in trading. Alternatively, a monopolist could relax its commitment problem and increase its profit by renting the good rather than selling it (Bulow, 1982), by introducing best-price provisions (Butz, 1990), or by introducing new updated versions of the durable good over time (Bulow, 1986; Levinthal and Purohit, 1989; Waldman, 1993, 1996; Choi, 1994; Fudenberg and Tirole, 1998; Lee and Lee, 1998).

Other studies have analyzed environments that preclude the market from fully deteriorating. These include environments with capacity constraints (Kahn, 1986; McAfee and Wiseman, 2008), with arrival of new traders (Sobel, 1991; Inderst, 2008b; Fuchs and Skrzypacz, 2010) or information (Daley and Green, 2020; Duraj, 2020; Lomys, 2021; Laiho and Salmi, 2021), with time-varying costs (Ortner, 2017), in which buyers' valuations are subject to idiosyncratic stochastic shocks (Biehl, 2001; Deb, 2014; Garrett, 2016), in which goods depreciate over time (Bond and Samuelson, 1987), and in which demand is discrete (Bagnoli, Salant, and Swierzbinski, 1989; von der Fehr and Kuhn, 1995; Montez, 2013).

[6] Other than durable-good monopoly, Liu, Mierendorff, Shi, and Zhong (2019) consider posted prices in dynamic auctions, and Gerardi and Maestri (2020) consider repeated production. Strulovici (2017) and Maestri (2017) study renegotiation with limited commitment.

Another approach to breaking the Coase Conjecture is to allow the seller to intratemporally screen, e.g., by producing a variety, see (Wang, 1998; Takeyama, 2002; Hahn, 2006; Inderst, 2008a), or if buyers can exercise an outside option (Board and Pycia, 2014). Recently, Nava and Schiraldi (2019) show that these results are consistent with the Coasean logic in the sense that the seller's limit payoff is the maximal static monopoly profit subject to the market-clearing condition.

In contrast to these papers, we consider the same basic game as in the classical seminal contributions. Instead, we examine alternative information structures.

Outline. In Section 2, we present the general model. In Section 3, we provide several examples to motivate our approach to mechanism design with limited commitment. In Section 4, we study the durable-good monopoly application. In Section 5, we conclude. Omitted proofs and additional details are in the Appendices.

2 Model

Primitives. There are three players: a *principal* (he), an *agent* (she), and a *mediator* (he).[7] *Time* is discrete and periods are indexed by $t \in \mathcal{T} := \{0, \dots, T\}$, where $2 \leq T \leq \infty$. Let $\mathcal{T}_0 := \mathcal{T} \setminus \{0\}$ and $\mathcal{T}_1 := \mathcal{T} \setminus \{0, 1\}$. At the beginning of period $t = 0$, the agent observes her *private information (type)* $\theta \in \Theta$, which is distributed according to a full-support probability distribution μ. The agent's type does not change over time. Each period $t \in \mathcal{T}_0$, as a result of the interaction between the players, an *allocation* $a_t \in A_t$ is determined; the set of admissible allocations may be time dependent—hence, its dependence on t. Each allocation set A_t contains the element $\varnothing$, corresponding to the non-participation allocation. For all $t \in \mathcal{T}_0$, let $A^t := \times_{\tau=1}^{t} A_\tau$ be the set of all allocation sequences of the form $a^t := (a_\tau)_{\tau=1}^{t}$. For all $t \in \mathcal{T}_1$, there is a correspondence $\mathcal{A}_t \colon A^{t-1} \rightrightarrows A_t$ such that, for all $a^{t-1} \in A^{t-1}$, $\mathcal{A}_t(a^{t-1}) \subseteq A_t$ describes the set of all allocations that the principal can offer in period t given the allocations he has offered through period $t-1$. This allows for the case in which past allocations restrict what the principal can offer the agent in the future. We assume that $\varnothing \in \mathcal{A}_t(a^{t-1})$ for all $t \in \mathcal{T}_1$ and $a^{t-1} \in A^{t-1}$.

The set of all *payoff-relevant pure outcomes* of the game is $\Theta \times A^T$. The

[7] We focus on the single-agent case throughout the paper, but there is no conceptual difficulty in extending the model to more agents (indeed, the example in Section C.2 considers two agents).

principal's (resp., agent's) *preferences* are represented by a bounded payoff function $U_P \colon \Theta \times A^T \to \mathbb{R}$ (resp., $U_A \colon \Theta \times A^T \to \mathbb{R}$). The mediator is indifferent over payoff-relevant pure outcomes—that is, his preferences are represented by a constant payoff function $U_M \colon \Theta \times A^T \to \mathbb{R}$. All players are expected-payoff maximizers. Thus, the mediator can commit to any strategy.

In each period $t \in \mathcal{T}_0$, the principal offers a *spot contract* $C_t := (M_t, \alpha_t)$ to the agent, where: (i) M_t is a set of *input messages* to the contract; (ii) $\alpha_t \colon M_t \to \Delta(A_t)$ is an *allocation rule*. We write $\alpha_t(m_t)$ for the probability distribution on A_t when the input message to the contract is m_t and $\alpha_t(a_t \,|\, m_t)$ for the probability of allocation a_t when the input message to the contract is m_t. We endow the principal with a class of input message sets $\mathcal{M} := \{M^i\}_{i \in \mathcal{I}}$, where $\mathcal{I}$ is an index set; each M^i contains element $\varnothing$, interpreted as non-participation message. For all $t \in \mathcal{T}_0$, let $\mathcal{C}_t := \{C_t := (M_t, \alpha_t) \in \mathcal{M} \times \Delta(A_t)^{M_t} : \alpha_t(\varnothing \,|\, \varnothing) = 1, \alpha_t(\varnothing \,|\, m_t) = 0 \text{ for all } m_t \neq \varnothing\}$ be the set of all *admissible spot contracts* in period t. Let $\mathcal{C} := \{\mathcal{C}_t\}_{t \in \mathcal{T}_0}$ be the set of all admissible spot contracts. Finally, for all $t \in \mathcal{T}_0$, let $\mathcal{M}^t := \times_{\tau=1}^t \mathcal{M}$ and $\mathcal{C}^t := \times_{\tau=1}^t \mathcal{C}_\tau$. Hereafter, we refer to spot contracts simply as contracts.

In period $t = 0$, the agent has a set of possible *private reports* to send to the mediator, denoted by R, with $|R| \geq |\Theta|$. In each period $t \in \mathcal{T}_0$, the principal (resp., the agent) has a set of possible *private signals* to receive from the mediator, denoted by $S_{P,t}$ (resp., $S_{A,t}$). For all $t \in \mathcal{T}_0$, let $S_P^t := \times_{\tau=1}^t S_{P,\tau}$ and $S_A^t := \times_{\tau=1}^t S_{A,\tau}$.

Timing. The timing of events within period $t = 0$ is the following:

0.1 The agent privately observes her type $\theta \in \Theta$.

0.2 The agent sends a private report $r \in R$ to the mediator.

The timing of events within each period $t \in \mathcal{T}_0$ is the following:

t.1 The mediator sends a private signal $s_{P,t} \in S_{P,t}$ to the principal.

t.2 The principal offers a contract $C_t \in \mathcal{C}_t$ to the agent with the the following property: if $t = 1$, then, $\sum_{a_t \in A_t} \alpha_t(a_t \,|\, m_t) = 1$ for all $m_t \in M_t$; if $t \in \mathcal{T}_1$, then $\sum_{a_t \in \mathcal{A}_t(a^{t-1})} \alpha_t(a_t \,|\, m_t) = 1$ for all $m_t \in M_t$.

t.3 After both the mediator and the agent observe C_t, the mediator sends a private signal $s_{A,t} \in S_{A,t}$ to the agent.

t.4 The agent sends input message $m_t \in M_t$ to the contract, where m_t is observed by the mediator, but not by the principal.

$t.5$ An allocation a_t is drawn from $\alpha_t(m_t)$, the allocation a_t is publicly observed, and the game proceeds to the next period.

For all $t \in \mathcal{T}$ and $n \in \{1, \ldots, 5\}$, we refer to $t.n$ as "stage $t.n$" of the game.

Histories. For all $t \in \mathcal{T}$, and $n \in \{1, \ldots, 5\}$, we denote by $h^{t.n}$ a history at the beginning of stage $t.n$ and by $H^{t.n}$ the set of all such histories. Then, we have $H^{0.1} = \{\emptyset\}$, $H^{0.2} = \Theta$, $H^{1.1} = \Theta \times R$, $H^{1.2} = \Theta \times R \times S_P^1$, $H^{1.3} = \Theta \times R \times S_P^1 \times \mathcal{C}^1$, $H^{1.4} = \Theta \times R \times S_P^1 \times \mathcal{C}^1 \times S_A^1$, $H^{1.5} = \Theta \times R \times S_P^1 \times \mathcal{C}^1 \times S_A^1 \times \mathcal{M}$, and, for all $t \in \mathcal{T}_1$,

$$
\begin{aligned}
H^{t.1} &= \Theta \times R \times S_P^{t-1} \times \mathcal{C}^{t-1} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
H^{t.2} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t-1} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
H^{t.3} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
H^{t.4} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t} \times S_A^{t} \times \mathcal{M}^{t-1} \times A^{t-1},
\end{aligned}
$$

and

$$H^{t.5} = \Theta \times R \times S_P^{t} \times \mathcal{C}^{t} \times S_A^{t} \times \mathcal{M}^{t} \times A^{t-1}.$$

For all $t \in \mathcal{T}_0$, we denote by h^t a history at the end of period t and by H^t the sets of all possible such histories, with $H^t = \Theta \times R \times S_P^t \times \mathcal{C}^t \times S_A^t \times \mathcal{M}^t \times A^t$. We denote by H^T the set of all terminal histories of the game.

Information Sets. For all $i \in \{P, A, M\}$, $t \in \mathcal{T}$, and $n \in \{1, \ldots, 5\}$, let $h_i^{t.n}$ denote player i's information set at the beginning of stage $t.n$ and by $H_i^{t.n}$ the set of all such information sets.[8]

Mediated Contract-Selection Game. The above defines an extensive-form game, which we dub the *mediated contract-selection game* (hereafter, MCS game) and denote by $\mathcal{G}$. Game $\mathcal{G}$ is common knowledge among the players.

Strategies. A behavioral strategy for the principal is a collection of functions $\sigma_P := (\sigma_P^{t.2})_{t=1}^T$, where $\sigma_P^{t.2} \colon H_P^{t.2} \to \Delta(\mathcal{C}_t)$. A behavioral strategy for the agent is a collection of functions $\sigma_A := \left(\sigma_A^{0.2}, (\sigma_A^{t.4})_{t=1}^T\right)$, where $\sigma_A^{0.2} \colon H_A^{0.2} \to \Delta(R)$ and $\sigma_A^{t.4} \colon H_A^{t.4} \to \Delta(M_t)$. A behavioral strategy for the mediator is a collection of

[8] In particular, we have: $H_P^{1.2}$ is the projection of $H^{1.2}$ on S_P^1 and, for all $t \in \mathcal{T}_1$, $H_P^{t.2}$ is the projection of $H^{t.2}$ on $S_P^t \times \mathcal{C}^{t-1} \times A^{t-1}$; $H_A^{0.2} = H^{0.2}$, $H_A^{1.4}$ is the projection of $H^{1.4}$ on $\Theta \times R \times \mathcal{C}^1 \times S_A^1$, and, for all $t \in \mathcal{T}_1$, $H_A^{t.4}$ is the projection of $H^{t.4}$ on $\Theta \times R \times \mathcal{C}^t \times S_A^t \times \mathcal{M}^{t-1} \times A^{t-1}$; $H_M^{1.1}$ is the projection of $H^{1.1}$ on R, $H_M^{1.3}$ is the projection of $H^{1.3}$ on $R \times S_P^1 \times \mathcal{C}^1$, and, for all $t \in \mathcal{T}_1$, $H_M^{t.1}$ is the projection of $H^{t.1}$ on $R \times S_P^{t-1} \times \mathcal{C}^{t-1} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}$ and $H_M^{t.3}$ is the projection of $H^{t.3}$ on $R \times S_P^t \times \mathcal{C}^t \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}$.

functions $\sigma_M \coloneqq (\sigma_M^{t.1}, \sigma_M^{t.3})_{t=1}^T$, where $\sigma_M^{t.1}\colon H_M^{t.1} \to \Delta\big(S_t^P\big)$ and $\sigma_M^{t.3}\colon H_M^{t.3} \to \Delta\big(S_t^A\big)$. A profile of behavioral strategies is $\sigma \coloneqq (\sigma_P, \sigma_A, \sigma_M)$.

A prior μ and a profile of behavioral strategies σ induce a probability distribution over payoff-relevant pure outcomes. We extend players' payoff functions from payoff-relevant pure outcomes $\big(\theta, a^T\big) \in \Theta \times A^T$ to outcomes $\nu \in \Delta\big(\Theta \times A^T\big)$ in the usual way. We denote by $U_P(\nu)$ (resp., $U_A(\nu)$) the principal's (resp., agent's) ex ante expected payoff at the beginning of the MCS game under outcome ν.

Beliefs. A principal's belief is a collection of functions $\beta_P \coloneqq (\beta_P^{t.2})_{t=1}^T$, where $\beta_P^{t.2}\colon H_P^{t.2} \to \Delta(H^{t.2})$. Similarly, an agent's belief is a collection of functions $\beta_A \coloneqq \Big(\beta_A^{0.2}, (\beta_A^{t.4})_{t=1}^T\Big)$, where $\beta_A^{0.2}\colon H_A^{0.2} \to \Delta(H^{0.2})$ and $\beta_A^{t.4}\colon H_A^{t.4} \to \Delta(H^{t.4})$. Since there are no optimality conditions on the mediator's strategy, there is no need to introduce beliefs for the mediator. A belief system is a pair $\beta \coloneqq (\beta_P, \beta_A)$.

Solution Concept. We refer to a profile of behavioral strategies and a belief system (σ, β) as an assessment. The equilibrium notion we adopt is *weak perfect Bayesian equilibrium* (hereafter, wPBE). An assessment (σ, β) is a wPBE of $\mathcal{G}$ if: (i) σ_P (resp., σ_A) is sequentially rational given (σ_M, σ_A) (resp., (σ_M, σ_P)) and β; and (ii) β is on-path consistent given σ (i.e., satisfies Bayes' rule where possible).

Our wPBE notion does not restrict the belief system out of the equilibrium path. However, all the wPBEs we study in this paper do not rely on "pathological" or ad hoc specifications of the belief system out of the equilibrium path. In particular, once our games are appropriately discretized in a way that they become finite games, the wPBEs we construct can be sustained as sequential equilibria.

We say that $\nu \in \Delta(\Theta \times A^T)$ is a *wPBE outcome* of $\mathcal{G}$ if there is a wPBE (σ, β) of $\mathcal{G}$ that (together with μ) induces ν. We denote by $\mathcal{E}$ (resp., $\mathcal{O}$) the set of all wPBEs (resp., wPBE outcomes) of $\mathcal{G}$.

Notation and Terminology. A contract $C_t \in \mathcal{C}_t$ is *constant* if, for some $a_t \in A_t$, we have $\alpha_t(a_t \,|\, m_t) = 1$ for all $m_t \in M_t \setminus \{\varnothing\}$. A contract $C_t \in \mathcal{C}_t$ is *direct* if $M_t = \{\varnothing\} \cup \Theta$. Suppose $\mathcal{E} \neq \emptyset$ and that $\arg\max_{\nu \in \mathcal{O}} U_P(\nu) \neq \emptyset$; we say that $\nu^* \in \Delta(\Theta \times A^T)$ is a *principal-optimal wPBE outcome* of $\mathcal{G}$ if $\nu^* \in \arg\max_{\nu \in \mathcal{O}} U_P(\nu)$.

Discussion. Our approach to mechanism design with limited commitment builds on the notion of (sequential) *communication equilibrium* in multistage games with communication (Forges, 1986; Myerson, 1986; Sugaya and Wolitzky, 2021, hereafter FMSW). As in that literature, we interpret the mediator as a fictitious player who has commitment power and designs the entire information structure of the game. We follow this approach because, by the revelation-principle results in FMSW (several versions depending on the specific solution concept), any *admissible information structure* of the underlying multistage game without a mediator and its equilibrium outcome can be represented by the mediated version of that game and its corresponding equilibrium outcome (and viceversa).

We call an information structure admissible if it satisfies the two following conditions. First, the agent's type remains private unless the agent wants to disclose it. Second, the exogenously given basic game—hence, the extensive form and the principal's limited commitment—is respected; in particular, in each period, the contract chosen by the principal and the realized allocation are publicly observed.

3 Motivating Examples

We construct three examples to motivate the mediator approach to mechanism design with limited commitment. The examples show that the set of all wPBE outcomes under our approach can be larger than that under the alternative approaches in the literature. We identify three reasons for why this can be the case.[9]

1. Information Storage and Delayed Disclosure. In the MCS game, the mediator can store information—in particular, the agent's report to the mediator—and disclose it to the principal only in the future. Example 1 in Section 3.1 shows that the principal's ex ante expected wPBE payoff under information storage and delayed disclosure can be greater than that when a given period's information can only be either (noisily) disclosed in the current period or never disclosed.

2. Deletion of Stored Information. In the MCS game, the information provided to the principal's future selves can depend on their contract choices. In particular, the mediator can punish a principal's deviation to a non-equilibrium contract by deleting any planned disclosure of stored information. Example 2 in Appendix C.1 shows that the deletion of stored information is an effective threat to prevent a principal's deviation. As a result, the principal's ex ante expected

[9]Of course, additional reasons may be possible.

wPBE payoff can be greater than that when such punishments are not possible.

3. Equilibrium Multiplicity and Equilibrium Selection. In the MCS game, continuation equilibrium selection can depend on the principal's contract choice. In particular, the mediator can punish a principal's deviation to a non-equilibrium contract by suboptimal equilibrium selection. Example 3 in Appendix C.2 shows that suboptimal continuation equilibrium selection is an effective threat to prevent a principal's deviation. As a result, the principal's ex ante expected wPBE payoff can be greater than that when such punishments are not possible.

3.1 Example 1

Let $\mathcal{T} = \{0,1,2,3\}$. The principal is a seller who owns one unit of a durable, indivisible good to which he assigns value 0 (normalization). The agent is a buyer whose private type $\theta \in \Theta = \{1,2\}$ corresponds to her valuation for the good. Let $\mu = \frac{9}{10}$ be the probability that $\theta = 2$ at $t = 0$. If the buyer participates in period t, an allocation for the period is a pair $(x_t, p_t) \in \{0,1\} \times \mathbb{R}$, where x_t indicates whether the good is traded ($x_t = 1$) or not ($x_t = 0$) and p_t is a transfer from the buyer to the seller. If $x_t = 1$ for some $t < 3$, by convention, allocation $(x_\tau, p_\tau) = (0,0)$ is implemented for all $\tau \in \mathcal{T}$ with $\tau > t$. Thus, the allocation set in period t is $A_t = \{\varnothing\} \cup (\{0,1\} \times \mathbb{R})$. If $a_t = \varnothing$, the seller's and the buyer's flow payoffs are 0; if $a_t = (x_t, p_t)$, flow payoffs are $\theta x_t - p_t$ for the buyer and p_t for the seller. The seller and the buyer share a common discount factor $\delta = \frac{1}{2}$. The seller's and the buyer's payoffs, $U_P(\theta, a)$ and $U_A(\theta, a)$, are the discounted sum of flow payoffs.

Information Storage and Delayed Disclosure. The following events occur in a seller-optimal wPBE outcome the MCS game:[10]

- *Period* $t = 0$. The buyer truthfully reports her type θ to the mediator.
- *Period* $t = 1$. The mediator recommends to the seller to offer a menu with two options: (i) trade with probability $\frac{17}{18}$ at price $\frac{59}{34}$; (ii) do not trade. The mediator recommends to type $\theta = 2$ to choose option (i) and to type $\theta = 1$ to choose option (ii). The seller and the buyer obey.
- *Period* $t = 2$. If no trade has occurred, the mediator recommends to the seller

[10] See Section 4 and Appendix H for the complete equilibrium characterization.

to offer a menu with two options: (i) trade with probability 1 at price 2; (ii) do not trade. The mediator recommends to type $\theta = 2$ to choose option (i) and to type $\theta = 1$ to choose option (ii). The seller and the buyer obey.

- *Period* $t = 3$. If no trade has occurred, the mediator fully discloses the buyer's type report θ to the seller. The seller offers to trade at price θ and type θ trades with probability 1 at price θ.

The mediator's key role in this wPBE is to store the buyer's report in period $t = 0$ and fully disclose it in period $t = 3$ to the seller. The mediator's recommendation to the seller in periods $t = 1$ and $t = 2$ does not depend on the buyer's report.

To clarify the role of information storage and delayed disclosure, we contrast the previous wPBE outcome with a seller-optimal wPBE outcome of the game in which a given period's information can only be either (noisily) disclosed in the current period or never disclosed. The following events occur in a seller-optimal wPBE outcome of the game studied by Doval and Skreta (2021) (hereafter, DS game):

- *Period* $t = 1$. Type $\theta = 2$ trades with probability $\frac{8}{9}$ at price $\frac{7}{4}$, while type $\theta = 1$ does not trade.

- *Period* $t = 2$. If no trade has occurred, type $\theta = 2$ trades with probability 1 at price $\frac{3}{2}$, while type $\theta = 1$ does not trade.

- *Period* $t = 3$. If no trade has occurred, both types trade with probability 1 at price 1.

Let $x_{\ell,t}$ (resp., $x_{h,t}$) denote the probability with which type $\theta = 1$ (resp., $\theta = 2$) trades in period t; moreover, let $p_{\ell,t}$ (resp., $p_{h,t}$) denote the transfer from type $\theta = 1$ (resp., $\theta = 2$) to the seller conditional on trade in period t. The next table summarizes the seller-optimal wPBE outcomes of the MCS and DS games.

	Period $t = 1$				Period $t = 2$				Period $t = 3$			
	$x_{\ell,1}$	$p_{\ell,1}$	$x_{h,1}$	$p_{h,1}$	$x_{\ell,2}$	$p_{\ell,2}$	$x_{h,2}$	$p_{h,2}$	$x_{\ell,3}$	$p_{\ell,3}$	$x_{h,3}$	$p_{h,3}$
MCS Game	0	0	$\frac{17}{18}$	$\frac{59}{34}$	0	0	$\frac{1}{18}$	2	1	1	0	0
DS Game	0	0	$\frac{8}{9}$	$\frac{7}{4}$	0	0	$\frac{1}{9}$	$\frac{3}{2}$	1	1	0	0

In both wPBE outcomes, type $\theta = 2$ trades with positive probability over multiple periods, whereas type $\theta = 1$ trades only in the last period. Crucially, however, price paths and trade probabilities are different in the two wPBEs: first, the price path is not decreasing over time in the MCS game, whereas it is so in the DS game; second, the probability of trading in period $t = 1$ (resp., $t = 2$) is greater (resp., less) in the MCS game than in the DS game.

The intuition behind the different equilibrium dynamics is the following. In the MCS game, in period $t = 3$, the seller can fully extract the surplus from the transaction thanks to the mediator's disclosure of the buyer's type report (made in period $t = 0$). Such surplus extraction is not possible without information storage.

In period $t = 2$, the seller offers price 2 in the MCS game and price $\frac{3}{2}$ in the DS game. In the MCS game, the seller has no incentive to lower the price below 2 because, even if type $\theta = 2$ does not buy the good, the seller can still sell the good at price 2 in period $t = 3$ thanks to the mediator's disclosure of the buyer's type report. Since the seller's (posterior) belief that $\theta = 2$ at the beginning of period $t = 2$ is less than $\frac{1}{2}$, the ability to store information and delay its disclosure to the seller is crucial to sustain a price of 2. Without such ability, sequential rationality would require the seller to post a price equal to 1 whenever his (posterior) belief that $\theta = 2$ at the beginning of period $t = 2$ is less than $\frac{1}{2}$. In other words, information storing and its delayed disclosure increase the seller's bargaining power in period $t = 2$.

In period $t = 1$, in the MCS game, the price is lower, but the probability of trade is greater, than in the DS game. The probability of trade in the DS game must be small enough for the seller's (posterior) belief that $\theta = 2$ at the beginning of period $t = 2$ to be no less than $\frac{1}{2}$. Only when this occurs the seller can sell to type $\theta = 2$ at a price greater than 1; otherwise, sequential rationality would require the seller to sell to both types with probability 1 at price 1.

The seller's ex ante expected payoff in a seller-optimal wPBE outcome of the MCS game is $U_P^* = \mu\big(\sum_{t=1}^{3} \delta^{t-1} x_{h,t} p_{h,t}\big) + (1-\mu)\big(\sum_{t=1}^{3} \delta^{t-1} x_{\ell,t} p_{\ell,t}\big) = \frac{558}{360}$, and in a seller-optimal wPBE outcome of the DS game is $\widehat{U}_P^* = \mu\big(\sum_{t=1}^{3} \delta^{t-1} x_{h,t} p_{h,t}\big) + (1-\mu)\big(\sum_{t=1}^{3} \delta^{t-1} x_{\ell,t} p_{\ell,t}\big) = \frac{540}{360}$. Since $U_P^* - \widehat{U}_P^* = \frac{1}{20} > 0$, information storage and delayed disclosure are beneficial from the seller's ex ante viewpoint.

4 Durable-Good Monopoly

We apply the mediator approach to mechanism design with limited commitment to the durable-good monopoly problem. The setting is as in Section 3.1, with the following generalizations: the time horizon is infinite ($T = \infty$); the common discount factor is $\delta \in (0,1)$; the buyer's private valuation for the good is $\theta \in \Theta = \{\theta_\ell, \theta_h\}$, where $0 < \theta_\ell < \theta_h$; the probability that $\theta = \theta_h$ at $t = 0$ is $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$.[11]

Hereafter, $\mathcal{G}$ refers to the durable-good monopoly MCS game and $\mathcal{E}$ (resp., $\mathcal{O}$) to all its wPBEs (resp., wPBE outcomes). Finally, we denote by $\pi := (\delta, \theta_\ell, \theta_h, \mu)$ the vector of exogenous parameters of the model.

Section Outline. We present our indirect approach to characterize the set $\mathcal{O}$ in Section 4.1. We characterize a seller-optimal wPBE outcome of $\mathcal{G}$ in Section 4.2 and discuss its mediated and unmediated implementation in Sections 4.3 and 4.4.

4.1 Indirect Approach

To characterize the set $\mathcal{O}$, we could rely on (some versions of) the revelation principle in FMSW. Informally, when applied to our setting, the FMSW revelation principle says that it is without loss to assume the following:

1. In period $t = 0$, the buyer's report to the mediator is her type, i.e., $R = \Theta$.
2. In each period $t \in \mathcal{T}_0$: the mediator's signal to the seller is a recommendation of an admissible contract, i.e., $S_{P,t} = \mathcal{C}_t$; the mediator's signal to the buyer is a recommendation of an input message to the contract, i.e., $S_{A,t} = M_t$.
3. In period $t = 0$, the buyer truthfully reports her type to the mediator; in each period $t \in \mathcal{T}_0$, the seller and the buyer obey the mediator's recommendation.

The revelation principle in FMSW simplifies the characterization of the set $\mathcal{O}$ relative to the original setting in which any indirect reports and signals are allowed. However, its literal application is still challenging. In particular, when the set of admissible contracts $\mathcal{C}$ is large, there are two challenges: (i) identifying the sequence of contracts that the seller offers on the equilibrium path; and (ii) verifying the seller's obedience to the on-path contracts for all possible deviations to off-path contracts. We circumvent these issues by following an indirect approach based on the insights we develop in the remaining part of this section.

[11] When $\mu \leq \frac{\theta_\ell}{\theta_h}$, a fully committed seller would post price θ_ℓ and trade with both types in period $t = 1$. This is also a seller-optimal wPBE outcome of the durable-good monopoly MCS game.

4.1.1 Outcome-Based Approach

We represent any outcome of $\mathcal{G}$ by a sequence of trade probabilities and expected transfers $(\boldsymbol{x}, \boldsymbol{p}) \coloneqq (x_{\ell,t}, x_{h,t}, p_{\ell,t}, p_{h,t})_{t=1}^{\infty}$, where $x_{\ell,t}$ (resp., $x_{h,t}$) denotes the probability with which the seller trades with type θ_ℓ (resp., θ_h) in period t, and $p_{\ell,t}$ (resp., $p_{h,t}$) denotes the expected transfer from type θ_ℓ (resp., θ_h) to the seller in period t. To form an outcome of $\mathcal{G}$, a sequence $(\boldsymbol{x}, \boldsymbol{p})$ must satisfy the obvious feasibility constraints: $x_{k,t} \geq 0$ for all $k \in \{\ell, h\}$ and $t \in \mathcal{T}_0$, and $\sum_{t=1}^{\infty} x_{k,t} \leq 1$ for all $k \in \{\ell, h\}$.

Given an outcome $(\boldsymbol{x}, \boldsymbol{p})$:

- For all $k \in \{\ell, h\}$, we denote by $x_{k,t}(\boldsymbol{x}, \boldsymbol{p})$ the probability with which type θ_k trades in period t in outcome $(\boldsymbol{x}, \boldsymbol{p})$ conditional on no trade before.
- For all $k \in \{\ell, h\}$, we denote by $p_{k,t}(\boldsymbol{x}, \boldsymbol{p})$ the expected transfer from type θ_k to the seller in period t in outcome $(\boldsymbol{x}, \boldsymbol{p})$ conditional on no trade before.
- We denote by $\mathcal{C}(\boldsymbol{x}, \boldsymbol{p}) \coloneqq (C_t(\boldsymbol{x}, \boldsymbol{p}))_{t=1}^{\infty}$ the sequence of direct contracts corresponding to outcome $(\boldsymbol{x}, \boldsymbol{p})$. That is, $C_t(\boldsymbol{x}, \boldsymbol{p})$ is the direct contract allocating
$$(x_t, p_t) = \begin{cases} (1, p_{k,t}(\boldsymbol{x}, \boldsymbol{p})) & \text{with probability } x_{k,t}(\boldsymbol{x}, \boldsymbol{p}) \\ (0, p_{k,t}(\boldsymbol{x}, \boldsymbol{p})) & \text{with probability } 1 - x_{k,t}(\boldsymbol{x}, \boldsymbol{p}) \end{cases}$$
if the input message to the direct contract is θ_k for all $k \in \{\ell, h\}$.

Thus, one can think of the mediator's strategy as directly recommending outcomes—or, equivalently, direct contracts to the seller and type reports to the direct contract to the buyer—instead of sending signals to the seller and the buyer.

4.1.2 Upper-Bound Problem

Let C^ℓ denote the constant contract allocating $(x, p) = (1, \theta_\ell)$. To form a wPBE outcome of $\mathcal{G}$, an outcome $(\boldsymbol{x}, \boldsymbol{p})$ must be such that: (i) the buyer's incentive compatibility constraint when sending a report to the mediator in period $t = 0$ is satisfied; (ii) in each period $t \in \mathcal{T}_0$, the buyer's expected continuation payoff is non-negative; (iii) in each period $t \in \mathcal{T}_0$, the seller's expected continuation payoff is non-negative; (iv) in each period $t \in \mathcal{T}_0$ such that trade has not yet occurred, the seller's expected continuation payoff must be no less than that from ending the game by offering the constant contract C^ℓ in period t. Requirements (i)–(iii) are obvious. To understand requirement (iv), it is enough to note that, in any wPBE of $\mathcal{G}$, if trade has not yet occurred by period t and the seller offers contract C^ℓ in period t, both buyer types accept the contract, and the game ends.

The next theorem formalizes the previous discussion by identifying necessary conditions on outcomes $(\boldsymbol{x}, \boldsymbol{p})$ to form a wPBE outcome of $\mathcal{G}$.

Theorem 1 (Upper-Bound Problem)**.** Consider any $(\boldsymbol{x}, \boldsymbol{p}) \in \mathcal{O}$. Then, there exists $(\sigma, \beta) \in \mathcal{E}$ that induces $(\boldsymbol{x}, \boldsymbol{p})$ and satisfies the following conditions:

1. For all $\theta, \theta' \in \Theta$, at information set $h_A^{0,2} = \{\theta\}$, the buyer's expected continuation payoff by playing $\sigma_A^{0,2}(\theta)$ is at least as high as that by playing $\sigma_A^{0,2}(\theta')$.
2. For all $t \in \mathcal{T}_0$ and $h_A^{t,4} \in H_A^{t,4}$, the buyer's expected continuation payoff is non-negative.
3. For all $t \in \mathcal{T}_0$ and $h_P^{t,2} \in H_P^{t,2}$, the seller's expected continuation payoff is non-negative.
4. For all $t \in \mathcal{T}_0$ and $h_P^{t,2} \in H_P^{t,2}$ such that trade has not yet occurred by period t, the seller's expected continuation payoff is not less than that from ending the game by offering the constant contract C^ℓ in period t.

Relaxed Revelation Principle. The outcome-based approach and Theorem 1 together can be interpreted as a *relaxed revelation principle*. They allow us to characterize candidate wPBE outcomes by solving a problem that is a much simpler relaxation of the original problem. First, the space of all outcomes is much more well-structured than the unrestricted set of all admissible contracts $\mathcal{C}$. Second, the set of incentive compatibility and obedience constraints is much smaller; in particular, for the seller's obedience, we can focus only on deviations to the constant contract C^ℓ, ignoring all the other obedience constraints.

4.1.3 Lower-Bound Problem

For all $\tilde{\mu} \in (0,1)$, let $(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu}))$ denote the Coasean outcome of $\mathcal{G}$ when the seller's initial belief that $\theta = \theta_h$ is $\tilde{\mu}$. That is, $(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu}))$ is the outcome that would arise in the essentially unique equilibrium of $\mathcal{G}$ in which the mediator replicates the information structure in Doval and Skreta (2021) when the seller's initial belief that $\theta = \theta_h$ is $\tilde{\mu}$. Moreover, let $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu})) := (C_t(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu})))_{t=1}^\infty$ be the corresponding sequence of direct contracts.

The next theorem identifies sufficient conditions on outcomes $(\boldsymbol{x}, \boldsymbol{p})$ to form a wPBE outcome of $\mathcal{G}$. Let $\tilde{H}_A^{1.4}$ be the projection of $H^{1.4}$ on $R \times \mathcal{C}^1 \times S_A^1$, and, for all $t \in \mathcal{T}_1$, let $\tilde{H}_A^{t.4}$ be the projection of $H^{t.4}$ on $R \times \mathcal{C}^t \times S_A^t \times \mathcal{M}^{t-1} \times A^{t-1}$. For all $t \in \mathcal{T}_0$, we denote by $\tilde{h}_A^{t.4}$ a typical element of $\tilde{H}_A^{t.4}$. Note that $H_A^{t.4} = \Theta \times \tilde{H}_A^{t.4}$ for all $t \in \mathcal{T}_0$.

Theorem 2 (Lower-Bound Problem)**.** Consider an outcome $(\boldsymbol{x}, \boldsymbol{p})$ of $\mathcal{G}$ and let (σ, β) be an assessment of $\mathcal{G}$ that induces $(\boldsymbol{x}, \boldsymbol{p})$ and such that β is on-path consistent given σ. Suppose that the following conditions hold:

1. For all $\theta, \theta' \in \Theta$, at information set $h_A^{0.2} = \{\theta\}$, the buyer's expected continuation payoff by playing $\sigma_A^{0.2}(\theta)$ is at least as high as that by playing $\sigma_A^{0.2}(\theta')$.

2. For all $t \in \mathcal{T}_0$ and $h_A^{t.4} \in H_A^{t.4}$, the buyer's expected continuation payoff is non-negative.

3. For all $t \in \mathcal{T}_0$, $\theta, \theta' \in \Theta$ and $\tilde{h}_A^{t.4} \in \tilde{H}_A^{t.4}$, at information set $h_A^{t.4} = \big(\theta, h_A^{\tilde{t}.4}\big)$, the buyer's expected continuation payoff by playing $\sigma_A^{t.4}\big(\theta, h_A^{\tilde{t}.4}\big)$ is at least as high as that by playing $\sigma_A^{t.4}\big(\theta', h_A^{\tilde{t}.4}\big)$.

4. For all $t \in \mathcal{T}_0$ and $h_P^{t.2} \in H_P^{t.2}$, the seller's expected continuation payoff is not less than that he would obtain by offering the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_t), \boldsymbol{p}^{\boldsymbol{c}}(\mu_t))$ thereafter, where μ_t is the seller's belief that $\theta = \theta_h$ at $h_P^{t.2}$ derived from (σ, β).

Then, $(\sigma, \beta) \in \mathcal{E}$ and $(\boldsymbol{x}, \boldsymbol{p}) \in \mathcal{O}$.

Conditions 1 and 2 in Theorem 2 correspond to conditions 1 and 2 in Theorem 1. Condition 3 in Theorem 2 requires the buyer's incentive compatibility constraint when sending an input message to the contract to be satisfied in each period. When conditions 1–3 in Theorem 2 hold, the buyer's sequential rationality follows.

Condition 4 in Theorem 2 says that if an outcome satisfies the seller's obedience with respect to deviations to contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_t), \boldsymbol{p}^{\boldsymbol{c}}(\mu_t))$, then the seller's sequential rationality follows. To see why it suffices to focus on such deviations, consider the following mediator's strategy at any $t \in \mathcal{T}_0$:

- Stage $t.1$: the mediator recommends the direct contract $C_t(\boldsymbol{x}, \boldsymbol{p})$ to the seller.
- Stage $t.3$: (i) if the seller offered contract $C_t(\boldsymbol{x}, \boldsymbol{p})$ at stage $t.2$, the mediator recommends the buyer to participate and truthfully report her type; (ii) if the seller offered contract $C_t \neq C_t(\boldsymbol{x}, \boldsymbol{p})$ at stage $t.2$, the mediator reverts to the

replication of the essentially unique equilibrium outcome in Doval and Skreta (2021), in which the seller can do no better and no worse than if he could only set prices in each period, and so the Coasean outcome obtains thereafter.

Given the mediator's strategy, the seller's most profitable deviation in period t is to the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_t), \boldsymbol{p}^{\boldsymbol{c}}(\mu_t))$. If an outcome $(\boldsymbol{x}, \boldsymbol{p})$ satisfies condition 4 in Theorem 2, it also satisfies the necessary conditions 3 and 4 in Theorem 1.

Mediated Implementation in Direct Contracts. The outcome-based approach and Theorem 2 together can be interpreted as *mediated implementation in direct contracts*. They allow us to characterize wPBE outcomes by solving a problem that is simpler than the original problem. In particular, for the seller's obedience, we can focus only on deviations to a well-defined sequence of contracts.

4.2 Seller-Optimal wPBE Outcomes

We denote by U_P^* the seller's ex ante expected payoff in a seller-optimal wPBE outcome of $\mathcal{G}$. Moreover, we denote by $\mathbb{R}^{n\infty}$ the set of all sequences with values in $\mathbb{R}^n$.

Upper-Bound Problem. By Theorem 1, to find a candidate for a seller-optimal wPBE outcome of $\mathcal{G}$, it suffices to solve the following linear program:

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P1}$$

$$\text{s.t.} \quad x_{\ell,t}, x_{h,t} \geq 0 \quad \forall t \in \mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{h,t} - p_{h,t}) \tag{ICℓ}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t} - p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t} - p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq 0 \quad \forall \tau \in \mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_h x_{h,t} - p_{h,t}) \geq 0 \quad \forall \tau \in \mathcal{T}_0 \tag{IRh}$$

$$\mu\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{h,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right) \tag{Oℓ}$$
$$+(1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right)\geq\theta_\ell \quad \forall\tau\in\mathcal{T}_0.$$

The inequalities (F1)–(F3) are the feasibility constraints. The inequalities (ICℓ) and (ICh) are the buyer's incentive compatibility constraints in period $t=0$ (condition 1 in Theorem 1). The inequalities (IRℓ) and (IRh) are the buyer's individual rationality (or participation) constraints for all $\tau\in\mathcal{T}_0$ (condition 2 in Theorem 1). The inequalities in (Oℓ) are the seller's obedience constraints for all $\tau\in\mathcal{T}_0$ with respect to the deviation to contract C^ℓ (condition 4 in Theorem 1). The inequalities in (Oℓ) also ensure that the seller's expected continuation payoff is non-negative for all $\tau\in\mathcal{T}_0$ (condition 3 in Theorem 1).

In Appendix B, by using linear programming duality arguments, we solve the linear program (P1) and establish the following result.

Theorem 3 (A Candidate for a Seller-Optimal wPBE Outcome)**.** The following characterizes a candidate for a seller-optimal wPBE outcome of $\mathcal{G}$. For all $\delta\in(0,1)$ and $\mu\in\left(\frac{\theta_\ell}{\theta_h},1\right)$, there exists a positive integer $\overline{T}\geq 2$ such that:[12]

1. The game ends in period $t=\overline{T}$.
2. Type θ_ℓ trades only (and with probability 1) in period $t=\overline{T}$ at price θ_ℓ.
3. Type θ_h trades with positive probability in all periods $t\in\left\{1,\dots,\overline{T}-1\right\}$ at price θ_h, except for period $t=1$, when he trades at price in (θ_ℓ,θ_h).
4. As $\delta\to 1$, $\overline{T}\to\infty$.

Hereafter, we denote by $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ the solution to program (P1) we describe in Theorem 3 and characterize in Appendix B. At $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ of $\mathcal{G}$, the Coase conjecture fails, as the following theorem formalizes (see Appendix D for the proof).

Theorem 4 (Failure of the Coase Conjecture)**.** For all $\mu\in\left(\frac{\theta_\ell}{\theta_h},1\right)$, at the solution $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ to program (P1), as $\delta\to 1$:

1. The seller's ex ante expected payoff is bounded away from θ_ℓ.
2. The outcome is bounded away from first-best efficiency.

[12]The value of $\overline{T}$ as well as that of all the other endogenous variables depend on the model's exogenous parameters π. Hereafter, we omit such dependence from the notation unless it is needed.

3. If, in addition, $\theta_\ell \to 0$, the seller's ex ante expected payoff converges to the commitment payoff.

Lower-Bound Problem. Let $U_P^c(\tilde{\mu};\delta)$ denote the seller's expected payoff from contracts $\mathcal{C}(\boldsymbol{x^c}(\tilde{\mu}), \boldsymbol{p^c}(\tilde{\mu}))$ when the seller's initial belief that $\theta = \theta_h$ is $\tilde{\mu}$. Moreover, let $(\boldsymbol{x}, \boldsymbol{p}) \coloneqq (x_{\ell,t}, x_{h,t}, p_{\ell,t}, p_{h,t})_{t=1}^{\infty}$ be an outcome of $\mathcal{G}$. For all $\tau \in \mathcal{T}_0$, let

$$\mu_\tau(\boldsymbol{x}, \boldsymbol{p}) \coloneqq \frac{\left(1 - \sum_{t=1}^{\tau-1} x_{h,t}\right)\mu}{\left(1 - \sum_{t=1}^{\tau-1} x_{\ell,t}\right)(1-\mu) + \left(1 - \sum_{t=1}^{\tau-1} x_{h,t}\right)\mu} \tag{1}$$

be the seller's belief that $\theta = \theta_h$ at the beginning of stage $\tau.2$ under outcome $(\boldsymbol{x}, \boldsymbol{p})$. Thus, under outcome $(\boldsymbol{x}, \boldsymbol{p})$, $U_P^c(\mu_\tau(\boldsymbol{x}, \boldsymbol{p});\delta)$ corresponds to the seller's expected continuation payoff at stage $\tau.2$ from contracts $\mathcal{C}(\boldsymbol{x^c}(\mu_\tau(\boldsymbol{x}, \boldsymbol{p})), \boldsymbol{p^c}(\mu_\tau(\boldsymbol{x}, \boldsymbol{p})))$.

By Theorem 2, if $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ also satisfies the following constraints

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_\ell x_{\ell,t}^* - p_{\ell,t}^*) \geq (\theta_\ell x_{h,\tau}^* - p_{h,\tau}^*)\left(1 - \sum_{t=1}^{\tau-1} x_{\ell,t}^*\right) \quad \forall \tau \in \mathcal{T}_0, \tag{ICℓ'}$$

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_h x_{h,t}^* - p_{h,t}^*) \geq (\theta_h x_{\ell,\tau}^* - p_{\ell,\tau}^*)\left(1 - \sum_{t=1}^{\tau-1} x_{h,t}^*\right) \quad \forall \tau \in \mathcal{T}_0, \tag{ICh'}$$

$$\begin{aligned} &\mu\left(\sum_{t=1}^{\tau-1} U_P^c(\mu_\tau(\boldsymbol{x}^*, \boldsymbol{p}^*);\delta) x_{h,t}^* + \sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{h,t}^*\right) \\ &+ (1-\mu)\left(\sum_{t=1}^{\tau-1} U_P^c(\mu_\tau(\boldsymbol{x}^*, \boldsymbol{p}^*);\delta) x_{\ell,t}^* + \sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{\ell,t}^*\right) \geq U_P^c(\mu_\tau(\boldsymbol{x}^*, \boldsymbol{p}^*);\delta) \quad \forall \tau \in \mathcal{T}_0, \end{aligned} \tag{O}$$

then $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a seller-optimal wPBE outcome of $\mathcal{G}$. The inequalities in (ICℓ') and (ICh') are the buyer's incentive compatibility constraints for all $\tau \in \mathcal{T}_0$ (condition 3 in Theorem 2). To understand the right-hand side of the inequalities in (ICℓ') (resp., (ICh')), note that if type θ_ℓ (resp., θ_h) deviates in period τ, then in period $\tau+1$ the mediator can disclose the buyer's true type to the seller, who then extracts the full surplus; as a result, type θ_ℓ's (resp., θ_h's) payoff from deviating is at most $\theta_\ell x_{h,\tau}^* - p_{h,\tau}^*$ (resp., $\theta_h x_{\ell,\tau}^* - p_{\ell,\tau}^*$). The inequalities in (O) are the seller's obedience constraints at stage $\tau.2$ for all $\tau \in \mathcal{T}_0$ with respect to the deviation to offering contracts $\mathcal{C}(\boldsymbol{x^c}(\mu_\tau(\boldsymbol{x}, \boldsymbol{p})), \boldsymbol{p^c}(\mu_\tau(\boldsymbol{x}, \boldsymbol{p})))$ thereafter (condition 4 in Theorem 2).

In Appendix E, we show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraints (ICℓ') and (ICh'). Moreover, we also show that, for all $\delta \in (0,1)$, there exists $\overline{\mu}(\delta) \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$ such that, for all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$, $\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h}$, and so $\mu_\tau(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h}$ for all $\tau > 2$. Hence, for all $\tau \in \mathcal{T}_1$ such that $\min\left\{\sum_{t=1}^{\tau-1} x_{\ell,t}, \sum_{t=1}^{\tau-1} x_{h,t}\right\} < 1$, we have $U_P^c(\mu_\tau(\boldsymbol{x}^*, \boldsymbol{p}^*); \delta) = \theta_\ell$. Therefore, since $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (Oℓ), it also satisfies constraint (O) for all $\tau \in \mathcal{T}_1$. Moreover, since $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ maximizes the seller's ex ante expected payoff in the relaxed upper-bound problem (P1), it also satisfies constraint (O) in period $\tau = 1$. The next theorem follows.

Theorem 5 (A Seller-Optimal wPBE Outcome)**.** For all $\delta \in (0, 1)$, there exists $\overline{\mu}(\delta) \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$ such that, for all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a seller-optimal wPBE outcome of $\mathcal{G}$. Moreover, $\overline{\mu}(\delta)$ satisfies the following properties: (i) $\overline{\mu}(\delta)$ is decreasing in δ; (ii) $\overline{\mu}(\delta) \to 1$ as $\delta \to 0$; and (iii) $\overline{\mu}(\delta)$ is bounded away from $\frac{\theta_\ell}{\theta_h}$ as $\delta \to 1$.

For $\mu > \overline{\mu}(\delta)$, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ need not satisfy constraint (O), as it can be that $U_P^c(\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*); \delta) > \theta_\ell$. However, in Appendix F we show that a sufficient condition for $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ to satisfy constraint (O) is that it satisfies the following constraint:

$$\mu\left(\sum_{t=1}^{\tau-1}(\theta_\ell + \varepsilon(\delta))x_{h,t}^* + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}^*\right) + (1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t}^* + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}^*\right) \geq \theta_\ell + \mu\varepsilon(\delta) \quad \forall \tau \in \mathcal{T}_0, \tag{O$'$}$$

where $\varepsilon(\delta) := U_P^c(\tilde{\mu}; \delta) - \theta_\ell$ and $\tilde{\mu}$ is some (well-chosen) element of $\left(\frac{\theta_\ell}{\theta_h}, 1\right)$. By the Coase conjecture, as $\delta \to 1$, $U_P^c(\tilde{\mu}; \delta) \to \theta_\ell$, and so $\varepsilon(\delta) \to 0$; that is, as $\delta \to 1$, constraint (O$'$) converges to constraint (Oℓ) in program (P1).

Next, consider the following linear program:

$$\max_{(\boldsymbol{x},\boldsymbol{p}) \in \mathbb{R}^{4\infty}} \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P2}$$
$$\text{s.t.} \quad (\text{F1}), (\text{F2}), (\text{F3}), (\text{IC}\ell), (\text{IC}h), (\text{IR}\ell), (\text{IR}h), (\text{IC}\ell'), (\text{IC}h'), (\text{O}').$$

Denote by $U_P^{\text{P2}}(\pi)$ the optimal value of the linear program (P2). Since constraint (O$'$) is more demanding than constraint (O), we have that $U_P^{\text{P2}}(\pi) \leq U_P^*$ for all $\delta \in (0, 1)$. In Appendix G, we characterize a solution to program (P2), which we denote by $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$, and establish the following theorem.

Theorem 6 (A Seller-Approximately Optimal wPBE Outcome)**.** For all $\mu \in (\overline{\mu}(\delta), 1)$, as $\delta \to 1$: (i) $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) \to (\boldsymbol{x}^*, \boldsymbol{p}^*)$, and (ii) $U_P^{\text{P2}}(\pi) \to U_P^*$.

By Theorems 4–6, the Coase conjecture fails at the seller-optimal mechanism.

Discussion. Our characterization of the seller-optimal information structure uncovers the following results of economic substance. In general, the seller finds it optimal if the information input by the buyer arrives precisely (i.e., without garbling) but with delay. Formally, the optimal information structure specifies an endogenous deadline $\overline{T}$ at which the buyer's private information report is made public; until then, no information—except whether a trade has occurred or not—is revealed.

Intuitively, delayed (but precise) information disclosure increases the seller's bargaining power. Relative to the case in which no such information arrives, the seller's incentive to offer a more aggressive price is stronger because even if the buyer did not buy at that aggressive price, full extraction would be possible once the time comes. Furthermore, once the seller becomes aggressive in some period, he can also be more aggressive in the previous period as the buyer has less continuation payoff conditional on no trading. In this sense, the aggressiveness of each seller's self is a "strategic complement" to each other. Indeed, except for the initial period, the seller offers the high, surplus-extracting price until the time of revelation.

As a result, equilibrium prices are different from the classical pattern of decreasing prices: our case may be interpreted as an initial "fire sale" followed by the rigid high price. Even in the limiting case of perfect patience, the seller extracts surplus from trade and trade is not efficient. Thus, we identify a novel reason why the Coase conjecture can fail.

4.3 Mediated Implementation

Let $\mathcal{C}(\boldsymbol{x}^*, \boldsymbol{p}^*) := \{C_t(\boldsymbol{x}^*, \boldsymbol{p}^*)\}_{t=1}^{\infty}$ be the sequence of direct contracts corresponding to outcome $(\boldsymbol{x}^*, \boldsymbol{p}^*)$. Then, we have the following.

Theorem 7 (Mediated Implementation of $(\boldsymbol{x}^*, \boldsymbol{p}^*)$)**.** Suppose $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$. Then, there exists a wPBE of $\mathcal{G}$ which satisfies the following properties:

1. The two buyer types fully separate from each other at stage 0.2.

2. The seller and the buyer always obey the mediator's recommendations.

3. For all $t \in \mathcal{T}_0$: at stage $t.1$, the mediator recommends contract $C_t(\boldsymbol{x}^*, \boldsymbol{p}^*)$ to

the seller; at stage $t.3$, the mediator recommends the buyer to participate and to truthfully report her type.

Such wPBE induces outcome $(\boldsymbol{x}^{*}, \boldsymbol{p}^{*})$.

That any wPBE in which properties 1–3 in Theorem 7 hold induces outcome $(\boldsymbol{x}^{*}, \boldsymbol{p}^{*})$ is obvious. Moreover, since $(\boldsymbol{x}^{*}, \boldsymbol{p}^{*})$ satisfies constraints (ICℓ), (ICh), (IRℓ), (IRh), (ICℓ'), (ICh'), and (O), it satisfies the sufficient conditions 1–4 in Theorem 2. Thus, Theorem 7 follows.

Now let $\mathcal{C}(\boldsymbol{x}^{*}\boldsymbol{(\delta)}, \boldsymbol{p}^{*}\boldsymbol{(\delta)}) := \{C_t(\boldsymbol{x}^{*}\boldsymbol{(\delta)}, \boldsymbol{p}^{*}(\delta))\}_{t=1}^{\infty}$ be the sequence of direct contracts corresponding to $(\boldsymbol{x}^{*}(\delta), \boldsymbol{p}^{*}(\delta))$. Then, we have the following.

Theorem 8 (Mediated Implementation of $(\boldsymbol{x}^{*}(\delta), \boldsymbol{p}^{*}(\delta))$)**.** Suppose $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$. Then, there exists a wPBE of $\mathcal{G}$ which satisfies the following properties:

1. The two buyer types fully separate from each other at stage 0.2.
2. The seller and the buyer always obey the mediator's recommendations.
3. For all $t \in \mathcal{T}_0$: at stage $t.1$, the mediator recommends contract $C_t(\boldsymbol{x}^{*}\boldsymbol{(\delta)}, \boldsymbol{p}^{*}(\delta))$ to the seller; at stage $t.3$, the mediator recommends the buyer to participate and to truthfully report her type.

Such wPBE induces outcome $(\boldsymbol{x}^{*}(\delta), \boldsymbol{p}^{*}(\delta))$.

That any wPBE in which properties 1–3 in Theorem 8 hold induces outcome $(\boldsymbol{x}^{*}(\delta), \boldsymbol{p}^{*}(\delta))$ is obvious. Since $(\boldsymbol{x}^{*}(\delta), \boldsymbol{p}^{*}(\delta))$ satisfies constraints (ICℓ), (ICh), (IRℓ), (IRh), (ICℓ'), and (ICh'), it satisfies conditions 1–3 in Theorem 2. Moreover, since $(\boldsymbol{x}^{*}(\delta), \boldsymbol{p}^{*}(\delta))$ satisfies constraint (O$'$), which is more stringent than constraint (O), it also satisfies condition 4 in Theorem 2. Thus, Theorem 8 follows.

4.4 Unmediated Implementation

So far, by considering the mediator, we allowed for arbitrary information structures. However, it is also interesting to understand whether the seller-optimal wPBE outcome of $\mathcal{G}$ is attainable without a mediator. Recently, Brzustowski et al. (2021) consider an unmediated durable-good monopoly problem where the seller can offer a long-term "smart" contract, though he may discard it and employ a new long-term contract in any future period, reflecting the idea of limited commitment. They construct an equilibrium where the seller initially offers a *d-abiding* contract, which is a contract that no future seller wants to discard, and show that the Coase conjecture fails if players are sufficiently patient. Following Brzustowski et al. (2021), in this section, we consider the unmediated durable-good monopoly problem with long-term smart contracts and argue that the mediated seller-optimal outcome is attainable if and only if $\mu \leq \overline{\mu}(\delta)$. However, even if $\mu > \overline{\mu}(\delta)$, we argue that an appropriately-modified version of our approach is useful to characterize the best unmediated equilibrium. Since it is well beyond the scope of this paper to formally introduce such an unmediated model where the seller can also offer long-term contracts, we keep the argument informal and concise.

Low Prior. Suppose $\mu \leq \overline{\mu}(\delta)$. Assume that the seller in period $t = 1$ can design the following *long-term smart contract*: first, the buyer is asked to report her type θ; this report is not observed by the seller but remains under cryptographic encryption, which is to be decrypted at some predetermined date $\overline{T}$; until $\overline{T}$, the contract specifies the same sequence of allocations as in the mediated solution. Due to limited commitment, the seller can discard this contract in any future period and propose a new long-term smart contract. However, discarding the original contract implies the loss of the encrypted information.

If the seller does not discard the original contract, the same outcome as in the mediated solution obtains. At the mediated solution, if $\mu \leq \overline{\mu}(\delta)$, the seller's obedience constraint in binding and so his continuation expected payoff is θ_ℓ in any period $t \geq 2$. Thus, even a fully-committed seller prefers obeying the mediator's recommendation to any deviation. Clearly, the same is true in the unmediated case.

High Prior. If $\mu > \overline{\mu}(\delta)$, in the mediated solution, we obtain both an upper and a lower bound for the seller's ex ante expected payoff, and show that they coincide as $\delta \to 1$. In particular, to attain the lower bound, the mediator actively selects an information structure and a continuation equilibrium after any deviation of the seller in a way that the Coasian outcome obtains (so that the deviating seller's continuation payoff converges to θ_ℓ as $\delta \to 1$). In this sense, the set of continuation outcomes for a deviating seller is carefully selected by the mediator.

In the unmediated environment, in each period, the seller has the same set of feasible continuation outcomes and so a larger set of possible deviations. Thus, the seller's ex ante expected payoff at a seller-optimal unmediated equilibrium can be lower than that at a seller-optimal wPBE outcome of $\mathcal{G}$. We next argue that, even in such cases, an appropriately-modified version of our approach can be useful.

Let $V_u^*(\mu)$ be the seller's ex ante expected payoff in the seller-optimal unmediated equilibrium when the seller's initial belief that $\theta = \theta_h$ is μ. The function $V_u^*\colon [0,1] \to \mathbb{R}$ can be characterized pointwise (i.e., for each $\mu \in [0,1]$) as follows:

$$V_u^*(\mu) := \max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P$_u$}$$

$$\text{s.t.}\quad \text{(F1), (F2), (F3), (IC}\ell\text{), (IC}h\text{), (IR}\ell\text{), (IR}h\text{)},$$

$$\mu\left(\sum_{t=1}^{\tau-1}V_u^*(\mu_\tau(\boldsymbol{x},\boldsymbol{p}))x_{h,t} + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right) \tag{0$_u$}$$

$$+ (1-\mu)\left(\sum_{t=1}^{\tau-1}V_u^*(\mu_\tau(\boldsymbol{x},\boldsymbol{p}))x_{\ell,t} + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right) \geq V_u^*(\mu_\tau(\boldsymbol{x},\boldsymbol{p}))\quad \forall\,\tau \in \mathcal{T}_0.$$

The same V_u^* appears in the seller's obedience constraint (0$_u$) because, in each period, the seller has the same set of feasible deviations as in the first period.

For any convex, increasing function $V\colon [0,1] \to \mathbb{R}$, let $W(V)$ be the value function of the constrained maximization problem (P$_u$) in which $V_u^*(\mu_\tau(\boldsymbol{x},\boldsymbol{p}))$ in constraint (0$_u$) is replaced by $V(\mu_\tau(\boldsymbol{x},\boldsymbol{p}))$ for all $\tau \in \mathcal{T}_0$. Then, V_u^* is a fixed point of the operator W: $V_u^* = W(V_u^*)$. Clearly, W is non-increasing in the sense that, if $V \leq V'$ (pointwise, i.e., $V(\mu) \leq V'(\mu)$ for all $\mu \in [0,1]$), then $W(V) \geq W(V')$.

We know that (i) $V_u^*(\mu) = \theta_\ell$ for $\mu \in \left[0, \frac{\theta_\ell}{\theta_h}\right]$, and (ii) $V_u^*(\mu)$ is equal to the value under the mediated solution for $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$. Let V^1 be the smallest convex, increasing function $V\colon [0,1] \to \mathbb{R}$ that satisfies (i) and (ii).[13] Clearly, $V^1 \leq V_u^*$.

Let $\widehat{V}^1 = W(V^1)$ be the value function of the constrained maximization problem (P$_u$) in which V^1 is used in constraint (0$_u$). By monotonicity of W, $V_u^* \leq \widehat{V}^1$. Now, we conjecture that there exists $\overline{\mu}^1 \in (\overline{\mu}(\delta), 1]$ such that $\widehat{V}^1(\mu) = V_u^*(\mu) > V^1(\mu)$ for all $\mu \in (\overline{\mu}(\delta), \overline{\mu}^1]$. The idea is that, if $\mu < \overline{\mu}^1$, then the optimal mechanism in the problem of $W(V^1)$ would be such that, right after the initial trade, the posterior μ_1 drops below $\overline{\mu}(\delta)$. Since V^1 and V_u^* already coincide for beliefs below $\overline{\mu}(\delta)$, the same mechanism continues to be feasible (given μ) even in the

[13]This V^1 is identified by $V^1(\mu) = V_u^*(\overline{\mu}(\delta)) + \frac{\partial V}{\partial \mu}(\overline{\mu}(\delta))(\mu - \overline{\mu}(\delta))$ for $\mu > \overline{\mu}(\delta)$.

problem of $W(V_u^*)$. Therefore, we must have $\widehat{V}^1(\mu) = V_u^*(\mu)$ for $\mu < \overline{\mu}^1$.[14]

This further suggests that the entire V_u^* can be characterized iteratively. Let V^2 be the smallest convex function such that $V^2(\mu) = \widehat{V}^1(\mu)$ $(= V_u^*(\mu))$ for $\mu < \overline{\mu}^1$, and let $\widehat{V}^2 = W(V^2)$. In the problem of $W(V^2)$, it is conjectured that there would be $\overline{\mu}^2 > \overline{\mu}^1$ such that $\widehat{V}^2(\mu) = V_u^*(\mu)$ for all $\mu < \overline{\mu}^2$. If the increasing sequence $\overline{\mu}^1, \overline{\mu}^2, \ldots$ converges to 1, then the entire V_u^* is characterized.

5 Conclusion

We propose a mediator approach to mechanism design with limited commitment. Our approach builds on the (sequential) communication equilibrium notion in multistage games with communication and enables a systematic study of equilibrium outcomes under all admissible information structures. We represent the information structure as a fictitious mediator and re-interpret the model as mechanism design by the committed mediator. We construct examples to explain why new equilibrium outcomes can arise when general information structures are considered.

In the durable-good monopoly application, trading outcomes and welfare consequences can substantially differ from those in the classical model with a fixed information structure. In the seller-optimal mechanism, the seller offers a discount to the high-valuation buyer in the initial period, followed by the high, surplus-extracting price until some endogenous deadline, when the buyer's information is revealed and fully extracted. As a result, the Coase conjecture fails. We also characterize mediated and unmediated implementation of the seller-optimal outcome.

To circumvent the key challenge regarding the complexity of the seller's contract space, we take an outcome-based approach and propose two simpler auxiliary problems whose values provide an upper and a lower bound to the value of the original problem. These bounds are enough to characterize seller-optimal equilibrium outcomes. We interpret the two auxiliary problems as relaxed revelation principle and mediated implementation in direct contracts.

Our approach can be used to characterize robust predictions in bargaining games with one-sided incomplete information—i.e., to characterize the set of all equilibrium outcomes that can arise under any admissible information structure.

[14] If $\delta \simeq 1$, $\overline{\mu}(\delta) \simeq 0.8$ and $\overline{\mu}^1 \simeq 0.956 > 0.8$. A more formal argument is available upon request.

Moreover, our approach is potentially useful in other problems with a partially committed principal. A first example is mechanism design with an informed principal, where the principal can perfectly commit to a mechanism, but only after observing a private signal, and so he is subject to sequential rationality given each signal realization. A second example is mechanism design with multiple principals, where each principal can perfectly commit to his mechanism but cannot control other principals' mechanisms. We plan to explore these ideas in future work.

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A Notation

Throughout the appendices, we use the following notation:

$$\Delta\theta := \theta_h - \theta_\ell, \tag{2}$$

$$r := \frac{\theta_h - \delta\theta_\ell}{\delta\Delta\theta}, \tag{3}$$

$$\rho := \delta r. \tag{4}$$

Note that $\Delta\theta > 0$ and that $r > 1$ and $\rho > 1$ for all $\delta \in (0, 1)$. Moreover, note that

$$(3) \implies 1 - r = -\frac{(1-\delta)\theta_h}{\delta\Delta\theta} \tag{5}$$

and

$$(4) \implies 1 - \rho = -\frac{(1-\delta)\theta_\ell}{\Delta\theta}. \tag{6}$$

B Proof of Theorem 3

B.1 Simplifying the Primal Program (P1)

In the primal program (P1): (i) we ignore constraint (ICℓ) and we will verify in Appendix B.5 that it is satisfied by the solution to the relaxed version of the program; (ii) constraints (ICh) and (IRℓ) for $\tau = 1$, together with the assumption that $\theta_h > \theta_\ell$, imply that constraint (IRh) is satisfied for $\tau = 1$; (iii) that the seller's payoff in a seller-optimal wPBE must be at least θ_ℓ implies that constraint (Oℓ) is satisfied for $\tau = 1$. As a result, the relaxed version of program (P1) is the following:

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \quad \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P1a}$$

$$\text{s.t.} \quad x_{\ell,t}, x_{h,t} \geq 0 \quad \forall\, t \in \mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t} - p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t} - p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq 0 \quad \forall\, \tau \in \mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_h x_{h,t} - p_{h,t}) \geq 0 \quad \forall \tau \in \mathcal{T}_1 \tag{IRh}$$

$$\mu\left(\sum_{t=1}^{\tau-1} \theta_\ell x_{h,t} + \sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{h,t}\right) \tag{Oℓ}$$

$$+ (1-\mu)\left(\sum_{t=1}^{\tau-1} \theta_\ell x_{\ell,t} + \sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{\ell,t}\right) \geq \theta_\ell \quad \forall \tau \in \mathcal{T}_1.$$

B.2 The Dual of Program (P1a)

Let $\boldsymbol{\xi} := \left(\alpha, \beta, \zeta, (\lambda_{\ell,t}, \lambda_{h,t+1}, \gamma_{t+1})_{t=1}^{\infty}\right) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$, where: α (resp., β) is the Lagrange multiplier associated to constraint (F2) (resp., (F3)) in program (P1a); ζ is the Lagrange multiplier associated to constraint (ICh) in program (P1a); for all $t \in \mathcal{T}_0$, $\lambda_{\ell,t}$ is the Lagrange multiplier associated to constraint (IRℓ) in period t in program (P1a); for all $t \in \mathcal{T}_0$, $\lambda_{h,t+1}$ is the Lagrange multiplier associated to constraint (IRh) in period $t+1$ in program (P1a); for all $t \in \mathcal{T}_0$, γ_{t+1} is the Lagrange multiplier associated to constraint (Oℓ) in period $t+1$ in program (P1a).

The dual program of program (P1a) is the following:

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}} \alpha + \beta - \sum_{t=2}^{\infty} \theta_\ell \gamma_t \tag{P1b}$$

$$\text{s.t.} \quad \alpha \geq 0, \quad \beta \geq 0, \quad \zeta \geq 0, \quad \lambda_{\ell,t}, \lambda_{h,t+1}, \gamma_{t+1} \geq 0 \quad \forall t \in \mathcal{T}_0 \tag{7}$$

$$\alpha \geq -\delta^{t-1}\theta_h \zeta + \sum_{\tau=1}^{t} \delta^{t-\tau}\theta_\ell \lambda_{\ell,\tau} + \sum_{\tau=t+1}^{\infty} (1-\mu)\theta_\ell \gamma_\tau \quad \forall t \in \mathcal{T}_0, \tag{8}$$

$$\beta \geq \delta^{t-1}\theta_h \zeta + \sum_{\tau=2}^{t} \delta^{t-\tau}\theta_h \lambda_{h,\tau} + \sum_{\tau=t+1}^{\infty} \mu\theta_\ell \gamma_\tau \quad \forall t \in \mathcal{T}_0, \tag{9}$$

$$\delta^{t-1}\zeta - \sum_{\tau=1}^{t} \delta^{t-\tau}\lambda_{\ell,\tau} + \sum_{\tau=2}^{t} \delta^{t-\tau}(1-\mu)\gamma_\tau + \delta^{t-1}(1-\mu) = 0 \quad \forall t \in \mathcal{T}_0, \tag{10}$$

$$-\delta^{t-1}\zeta - \sum_{\tau=2}^{t} \delta^{t-\tau}\lambda_{h,\tau} + \sum_{\tau=2}^{t} \delta^{t-\tau}\mu\gamma_\tau + \delta^{t-1}\mu = 0 \quad \forall t \in \mathcal{T}_0, \tag{11}$$

where constraints (10) and (11) hold with equality as $p_{\ell,t}$ and $p_{h,t}$ are unrestricted (i.e., they can be positive or negative) for all $t \in \mathcal{T}_0$ in the primal program (P1a).

B.3 A Candidate Solution to the Dual Program (P1b)

We recover a candidate solution $\boldsymbol{\xi}^* := \left(\alpha^*, \beta^*, \zeta^*, \left(\lambda^*_{\ell,t}, \lambda^*_{h,t+1}, \gamma^*_{t+1}\right)_{t=1}^{\infty}\right) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$ to program (P1b).

Step 1. Solve equations (11) at successive values of t, starting with $t = 1$, to find

$$\zeta^* = \mu \qquad \text{and} \qquad \lambda^*_{h,t} = \mu \gamma^*_t \quad \text{for all } t \in \mathcal{T}_1. \tag{12}$$

Next, solve equations (10) at successive values of t, starting with $t = 1$, to find

$$\lambda^*_{\ell,1} = 1 \qquad \text{and} \qquad \lambda^*_{\ell,t} = (1-\mu)\gamma^*_t \quad \text{for all } t \in \mathcal{T}_1. \tag{13}$$

Step 2. Given equations (12) and (13), the dual program (P1b) simplifies to:

$$\min_{(\alpha,\beta,(\gamma_t)_{t=2}^{\infty}) \in \mathbb{R}^2 \times \mathbb{R}^{\infty}} \alpha + \beta - \sum_{t=2}^{\infty} \theta_\ell \gamma_t \tag{P1c}$$

$$\text{s.t.} \quad \alpha \geq 0, \quad \beta \geq 0, \quad \gamma_t \geq 0 \ \ \forall t \in \mathcal{T}_1 \tag{14}$$

$$\alpha \geq \delta^{t-1}(\theta_\ell - \mu\theta_h) + (1-\mu)\theta_\ell \left[\sum_{\tau=2}^{t} \delta^{t-\tau}\gamma_\tau + \sum_{\tau=t+1}^{\infty} \gamma_\tau\right] \quad \forall t \in \mathcal{T}_0, \tag{15}$$

$$\beta \geq \delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t} \delta^{t-\tau}\theta_h\gamma_\tau + \sum_{\tau=t+1}^{\infty} \theta_\ell\gamma_\tau\right] \quad \forall t \in \mathcal{T}_0. \tag{16}$$

Step 3. As program (P1c) is a minimization problem, its solutions must satisfy

$$\alpha^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}(\theta_\ell - \mu\theta_h) + (1-\mu)\theta_\ell\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\gamma^*_\tau + \sum_{\tau=t+1}^{\infty}\gamma^*_\tau\right]\right\}\right\} \tag{17}$$

and

$$\beta^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma^*_\tau + \sum_{\tau=t+1}^{\infty}\theta_\ell\gamma^*_\tau\right]\right\}\right\}. \tag{18}$$

Step 4. Guess that, for some $\overline{T} \in \mathcal{T}_1$, $\gamma^*_t = 0$ for all $t > \overline{T}$. Moreover, guess that

$$\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma^*_\tau + \sum_{\tau=t+1}^{\infty}\theta_\ell\gamma^*_\tau\right] = \delta^{t}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t+1}\delta^{t+1-\tau}\theta_h\gamma^*_\tau + \sum_{\tau=t+2}^{\infty}\theta_\ell\gamma^*_\tau\right] \tag{19}$$

for all $t \in \{1, \ldots, \overline{T} - 1\}$. Solve equations (19) for γ^*_t at successive values of t, starting with $t = 1$, to find

$$\gamma^*_t = \frac{(1-\delta)\theta_h}{\theta_h - \theta_\ell}\left(\frac{\theta_h - \delta\theta_\ell}{\theta_h - \theta_\ell}\right)^{t-2} = \frac{(1-\delta)\theta_h}{\Delta\theta}\rho^{t-2} \quad \text{for all } t \in \{2, \ldots, \overline{T}\}, \tag{20}$$

where the last equality holds by definitions (2) and (4).

Step 5. Use equation (18), the guess in equations (19), and the guess that $\gamma_t^* = 0$ for all $t > \overline{T}$, to obtain

$$\beta^* = \mu\theta_h + \mu\theta_\ell \left(\sum_{t=2}^{\overline{T}} \gamma_t^* \right). \tag{21}$$

Step 6. By using equations (17) and (20), definition (3), and the guess that $\gamma_t^* = 0$ for all $t > \overline{T}$, constraint (15) becomes

$$\alpha^* = \max_{t\in\{1,\ldots,\overline{T}\}} \left\{ \delta^{t-1}(\theta_\ell - \mu\theta_h) + \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta} \left(\delta^{t-2}\frac{1-r^{t-1}}{1-r} + \frac{\rho^{t-1}-\rho^{\overline{T}-1}}{1-\rho} \right) \right\}, \tag{22}$$

assuming that the right-hand side of equation (22) is non-negative (which we will show to be the case in Step 8 of this section). The maximand on the right-hand side of equation (22) simplifies as follows:

$$\begin{aligned}
&\delta^{t-1}(\theta_\ell - \mu\theta_h) + \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta} \left(\delta^{t-2}\frac{1-r^{t-1}}{1-r} + \frac{\rho^{t-1}-\rho^{\overline{T}-1}}{1-\rho} \right) \\
&= \delta^{t-1}(\theta_\ell - \mu\theta_h) - \delta^{t-1}(1-\mu)\theta_\ell\big(1-r^{t-1}\big) - (1-\mu)\theta_h\Big(\rho^{t-1} - \rho^{\overline{T}-1}\Big) \\
&= (1-\mu)\theta_h\rho^{\overline{T}-1} - \delta^{t-1}\Delta\theta\big(\mu + (1-\mu)r^{t-1}\big),
\end{aligned} \tag{23}$$

where: the first equality holds by implications (5) and (6); the second equality holds by definition (4). Thus, by equation (23), equation (22) is equivalent to

$$\alpha^* = \max_{t\in\{1,\ldots,\overline{T}\}} \Big\{ (1-\mu)\theta_h\rho^{\overline{T}-1} - \delta^{t-1}\Delta\theta\big(\mu + (1-\mu)r^{t-1}\big) \Big\}. \tag{24}$$

Let

$$t^* := \operatorname*{arg\,max}_{t\in\{1,\ldots,\overline{T}\}} \Big\{ (1-\mu)\theta_h\rho^{\overline{T}-1} - \delta^{t-1}\Delta\theta\big(\mu + (1-\mu)r^{t-1}\big) \Big\}. \tag{25}$$

Moreover, let $\underline{t}$ be defined as follows: $\underline{t} \in \mathbb{R}$ such that $\delta^{\underline{t}-1}\Delta\theta\big(\mu + (1-\mu)r^{\underline{t}-1}\big) = \delta^{\underline{t}}\Delta\theta\big(\mu + (1-\mu)r^{\underline{t}}\big)$. Since the maximand on the right-hand side of equation (25) is concave in t and $\overline{T}$ is yet to be determined (and so can be chosen arbitrarily large),

$$t^* = \inf\left\{ t \in \mathcal{T}_0 : t \geq \underline{t} \right\}. \tag{26}$$

Note that

$$\begin{aligned}
&\delta^{\underline{t}-1}\Delta\theta\big(\mu + (1-\mu)r^{\underline{t}-1}\big) = \delta^{\underline{t}}\Delta\theta\big(\mu + (1-\mu)r^{\underline{t}}\big) \\
&\qquad\qquad \iff \mu(1-\delta) = -r^{\underline{t}-1}(1-\mu)(1-\delta r)
\end{aligned}$$

$$\begin{aligned}
&\iff \mu(1-\delta) = r^{\underline{t}-1}(1-\mu)\frac{(1-\delta)\theta_\ell}{\Delta\theta} \\
&\iff r^{\underline{t}-1} = \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} && (27)\\
&\iff \underline{t} = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r}, && (28)
\end{aligned}$$

where the second equivalence holds by implication (6). Since $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, $\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} > 1$, and so $\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right) > 0$. Moreover, $r > 1$, and so $\ln r > 0$. Thus, $\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r} > 1$, which implies, together with equations (26) and (28), that $t^* \geq 2$.

Step 7. The choice of $\overline{T}$ is part of the choice of the Lagrange multipliers $(\gamma_t^*)_{t=2}^\infty$. Thus, $\overline{T}$ must be chosen to minimize the objective function of program (P1b).

Let $U_P^{\text{P1b}}(\pi)$ denote the optimal value of program (P1b). Note that

$$\begin{aligned}
U_P^{\text{P1b}}(\pi) &= \alpha^* + \beta^* - \sum_{t=2}^{\infty} \theta_\ell \gamma_t^* \\
&= \alpha^* + \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{\overline{T}} \gamma_t^*\right) - \sum_{t=2}^{\overline{T}} \theta_\ell\gamma_t^* \\
&= \alpha^* + \mu\theta_h - (1-\mu)\theta_\ell\left(\sum_{t=2}^{\overline{T}} \frac{(1-\delta)\theta_h}{\Delta\theta}\rho^{t-2}\right) \\
&= \alpha^* + \mu\theta_h - \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta}\frac{1-\rho^{\overline{T}-1}}{1-\rho} && (29)\\
&= \alpha^* + \mu\theta_h + (1-\mu)\theta_h\left(1-\rho^{\overline{T}-1}\right) \\
&= (1-\mu)\theta_h\rho^{\overline{T}-1} - \delta^{t^*-1}\Delta\theta\big(\mu + (1-\mu)r^{t^*-1}\big) + \theta_h - (1-\mu)\theta_h\rho^{\overline{T}-1} \\
&= \theta_h - \delta^{t^*-1}\Delta\theta\left(\mu + (1-\mu)r^{t^*-1}\right),
\end{aligned}$$

where: the second equality holds by equation (21) and the guess that $\gamma_t^* = 0$ for all $t > \overline{T}$; the third equality holds by equation (20); the fifth equality holds by implication (6); the sixth equality follows from equation (24) and the definition of t^*. Since $U_P^{\text{P1b}}(\pi)$ does not depend on $\overline{T}$, we take

$$\overline{T} = t^*. \tag{30}$$

Step 8. We show that $\alpha^* \geq 0$. From the two previous steps, we have

$$\alpha^* = (1-\mu)\theta_h\rho^{\overline{T}-1} - \delta^{t^*-1}\Delta\theta\big(\mu + (1-\mu)r^{t^*-1}\big)$$

$$
\begin{aligned}
&= (1-\mu)\theta_h \rho^{\overline{T}-1} - \delta^{\overline{T}-1}\Delta\theta\big(\mu + (1-\mu) r^{\overline{T}-1}\big) \\
&= \delta^{\overline{T}-1}\Big[(1-\mu)\theta_\ell r^{\overline{T}-1} - \mu\Delta\theta\Big] \\
&\geq \delta^{\underline{t}-1}\Big[(1-\mu)\theta_\ell r^{\underline{t}-1} - \mu\Delta\theta\Big] \\
&= 0,
\end{aligned}
$$

where: the first equality holds by equation (24) and definition (25); the second equality holds by equation (30); the third equality holds by definition (4); the inequality holds by equation (24), definition (25), and equation (30); the last equality holds by equation (27).

Step 9. Summing up, a candidate solution $\boldsymbol{\xi}^* := \big(\alpha^*, \beta^*, \zeta^*, \big(\lambda^*_{\ell,t}, \lambda^*_{h,t+1}, \gamma^*_{t+1}\big)_{t=1}^{\infty}\big) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$ to program (P1b) is as follows. For

$$
\overline{T} = \inf\left\{t \in \mathcal{T}_1 : t \geq \underline{t}\right\}, \qquad \text{where} \qquad \underline{t} = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r},
$$

we have:

$$
\begin{aligned}
\alpha^* &= \delta^{\overline{T}-1}\Big[(1-\mu)\theta_\ell r^{\overline{T}-1} - \mu\Delta\theta\Big], \\
\beta^* &= \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{\overline{T}} \gamma^*_t\right) \\
\zeta^* &= \mu \qquad (31) \\
\lambda^*_{\ell,t} &= \begin{cases} 1 & \text{if } t = 1 \\ (1-\mu)\gamma^*_t & \text{otherwise} \end{cases}, \qquad (32) \\
\lambda^*_{h,t} &= \mu\gamma^*_t \quad \text{for all } t \in \mathcal{T}_0, \qquad (33) \\
\gamma^*_t &= \begin{cases} \frac{(1-\delta)\theta_h}{\Delta\theta}\rho^{t-2} & \text{if } t \in \left\{2, \ldots, \overline{T}\right\} \\ 0 & \text{if } t > \overline{T} \end{cases}. \qquad (34)
\end{aligned}
$$

All elements of $\boldsymbol{\xi}^*$ are clearly non-negative. Moreover, we have

$$
U_P^{\mathrm{P1b}}(\pi) = \theta_h - \delta^{\overline{T}-1}\Delta\theta\Big(\mu + (1-\mu) r^{\overline{T}-1}\Big), \qquad (35)
$$

where the equality holds by equations (29) and (30).

B.4 A Solution to the Primal Program (P1a)

We recover a solution $(\boldsymbol{x}^*, \boldsymbol{p}^*) := \left(x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t}\right)_{t=1}^{\infty} \in \mathbb{R}^{4\infty}$ to program (P1a).

Step 1. Since $\lambda^*_{\ell,t} > 0$ for all $t \in \{1, \ldots, \overline{T}\}$ (see equation (32)), constraint (IRℓ) is binding for all such t. Thus,

$$p^*_{\ell,t} = \theta_\ell x^*_{\ell,t} \quad \text{for all } t \in \{1, \ldots, \overline{T}\}. \tag{36}$$

Since $\lambda^*_{h,t} > 0$ for all $t \in \{2, \ldots, \overline{T}\}$ (see equation (33)), constraint (IRh) is binding for all such t. Thus,

$$p^*_{h,t} = \theta_h x^*_{h,t} \quad \text{for all } t \in \{2, \ldots, \overline{T}\}. \tag{37}$$

Moreover, guess that the solution to program (P1a) is such that

$$\sum_{t=1}^{\overline{T}-1} x^*_{h,t} = 1. \tag{38}$$

and

$$x^*_{h,t} = p^*_{h,t} = 0 \quad \text{for all } t \geq \overline{T}. \tag{39}$$

Step 2. From constraint (Oℓ) binding for $t = \overline{T}$ (as $\gamma^*_{\overline{T}} > 0$, see equation (34)) and the conjectures in equations (38) and (39), we have that $\sum_{t=\overline{T}}^{\infty} \delta^{t-\overline{T}} p_{\ell,t} = \theta_\ell$, from which we guess that

$$p^*_{\ell,\overline{T}} = \theta_\ell. \tag{40}$$

From equations (36) and (40), we have that

$$x^*_{\ell,\overline{T}} = 1, \tag{41}$$

and so $x^*_{\ell,t} = p^*_{\ell,t} = 0$ for all $t \in \{1, \ldots, \overline{T} - 1\}$. Finally, guess that

$$p^*_{l,t} = 0 \quad \text{for all } t > \overline{T}. \tag{42}$$

Step 3. Since $\zeta^* > 0$ (see equation (31)), constraint (ICh) is binding. This, together with equations (37), (40), and (41), implies that $\theta_h x^*_{h,1} - p^*_{h,1} = \delta^{\overline{T}-1}\Delta\theta$ or, equivalently,

$$p^*_{h,1} = \theta_h x^*_{h,1} - \delta^{\overline{T}-1}\Delta\theta.$$

Step 4. To find $x^*_{h,t}$ for all $t \in \{2, \ldots, \overline{T} - 1\}$, we use that constraint (Oℓ) is binding for all such t (as $\gamma_t^* > 0$ for all such t, see equation (34)). From constraint (Oℓ) binding for $t = \overline{T}-1$, we obtain $\mu\big[\theta_\ell\big(1-x^*_{h,\overline{T}-1}\big)+\theta_h x^*_{h,\overline{T}-1}\big]+(1-\mu)\delta\theta_\ell = \theta_\ell$ or, equivalently

$$x^*_{h,\overline{T}-1} = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}.$$

Similarly, solving for constraint (Oℓ) binding backwards for all $t \in \{2,\dots,\overline{T}-2\}$, starting with $t = \overline{T}-2$, we obtain

$$x^*_{h,t} = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\left(\frac{\theta_h-\delta\theta_\ell}{\Delta\theta}\right)^{\overline{T}-t-1} = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{\overline{T}-t-1},$$

where the second equality holds by definition (4). Finally, using the guess in equation (38), we have

$$\begin{aligned} x^*_{h,1} &= 1-\sum_{t=2}^{\overline{T}-1} x^*_{h,t} \\ &= 1-\sum_{t=2}^{\overline{T}-1}\frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{\overline{T}-t-1} \\ &= 1-\frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\frac{1-\rho^{\overline{T}-2}}{1-\rho} \\ &= 1+\frac{1-\mu}{\mu}\Big(1-\rho^{\overline{T}-2}\Big) \\ &= \frac{1}{\mu}\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big), \end{aligned} \tag{43}$$

where the fourth equality holds by implication (6).

Step 5. It remains to show that $x^*_{h,1} \geq 0$ or, equivalently, that

$$(1-\mu)\rho^{\overline{T}-2} \leq 1. \tag{44}$$

To begin, note that $(1-\mu)\rho^{\overline{T}-2} \leq (1-\mu)\rho^{\underline{t}-1} = (1-\mu)\delta^{\underline{t}-1}r^{\underline{t}-1} = \delta^{\underline{t}-1}\frac{\mu\Delta\theta}{\theta_\ell}$, where: the inequality holds because $\rho > 1$ and the fact that $\overline{T} = \inf\{t \in \mathcal{T}_1 : t \geq \underline{t}\}$ $\implies \underline{t}-1 \geq \overline{T}-2$; the first equality holds by definition (3); the second equality holds by equation (27). Thus, a sufficient condition for inequality (44) to hold is that

$$\delta^{\underline{t}-1}\frac{\mu\Delta\theta}{\theta_\ell} \leq 1. \tag{45}$$

Since $\underline{t} = 1$ for $\mu = \frac{\theta_\ell}{\theta_h}$ (see equation (28)), the left-hand side of inequality (45) evaluated at $\mu = \frac{\theta_\ell}{\theta_h}$ equals $\frac{\Delta\theta}{\theta_h}$, which is less than 1; thus, to show that inequality

(44) holds, it suffices to show that $\mu\delta^{\underline{t}-1}$ is non-increasing in μ. Note that

$$\begin{aligned}\frac{\partial}{\partial\mu}\left[\mu\delta^{\underline{t}-1}\right] &= \delta^{\underline{t}-1} + \mu\delta^{\underline{t}-1}\ln\delta\frac{\partial}{\partial\mu}\left[\underline{t}-1\right] \\ &= \delta^{\underline{t}-1} + \mu\delta^{\underline{t}-1}\ln\delta\frac{(1-\mu)\theta_\ell}{\mu\Delta\theta}\frac{\Delta\theta(1-\mu)\theta_\ell+\mu\theta_\ell\Delta\theta}{[(1-\mu)\theta_\ell)]^2}\frac{1}{\ln r} \\ &= \delta^{\underline{t}-1}\left(1+\frac{\ln\delta}{(1-\mu)\ln r}\right),\end{aligned}$$

where the second equality holds by equation (28). Thus, to show that $\mu\delta^{\underline{t}-1}$ is non-increasing in μ, it suffices to show that

$$1+\frac{\ln\delta}{(1-\mu)\ln r} \leq 0. \tag{46}$$

Since the left-hand side of inequality (46) is decreasing in μ (as $\delta \in (0,1)$, and so $\ln\delta < 0$), it suffices to show that inequality (46) is satisfied for $\mu = \frac{\theta_\ell}{\theta_h}$, i.e., that $1+\frac{\ln\delta}{\frac{\Delta\theta}{\theta_h}\ln r} \leq 0$ or, equivalently,

$$\frac{\Delta\theta}{\theta_h}\ln r + \ln\delta \leq 0. \tag{47}$$

Since inequality (47) is satisfied with equality if $\delta = 1$ (as $r = 1$ if $\delta = 1$, see definition (3)), it suffices to show that its left-hand side is increasing in δ (so that, for $\delta < 1$, the left-hand side is negative). To see this, note that

$$\frac{\partial}{\partial\delta}\left[\frac{\Delta\theta}{\theta_h}\ln r + \ln\delta\right] = \frac{\Delta\theta}{\theta_h}\frac{1}{r}\frac{\partial r}{\partial\delta} + \frac{1}{\delta} = -\frac{\Delta\theta}{\theta_h}\frac{\delta\Delta\theta}{\theta_h-\delta\theta_\ell}\frac{\theta_h}{\delta^2\Delta\theta} + \frac{1}{\delta} = \frac{(1-\delta)\theta_\ell}{\delta(\theta_h-\delta\theta_\ell)} > 0,$$

where the second equality holds by using definition (3).

Remark 1. The previous argument shows that: (i) $x^*_{h,1} > 0$ (not only that $x^*_{h,1} \geq 0$); and (ii) $1+\frac{\ln\delta}{(1-\mu)\ln r} < 0$ for all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, and so $\mu\delta^{\underline{t}-1}$ is decreasing in μ.

Step 6. Summing up, a candidate solution $(\boldsymbol{x}^*, \boldsymbol{p}^*) := \left(x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t}\right)_{t=1}^{\infty} \in \mathbb{R}^{4\infty}$ to program (P1a) is as follows. For

$$\overline{T} = \inf\left\{t \in \mathcal{T}_1 : t \geq \underline{t}\right\}, \tag{48}$$

where

$$\underline{t} = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r}, \tag{49}$$

we have:

$$x^*_{\ell,t} = \begin{cases} 1 & \text{if } t = \overline{T} \\ 0 & \text{otherwise} \end{cases}, \tag{50}$$

$$x^*_{h,t} = \begin{cases} \frac{1}{\mu}\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big) & \text{if } t = 1 \\ \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{\overline{T}-t-1} & \text{if } t \in \Big\{2,\ldots,\overline{T}-1\Big\} \\ 0 & \text{otherwise} \end{cases}, \tag{51}$$

$$p^*_{\ell,t} = \theta_\ell x^*_{\ell,t} \quad \text{for all } t \in \mathcal{T}_0, \tag{52}$$

$$p^*_{h,t} = \begin{cases} \theta_h x^*_{h,1} - \delta^{\overline{T}-1}\Delta\theta & \text{if } t = 1 \\ \theta_h x^*_{h,t} & \text{otherwise} \end{cases}. \tag{53}$$

All elements of $\boldsymbol{x}^*$ are clearly non-negative.

Let $U_P^{\mathrm{P1a}}(\pi)$ be the optimal value of the primal program (P1a). Thus, we have

$$\begin{aligned} U_P^{\mathrm{P1a}}(\pi) &= \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p^*_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p^*_{\ell,t}\right) \\ &= \mu\left(\sum_{t=1}^{\overline{T}-1}\delta^{t-1}p^*_{h,t}\right) + \delta^{\overline{T}-1}(1-\mu)p^*_{\ell,\overline{T}} \\ &= \mu\left[\frac{\theta_h}{\mu}\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big) - \delta^{\overline{T}-1}\Delta\theta + \sum_{t=2}^{\overline{T}-1}\delta^{t-1}\frac{(1-\delta)(1-\mu)\theta_\ell\theta_h}{\mu\Delta\theta}\rho^{\overline{T}-t-1}\right] \\ &+ \delta^{\overline{T}-1}(1-\mu)\theta_\ell \\ &= \theta_h\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big) - \delta^{\overline{T}-1}\mu\Delta\theta \\ &+ \delta^{\overline{T}-2}\frac{(1-\delta)(1-\mu)\theta_\ell\theta_h}{\Delta\theta}\frac{1-r^{\overline{T}-2}}{1-r} + \delta^{\overline{T}-1}(1-\mu)\theta_\ell \\ &= \theta_h\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big) - \delta^{\overline{T}-1}\mu\Delta\theta \\ &- \delta^{\overline{T}-1}(1-\mu)\theta_\ell\Big(1-r^{\overline{T}-2}\Big) + \delta^{\overline{T}-1}(1-\mu)\theta_\ell \\ &= \theta_h\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big) - \delta^{\overline{T}-1}\mu\Delta\theta + \delta^{\overline{T}-1}(1-\mu)\theta_\ell r^{\overline{T}-2}, \end{aligned} \tag{54}$$

where: the second and third equalities hold by equations (50)–(53); the fourth equality holds because, by definition (4), $\delta^{t-1}\rho^{\overline{T}-t-1} = \delta^{\overline{T}-2}r^{\overline{T}-t-1}$; the second-to-last equality holds by using implication (5).

Step 7. We show that $U_P^{\mathrm{P1a}}(\pi) = U_P^{\mathrm{P1b}}(\pi)$, so that, by weak duality, the candidate solution to program (P1a) is indeed a solution to the program.

To obtain the desired result, note that

$$\begin{aligned}
U_P^{\mathrm{P1b}}(\pi) - U_P^{\mathrm{P1a}}(\pi) &= \theta_h - \delta^{\overline{T}-1}\Delta\theta\Big(\mu + (1-\mu)r^{\overline{T}-1}\Big) \\
&\quad - \theta_h\Big(1-(1-\mu)\rho^{\overline{T}-2}\Big) + \delta^{\overline{T}-1}\mu\Delta\theta - \delta^{\overline{T}-1}(1-\mu)\theta_\ell r^{\overline{T}-2} \\
&= -(1-\mu)\Delta\theta\rho^{\overline{T}-1} + (1-\mu)\theta_h\rho^{\overline{T}-2} - (1-\mu)\frac{\rho^{\overline{T}-1}}{r}\theta_\ell \\
&= (1-\mu)\theta_h\rho^{\overline{T}-2} - (1-\mu)\rho^{\overline{T}-1}\left(\Delta\theta + \frac{\theta_\ell}{r}\right) \\
&= (1-\mu)\theta_h\rho^{\overline{T}-2} - (1-\mu)\rho^{\overline{T}-1}\left(\frac{\theta_h\Delta\theta}{\theta_h - \delta\theta_\ell}\right) \\
&\propto 1 - \rho\frac{\Delta\theta}{\theta_h - \delta\theta_\ell} \\
&= 1 - \rho\frac{1}{\rho} \\
&= 0,
\end{aligned}$$

where: the first equality holds by equations (35) and (54); the second equality holds by using definition (4); the fourth equality holds by using definition (3); the second-to-last equality holds by using definition (4).

B.5 A Solution to the Primal Program (P1)

We show that the solution $(\boldsymbol{x}^*, \boldsymbol{p}^*) \in \mathbb{R}^{4\infty}$ to program (P1a) described in Step 6 of Appendix B.4 satisfies constraint (ICℓ) in the primal program (P1), so that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is also a solution to program (P1).

By equations (50) and (52), the left-hand side of constraint (ICℓ) evaluated at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ equals 0. By equations (51) and (53), and since $\theta_h > \theta_\ell$, the right-hand side of constraint (ICℓ) evaluated at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is smaller than $\theta_\ell x_{h,1}^* - p_{h,1}^*$. Thus, to show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICℓ), it suffices to show that $\theta_\ell x_{h,1}^* - p_{h,1}^* \leq 0$. To begin, note that

$$\begin{aligned}
\theta_\ell x_{h,1}^* - p_{h,1}^* \leq 0 &\iff \theta_\ell x_{h,1}^* - \theta_h x_{h,1}^* + \delta^{\overline{T}-1}\Delta\theta \leq 0 \\
&\iff \Delta\theta\Big(\delta^{\overline{T}-1} - x_{h,1}^*\Big) \leq 0 \\
&\iff \delta^{\overline{T}-1} \leq x_{h,1}^* \\
&\iff \mu\delta^{\overline{T}-1} \leq 1 - (1-\mu)\rho^{\overline{T}-2},
\end{aligned} \tag{55}$$

where: the first equivalence holds by equation (53); the third equivalence holds because $\Delta\theta > 0$; the fourth equivalence holds by equation (51). Since $\overline{T} = \inf\{t \in \mathcal{T}_1 : t \geq \underline{t}\} \implies \overline{T} - 2 \leq \underline{t} - 1 \leq \overline{T} - 1$, $\delta < 1$, and $\rho > 1$, we have that

$$\mu\delta^{\overline{T}-1} \leq \mu\delta^{\underline{t}-1} \qquad \text{and} \qquad 1-(1-\mu)\rho^{\underline{t}-1} \leq 1-(1-\mu)\rho^{\overline{T}-2}. \tag{56}$$

From the equivalences in (55) and the inequalities in (56), we have that $\mu\delta^{\underline{t}-1} \leq 1-(1-\mu)\rho^{\underline{t}-1} \implies \theta_\ell x^*_{h,1} - p^*_{h,1} \leq 0$. Thus, to show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICℓ') for $\tau = 1$, it is enough to show that $\mu\delta^{\underline{t}-1} \leq 1-(1-\mu)\rho^{\underline{t}-1}$. Now note that

$$\begin{aligned}
\mu\delta^{\underline{t}-1} \leq 1-(1-\mu)\rho^{\underline{t}-1} &\iff \mu\delta^{\underline{t}-1} \leq 1-(1-\mu)\delta^{\underline{t}-1}r^{\underline{t}-1} \\
&\iff \mu\delta^{\underline{t}-1} \leq 1-(1-\mu)\delta^{\underline{t}-1}\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} \\
&\iff \mu\delta^{\underline{t}-1}\left(1+\frac{\mu\Delta\theta}{\theta_\ell}\right) \leq 1 \\
&\impliedby \mu\delta^{\underline{t}-1}\left(1+\frac{\Delta\theta}{\theta_\ell}\right) \leq 1,
\end{aligned} \tag{57}$$

where: the first equivalence holds by definition (4); the second equivalence holds by equation (27). Since $\mu\delta^{\underline{t}-1}$ is non-increasing in μ (see Step 5 in Appendix B.4), it suffices to show that inequality (57) is satisfied for $\mu = \frac{\theta_\ell}{\theta_h}$. Since $\underline{t} = 1$ if $\mu = \frac{\theta_\ell}{\theta_h}$, and $1 + \frac{\Delta\theta}{\theta_\ell} = \frac{\theta_h}{\theta_\ell}$, the desired result follows.

Remark 2. Since $\mu\delta^{\underline{t}-1}$ is decreasing (see Remark 1), the previous argument shows that $x^*_{h,1} > \delta^{\overline{T}-1}$ (or, equivalently, $0 > \theta_\ell x^*_{h,1} - p^*_{h,1}$) for all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$. That is, constraint (ICℓ) is not binding at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$.

B.6 Completing the Proof of Theorem 3

At the solution $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ to the primal program (P1) described in Step 6 of Appendix B.4, properties 1, 2, and 3 in Theorem 3 hold true. Moreover, from equations (48) and (49), we have that $\overline{T}$ is finite for any $\delta \in (0,1)$ and, in addition,

$$\lim_{\delta\to 1}\overline{T} \geq \lim_{\delta\to 1}\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r} = \lim_{\delta\to 1}\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln\left(\frac{\theta_h-\delta\theta_\ell}{\delta\Delta\theta}\right)} = \infty,$$

where: the inequality holds by equations (48) and (49); the first equality holds by definitions (2) and (3). Thus, property 4 in Theorem 3 holds true at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$. ■

Supplement to

Supplement to "A Mediator Approach to Mechanism Design with Limited Commitment"

Niccolò Lomys Takuro Yamashita

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C Example 2 and Example 3

C.1 Example 2

Let $\mathcal{T} = \{0, 1, 2, 3\}$. The agent's type θ is uniformly distributed over $\Theta = \{-1, 1\}$. The allocation sets are $A_1 = A_3 = \{\varnothing, -1, 1\}$ and $A_2 = \{\varnothing, e, n\}$, where $a_2 = e$ is interpreted as "extract" and $a_2 = n$ is interpreted as "not extract". Non-participation in any period t is an irreversible option for the agent (i.e., it implies non-participation in all future periods). The principal's payoff is

$$U_P(\theta, a) = \begin{cases} 0 & \text{if } a_1 = \varnothing \text{ or } a_2 = \varnothing \\ \mathbb{1}_{\{a_2 = e\}} & \text{if } a_1 \neq \varnothing, a_2 \neq \varnothing, \text{ and } a_3 = \varnothing \\ \mathbb{1}_{\{a_2 = e\}} - \theta a_3 \lambda & \text{otherwise} \end{cases}$$

and the agent's payoff is

$$U_A(\theta, a) = \begin{cases} 0 & \text{if } a_1 = \varnothing \\ \theta a_1 \lambda - K & \text{if } a_1 \neq \varnothing \text{ and } a_2 = \varnothing \\ -1 - K & \text{if } a_1 \neq \varnothing, a_2 = e, \text{ and } a_3 = \varnothing \\ \theta a_1 \lambda - K & \text{if } a_1 \neq \varnothing, a_2 = n, \text{ and } a_3 = \varnothing \\ -1 & \text{if } a_1 \neq \varnothing, a_2 = e, \text{ and } a_3 \neq \varnothing \\ \theta a_1 \lambda + \theta a_3 & \text{if } a_1 \neq \varnothing, a_2 = n, \text{ and } a_3 \neq \varnothing \end{cases},$$

where $\lambda \in (1, 2)$ and $K > 0$ is high enough for participation in periods 2 and 3 to be optimal for the agent.

Deletion of Stored Information. If the principal could commit to the allocation sequence $(a_1, a_2, a_3) = (\theta, n, -\theta)$, the agent would be truthful and the principal's payoff would be $\lambda > 0$. However, without commitment and without the possibility of storing information, the principal's ex ante expected payoff in any wPBE of the resulting game is 0. To establish this point, it suffices to show that, in any wPBE, the agent never participate to any contract in period $t = 1$. By contradiction, suppose the agent participates in period $t = 1$. Then, in period $t = 2$, the principal offers a constant contract allocating $a_2 = e$ (and the agent participates), because it increases the principal's payoff without affecting the agent's incentive constraint in period $t = 3$; thus, the agent's flow payoff in period $t = 2$ is -1. In period $t = 3$, the principal's expected flow payoff is non-negative because he can always offer any constant contract guaranteeing 0 expected flow payoff; thus, the agent's flow payoff in period $t = 3$ is non-positive. Therefore, the agent's expected payoff from participating in period $t = 1$ is -1. This is a contradiction because the agent can always guarantee herself a payoff of 0 by not participating in period $t = 1$.

In contrast, the MCS game has a wPBE in which the principal's ex ante expected payoff is λ. As in Example 1, the mediator again stores the agent's type report in period $t = 0$ and fully disclose it in period $t = 3$ to the principal, but his role is different. In particular, the mediator's role is now to prevent the principal deviation to the constant contract allocating $a_2 = e$ in period $t = 2$; this is achieved by the threat of not disclosing the agent's type report in period $t = 3$ if the principal deviates in period $t = 2$. The next claim and its proof formalize these ideas.

Claim 1. The MCS game has a wPBE in which the principal's ex ante expected payoff is λ.

Proof. We begin by describing a candidate wPBE of the MCS game. On-path events are the following:

- *Period* $t = 0$*.* The agent truthfully report her type θ to the mediator.
- *Period* $t = 1$*.* The mediator sends signal $s_{P,1} = \theta$ with probability α and signal $s_{P,1} = -\theta$ with probability $1 - \alpha$ to the principal, where $(1 + \lambda)/2\lambda < \alpha < (2\lambda - 1)/2\lambda$. The principal offers a constant contract allocating $a_1 = s_{P,1}$. The agent participates to the contract.
- *Period* $t = 2$*.* The principal offers a constant contract allocating $a_2 = n$. The agent participates to the contract.
- *Period* $t = 3$*.* The mediator fully discloses the agent's type report θ (made

in period $t = 0$) to the principal. The principal offers a constant contract allocating $a_3 = -\theta$. The agent participates to the contract.

Off-path events are the following:

- If the principal observes any off-path event by period $t = 2$, then he offers a constant contract allocating $a_2 = e$.
- If the principal deviates in period $t = 1$ or $t = 2$ (which is observed by the mediator by assumption), then the mediator does not reveal the agent's type report θ (made in period $t = 0$) to the principal in period $t = 3$.
- If the principal observes any off-path event by period $t = 3$, then he offers any best responding contract in period $t = 3$.

The principal's ex ante expected payoff in the candidate wPBE is λ. Thus, it remains to show that the candidate wPBE is indeed a wPBE of the MCS game. We begin with the agent's incentives. The agent's participation in periods $t = 2$ and $t = 3$ is optimal. In period $t = 0$, the agent's expected continuation payoff from truthfully reporting her type to the mediator is $\lambda(\alpha - (1 - \alpha)) - 1$, whereas her expected continuation payoff from lying to the mediator is $\lambda(-\alpha + (1 - \alpha)) + 1$; since the former is greater than the latter for $\alpha > (1 + \lambda)/2\lambda$, the agent finds it optimal to be truthful. In period $t = 1$, the agent's expected continuation payoff from participating to the mechanism is $\lambda(\alpha - (1 - \alpha)) - 1$, whereas the agent's expected continuation payoff from non-participation is 0; since the former is greater than the latter for $\alpha > (1 + \lambda)/2\lambda$, the agent finds it optimal to participate.

Next, we consider the principal's incentive. First, in period $t = 3$, the principal has no incentive to deviate since the allocation $a_3 = -\theta$ is the best one from the period-3 principal's viewpoint. The principal's continuation payoff from following the candidate equilibrium strategy in period $t = 1$ and $t = 2$ is λ. If the principal deviates in period $t = 1$ or $t = 2$, then the mediator does not reveal the agent's type report to the principal in period $t = 3$. Accordingly, at the beginning of period $t = 3$, the principal has only the partial information $s_{P,1}$ about the agent's type θ. Thus, the principal in period $t = 3$ can do no better than offering a constant contract allocating $a_3 = s_{P,1}$. Therefore, the principal's expected continuation payoff from deviating in period $t = 1$ or $t = 2$ is at most $1 + \lambda(2\alpha - 1)$. Since $\lambda > 1 + \lambda(2\alpha - 1)$ for $\alpha < (2\lambda - 1)/2\lambda$, the principal has no incentive to deviate from the candidate equilibrium strategy in periods $t = 1$ and $t = 2$. ■

C.2 Example 3

Let $\mathcal{T} = \{0, 1, 2\}$. There are two agents, denoted by $i = 1, 2$. In each period $t \in \mathcal{T}_0$, the principal allocates $a_t = (a_{1,t}, a_{2,t}) \in A_t = \{\varnothing, -1, 1\}^2$, where $a_{i,t} = \varnothing$ if and only if agent i does not participate in period t. Each agent i has a type $\theta_i \in \Theta = \{-1, 1\}$, equally likely and independent across players. The principal's payoff is

$$U_P((\theta_1, \theta_2), a) = \begin{cases} 0 & \text{if } a_{1,1} = \varnothing \text{ or } a_{2,1} = \varnothing \\ \sum_i a_{i,1}\theta_i & \text{if } a_{1,1} \neq \varnothing, a_{2,1} \neq \varnothing, \text{ and } a_{1,2} = \varnothing \text{ or } a_{2,2} = \varnothing \\ \sum_i (a_{i,1}\theta_i + a_{i,2}\varepsilon) & \text{otherwise} \end{cases},$$

where $\varepsilon \in (0, 1)$. Agent i's payoff is

$$U_i(\theta_i, a) = \begin{cases} 0 & \text{if } a_{i,1} = \varnothing \\ 1 & \text{if } a_{i,1} \neq \varnothing, \text{ and } a_{1,2} = \varnothing \text{ or } a_{2,2} = \varnothing \\ 2 - a_{i,1}\theta_i & \text{otherwise} \end{cases}.$$

Each agent finds it optimal to participate in the contract in period $t = 1$. The best continuation event from the period-2 principal's viewpoint is such that both agents participate in period $t = 2$ and the allocation is $(a_{1,2}, a_{2,2}) = (1, 1)$; the worst continuation event from the period-2 principal's viewpoint is such that (at least) one agent does not participate in period $t = 2$. If (at least) one agent does not participate in period $t = 2$, the principal and agent i have the same preferences regarding $a_{i,1}$ for $i = 1, 2$; in contrast, if both agents participate in period $t = 2$, the principal and agent i have opposite preferences regarding $a_{i,1}$ for $i = 1, 2$.

Equilibrium Multiplicity and Equilibrium Selection. The MCS game has a wPBE in which the principal's ex ante expected payoff is 2; such wPBE requires the worst continuation equilibrium selection from the period-2 principal's viewpoint. The next claim and its proof formalize these ideas.

Claim 2. The MCS game has a wPBE in which the principal's ex ante expected payoff is 2.

Proof. Regardless of what happens in period $t = 1$, a continuation equilibrium in period $t = 2$ has no agent participating in whatever contract is offered by the principal: if agent $-i$ does not participate, then agent i's participation is payoff irrelevant and so it is a best response for i not to participate. Given this, each agent's continuation payoff in period $t = 1$ (upon participation, which is optimal for each agent) is 1; thus, for $i = 1, 2$, the principal offers a direct contract with allocation rule $\alpha_{i,1}(a_{i,1} = m_{i,1} \,|\, m_{i,1}) = 1$, and agents are truthful. Thus, the principal's ex ante expected payoff in this wPBE is 2. ■

The next claim and its proof show that if the best continuation equilibrium from the period-2 principal viewpoint is selected, then the principal's ex ante expected payoff in a wPBE cannot exceed 2ε. Since $\varepsilon \in (0, 1)$, so that $2\varepsilon < 2$, the principal-optimal wPBE outcome necessarily requires suboptimal continuation equilibrium selection from the period-2 principal viewpoint.

Claim 3. If the best continuation equilibrium from the period-2 principal's viewpoint is selected, then the principal's ex ante expected wPBE payoff is at most 2ε.

Proof. Regardless of what happens in period $t = 1$, the best continuation equilibrium from the period-2 principal's viewpoint is as follows: for $i = 1, 2$, the principal offers a constant contract allocating $a_{i,2} = 1$ if agent i participates; both agents participate. Given this, each agent i's continuation payoff in period $t = 1$ (upon participation, which is optimal for each agent) is $2 - a_{i,1}\theta_i$; thus, for $i = 1, 2$, the principal finds it optimal to offer any constant contract in period $t = 1$. As a result, the principal's ex ante expected payoff in this wPBE is 2ε. ■

D Proof of Theorem 4

Step 1. Let $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ be the solution to program (P1) we characterize in Appendix B (see Step 6 of Appendix B.4). To begin, note that

$$\overline{T} = \inf\{t \in \mathcal{T}_1 : t \geq \underline{t}\} \Longrightarrow \underline{t} - 1 \leq \overline{T} - 1 \leq \underline{t}. \tag{58}$$

From implication (58) and since $\delta \in (0, 1)$, we have $\delta^{\underline{t}} \leq \delta^{\overline{T}-1} \leq \delta^{\underline{t}-1}$. Thus,

$$\lim_{\delta \to 1} \delta^{\overline{T}-1} = \lim_{\delta \to 1} \delta^{\underline{t}-1} = \lim_{\delta \to 1} \delta^{\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln\left(\frac{\theta_h - \delta\theta_\ell}{\delta\Delta\theta}\right)}} = \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}}, \tag{59}$$

where: the first equality holds by the sandwich theorem for the limits of functions; the second equality holds by equation (28) and definition (3).

From implication (58) and since $r > 1$, we have $r^{\underline{t}-1} \leq r^{\overline{T}-1} \leq r^{\underline{t}}$. Moreover, as $\delta \to 1$, $r \to 1$ (see definition (3)). Thus,

$$\lim_{\delta \to 1} r^{\overline{T}-1} = \lim_{\delta \to 1} r^{\underline{t}-1} = \frac{\mu \Delta\theta}{(1-\mu)\theta_\ell}, \tag{60}$$

where: the first equality holds by the sandwich theorem for the limits of functions; the second equality holds by equation (27).

Step 2. To establish part 1 of Theorem 4, we need to show that $\lim_{\delta \to 1} U_P^{\mathrm{P1}}(\pi) > \theta_\ell$. Note that

$$\begin{aligned}
\lim_{\delta \to 1} U_P^{\mathrm{P1}}(\pi) > \theta_\ell \iff & \lim_{\delta \to 1} U_P^{\mathrm{P1b}}(\pi) > \theta_\ell \\
\iff & \lim_{\delta \to 1} \left[\theta_h - \delta^{\overline{T}-1} \Delta\theta \Big(\mu + (1-\mu) r^{\overline{T}-1}\Big)\right] > \theta_\ell \\
\iff & \theta_h - \Delta\theta \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} \left(\mu + (1-\mu)\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right) > \theta_\ell \qquad (61) \\
\iff & \Delta\theta - \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} \frac{\mu\Delta\theta\theta_h}{\theta_\ell} > 0 \\
\iff & \frac{\theta_\ell}{\mu\theta_h} - \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} > 0, \qquad (62)
\end{aligned}$$

where: the first equivalence holds because, by the analysis in Appendix B, $U_P^{\mathrm{P1}}(\pi) = U_P^{\mathrm{P1b}}(\pi)$ holds true at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$; the second equivalence holds by equation (35); the third equivalence holds by equations (59) and (60). The left-hand side of inequality (62) evaluated at $\mu = \frac{\theta_\ell}{\theta_h}$ equals 0. Thus, to obtain the desired result, it is enough to show that the left-hand side of inequality (62) is increasing in μ. Note that

$$\frac{\partial}{\partial\mu}\left[\frac{\theta_\ell}{\mu\theta_h} - \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}}\right] = \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\theta_\ell}{\theta_h}} - 1. \tag{63}$$

The right-hand side of equation (63) evaluated at $\mu = \frac{\theta_\ell}{\theta_h}$ equals 0. Moreover,

$$\frac{\partial}{\partial\mu}\left[\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\theta_\ell}{\theta_h}} - 1\right] = \frac{\theta_\ell\Delta\theta}{[(1-\mu)\theta_\ell]^2} > 0.$$

Thus, the right-hand side of equation (63) is positive for all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$. The desired result follows.

Step 3. From the analysis in Appendix B (see step 6 of Appendix B.4), the following hold true at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$: the payoff of type θ_ℓ is 0; the expected payoff of type θ_h is $\delta^{\overline{T}-1}\Delta\theta$. Thus, the expected payoff of the buyer is $\mu\delta^{\overline{T}-1}\Delta\theta$ and the expected total surplus, denoted by $S(\pi)$, is

$$\begin{aligned} S(\pi) &:= U_P^{\text{P1}}(\pi) + \mu\delta^{\overline{T}-1}\Delta\theta \\ &= \theta_h - (1-\mu)\Delta\theta\delta^{\overline{T}-1}r^{\overline{T}-1}, \end{aligned} \tag{64}$$

where the equality holds by equation (35).

To establish part 2 of Theorem 4, we need to show that $\lim_{\delta\to 1} S(\pi) < \mu\theta_h + (1-\mu)\theta_\ell$ ($\mu\theta_h + (1-\mu)\theta_\ell$ is the expected total surplus at the first best). Note that

$$\begin{aligned} \lim_{\delta\to 1} S(\pi) < \mu\theta_h + (1-\mu)\theta_\ell &\iff \lim_{\delta\to 1}\left[\theta_h - (1-\mu)\Delta\theta\delta^{\overline{T}-1}r^{\overline{T}-1}\right] < \mu\theta_h + (1-\mu)\theta_\ell \\ &\iff \theta_h - (1-\mu)\Delta\theta\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{\frac{\theta_\ell}{\theta_h}} < \mu\theta_h + (1-\mu)\theta_\ell \\ &\iff (1-\mu)\Delta\theta < (1-\mu)\Delta\theta\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{\frac{\theta_\ell}{\theta_h}} \\ &\iff 1 < \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} \\ &\iff \theta_\ell < \mu\theta_h, \end{aligned}$$

where: the first equivalence holds by equation (64); the second equivalence holds by equations (59) and (60). Since $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, the desired result follows.

Step 4. For all $k \in \mathbb{N}$, let $\theta_\ell^k := \frac{\theta_h}{k}$. Then, $\lim_{k\to\infty} \theta_\ell^k = 0$ and $\Delta\theta^k := \theta_h - \theta_\ell^k = \frac{\theta_h(k-1)}{k}$. Moreover,

$$\begin{aligned} \lim_{k\to\infty} U_P^{\text{P1}}(\pi;\theta_\ell^k) &:= \lim_{k\to\infty} \theta_h\left(1 - \left(\frac{\mu\Delta\theta^k}{(1-\mu)\theta_\ell^k}\right)^{-\frac{\Delta\theta^k}{\theta_h}}\frac{\mu\Delta\theta^k}{\theta_\ell^k}\right) \\ &= \lim_{k\to\infty} \theta_h\left(1 - (1-\mu)\left(\frac{\mu(k-1)}{(1-\mu)}\right)^{\frac{1}{k}}\right) \\ &= \lim_{k\to\infty} \theta_h\left(1 - (1-\mu)\exp\left(\frac{1}{k}\ln\frac{\mu(k-1)}{(1-\mu)}\right)\right) \\ &= \theta_h(1-(1-\mu)\exp(0)) \\ &= \mu\theta_h. \end{aligned}$$

Since $\mu\theta_h$ is the commitment payoff, part 3 of Theorem 4 follows. ■

E Proof of Theorem 5

Let $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ be the solution to program (P1) we characterize in Appendix B (see Step 6 of Appendix B.4). By Theorem 2, if $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraints (ICℓ'), (ICh'), and (O), then $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a wPBE outcome of $\mathcal{G}$.

Step 1. We show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICℓ') for all $\tau \in \mathcal{T}_0$.

For all $\tau \in \mathcal{T}_0$, $p^*_{\ell,t} = \theta_\ell x^*_{\ell,t}$ (see equation (52)). Thus, the left-hand side of constraint (ICℓ') equals 0 for all $\tau \in \mathcal{T}_0$.

For all $\tau \in \mathcal{T}_1$, $\theta_\ell x^*_{h,\tau} - p^*_{h,\tau} \leq \theta_h x^*_{h,\tau} - p^*_{h,\tau} = 0$, where the inequality holds because $\theta_\ell < \theta_h$ and the equality holds by equality (53). Moreover, $1 - \sum_{t=1}^{\tau-1} x^*_{\ell,t} \geq 0$ for all $\tau \in \mathcal{T}_0$. Thus, the right-hand side of constraint (ICℓ') is non-positive for all $\tau \in \mathcal{T}_1$, and so $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICℓ') for all $\tau \in \mathcal{T}_1$.

To show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICℓ') for $\tau = 1$, we need to show that $\theta_\ell x^*_{h,1} - p^*_{h,1} \leq 0$, which we have already established in Appendix B.5.

Step 2. We show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICh') for all $\tau \in \mathcal{T}_0$.

For all $\tau \neq \overline{T}$, $x^*_{\ell,\tau} = 0$ and $p^*_{\ell,\tau} = 0$ (see equations (50) and (52)). Thus, the right-hand side of constraint (ICh') equals 0 for all $\tau \neq \overline{T}$. Moreover, $\sum_{t=1}^{\overline{T}-1} x^*_{h,t} = 1$ (see equation (43)), and so the right-hand side of constraint (ICh') equals 0 also for $\tau = \overline{T}$. Since $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (IRh) for all $\tau \in \mathcal{T}_0$, that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ also satisfies constraint (ICh') for all $\tau \in \mathcal{T}_0$ follows.

Step 3. We show that, for all $\delta \in (0,1)$, there exists $\overline{\mu}(\delta) \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$ such that, for all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint constraint (O).

A sufficient condition for constraint (O) to hold is that $\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h}$ (see the discussion in Section 4.2 before the statement of Theorem 5). To begin, note that

$$
\begin{aligned}
\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h} &\iff \frac{(1 - x^*_{h,1})\mu}{(1 - x^*_{h,1})\mu + (1-\mu)} \leq \frac{\theta_\ell}{\theta_h} \\
&\iff \frac{-(1-\mu)\left(1 - \rho^{\overline{T}-2}\right)}{-(1-\mu)\left(1-\rho^{\overline{T}-2}\right) + (1-\mu)} \leq \frac{\theta_\ell}{\theta_h} \\
&\iff -\frac{1-\rho^{\overline{T}-2}}{\rho^{\overline{T}-2}} \leq \frac{\theta_\ell}{\theta_h} \\
&\iff \frac{\Delta\theta}{\theta_h}\rho^{\overline{T}-2} \leq 1
\end{aligned}
$$

$$\iff \ln\left(\frac{\Delta\theta}{\theta_h}\right) + (\overline{T} - 2)\ln\rho \le 0, \tag{65}$$

where: the first equivalence holds by definition (1); the second equivalence holds by the fourth equality in equation (43). Since $\rho > 1$, $\ln\rho > 0$; moreover, $\overline{T} = \inf\{t \in \mathcal{T}_1 : t \ge \underline{t}\} \Longrightarrow \underline{t} - 1 \ge \overline{T} - 2$. Therefore, we have

$$\ln\left(\frac{\Delta\theta}{\theta_h}\right) + (\underline{t} - 1)\ln\rho \le 0 \Longrightarrow \ln\left(\frac{\Delta\theta}{\theta_h}\right) + (\overline{T} - 2)\ln\rho \le 0. \tag{66}$$

Moreover, note that

$$\ln\left(\frac{\Delta\theta}{\theta_h}\right) + (\underline{t} - 1)\ln\rho \le 0 \iff \ln\left(\frac{\Delta\theta}{\theta_h}\right) + \left(\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r}\right)\ln(\delta r) \le 0$$

$$\iff \ln\left(\frac{\Delta\theta}{\theta_h}\right) + [\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)]\frac{\ln(\delta r)}{\ln r} \le 0, \tag{67}$$

where the first equivalence holds by equation (28) and definition (4). Summing up, from implications and equivalences (65)–(67), a sufficient condition for $\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \le \frac{\theta_\ell}{\theta_h}$ to hold is that

$$\ln\left(\frac{\Delta\theta}{\theta_h}\right) + [\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)]\frac{\ln(\delta r)}{\ln r} \le 0. \tag{68}$$

Let $\overline{\mu}(\delta)$ be the value of $\mu \in (0,1)$ that satisfy inequality (68) with equality (note that such $\overline{\mu}(\delta)$ always exists). Since $\delta r > 1$ and $r > 1$, we have

$$\frac{\ln(\delta r)}{\ln r} > 0. \tag{69}$$

Thus, the left-hand side of inequality (68) is increasing in $\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)$. In turn, $\frac{\partial}{\partial\mu}[\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)] = \frac{1}{\mu(1-\mu)} > 0$ for all $\mu \in (0,1)$, so that the left-hand side of inequality (68) is increasing in μ. Therefore, inequality (68) is satisfied by all $\mu \in (0, \overline{\mu}(\delta)]$. Since $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, we have $\mu\Delta\theta > (1-\mu)\theta_\ell$, and so $\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell) > 0$. Thus, the left-hand side of inequality (67) is increasing in $\frac{\ln(\delta r)}{\ln r}$. In turn, $\lim_{\delta\to 1}\frac{\partial}{\partial\delta}\frac{\ln(\delta r)}{\ln r} = \lim_{\delta\to 1}\left[\ln\left(\frac{\theta_h - \delta\theta_\ell}{\Delta\theta}\right) + \frac{\delta\theta_\ell}{\theta_h - \delta\theta_\ell}\ln\delta\right] = 0$, where the first equality holds by definition (3) and (4). Moreover, $\frac{\partial^2}{\partial\delta^2}\frac{\ln(\delta r)}{\ln r} = \frac{\theta_\ell\theta_h}{(\theta_h - \delta\theta_\ell)^2}\ln\delta < 0$ for all $\delta \in (0,1)$. Thus, $\frac{\ln(\delta r)}{\ln r}$ is increasing in δ. Hence,

$$\frac{\partial}{\partial\delta}\frac{\ln(\delta r)}{\ln r} > 0 \tag{70}$$

for all $\delta \in (0,1)$, so that the left-hand side of inequality (68) is increasing in δ.

Step 4. We show that $\overline{\mu}(\delta)$ satisfies properties (i)–(iii) in the statement of Theorem 5. By the Implicit Function Theorem, $\overline{\mu}(\delta)$ is continuous in δ and $\frac{\partial \overline{\mu}(\delta)}{\partial \delta} = -\frac{\frac{\ln(\delta r)}{\ln r}\frac{\partial}{\partial \mu}[\ln(\mu\Delta\theta)-\ln((1-\mu)\theta_\ell)]}{[\ln(\mu\Delta\theta)-\ln((1-\mu)\theta_\ell)]\frac{\partial}{\partial\delta}\frac{\ln(\delta r)}{\ln r}} < 0$, where the inequality holds by inequalities (69) and (70). As a result, $\overline{\mu}(\delta)$ is decreasing in δ.

Consider the left-hand side of inequality (68). Note that

$$
\begin{aligned}
&\lim_{\delta\to 0}\left[\ln\left(\frac{\Delta\theta}{\theta_h}\right) + [\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)]\frac{\ln(\delta r)}{\ln r}\right] \\
&\qquad = \ln\left(\frac{\Delta\theta}{\theta_h}\right) + [\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)]\lim_{\delta\to 0}\frac{\ln\frac{\theta_h-\delta\theta_\ell}{\Delta\theta}}{\ln\frac{\theta_h-\delta\theta_\ell}{\delta\Delta\theta}} \\
&\qquad = \ln\left(\frac{\Delta\theta}{\theta_h}\right) \\
&\qquad < 0,
\end{aligned}
$$

where: the first equality holds by definition (3); the inequality holds because $\Delta\theta < \theta_h$ (so that $\frac{\Delta\theta}{\theta_h} < 1$). That $\overline{\mu}(\delta) \to 1$ as $\delta \to 0$ follows.

Consider again the left-hand side of inequality (68). By definition (3), it is immediate that $\lim_{\delta\to 1}\ln r = 0$ and $\lim_{\delta\to 1}\ln\delta r = 0$, where the first equalities hold by definition (3). By using de L'Hôpital's rule, we have $\lim_{\delta\to 1}\frac{\ln(\delta r)}{\ln r} = \lim_{\delta\to 1}\frac{\frac{\partial}{\partial\delta}(\ln(\delta r))}{\frac{\partial}{\partial\delta}(\ln r)} = \lim_{\delta\to 1}\frac{\delta\theta_\ell-(1-\delta)\theta_h}{\delta\theta_h} = \frac{\theta_\ell}{\theta_h}$. Hence,

$$
\begin{aligned}
&\lim_{\delta\to 1}\left\{\ln\left(\frac{\Delta\theta}{\theta_h}\right) + [\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)]\frac{\ln(\delta r)}{\ln r}\right\} \leq 0 \\
&\qquad \iff \ln\left(\frac{\theta_h}{\Delta\theta}\right) \geq [\ln(\mu\Delta\theta) - \ln((1-\mu)\theta_\ell)]\frac{\theta_\ell}{\theta_h} \\
&\qquad \iff \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}} \geq \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} \\
&\qquad \iff \mu \leq \frac{\left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}{\frac{\Delta\theta}{\theta_\ell} + \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}.
\end{aligned}
$$

Since

$$
\frac{\left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}{\frac{\Delta\theta}{\theta_\ell} + \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}} > \frac{\theta_\ell}{\theta_h} \iff \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}} > 1
$$

and $\theta_h > \Delta\theta$, that $\overline{\mu}(\delta)$ is bounded away from $\frac{\theta_\ell}{\theta_h}$ as $\delta \to 1$ follows. ■

F Constraint (O′) is Sufficient for Constraint (O)

Constraint (O) can be expressed equivalently as

$$\begin{aligned} &\mu\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p^*_{h,t}\right)+(1-\mu)\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p^*_{\ell,t}\right) \\ &\qquad\geq\left[1-\mu\left(\sum_{t=1}^{\tau-1}x^*_{h,t}\right)-(1-\mu)\left(\sum_{t=1}^{\tau-1}x^*_{\ell,t}\right)\right]U^c_P(\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*);\delta)\quad\forall\tau\in\mathcal{T}_0. \end{aligned} \tag{71}$$

$U^c_P(\hat{\mu};\delta)$ is convex and increasing in $\hat{\mu}$ on $\left[\frac{\theta_\ell}{\theta_h},1\right]$. Moreover, $U^c_P(\hat{\mu};\delta)=\theta_\ell$ for all $\hat{\mu}\in\left[0,\frac{\theta_\ell}{\theta_h}\right]$ and so $U^c_P(\hat{\mu};\delta)>\theta_\ell$ for all $\left(\frac{\theta_\ell}{\theta_h},1\right]$. Therefore, $U^c_P(\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*);\delta)<\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)U^c_P(1;\delta)+(1-\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*))U^c_P(0;\delta)=\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)U^c_P(1;\delta)+(1-\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*))\theta_\ell$ for all $\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)\in(0,\mu]$. Moreover, since U^c_P is continuous, there exists $\overline{\mu}\in(\mu,1)$ such that, for all $\tilde{\mu}\in(\overline{\mu},1)$, we have $U^c_P(\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*);\delta)\leq\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)U^c_P(\tilde{\mu};\delta)+(1-\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*))\theta_\ell$ for all $\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)\in[0,\mu]$, and so

$$U^c_P(\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*);\delta)\leq\theta_\ell+\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)\big(U^c_P(\tilde{\mu};\delta)-\theta_\ell\big). \tag{72}$$

for all $\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)\in[0,\mu]$. Let $\varepsilon(\delta):=U^c_P(\tilde{\mu};\delta)-\theta_\ell$. From equations (71) and (72), a sufficient condition for constraint (O) to be satisfied is

$$\begin{aligned} &\mu\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p^*_{h,t}\right)+(1-\mu)\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p^*_{\ell,t}\right) \\ &\qquad\geq\left[1-\mu\left(\sum_{t=1}^{\tau-1}x^*_{h,t}\right)-(1-\mu)\left(\sum_{t=1}^{\tau-1}x^*_{\ell,t}\right)\right]\big(\theta_\ell+\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)\varepsilon(\delta)\big)\quad\forall\tau\in\mathcal{T}_0. \end{aligned} \tag{73}$$

From definition (1), it follows that equation (73) is equivalent to constraint (O′) in period τ, establishing the desired result. ■

G Proof of Theorem 6

Throughout Appendix G, we use the following notation:

$$r(\delta):=\frac{\theta_h-\delta(\theta_\ell+\varepsilon(\delta))}{\delta(\Delta\theta-\varepsilon(\delta))} \tag{74}$$

$$\rho(\delta):=\delta r(\delta). \tag{75}$$

Since $U_P^c(\tilde{\mu};\delta) < \theta_h$ for all $\delta \in (0,1)$, we have $\Delta\theta - \varepsilon(\delta) = \theta_h - \theta_\ell - (U_P^c(\tilde{\mu};\delta) - \theta_\ell) > 0$. Moreover, note that $r(\delta) > 1$, $\rho(\delta) > 1$ for all $\delta \in (0,1)$,

$$(74) \implies 1 - r(\delta) = -\frac{(1-\delta)\theta_h}{\delta(\Delta\theta - \varepsilon(\delta))}, \tag{76}$$

and

$$(75) \implies 1 - \rho(\delta) = -\frac{(1-\delta)(\theta_\ell + \varepsilon(\delta))}{\Delta\theta - \varepsilon(\delta)}. \tag{77}$$

G.1 Simplifying Program (P2)

Consider the following relaxation of program (P2):

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P2a}$$

$$\text{s.t.} \quad x_{\ell,t}, x_{h,t} \geq 0 \quad \forall t \in \mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{h,t} - p_{h,t}) \tag{ICℓ}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t} - p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t} - p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq 0 \quad \forall \tau \in \mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_h x_{h,t} - p_{h,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 \tag{IRh}$$

$$\mu\left(\sum_{t=1}^{\tau-1}(\theta_\ell + \varepsilon(\delta))x_{h,t} + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right) \tag{O$'$}$$

$$+ (1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t} + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right) \geq \theta_\ell + \mu\varepsilon(\delta) \quad \forall \tau \in \mathcal{T}_0.$$

Program (P2a) is obtained from program (P2) by ignoring constraints (ICℓ') and (ICh'). Below we solve program (P2a) and show that its solution satisfies the omitted constraint of program (P2). The approach to solve program (P2a) mimics that to solve program (P1) in Appendix B.

G.2 Simplifying the Primal Program (P2a)

In the primal program (P2a): (i) we ignore constraint (ICℓ) and constraint (O$'$) for $\tau = 1$, and we will verify in Appendix G.6 that they are satisfied by the solution to the relaxed version of the program; (ii) constraints (ICh) and (IRℓ) for $\tau = 1$, together with the assumption that $\theta_h > \theta_\ell$, imply that constraint (IRh) is satisfied for $\tau = 1$. As a result, the relaxed version of program (P2a) is the following:

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right)+(1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P2b}$$

$$\text{s.t.}\quad x_{\ell,t}, x_{h,t} \geq 0 \quad \forall\tau\in\mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t}-p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t}-p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t}-p_{\ell,t}) \geq 0 \quad \forall\tau\in\mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_h x_{h,t}-p_{h,t}) \geq 0 \quad \forall\tau\in\mathcal{T}_1 \tag{IRh}$$

$$\begin{aligned}&\mu\left(\sum_{t=1}^{\tau-1}(\theta_\ell+\varepsilon(\delta))x_{h,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right)\\ &\quad+(1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right)\geq\theta_\ell+\mu\varepsilon(\delta) \quad \forall\tau\in\mathcal{T}_1.\end{aligned} \tag{O$'$}$$

G.3 The Dual of Program (P2b)

Let $\boldsymbol{\xi} := \left(\alpha,\beta,\zeta,(\lambda_{\ell,t},\lambda_{h,t+1},\gamma_{t+1})_{t=1}^{\infty}\right) \in \mathbb{R}^3\times\mathbb{R}^{3\infty}$, where: α (resp., β) is the Lagrange multiplier associated to constraint (F2) (resp., (F3)) in program (P2b); ζ is the Lagrange multiplier associated to constraint (ICh) in program (P2b); for all $t\in\mathcal{T}_0$, $\lambda_{\ell,t}$ is the Lagrange multiplier associated to constraint (IRℓ) in period t in program (P2b); for all $t\in\mathcal{T}_0$, $\lambda_{h,t+1}$ is the Lagrange multiplier associated to constraint (IRh) in period $t+1$ in program (P2b); for all $t\in\mathcal{T}_0$, γ_{t+1} is the Lagrange multiplier associated to constraint (O$'$) in period $t+1$ in program (P2b).

The dual program of program (P2b) is the following:

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^3\times\mathbb{R}^{3\infty}} \alpha + \beta - \sum_{t=2}^{\infty}(\theta_\ell + \mu\varepsilon(\delta))\gamma_t \tag{P2c}$$

$$\text{s.t.}\quad \alpha \geq 0, \quad \beta \geq 0, \quad \zeta \geq 0, \quad \lambda_{\ell,t}, \lambda_{h,t+1}, \gamma_{t+1} \geq 0 \;\; \forall t \in \mathcal{T}_0 \tag{78}$$

$$\alpha \geq -\delta^{t-1}\theta_h\zeta + \sum_{\tau=1}^{t}\delta^{t-\tau}\theta_\ell\lambda_{\ell,\tau} + \sum_{\tau=t+1}^{\infty}(1-\mu)\theta_\ell\gamma_\tau \;\; \forall t \in \mathcal{T}_0, \tag{79}$$

$$\beta \geq \delta^{t-1}\theta_h\zeta + \sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\lambda_{h,\tau} + \sum_{\tau=t+1}^{\infty}\mu(\theta_\ell + \varepsilon(\delta))\gamma_\tau \;\; \forall t \in \mathcal{T}_0, \tag{80}$$

$$\delta^{t-1}\zeta - \sum_{\tau=1}^{t}\delta^{t-\tau}\lambda_{\ell,\tau} + \sum_{\tau=2}^{t}\delta^{t-\tau}(1-\mu)\gamma_\tau + \delta^{t-1}(1-\mu) = 0 \;\; \forall t \in \mathcal{T}_0, \tag{81}$$

$$-\delta^{t-1}\zeta - \sum_{\tau=2}^{t}\delta^{t-\tau}\lambda_{h,\tau} + \sum_{\tau=2}^{t}\delta^{t-\tau}\mu\gamma_\tau + \delta^{t-1}\mu = 0 \;\; \forall t \in \mathcal{T}_0, \tag{82}$$

where constraints (81) and (82) hold with equality as $p_{\ell,t}$ and $p_{h,t}$ are unrestricted (i.e., they can be positive or negative) for all $t \in \mathcal{T}_0$ in the primal program (P2b).

G.4 A Candidate Solution to the Dual Program (P2c)

We recover a candidate solution $\boldsymbol{\xi}^* := \left(\alpha^*, \beta^*, \zeta^*, \left(\lambda^*_{\ell,t}, \lambda^*_{h,t+1}, \gamma^*_{t+1}\right)_{t=1}^{\infty}\right) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$ to program (P2c).

Step 1. Solve equations (82) at successive values of t, starting with $t = 1$, to find

$$\zeta^* = \mu \qquad \text{and} \qquad \lambda^*_{h,t} = \mu\gamma^*_t \quad \text{for all } t \in \mathcal{T}_1. \tag{83}$$

Next, solve equations (81) at successive values of t, starting with $t = 1$, to find

$$\lambda^*_{\ell,1} = 1 \qquad \text{and} \qquad \lambda^*_{\ell,t} = (1-\mu)\gamma^*_t \quad \text{for all } t \in \mathcal{T}_1. \tag{84}$$

Step 2. Given equations (83) and (84), the dual program (P2c) simplifies to:

$$\min_{(\alpha,\beta,(\gamma_t)_{t=2}^{\infty})\in\mathbb{R}^2\times\mathbb{R}^{\infty}} \alpha + \beta - \sum_{t=2}^{\infty}(\theta_\ell + \mu\varepsilon(\delta))\gamma_t \tag{P2d}$$

$$\text{s.t.}\quad \alpha \geq 0, \quad \beta \geq 0, \quad \gamma_t \geq 0 \;\; \forall t \in \mathcal{T}_1 \tag{85}$$

$$\alpha \geq \delta^{t-1}(\theta_\ell - \mu\theta_h) + (1-\mu)\theta_\ell\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\gamma_\tau + \sum_{\tau=t+1}^{\infty}\gamma_\tau\right] \quad \forall t \in \mathcal{T}_0, \tag{86}$$

$$\beta \geq \delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma_\tau + \sum_{\tau=t+1}^{\infty}(\theta_\ell+\varepsilon(\delta))\gamma_\tau\right] \quad \forall t \in \mathcal{T}_0. \tag{87}$$

Step 3. As program (P2d) is a minimization problem, its solutions must satisfy

$$\alpha^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}(\theta_\ell-\mu\theta_h)+(1-\mu)\theta_\ell\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\gamma_\tau^* + \sum_{\tau=t+1}^{\infty}\gamma_\tau^*\right]\right\}\right\} \tag{88}$$

and

$$\beta^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma_\tau^* + \sum_{\tau=t+1}^{\infty}(\theta_\ell+\varepsilon(\delta))\gamma_\tau^*\right]\right\}\right\}. \tag{89}$$

Step 4. Guess that, for some $\overline{T}(\delta) \in \mathcal{T}_1$, $\gamma_t^* = 0$ for all $t > \overline{T}(\delta)$. Moreover, guess that

$$\begin{aligned}\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma_\tau^* + \sum_{\tau=t+1}^{\infty}(\theta_\ell+\varepsilon(\delta))\gamma_\tau^*\right]& \\ = \delta^{t}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t+1}\delta^{t+1-\tau}\theta_h\gamma_\tau^* + \sum_{\tau=t+2}^{\infty}(\theta_\ell+\varepsilon(\delta))\gamma_\tau^*\right]&\end{aligned} \tag{90}$$

for all $t \in \{1,\ldots,\overline{T}(\delta)-1\}$. Solve equations (90) for γ_t^* at successive values of t, starting with $t=1$, to find

$$\gamma_t^* = \frac{(1-\delta)\theta_h}{\theta_h-\theta_\ell-\varepsilon(\delta)}\left(\frac{\theta_h-\delta(\theta_\ell+\varepsilon(\delta))}{\theta_h-\theta_\ell-\varepsilon(\delta)}\right)^{t-2} = \frac{(1-\delta)\theta_h}{\Delta\theta-\varepsilon(\delta)}\rho(\delta)^{t-2} \tag{91}$$

for all $t \in \{2,\ldots,\overline{T}(\delta)\}$, where the last equality holds by definition (75).

Step 5. Use equation (89), the guess in equation (90), and the guess that $\gamma_t^* = 0$ for all $t > \overline{T}(\delta)$, to obtain

$$\beta^* = \mu\theta_h + \mu(\theta_\ell+\varepsilon(\delta))\left(\sum_{t=2}^{\overline{T}(\delta)}\gamma_t^*\right). \tag{92}$$

Step 6. By using equations (88) and (91), definition (74), and the guess that $\gamma_t^* = 0$ for all $t > \overline{T}(\delta)$, constraint (86) becomes

$$\begin{aligned}\alpha^* = \max_{t\in\{1,\ldots,\overline{T}(\delta)\}}&\Bigg\{\delta^{t-1}(\theta_\ell-\mu\theta_h) \\ &+ \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta-\varepsilon(\delta)}\left(\delta^{t-2}\frac{1-r(\delta)^{t-1}}{1-r(\delta)} + \frac{\rho(\delta)^{t-1}-\rho(\delta)^{\overline{T}(\delta)-1}}{1-\rho(\delta)}\right)\Bigg\}\end{aligned} \tag{93}$$

assuming that the right-hand side of equation (93) is non-negative (which we will show to be the case in Step 8 of this section). The maximand on the right-hand side of equation (93) simplifies as follows:

$$
\begin{aligned}
&\delta^{t-1}(\theta_\ell-\mu\theta_h)+\frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta-\varepsilon(\delta)}\left(\delta^{t-2}\frac{1-r(\delta)^{t-1}}{1-r(\delta)}+\frac{\rho(\delta)^{t-1}-\rho(\delta)^{\overline{T}(\delta)-1}}{1-\rho(\delta)}\right)\\
&=\delta^{t-1}(\theta_\ell-\mu\theta_h)-\delta^{t-1}(1-\mu)\theta_\ell\big(1-r(\delta)^{t-1}\big) \qquad (94)\\
&-\frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell+\varepsilon(\delta)}\Big(\rho(\delta)^{t-1}-\rho(\delta)^{\overline{T}(\delta)-1}\Big)\\
&=\frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell+\varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-1}-\delta^{t-1}\mu\Delta\theta-\delta^{t-1}\frac{(1-\mu)\theta_\ell(\Delta\theta-\varepsilon(\delta))}{\theta_\ell+\varepsilon(\delta)}r(\delta)^{t-1},
\end{aligned}
$$

where: the first equality holds by implications (76) and (77); the second equality holds by definition (4). Thus, by equation (94), equation (93) is equivalent to

$$
\begin{aligned}
\alpha^*=\max_{t\in\{1,\dots,\overline{T}(\delta)\}}\Bigg\{&\frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell+\varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-1} \qquad (95)\\
&-\delta^{t-1}\mu\Delta\theta-\delta^{t-1}\frac{(1-\mu)\theta_\ell(\Delta\theta-\varepsilon(\delta))}{\theta_\ell+\varepsilon(\delta)}r(\delta)^{t-1}\Bigg\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
t^*:=\operatorname*{arg\,max}_{t\in\{1,\dots,\overline{T}(\delta)\}}\Bigg\{&\frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell+\varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-1} \qquad (96)\\
&-\delta^{t-1}\mu\Delta\theta-\delta^{t-1}\frac{(1-\mu)\theta_\ell(\Delta\theta-\varepsilon(\delta))}{\theta_\ell+\varepsilon(\delta)}r(\delta)^{t-1}\Bigg\}.
\end{aligned}
$$

Moreover, let $\underline{t}(\delta)$ be defined as follows: $\underline{t}(\delta)\in\mathbb{R}$ such that

$$
\begin{aligned}
&\delta^{\underline{t}(\delta)-1}\mu\Delta\theta+\delta^{\underline{t}(\delta)-1}\frac{(1-\mu)\theta_\ell(\Delta\theta-\varepsilon(\delta))}{\theta_\ell+\varepsilon(\delta)}r(\delta)^{\underline{t}(\delta)-1}\\
&\qquad=\delta^{\underline{t}(\delta)}\mu\Delta\theta+\delta^{\underline{t}(\delta)}\frac{(1-\mu)\theta_\ell(\Delta\theta-\varepsilon(\delta))}{\theta_\ell+\varepsilon(\delta)}r(\delta)^{\underline{t}(\delta)}.
\end{aligned}
$$

Since the maximand on the right-hand side of equation (96) is concave in t and $\overline{T}(\delta)$ is yet to be determined (and so can be chosen arbitrarily large),

$$
t^*=\inf\left\{t\in\mathcal{T}_0:t\geq\underline{t}(\delta)\right\}. \qquad (97)
$$

Note that

$$
\delta^{\underline{t}(\delta)-1}\mu\Delta\theta+\delta^{\underline{t}(\delta)-1}\frac{(1-\mu)\theta_\ell(\Delta\theta-\varepsilon(\delta))}{\theta_\ell+\varepsilon(\delta)}r(\delta)^{\underline{t}(\delta)-1}
$$

$$= \delta^{\underline{t}(\delta)}\mu\Delta\theta + \delta^{\underline{t}(\delta)}\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} r(\delta)^{\underline{t}(\delta)}$$

$$\Longleftrightarrow \mu(1-\delta)\Delta\theta = -\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)}(1-\delta r(\delta)) r(\delta)^{\underline{t}(\delta)-1}$$

$$\Longleftrightarrow \mu(1-\delta)\Delta\theta = \frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} \frac{\theta_\ell + \varepsilon(\delta)}{(1-\delta)(\Delta\theta - \varepsilon(\delta))} r(\delta)^{\underline{t}(\delta)-1}$$

$$\Longleftrightarrow r(\delta)^{\underline{t}(\delta)-1} = \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} \tag{98}$$

$$\Longleftrightarrow \underline{t}(\delta) = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r(\delta)}, \tag{99}$$

where the second equivalence holds by implication (77). Since $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, $\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} > 1$, and so $\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right) > 0$. Moreover, $r(\delta) > 1$, and so $\ln r(\delta) > 0$. Thus, $\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r(\delta)} > 1$, which implies, together with equations (97) and (99), that $t^* \geq 2$.

Step 7. The choice of $\overline{T}(\delta)$ is part of the choice of the Lagrange multipliers $(\gamma_t^*)_{t=2}^{\infty}$. Thus, $\overline{T}(\delta)$ must be chosen to minimize the objective function of program (P2c).

Let $U_P^{\text{P2c}}(\pi)$ denote the optimal value of program (P2c). Note that

$$\begin{aligned}
U_P^{\text{P2c}}(\pi) &= \alpha^* + \beta^* - \sum_{t=2}^{\infty}(\theta_\ell + \mu\varepsilon(\delta))\gamma_t^* \\
&= \alpha^* + \mu\theta_h + \mu(\theta_\ell + \varepsilon(\delta))\left(\sum_{t=2}^{\overline{T}(\delta)}\gamma_t^*\right) - (\theta_\ell + \mu\varepsilon(\delta))\left(\sum_{t=2}^{\overline{T}(\delta)}\gamma_t^*\right) \\
&= \alpha^* + \mu\theta_h - (1-\mu)\theta_\ell\left(\sum_{t=2}^{\overline{T}(\delta)}\frac{(1-\delta)\theta_h}{\Delta\theta - \varepsilon(\delta)}\rho(\delta)^{t-2}\right) \\
&= \alpha^* + \mu\theta_h - \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta - \varepsilon(\delta)}\frac{1-\rho(\delta)^{\overline{T}(\delta)-1}}{1-\rho(\delta)} \qquad (100) \\
&= \alpha^* + \mu\theta_h + \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell + \varepsilon(\delta)}\left(1 - \rho(\delta)^{\overline{T}(\delta)-1}\right) \\
&= \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell + \varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-1} - \delta^{t^*-1}\mu\Delta\theta - \delta^{t^*-1}\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)}r(\delta)^{t^*-1} \\
&\quad + \mu\theta_h + \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell + \varepsilon(\delta)}\left(1 - \rho(\delta)^{\overline{T}(\delta)-1}\right) \\
&= \frac{\theta_h(\theta_\ell + \mu\varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} - \delta^{t^*-1}\mu\Delta\theta - \delta^{t^*-1}\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)}r(\delta)^{t^*-1},
\end{aligned}$$

where: the second equality holds by equation (92) and the guess that $\gamma_t^* = 0$ for all $t > \overline{T}(\delta)$; the third equality holds by equation (91); the fifth equality holds by implication (77); the sixth equality holds by equation (95) and the definition of t^*. Since $U_P^{\text{P2c}}(\pi)$ does not depend on $\overline{T}(\delta)$, we take

$$\overline{T}(\delta) = t^*. \tag{101}$$

Step 8. We show that $\alpha^* \geq 0$. From the two previous steps, we have

$$\begin{aligned}
\alpha^* &= \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell + \varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-1} - \delta^{t^*-1}\mu\Delta\theta - \delta^{t^*-1}\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)}r(\delta)^{t^*-1} \\
&= \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell + \varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-1} - \delta^{\overline{T}(\delta)-1}\mu\Delta\theta - \delta^{\overline{T}(\delta)-1}\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)}r(\delta)^{\overline{T}(\delta)-1} \\
&= \delta^{\overline{T}(\delta)-1}\Big[(1-\mu)\theta_\ell r(\delta)^{\overline{T}(\delta)-1} - \mu\Delta\theta\Big] \\
&\geq \delta^{\underline{t}(\delta)-1}\Big[(1-\mu)\theta_\ell r(\delta)^{\underline{t}(\delta)-1} - \mu\Delta\theta\Big] \\
&= 0
\end{aligned}$$

where: the first equality holds by equation (95) and definition (96); the second equality holds by equation (101); the third equality holds by definition (75); the inequality holds by equation (95), definition (96), and equation (101); the last equality holds by equation (98).

Step 9. Summing up, a candidate solution $\boldsymbol{\xi}^* := \left(\alpha^*, \beta^*, \zeta^*, \left(\lambda^*_{\ell,t}, \lambda^*_{h,t+1}, \gamma^*_{t+1}\right)_{t=1}^{\infty}\right) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$ to program (P2c) is as follows. For

$$\overline{T}(\delta) = \inf\left\{t \in \mathcal{T}_1 : t \geq \underline{t}(\delta)\right\}, \qquad \text{where} \qquad \underline{t}(\delta) = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r(\delta)},$$

we have:

$$\alpha^* = \delta^{\overline{T}(\delta)-1}\Big[(1-\mu)\theta_\ell r(\delta)^{\overline{T}(\delta)-1} - \mu\Delta\theta\Big], \tag{102}$$

$$\beta^* = \mu\theta_h + \mu(\theta_\ell + \varepsilon(\delta))\left(\sum_{t=2}^{\overline{T}(\delta)}\gamma^*_t\right) \tag{103}$$

$$\zeta^* = \mu \tag{104}$$

$$\lambda^*_{\ell,t} = \begin{cases} 1 & \text{if } t = 1 \\ (1-\mu)\gamma^*_t & \text{otherwise} \end{cases}, \tag{105}$$

$$\lambda^*_{h,t} = \mu\gamma^*_t \quad \text{for all } t \in \mathcal{T}_0, \tag{106}$$

$$\gamma_t^* = \begin{cases} \frac{(1-\delta)\theta_h}{\Delta\theta - \varepsilon(\delta)} \rho(\delta)^{t-2} & \text{if } t \in \left\{2, \ldots, \overline{T}(\delta)\right\} \\ 0 & \text{if } t > \overline{T}(\delta) \end{cases}. \tag{107}$$

All elements of $\boldsymbol{\xi}^*$ are clearly non-negative. Moreover, we have

$$\begin{aligned} U_P^{\mathrm{P2c}}(\pi) = & \frac{\theta_h(\theta_\ell + \mu\varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} - \delta^{\overline{T}(\delta)-1}\mu\Delta\theta \\ & - \delta^{\overline{T}(\delta)-1} \frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} r(\delta)^{\overline{T}(\delta)-1}, \end{aligned} \tag{108}$$

where the equality holds by equations (100) and (101).

G.5 A Solution to the Primal Program (P2b)

We recover a solution $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) \coloneqq \left(x^*_{\ell,t}(\delta), x^*_{h,t}(\delta), p^*_{\ell,t}(\delta), p^*_{h,t}(\delta)\right)_{t=1}^{\infty} \in \mathbb{R}^{4\infty}$ to program (P2b).

Step 1. Since $\lambda^*_{\ell,t} > 0$ for all $t \in \left\{1, \ldots, \overline{T}(\delta)\right\}$ (see equation (105)), constraint (IRℓ) is binding for all such t. Thus,

$$p^*_{\ell,t}(\delta) = \theta_\ell x^*_{\ell,t}(\delta) \quad \text{for all } t \in \left\{1, \ldots, \overline{T}(\delta)\right\}. \tag{109}$$

Since $\lambda^*_{h,t} > 0$ for all $t \in \left\{2, \ldots, \overline{T}(\delta)\right\}$ (see equation (106)), constraint (IRh) is binding for all such t. Thus,

$$p^*_{h,t}(\delta) = \theta_h x^*_{h,t}(\delta) \quad \text{for all } t \in \left\{2, \ldots, \overline{T}(\delta)\right\}. \tag{110}$$

Moreover, guess that the solution to program (P2b) is such that

$$\sum_{t=1}^{\overline{T}(\delta)-1} x^*_{h,t}(\delta) = 1. \tag{111}$$

and

$$x^*_{h,t}(\delta) = p^*_{h,t}(\delta) = 0 \quad \text{for all } t \geq \overline{T}(\delta). \tag{112}$$

Step 2. From constraint (O′) binding for $t = \overline{T}(\delta)$ (as $\gamma^*_{\overline{T}(\delta)} > 0$, see equation (107)) and the conjectures in equations (111) and (112), we have that

$$\sum_{t=\overline{T}(\delta)}^{\infty} \delta^{t-\overline{T}(\delta)} p_{\ell,t}(\delta) = \theta_\ell,$$

from which we guess that

$$p^*_{\ell,\overline{T}(\delta)}(\delta) = \theta_\ell. \tag{113}$$

From equations (36) and (40), we have that

$$x^*_{\ell,\overline{T}(\delta)}(\delta) = 1, \tag{114}$$

and so $x^*_{\ell,t}(\delta) = p^*_{\ell,t}(\delta) = 0 \quad \text{for all } t \in \{1, \dots, \overline{T}(\delta) - 1\}$. Finally, we guess that

$$p^*_{l,t}(\delta) = 0 \quad \text{for all } t > \overline{T}(\delta). \tag{115}$$

Step 3. Since $\zeta^* > 0$ (see equation (104)), constraint (ICh) is binding. This, together with equations (110), (113), and (114), implies that $\theta_h x^*_{h,1}(\delta) - p^*_{h,1}(\delta) = \delta^{\overline{T}(\delta)-1}\Delta\theta$ or, equivalently,

$$p^*_{h,1}(\delta) = \theta_h x^*_{h,1}(\delta) - \delta^{\overline{T}(\delta)-1}\Delta\theta.$$

Step 4. To find $x^*_{h,t}(\delta)$ for all $t \in \{2, \dots, \overline{T}(\delta)\}$, we use that constraint (O′) is binding for all such t (as $\gamma^*_t > 0$ for all such t, see equation (107)). From constraint (O′) binding for $t = \overline{T}(\delta) - 1$, we obtain $\mu\big[(\theta_\ell + \varepsilon(\delta))\big(1 - x^*_{h,\overline{T}(\delta)-1}(\delta)\big) + \theta_h x^*_{h,\overline{T}(\delta)-1}(\delta)\big] + (1-\mu)\delta\theta_\ell = \theta_\ell + \mu\varepsilon(\delta)$ or, equivalently

$$x^*_{h,\overline{T}(\delta)-1}(\delta) = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta - \varepsilon(\delta))}.$$

Similarly, solving for constraint (O′) binding backwards for all $t \in \{2, \dots, \overline{T}(\delta) - 2\}$, starting with $t = \overline{T}(\delta) - 2$, we obtain

$$x^*_{h,t}(\delta) = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta - \varepsilon(\delta))}\left(\frac{\theta_h - \delta\theta_\ell}{\Delta\theta - \varepsilon(\delta)}\right)^{\overline{T}(\delta)-t-1} = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta - \varepsilon(\delta))}\rho(\delta)^{\overline{T}(\delta)-t-1},$$

where the second equality holds by definition (75). Finally, using the guess in equation (111), we have

$$\begin{aligned} x^*_{h,1}(\delta) &= 1 - \sum_{t=2}^{\overline{T}(\delta)-1} x^*_{h,t}(\delta) \\ &= 1 - \sum_{t=2}^{\overline{T}(\delta)-1} \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta - \varepsilon(\delta))}\rho(\delta)^{\overline{T}(\delta)-t-1} \\ &= 1 - \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta - \varepsilon(\delta)}\frac{1 - \rho(\delta)^{\overline{T}(\delta)-2}}{1 - \rho(\delta)} \end{aligned} \tag{116}$$

$$
\begin{aligned}
&= 1 + \frac{(1-\mu)\theta_\ell}{\mu(\theta_\ell + \varepsilon(\delta))} - \frac{(1-\mu)\theta_\ell}{\mu(\theta_\ell + \varepsilon(\delta))}\rho(\delta)^{\overline{T}(\delta)-2} \\
&= \frac{1}{\mu(\theta_\ell + \varepsilon(\delta))}\left[\theta_\ell + \mu\varepsilon(\delta) - (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right],
\end{aligned}
$$

where the fourth equality holds by implication (77).

Step 5. Summing up, a candidate solution

$$
(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) := \left(x^*_{\ell,t}(\delta), x^*_{h,t}(\delta), p^*_{\ell,t}(\delta), p^*_{h,t}(\delta)\right)_{t=1}^{\infty} \in \mathbb{R}^{4\infty}
$$

to program (P2b) is as follows. For

$$
\overline{T}(\delta) = \inf\left\{t \in \mathcal{T}_1 : t \geq \underline{t}(\delta)\right\}, \tag{117}
$$

where

$$
\underline{t}(\delta) = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r(\delta)}, \tag{118}
$$

we have:

$$
x^*_{\ell,t}(\delta) = \begin{cases} 1 & \text{if } t = \overline{T}(\delta) \\ 0 & \text{otherwise} \end{cases}, \tag{119}
$$

$$
x^*_{h,t}(\delta) = \begin{cases} \frac{1}{\mu(\theta_\ell+\varepsilon(\delta))}\left[\theta_\ell + \mu\varepsilon(\delta) - (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right] & \text{if } t = 1 \\ \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta-\varepsilon(\delta))}\rho(\delta)^{\overline{T}(\delta)-t-1} & \text{if } t \in \left\{2, \ldots, \overline{T}(\delta)-1\right\} \\ 0 & \text{otherwise} \end{cases}, \tag{120}
$$

$$
p^*_{\ell,t}(\delta) = \theta_\ell x^*_{\ell,t}(\delta) \quad \text{for all } t \in \mathcal{T}_0, \tag{121}
$$

$$
p^*_{h,t}(\delta) = \begin{cases} \theta_h x^*_{h,1}(\delta) - \delta^{\overline{T}(\delta)-1}\Delta\theta & \text{if } t = 1 \\ \theta_h x^*_{h,t}(\delta) & \text{otherwise} \end{cases}. \tag{122}
$$

Except for $x^*_{h,1}(\delta)$, all elements of $\boldsymbol{x}^*$ are clearly non-negative. We will show in step 7 of this section that $x^*_{h,1}(\delta) \geq 0$ if δ is sufficiently high.

Let $U_P^{\mathrm{P2b}}(\pi)$ be the optimal value of the primal program (P2b). Thus, we have

$$
\begin{aligned}
U_P^{\mathrm{P2b}}(\pi) &= \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p^*_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p^*_{\ell,t}\right) \\
&= \mu\left(\sum_{t=1}^{\overline{T}(\delta)-1}\delta^{t-1}p^*_{h,t}\right) + \delta^{\overline{T}(\delta)-1}(1-\mu)p^*_{\ell,\overline{T}(\delta)}
\end{aligned}
$$

$$
\begin{aligned}
&= \mu\Bigg[\frac{\theta_h}{\mu(\theta_\ell+\varepsilon(\delta))}\left[\theta_\ell+\mu\varepsilon(\delta)-(1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right] \\
&- \delta^{\overline{T}(\delta)-1}\Delta\theta + \sum_{t=2}^{\overline{T}(\delta)-1}\delta^{t-1}\frac{(1-\delta)(1-\mu)\theta_\ell\theta_h}{\mu(\Delta\theta-\varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-t-1}\Bigg] \\
&+ \delta^{\overline{T}(\delta)-1}(1-\mu)\theta_\ell \\
&= \frac{\theta_h}{\theta_\ell+\varepsilon(\delta)}\left[\theta_\ell+\mu\varepsilon(\delta)-(1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right] - \delta^{\overline{T}(\delta)-1}\mu\Delta\theta \qquad (123) \\
&+ \delta^{\overline{T}(\delta)-2}\frac{(1-\delta)(1-\mu)\theta_\ell\theta_h}{\Delta\theta-\varepsilon(\delta)}\frac{1-r(\delta)^{\overline{T}(\delta)-2}}{1-r(\delta)} + \delta^{\overline{T}(\delta)-1}(1-\mu)\theta_\ell \\
&= \frac{\theta_h}{\theta_\ell+\varepsilon(\delta)}\left[\theta_\ell+\mu\varepsilon(\delta)-(1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right] - \delta^{\overline{T}(\delta)-1}\mu\Delta\theta \\
&+ \delta^{\overline{T}(\delta)-1}(1-\mu)\theta_\ell r(\delta)^{\overline{T}(\delta)-2},
\end{aligned}
$$

where: the second and third equalities hold by equations (119)–(122); the fourth equality holds because, by definition (4), $\delta^{t-1}\rho(\delta)^{\overline{T}(\delta)-t-1} = \delta^{\overline{T}(\delta)-2}r(\delta)^{\overline{T}(\delta)-t-1}$; the last equality holds by using implication (76).

Step 6. We show that $\boldsymbol{x}^*(\delta) \to \boldsymbol{x}^*$ as $\delta \to 1$ which implies, from equations (50)–(53) and (119)–(122), that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) \to (\boldsymbol{x}^*, \boldsymbol{p}^*)$ as $\delta \to 1$. This step establishes part (i) of Theorem 6 provided that $\boldsymbol{x}^*(\delta)$ is a solution to program (P2), which we show to be the case in Appendix G.7.

To begin, we show that $\lim_{\delta\to 1}\left[\overline{T}(\delta)-\overline{T}\right] = 0$. From equations (117) and (48), it follows that it is enough to show that $\lim_{\delta\to 1}[\underline{t}(\delta)-\underline{t}] = 0$. The desired result follows by observing the following:

$$
\begin{aligned}
\lim_{\delta\to 1}[\underline{t}(\delta)-\underline{t}] = 0 &\iff \lim_{\delta\to 1}\left[\frac{1}{\ln r(\delta)}-\frac{1}{\ln r}\right] = 0 \\
&\iff \lim_{\delta\to 1}\left[\frac{\ln r-\ln r(\delta)}{\ln r\ln r(\delta)}\right] = 0 \\
&\iff \lim_{\delta\to 1}\ln\left[\frac{r}{r(\delta)}\right] = 0 \\
&\iff \lim_{\delta\to 1}\ln\left(\frac{\frac{\theta_h-\delta\theta_\ell}{\delta\Delta\theta}}{\frac{\theta_h-\delta(\theta_\ell+\varepsilon(\delta))}{\delta(\Delta\theta-\varepsilon(\delta))}}\right) = 0 \\
&\iff \ln 1 = 0,
\end{aligned}
$$

where: the first equivalence holds by equations (49) and (118); the fourth equivalence holds by definitions (3) and (74).

Since $\lim_{\delta \to 1} \left[\overline{T}(\delta) - \overline{T}\right] = 0$, that $\lim_{\delta \to 1}[x^*_{\ell,t}(\delta) - x^*_{\ell,t}] = 0$ for all $t \in \mathcal{T}_0$ immediately follows from equations (50) and (119).

Next, note that

$$
\begin{aligned}
\lim_{\delta \to 1} x^*_{h,1} &= \lim_{\delta \to 1} \frac{1}{\mu}\left(1 - (1-\mu)\rho^{\overline{T}-2}\right) \\
&= \frac{1}{\mu} - \frac{1-\mu}{\mu} \lim_{\delta \to 1} \frac{1}{\rho} \delta^{\overline{T}-1} r^{\overline{T}-1} \\
&= \frac{1}{\mu} - \frac{1-\mu}{\mu} \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell},
\end{aligned}
\tag{124}
$$

where: the first equality holds by equation (51); the third equality holds by equations (59) and (60). Moreover, note that

$$
\overline{T}(\delta) = \inf\left\{t \in \mathcal{T}_1 : t \geq \underline{t}(\delta)\right\} \Longrightarrow \underline{t}(\delta) - 1 \leq \overline{T}(\delta) - 1 \leq \underline{t}(\delta). \tag{125}
$$

From implication (125) and since $\delta \in (0,1)$, we have $\delta^{\underline{t}}(\delta) \leq \delta^{\overline{T}(\delta)-1} \leq \delta^{\underline{t}(\delta)-1}$. Thus,

$$
\lim_{\delta \to 1} \delta^{\overline{T}(\delta)-1} = \lim_{\delta \to 1} \delta^{\underline{t}(\delta)-1} = \lim_{\delta \to 1} \delta^{\frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln \frac{\theta_h - \delta(\theta_\ell + \varepsilon(\delta))}{\delta(\Delta\theta - \varepsilon(\delta))}}} = \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}}, \tag{126}
$$

where: the first equality holds by the sandwich theorem for the limits of functions; the second equality holds by equation (99) and definition (74); the third equality holds as $\lim_{\delta \to 1} \varepsilon(\delta) = 0$. From implication (125) and since $r(\delta) > 1$, we have $r(\delta)^{\underline{t}(\delta)-1} \leq r(\delta)^{\overline{T}(\delta)-1} \leq r(\delta)^{\underline{t}(\delta)}$. Moreover, as $\delta \to 1$, $r \to 1$ (see definition (74)). Thus,

$$
\lim_{\delta \to 1} r(\delta)^{\overline{T}(\delta)-1} = \lim_{\delta \to 1} r(\delta)^{\underline{t}(\delta)-1} = \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}, \tag{127}
$$

where: the first equality holds by the sandwich theorem for the limits of functions; the second equality holds by equation (98). Therefore, we have

$$
\begin{aligned}
\lim_{\delta \to 1} x^*_{h,1}(\delta) &= \lim_{\delta \to 1} \frac{1}{\mu(\theta_\ell + \varepsilon(\delta))}\left[\theta_\ell + \mu\varepsilon(\delta) - (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right] \\
&= \frac{1}{\mu} - \frac{1-\mu}{\mu} \lim_{\delta \to 1} \frac{1}{\rho(\delta)} \delta^{\overline{T}(\delta)-1} r(\delta)^{\overline{T}(\delta)-1} \\
&= \frac{1}{\mu} - \frac{1-\mu}{\mu} \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell},
\end{aligned}
\tag{128}
$$

where: the first equality holds by equation (120); the third equality holds by equations (126) and (127). That $\lim_{\delta \to 1}[x^*_{h,1}(\delta) - x^*_{h,1}] = 0$ follows from equations (124) and (128).

Since $\rho^{\overline{T}-1} = \delta^{\overline{T}-1} r^{\overline{T}-1}$, from equations (59) and (60), it follows that $\lim_{\delta\to 1} \rho^{\overline{T}-1}$ is finite. In addition, for all $t \in \mathcal{T}_1$, $\lim_{\delta\to 1} \rho^t = 1$. Therefore, for all $t \in \mathcal{T}_1$,

$$\lim_{\delta\to 1} x^*_{h,t} = \lim_{\delta\to 1} \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{\overline{T}-t-1} = \lim_{\delta\to 1} \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\frac{1}{\rho^t}\rho^{\overline{T}-1} = 0, \quad (129)$$

where the first equality holds by equation (51). Moreover, since $\rho(\delta)^{\overline{T}(\delta)-1} = \delta^{\overline{T}(\delta)-1} r(\delta)^{\overline{T}(\delta)-1}$, from equations (126) and (127), it follows that $\lim_{\delta\to 1} \rho(\delta)^{\overline{T}(\delta)-1}$ is finite. In addition, for all $t \in \mathcal{T}_1$, $\lim_{\delta\to 1} \rho(\delta)^t = 1$. Therefore, for all $t \in \mathcal{T}_1$,

$$\begin{aligned}
\lim_{\delta\to 1} x^*_{h,t}(\delta) &= \lim_{\delta\to 1} \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta-\varepsilon(\delta))}\rho(\delta)^{\overline{T}(\delta)-t-1} \\
&= \lim_{\delta\to 1} \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu(\Delta\theta-\varepsilon(\delta))}\frac{1}{\rho(\delta)^t}\rho(\delta)^{\overline{T}(\delta)-1} \\
&= 0,
\end{aligned} \quad (130)$$

where the first equality holds by equation (120). That $\lim_{\delta\to 1}[x^*_{h,t}(\delta) - x^*_{h,t}] = 0$ for all $t \in \mathcal{T}_1$ follows from equation (129) and (130).

Step 7. We show that $x^*_{h,1}(\delta) \geq 0$ if δ is sufficiently high.

Note that: $x^*_{h,1}(\delta)$ is continuous in δ; and $\lim_{\delta\to 1}[x^*_{h,1}(\delta) - x^*_{h,1}] = 0$. Moreover, in step 5 of Appendix B.4 we show that $x^*_{h,1} > 0$ (see Remark 1). Therefore, there exists $\underline{\delta}_1 \in (0,1)$ such that $x^*_{h,1}(\delta) \geq 0$ for all $\delta \in (\underline{\delta}_1, 1)$.

Step 8. We show that $U_P^{\text{P2b}}(\pi) = U_P^{\text{P2c}}(\pi)$, so that, by weak duality, the candidate solution to program (P2b) is indeed a solution to the program for all $\delta \in (\underline{\delta}_1, 1)$.

To obtain the desired result, note that

$$\begin{aligned}
U_P^{\text{P2c}}(\pi) - U_P^{\text{P2b}}(\pi) &= \frac{\theta_h(\theta_\ell + \mu\varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} - \delta^{\overline{T}(\delta)-1}\mu\Delta\theta \\
&\quad - \delta^{\overline{T}(\delta)-1}\frac{(1-\mu)\theta_\ell(\Delta\theta - \varepsilon(\delta))}{\theta_\ell + \varepsilon(\delta)} r(\delta)^{\overline{T}(\delta)-1} \\
&\quad - \frac{\theta_h}{\theta_\ell + \varepsilon(\delta)}\left[\theta_\ell + \mu\varepsilon(\delta) - (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\right] + \delta^{\overline{T}(\delta)-1}\mu\Delta\theta \\
&\quad - \delta^{\overline{T}(\delta)-1}(1-\mu)\theta_\ell r(\delta)^{\overline{T}(\delta)-2} \\
&= \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell + \varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-2} \\
&\quad - (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-1}\left(\frac{\Delta\theta - \varepsilon(\delta)}{\theta_\ell + \varepsilon(\delta)} + \frac{1}{r(\delta)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell+\varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-2} \\
&- (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-1}\frac{\theta_h(\Delta\theta-\varepsilon(\delta))}{(\theta_\ell+\varepsilon(\delta))(\theta_h-\delta(\theta_\ell+\varepsilon(\delta)))} \\
&= \frac{(1-\mu)\theta_\ell\theta_h}{\theta_\ell+\varepsilon(\delta)}\rho(\delta)^{\overline{T}(\delta)-2} - (1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-1}\frac{\theta_h}{\theta_\ell+\varepsilon(\delta)}\frac{1}{\rho(\delta)} \\
&= 0
\end{aligned}
$$

where: the first equality holds by equations (108) and (123); the second to fourth equalities hold by definitions (74) and (75).

Step 9. We show that $\lim_{\delta\to 1}[U_P^{\text{P2b}}(\pi) - U_P^{\text{P1}}(\pi)] = 0$. This step establishes part (i) of Theorem 6 provided that $\boldsymbol{x}^*(\delta)$ is a solution to program (P2), which we show to be the case in Appendix G.7.

Note that

$$
\begin{aligned}
\lim_{\delta\to 1} U_P^{\text{P2b}}(\pi) &= \lim_{\delta\to 1}\frac{\theta_h}{\theta_\ell+\varepsilon(\delta)}\Big[\theta_\ell+\mu\varepsilon(\delta)-(1-\mu)\theta_\ell\rho(\delta)^{\overline{T}(\delta)-2}\Big] \\
&- \lim_{\delta\to 1}\delta^{\overline{T}(\delta)-1}\mu\Delta\theta + \lim_{\delta\to 1}\delta^{\overline{T}(\delta)-1}(1-\mu)\theta_\ell r(\delta)^{\overline{T}(\delta)-2} \\
&= \lim_{\delta\to 1}\frac{\theta_h}{\theta_\ell+\varepsilon(\delta)}\Big[\theta_\ell+\mu\varepsilon(\delta)-(1-\mu)\theta_\ell\frac{1}{\rho(\delta)}\rho(\delta)^{\overline{T}(\delta)-1}\Big] \\
&- \lim_{\delta\to 1}\delta^{\overline{T}(\delta)-1}\mu\Delta\theta + \lim_{\delta\to 1}\delta^{\overline{T}(\delta)-1}(1-\mu)\theta_\ell\frac{1}{r(\delta)}r(\delta)^{\overline{T}(\delta)-1} \qquad (131)\\
&= \theta_h - (1-\mu)\theta_h\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}}\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} \\
&- \mu\Delta\theta\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} + (1-\mu)\theta_\ell\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}}\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} \\
&= \theta_h - \Delta\theta\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}}\left(\mu+(1-\mu)\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right),
\end{aligned}
$$

where: the first equality holds by equation (123); the second equality holds because $\lim_{\delta\to 1}\varepsilon(\delta)=0$, $\lim_{\delta\to 1}r(\delta)=0$, $\lim_{\delta\to 1}\rho(\delta)=1$, and by equations (126) and (127). The desired result follows from equivalence (61) and equation (131).

G.6 A Solution to the Primal Program (P2a)

We show that the solution $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) \in \mathbb{R}^{4\infty}$ to program (P2b) described in Step 5 of Appendix G.5 satisfies constraint (ICℓ) and constraint (O$'$) for $\tau = 1$ in the primal program (P2a), so that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ is also a solution to program

(P2a) if δ is sufficiently high.

Step 1. By equations (119) and (121), the left-hand side of constraint (ICℓ) evaluated at $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ equals 0. By equations (120) and (122), and since $\theta_h > \theta_\ell$, the right-hand side of constraint (ICℓ) evaluated at $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ is smaller than $\theta_\ell x^*_{h,1}(\delta) - p^*_{h,1}(\delta)$. Thus, to show that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ satisfies constraint (ICℓ), it suffices to show that $0 \geq \theta_\ell x^*_{h,1}(\delta) - p^*_{h,1}(\delta)$. Note that: $x^*_{h,1}(\delta)$ and $p^*_{h,1}(\delta)$ are continuous in δ; $\lim_{\delta\to 1}[x^*_{h,1}(\delta) - x^*_{h,1}] = 0$; and $\lim_{\delta\to 1}[p^*_{h,1}(\delta) - p^*_{h,1}] = 0$. Moreover, in Appendix B.5, we show that $0 > \theta_\ell x^*_{h,1} - p^*_{h,1}$ (see Remark 2). Therefore, there exists $\underline{\delta}_2 \in (0,1)$ such that $0 \geq \theta_\ell x^*_{h,1}(\delta) - p^*_{h,1}(\delta)$ for all $\delta \in (\underline{\delta}_2, 1)$.

Step 2. In step 9 of Appendix G.5, we show that $\lim_{\delta\to 1}[U_P^{\text{P2b}}(\pi) - U_P^{\text{P1}}(\pi)] = 0$. In step 2 of Appendix D, we show that $\lim_{\delta\to 1} U_P^{\text{P1}}(\pi) > \theta_\ell$. Moreover, note that: (i) the left-hand side of constraint (O$'$) for $\tau = 1$ is equal to $U_P^{\text{P2b}}(\pi)$; and (ii) the right-hand side of constraint (O$'$) converges to θ_ℓ as $\delta \to 1$ (as $\varepsilon(\delta) \to 0$). Therefore, there exists $\underline{\delta}_3 \in (0,1)$ such that, for all $\delta \in (\underline{\delta}_3, 1)$, $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ satisfies constraint (O$'$) for $\tau = 1$ in the primal program (P2a).

Step 3. We conclude that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ is a solution to program (P2a) for all $\delta \in (\underline{\delta}, 1)$, where $\underline{\delta} := \max\{\underline{\delta}_1, \underline{\delta}_2, \underline{\delta}_3\}$.

G.7 A Solution to the Primal Program (P2)

We show that the solution $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) \in \mathbb{R}^{4\infty}$ to program (P2a) described in Step 5 of Appendix G.5 satisfies constraints (ICℓ') and (ICh') if $\delta \in (\underline{\delta}, 1)$.

Step 1. We show that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ satisfies constraint (ICℓ') for all $\tau \in \mathcal{T}_0$.

For all $\tau \in \mathcal{T}_0$, $p^*_{\ell,t}(\delta) = \theta_\ell x^*_{\ell,t}(\delta)$ (see equation (121)). Thus, the left-hand side of constraint (ICℓ') equals 0 for all $\tau \in \mathcal{T}_0$.

For all $\tau \in \mathcal{T}_1$, $\theta_\ell x^*_{h,\tau}(\delta) - p^*_{h,\tau}(\delta) \leq \theta_h x^*_{h,\tau}(\delta) - p^*_{h,\tau}(\delta) = 0$, where the inequality holds because $\theta_\ell < \theta_h$ and the equality holds by equality (122). Moreover, $1 - \sum_{t=1}^{\tau-1} x^*_{\ell,t}(\delta) \geq 0$ for all $\tau \in \mathcal{T}_0$. Thus, the right-hand side of constraint (ICℓ') is non-positive for all $\tau \in \mathcal{T}_1$, and so $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ satisfies constraint (ICℓ') for all $\tau \in \mathcal{T}_1$.

To show that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ satisfies constraint (ICℓ') for $\tau = 1$, we need to show that $\theta_\ell x^*_{h,1}(\delta) - p^*_{h,1}(\delta) \leq 0$. In Appendix G.6, we have already shown that the previous inequality holds true for all $\delta \in (\underline{\delta}_2, 1)$.

Step 2. We show that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ satisfies constraint (ICh') for all $\tau \in \mathcal{T}_0$.

For all $\tau \neq \overline{T}(\delta)$, $x^*_{\ell,\tau}(\delta) = 0$ and $p^*_{\ell,\tau}(\delta) = 0$ (see equations (119) and (121)). Thus, the right-hand side of constraint (ICh') equals 0 for all $\tau \neq \overline{T}(\delta)$. Moreover, $\sum_{t=1}^{\overline{T}(\delta)-1} x^*_{h,t}(\delta) = 1$ (see equation (116)), and so the right-hand side of constraint (ICh') equals 0 also for $\tau = \overline{T}(\delta)$. Since $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (IRh) for all $\tau \in \mathcal{T}_0$, that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ also satisfies constraint (ICh') for all $\tau \in \mathcal{T}_0$ follows. ■

H Calculations for Section 3.1

Consider the parametric specification in Example 1: $\delta = \frac{1}{2}$, $\theta_\ell = 1$, $\theta_h = 1$, and $\mu = \frac{9}{10}$. Under such specification, by Theorems 1 and 3 and the solution to program (P1) in Appendix B, a candidate for a seller-optimal wPBE outcome of $\mathcal{G}$ is as follows (see Step 6 of Appendix B.4).

1. $\underline{t} = 1 + \frac{\ln\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\ln r} = 1 + \frac{\ln 9}{\ln 3} = 3$, and so $\overline{T} = \inf\{t \in \mathcal{T}_1 : t \geq \underline{t}\} = 3$.
2. $x^*_{\ell,1} = 0$, $x^*_{\ell,2} = 0$, and $x^*_{\ell,3} = 1$.
3. $x^*_{h,1} = \frac{1}{\mu}\left(1 - (1-\mu)\rho^{\overline{T}-2}\right) = \frac{10}{9}\left(1 - \frac{3}{20}\right) = \frac{17}{18}$, $x^*_{h,2} = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta} = \frac{1}{18}$, and $x^*_{h,3} = 0$.
4. $p^*_{\ell,1} = 0$, $p^*_{\ell,2} = 0$, and $p^*_{\ell,3} = 1$.
5. $p^*_{h,1} = \theta_h x^*_{h,1} - \delta^{\overline{T}-1}\Delta\theta = 2\frac{17}{9} - \frac{1}{4} = \frac{59}{36}$, $p^*_{h,2} = \theta_h x^*_{h,2} = 2\frac{1}{18} = \frac{1}{9}$, and $p^*_{h,3} = 0$.
6. From 2 and 4, the transfers from type $\theta = 1$ to the principal conditional on trade in period t, denoted by $p_{\ell,t}$, are $p_{\ell,1} = 0$, $p_{\ell,2} = 0$, and $p_{\ell,3} = 1$. From 3 and 5, the transfers from type $\theta = 2$ to the principal conditional on trade in period t, denoted by $p_{h,t}$, are $p_{h,1} = \frac{59}{34}$, $p_{h,2} = 2$, and $p_{h,3} = 0$.

Steps 1–6 show that the entries of the first row in Table 3.1 in Section 3.1 describe a candidate for a seller-optimal wPBE outcome of the MCS game. To show that this is indeed a wPBE outcome of the MCS game, by Theorems 2 and 5, it is enough to show that $\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{1}{2}$ (see also Section E). Note that

$$\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) := \frac{\left(1 - x^*_{h,1}\right)\mu}{\left(1 - x^*_{\ell,1}\right)(1-\mu) + \left(1 - x^*_{h,1}\right)\mu} = \frac{\frac{1}{18}\frac{9}{10}}{1\frac{1}{10} + \frac{1}{18}\frac{9}{10}} = \frac{1}{3} < \frac{1}{2},$$

from which the desired result follows. ■