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Does dynamic market competition with technological innovation leave no one behind?: An axiomatic study in OLG frameworks

Yongsheng Xu
(Department of Economics, Andrew Young School of Policy Studies,
Georgia State University)
and
Naoki Yoshihara
(Department of Economics, University of Massachusetts Amherst;
Institute of Economic Research, Hitotsubashi University, and
School of Management, Kochi University of Technology)

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Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603 Japan

Does dynamic market competition with technological innovation leave no one behind?: An axiomatic study in OLG frameworks*

Yongsheng Xu[†] and Naoki Yoshihara[‡]

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Abstract

The *Hicksian optimism*, a neoclassical economic creed, says that the consistent implementation of 'Pareto-efficient policies' sequentially would eventually improve the welfare of every individual from the initial position in the long run. In this paper, we formulate the Hicksian optimism as an axiom and then examine whether the market mechanism with the consistent application of technological progress policies can fulfill the Hicksian optimism. We show in a simple Overlapping Generations model that the market mechanism with technological progress unavoidably leaves some individuals behind. This negative result holds for a broad class of intertemporal resource allocation mechanisms.

Keywords: dynamic market competition with technological progress, Hicksian Optimism, Walrasian allocation rule, weak Pareto efficiency, individual rationality, overlapping generations economy

JEL Classification Numbers: D30, D51, D60, O33, P10, P20, P40

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[†]Department of Economics, Andrew Young School of Policy Studies, Georgia State University, Atlanta, GA 30303, U.S.A. Email: yxu3@gsu.edu

[‡]Department of Economics, University of Massachusetts Amherst, Crotty Hall, Amherst, MA, 01002, USA; The Institute of Economic Research, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-0004, Japan; and School of Management, Kochi University of Technology, Tosayamada, Kami-city, Kochi 782-8502, Japan. Email: nyoshihara@econs.umass.edu

1 Introduction

1.1 Three Basic Functions of Market Mechanism

The market mechanism is the central subject in economics. Economics from the age of Adam Smith till the present day has viewed the market mechanism as having the following three basic functions:

i) it is a decentralized resource allocation mechanism that improves the economic welfare of consumers as a result of market exchange;

ii) it promotes the technological and social division of labor via specialization as motivated by the principle of comparative advantage, and consequently, it increases efficiency of production activities in the economy as a whole; and

iii) it promotes technical progress in order to enhance the productivity of the economy as a whole through a dynamic process of competition among the producers,[1] and consequently it improves individual welfare progressively.

Among the three functions of the market mechanism mentioned above, the first function is illustrated and explained in the Walrasian general equilibrium theory as the Fundamental Theorems of Welfare Economics, while the second function is captured and characterized in the Heckshar-Ohlin international trade theory as the Fundamental Theorems of International Trade (in particular, the Heckshar-Ohlin theorem).

In contrast, there are no basic results or theoretically rigorous analysis in the literature of contemporary economics that supports the argument for the third function of the market mechanism, despite some attempts to justify the market economy on this ground.[2] The third function of the market mechanism contains two main points. First, the market mechanism can enhance the productivity of the economy as a whole through promoting technical progress in the dynamic process of competition among producers. Second, assuming that the dynamic competition in the market economy promotes technical innovation, the market mechanism can ensure the progressive improvement of individual welfare.

Regarding the first point, it can be argued that the market competition does not necessarily promote technological innovation, as the knowledge of a new technology can be regarded as a pure public good. Indeed, it is easily diffused once it is discovered (or invented) and publicized. As a consequence, a competitive producer may not have enough motive for R&D investment, as the rival producers can easily freeride on the successful result of an R&D before the competitive producer fully reaps the benefits of the R&D investment. See, for example, Schumpeter (1939, 1942), and the evidence presented in the study by Blundell, Griffith and van Reenen (1999) where they find, in the study of

[1] Producers are motivated by capturing extra profits via the access to the most advanced technology.

[2] See, for example, Kornai (2013).

British manufacturing firms, that the monopolist invests more in its R&D. It is therefore further suggested that introducing the patent system would be a desirable incentive scheme to encourage R&D investment, though such a scheme regulates market competition by allowing the innovator to be a monopolist for some restricted period of time. On the other hand, we also have the argument that the market competition promotes technological innovation: more competition tends to diminish profits, which gives firms more incentive to innovate to capture profits (see, for example, Arrow (1962)). The recent literature on the topic has illustrated a complicated picture between competition and innovation (see, among others, Aghion et al. (2005), Igami and Uetake (2020)). For example, Aghion et al. (2005) have shown an inverted U-shaped relationship between innovation and competition, where some certain intermediate levels of competition give rise to the highest level of innovation.

However, even setting aside the incentive problem of R&D investment, the question that the market mechanism can ensure the progressive improvement of individual welfare remains. This is because the introduction of a new technology often involves a radical change of economic structure, which leads to the division of the population into the "winners" and the "losers".[3]

The issue concerning the winners and losers seems to be easily dealt with by the Kaldor-Hicks compensation test (Hicks (1939), Kaldor (1939)). Indeed, if a technical progress due to innovation can be formulated as an expansion of the production possibility set available in the economy à la Schumpeter (1942), then a policy encouraging such innovations would always be socially desirable with respect to the (weaker) Kaldor-Hicks hypothetical compensation principle. This is because, a change from a competitive equilibrium under one economy to another competitive equilibrium under possibly a different economy involves an *expansion* of the production possibility set in terms of set inclusion, and consequently, the aggregate sum of all consumers' 'upper contour sets' definitely has a non-empty intersection with the new production possibility set. Therefore, even if some consumer becomes a "loser" with this change, this consumer can be (hypothetically) compensated via a suitable shift of an aggregate supply of commodity bundles, and a suitable redistribution of the goods.

Such a compensation is, however, just hypothetical and the Kaldor-Hicks compensation principle provides no mechanism to implement such a compensation, and as a result, a loser would be left as a "loser" in reality. Therefore, judging such possible improvements of individual welfare by means of the Kaldor-Hicks compensation principle alone would not be much help if the hypothetical compensation cannot be actually implemented.

Nevertheless, neoclassical economics has attempted to justify the application of the (somewhat modified) hypothetical compensation principle to evaluate possible improvement of social welfare by appealing to the so-called *Hicksian Optimism*. As stated by Corden (1984, page 68), "if Pareto efficient policies are being pursued consistently over a long period, the *chances* are that eventually–

[3] See Frey (2019) for a fascinating historical account of the consequences of the introduction of new technologies.

though not at every particular step–everyone will be better off." He labels this principle as the *Hicksian Optimism* and attributes it to Hicks (1941) where Hicks writes:

> " ... then, although we could not say that all the inhabitants of that community would be necessarily better off than they would have been if the community had been organized on some different principle, nevertheless there would be a strong probability that almost all of them would be better off after the lapse of a sufficient length of time. Substantially, that is the creed of classical economics; if the 'improvements' are properly defined, it would appear to be a creed that is soundly based."

Therefore, if we follow through the above idea of the Hicksian optimism, we may conjecture that the innovation and technological progress in the market place can eventually benefit every individual in the society[4], laying a foundation for the third function of the market mechanism, and for achieving some goals like "inclusive economic growth","shared prosperity", and "no one being left behind" envisioned and set forth in the 2030 Agenda for Sustainable Development as pledged by 193 United Nations Member States.[5]

The purpose of this paper is to examine the above conjecture. While doing so, we propose a theoretical framework and introduce the notion of Hicksian optimism in the context. We then examine whether the expansions of the society's production possibility sets due to innovation and technological progress can indeed benefit every member of the society *eventually* (after a long period of time).

1.2 Organization of the Analysis of the Paper

We introduce and study a *simple overlapping generations economy.* Time is discrete and infinite. Each generation is a single individual and lives for two periods: works at young and retires at old (and then dies). Each generation has a common well-behaved utlity function of lifetime consumption activities and is endowed with one unit of labor endowment at young. In each period, there is a convex cone production possibility set. Due to economic development and technological progress made in the economy, production possibility sets can vary across periods of time.

An economic system is viewed as an *intertemporal allocation rule* that applies a desired allocation policy consistently and sequentially to each and every period of a simple overlapping generations economy in order to specify a subset of desirable intertemporal allocations feasible for each simple overlapping generations economy. A prominent example of such an allocation rule is the

[4] Kandori (2023) presents a slightly weaker version of the Hicksian optimism and observes that "... the experiences of the twentieth century taught us that, if we *adoopt the compensation principle* and *apply it consistently, many people will be better off in the long run."*

[5] See https://sustainabledevelopment.un.org/post2015/transformingourworld.

Walrasian intertemporal allocation rule, which specifies the set of Walrasian competitive equilibrium allocations for each and every period of a simple overlapping generations economy.

We then formulate the notion of *Hicksian optimism* in our setting by requiring an intertemporal allocation rule to choose a feasible intertemporal allocation that *eventually* improves every later generation's welfare from the first generation, if the simple overlapping generations economy is growing and transforming to a stationary overlalpping genrations economy. See Section 3 for a formal definition of the *axiom of Hicksian Optimism*. We show that an efficient and individually rational intertemporal allocation rule cannot satisfy the axiom of Hicksian Optimism. As a corollary of this result, the Walrasian intertemporal allocation rule is also shown to violate Hicksian Optimism.

Why use the OLG economic model in this paper? To test whether the market mechanism works as predicted by the idea of Hicksian optimism, it is appropriate to analyze it in a context of intertemporal resource allocation problems. It may be noted that a typical analysis using intertemporal economic models assumes a representative agent who exists perpetually over an infinite horizon of periods, and evaluates economic mechanisms based on the representative agent's welfare information, which is the lifetime welfare of the agent measured over an infinite horizon of periods. However, this standard approach is not appropriate when evaluating the performance of the economic system from the perspective of Hicksian optimism. This is because the idea of Hicksian optimism implicitly presumes the existence of multiple individuals in each period and compares the one-period welfare allocation of all individuals present in the initial period economy with the one-period welfare allocation of all later individuals present in the eventual economy from the perspective of Pareto improvement. Therefore, to implement the Hicksian optimism test in such a setting, while preserving the intrinsic features of intertemporal resource allocation problems such as involving the choice problem of consumption and saving, the adoption of an OLG framework would seem to be the most appropriate: it allows us to study the problem at hands in a simplest economic model with a few each finitely living individuals in each period over an infinite horizon of periods.

The remainder of the paper is organized as follows. Section 2 presents our basic model. Section 3 formulates the notion of Hicksian optimism. Section 4 examines whether any efficient and individually rational intertemporal allocation rule can satisfy Hicksian Optimism. We conclude in Section 5.

2 A Basic Model

Time is discrete $t = 1, 2, \cdots$ and let the time periods be indexed by $\mathcal{T} = \{1, 2, \cdots, \}$. Consider a simple Overlapping Generations (OLG) economy in which each generation $t = 1, 2, \ldots,$ is a single individual, lives for two periods, works only at young and retires at old and then dies. In the initial period $t = 1$, there are two individuals: one is the generation $t = 0$ at the old age (and dies at the end of the period $t = 1$) and the other is generation $t = 1$ at the young age. There are $n \geqq 2$ commodities that are produced in this economy and can be used as consumption goods and/or capital goods.

Each generation $t = 1, 2, \ldots,$ at the young age is endowed with one unit of labor endowment, $\omega_l = 1$. The generation $t = 0$ at the old age is endowed with an endowment vector $\omega_a^0 \in \mathbb{R}_+^n$.[6] ω_a^0 is the saving of the generation $t = 0$ at the young age in period $t = 0$ (outside this model analysis). For each generation $t = 1, 2, \ldots,$ $x_b^t \in \mathbb{R}_+^n$ represents a consumption bundle that is consumed by generation t at young and $x_a^t \in \mathbb{R}_+^n$ represents a consumption bundle that is consumed by the generation t at old. We shall use $\omega_a^t \in \mathbb{R}_+^n$ to denote a commodity bundle that is saved by the generation t at young for the sake of the consumption activity at old. $l^t \in [0, 1]$ denotes the amount of labor supplied by the generation t at young.

Each generation $t = 0, 1, 2, \ldots,$ has a common *utility function of lifetime consumption activities* as follows: $u : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ such that there are two *one-period utility functions*, $v_b : \mathbb{R}_+^n \to \mathbb{R}$ and $v_a : \mathbb{R}_+^n \to \mathbb{R}$, and a real-valued function $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying $u(x_b, x_a) = U(v_b(x_b), v_a(x_a))$ for all $(x_b, x_a) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$. We assume that $U(\cdot, \cdot)$ is continuous and is increasing in each of its arguments, and, for $i = a, b$, v_i is continuous, quasi-concave and increasing on $\mathbb{R}_+^n$.[7]

In each period $t \in \mathcal{T}$, there is a production possibility set $Y^t \subseteq \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n$ associated with a generic element $y \equiv \left(-y_l, -\underline{y}, \overline{y}\right) \in Y^t$, where $y_l \in \mathbb{R}_+$ represents a direct labor input, $\underline{y} \in \mathbb{R}_+^n$ represents a vector of commodity inputs and $\overline{y} \in \mathbb{R}_+^n$ represents a vector of gross outputs. Assume that Y^t is a closed convex cone with $\mathbf{0} \in Y^t$ and satisfies the following properties:

A1. There are two subsets Y_b^t and $\underline{Y}^t$ of Y^t such that[8] $co\left(Y_b^t \cup \underline{Y}^t\right) = Y^t$ and
(i) for any $y = \left(-y_l, -\underline{y}, \overline{y}\right) \in \underline{Y}^t$, if $\overline{y} \geq \mathbf{0}$, then $\underline{y} \geq \mathbf{0}$ and $y_l > 0$;
(ii) $Y_b^t \equiv \cup_{\lambda \in \mathbb{R}_+} \left\{\lambda\left(-1, \mathbf{0}, \omega_b^t\right)\right\}$ for some $\omega_b^t \in \mathbb{R}_+^n$ such that, for some $y' = \left(-1, -\underline{y}', \overline{y}'\right) \in \underline{Y}^t$, $\overline{y}' - \underline{y}' > \omega_b^t$ holds,
A2. For any $d \in \mathbb{R}_+^n$, there exists $y = \left(-y_l, -\underline{y}, \overline{y}\right) \in Y^t$ such that $\overline{y} - \underline{y} \geqq d$,
A3. For any $y = \left(-y_l, -\underline{y}, \overline{y}\right) \in Y^t$ and any $\left(-y_l', -\underline{y}'\right) \in \mathbb{R}_- \times \mathbb{R}_-^n$, if $\left(-y_l', -\underline{y}'\right) \leqq \left(-y_l, -\underline{y}\right)$, then $\left(-y_l', -\underline{y}', \overline{y}\right) \in Y^t$.

A3 is the standard property of *free disposal*. **A2** represents the standard property of *productiveness* and requires that any final demand vector can be produced as a net output vector via some production activities. **A1** implies that there are two types of production techniques: one type, represented by Y_b^t, is perfectly labor intensive in which a non-negative vector of gross outputs can be produced only by labor inputs, and the other type, represented by $\underline{Y}^t$, requires

[6] For any positive integer m, $\mathbb{R}^m$ (resp. $\mathbb{R}_+^m$, $\mathbb{R}_{++}^m$ and $\mathbb{R}_-^m$) denotes the m-fold Cartesian product of $\mathbb{R} = (-\infty, +\infty)$ (resp. $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$). For any $a, b \in \mathbb{R}^m$, $a \geqq b$ denotes $[a_1 \geqq b_1, \cdots, a_m \geqq b_m]$, $a \geq b$ denotes $[a \geqq b$ and $a \neq b]$, and $a > b$ denotes $[a_1 > b_1, \cdots, a_m > b_m]$.

[7] v_i is increasing if for all x_i and x_i' with $x_i \geqq x_i'$, we have $v_i(x_i) \geqq v_i(x_i')$ and $v_i(x_i) > v_i(x_i')$ for all x_i and x_i' with $x_i > x_i'$.

[8] For any set $K \subseteq \mathbb{R}^k$, $co(K)$ denotes the *convex hull of* K.

both labor input and non-labor inputs to produce a non-zero and non-negative vector of gross outputs. **A1** further requires that the net output productivity of the perfectly labor intensive technique Y_b^t is *inferior to* the technique contained in $\underline{Y}^t$: the net outout vector ω_b^t produced by one unit of labor in Y_b^t is dominated by a net output vector produced with one unit of labor input in $\underline{Y}^t$. It may be noted that the generation t at the young age can produce and consume ω_b^t by using one unit of the labor endowment in an autarkic economy.

An *economy* in period $t \in \mathcal{T}$, to be denoted by E_t, is a triple $\langle Y^t; \omega_l; u\rangle$. Note that in the economy E_t, there are two individuals, the generation $t-1$ at the old age and the generation t at the young age. Let $\mathcal{E}^t$ be the set of all available economies in period $t \in \mathcal{T}$. Let $\mathcal{E} \equiv \cup_{t\in\mathcal{T}}\mathcal{E}^t$.

For an economy $E_t \in \mathcal{E}^t$ associated with the savings bundle $\omega_a^{t-1} \in \mathbb{R}_+^n$ of generation $t-1$ at the old age, a profile $\left(x_a^{t-1}; \left(x_b^t, x_a^t\right), l^t, \omega_a^t; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)\right)$ is called a *temporary allocation in period* t if and only if $x_a^{t-1} \in \mathbb{R}_+^n$; $\left(x_b^t, x_a^t\right) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, $l^t \in [0,1]$, and $\omega_a^t \in \mathbb{R}_+^n$; and $\left(-y_l^t, -\underline{y}^t, \overline{y}^t\right) \in Y^t$. Moreover, a temporary allocation $\left(x_a^{t-1}; \left(x_b^t, x_a^t\right), l^t, \omega_a^t; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)\right)$ is *feasible in period* t if and only if $x_a^{t-1} + x_b^t + \omega_a^t \leqq \overline{y}^t - \underline{y}^t + \omega_a^{t-1}$ with $\underline{y}^t \leqq \omega_a^{t-1}$ and $y_l^t \leqq l^t$. Denote the set of temporary feasible allocations for the economy E_t with ω_a^{t-1} by $F(E_t; \omega_a^{t-1})$.

Define a *general OLG economy* by $(E_t = \langle Y^t; \omega_l; u\rangle)_{t=1}^{\infty}$. Let $\mathcal{E}^\infty$ be the set of all general OLG economies with generic element $E^\infty = (E_1, \cdots, E_t, \cdots)$. Given a general OLG economy $E^\infty = (E_1, \cdots, E_t, \cdots) \in \mathcal{E}^\infty$ associated with the savings bundle $\omega_a^0 \in \mathbb{R}_+^n$ of generation $t = 0$ at the old age, a profile

$$\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right)$$

is called an *intertemporal allocation* if and only if for each period $t \in \mathcal{T}$, $\left(x_a^{t-1}; \left(x_b^t, x_a^t\right), l^t, \omega_a^t; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)\right)$ is a temporary allocation in period t associated with the saving bundle $\omega_a^{t-1} \in \mathbb{R}_+^n$ of generation $t-1$ at the old age. Moreover, given a general OLG economy $E^\infty = (E_1, \cdots, E_t, \cdots) \in \mathcal{E}^\infty$ associated with the savings bundle $\omega_a^0 \in \mathbb{R}_+^n$ of generation $t = 0$ at the old age, an intertemporal allocation $\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right)$ is *feasible for the economy* E^∞ *with* $\omega_a^0 \in \mathbb{R}_+^n$ if and only if

$$\begin{aligned} x_a^0 + x_b^1 + \omega_a^1 &\leqq \overline{y}^1 - \underline{y}^1 + \omega_a^0 \text{ with } \underline{y}^1 \leqq \omega_a^0 \text{ and } y_l^1 \leqq l^1; \\ \text{and } x_a^{t-1} + x_b^t + \omega_a^t &\leqq \overline{y}^t - \underline{y}^t + \omega_a^{t-1} \text{ with } \underline{y}^t \leqq \omega_a^{t-1} \text{ and } y_l^t \leqq l^t \quad (\forall t \in \mathcal{T}). \end{aligned}$$

Denote the set of intertemporal feasible allocations for the economy E^∞ with $\omega_a^0 \in \mathbb{R}_+^n$ by $F(E^\infty; \omega_a^0)$.

Consider a mapping $\varphi : \mathcal{E} \times \mathbb{R}_+^n \twoheadrightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times [0,1] \times \mathbb{R}_+^n \times \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n$ such that for each $E \in \mathcal{E}$ and each $\omega_a \in \mathbb{R}_+^n$, $\varnothing \neq \varphi(E, \omega_a) \subseteq F(E; \omega_a)$. In the following discussion, as a slight abuse of notation for simplicity, let us simply denote $\varphi(E)$ instead of $\varphi(E, \omega_a)$. For any given $t \in \mathcal{T}$, a mapping φ is a *temporal allocation rule in period* t if and only if for each economy $E_t \in \mathcal{E}^t$

associated with $\omega_a^{t-1} \in \mathbb{R}_+^n$, $\varnothing \neq \varphi(E_t) \subseteq F(E_t;\omega_a^{t-1})$. A mapping φ is an *intertemporal allocation rule* if and only if for each and every $t \in \mathcal{T}$, φ is a temporal allocation rule in period t such that for each general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$, $\varnothing \neq (\varphi(E_1), \cdots, \varphi(E_t), \cdots) \subseteq F(E^\infty;\omega_a^0)$ holds. Denote $\varphi(E^\infty) \equiv (\varphi(E_1), \cdots, \varphi(E_t), \cdots)$ for each general OLG economy E^∞ when φ is an intertemporal allocation rule.

In this way, an intertemporal allocation rule φ is defined to implement a 'φ-optimal' (temporal) allocation policy sequentially at each and every period, rather than specifying, once at the initial period, an intertemporal allocation policy that will be applied to the whole infinite horizon of periods. In that it does not require a planner with perfect foresight, this particular form of intertemporal allocation rules would be more realistic in the context of policy decisions. This is also in line with the main objective of this paper, which is concerned with the long-run social welfare effects of the consistent implementation of Pareto-efficient policies in a sequential manner, as was the concern of Cordon (1984).

The market mechanism is a prominent example of an (intertemporal) allocation rule: it selects Walrasian competitive equilibrium allocations for each general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$. Given a general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$, a sequence of commodity prices and wages, $\left\{\left(\boldsymbol{p}_{t-1}, w_t\right)\right\}_{t=1}^\infty$ with $\boldsymbol{p}_{t-1} \in \mathbb{R}_+^n$ and $w_t \in \mathbb{R}_+$ ($\forall t \in \mathcal{T}$), and an intertemporal allocation $\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right)$ constitute a *Walrasian competitive equilibrium for* E^∞ if and only if the following conditions are satisfied:

(i) for each $t \in \mathcal{T}$, $\left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)$ is a solution to the following profit maximization problem:

$$\max_{\left(-y_l'^t, -\underline{y}'^t, \overline{y}'^t\right)\in Y^t} \boldsymbol{p}_t\overline{y}'^t - \boldsymbol{p}_{t-1}\underline{y}'^t - w_t y_l'^t \ ;$$

(ii) for each generation $t = 1, 2, \ldots,$, $\left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)$ is a solution to the following utility maximization problem:

$$\max_{((x_b, x_a), l, \omega_a)} u\left(x_b, x_a\right)$$

subject to:

$$\begin{aligned} \boldsymbol{p}_t x_b + \boldsymbol{p}_t \omega_a &\leqq w_t l; \\ \boldsymbol{p}_{t+1} x_a &\leqq \boldsymbol{p}_t \omega_a + \pi^{t+1}; \\ l &\in [0,1] \text{ and } \omega_a \in \mathbb{R}_+^m, \end{aligned}$$

where

$$\pi^t \equiv \max_{\left(-y_l'^t, -\underline{y}'^t, \overline{y}'^t\right)\in Y^t} \boldsymbol{p}_t\overline{y}'^t - \boldsymbol{p}_{t-1}\underline{y}'^t - w_t y_l'^t \ \ (\forall t \in \mathcal{T});$$

(iii) for generation $t = 0$, x_a^0 is a solution to the following utility maximization problem:

$$\max v_a\left(x_a\right)$$

subject to:

$$\boldsymbol{p}_1 x_a \leqq \boldsymbol{p}_0 \omega_a^0 + \pi^1;$$

(iv) $\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right) \in F(E^\infty; \omega_a^0)$.

An intertemporal allocation rule φ^W is called *Walrasian* if and only if for each general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$, and each

$$\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right) \in \varphi^W(E^\infty),$$

there exists a sequence of prices and wages $\left\{\left(\boldsymbol{p}_{t-1}, w_t\right)\right\}_{t=1}^\infty$ such that the pair of the allocation and the sequence constitutes a Walrasian competitive equilibrium for E^∞.

3 Hicksian Optimism

A movement from one Pareto efficient allocation to another Pareto efficient allocation is very likely to make someone worse off and yet someone else better off, and the Pareto principle itself cannot be used for making welfare judgments in such a movement. As discussed in Section 1, however, neoclassical economists believe that "if Pareto efficient policies are being pursued consistently over a long period, the *chances* are that eventually–though not at every particular step–everyone will be better off" (Corden, 1984, page 68), which is termed as the *Hicksian Optimism.*

What is the meaning of "Pareto efficient policies"? According to Corden (1984), it means "policy shifts in the direction of improved Pareto efficient rankings," that is, a Pareto efficient policy is to make a utility possibility frontier of the society shift upward to a *north-east* direction. The Hicksian Optimism presumes that such policies are consistently applied over a long period, so that they lead gradually from a more *south-west* utility possibility frontier to a more *north-east* utility possibility frontier. Typical examples of such policies would be to improve productive efficiency, such as reorganizations of production or promoting R&D investments, as Hicks (1941) mentioned.

Inspired by the above ideas, in what follows, we shall first introduce and define the 'improvements' that Hicks had in mind and then formulate the principle, the *Hicksian Optimism*, as an axiom of (intertemporal) allocation rules. For this purpose, we consider a general OLG economy, $E^\infty = (E_t = \langle Y^t; \omega_l; u\rangle)_{t=1}^\infty \in \mathcal{E}^\infty$, and a period economy $E^* = \langle Y^*; \omega_l; u\rangle \in \mathcal{E}$. We say that the general OLG economy $E^\infty = (E_t)_{t=0}^\infty$ associated with $\omega_a^0 \in \mathbb{R}_+^n$ is *growing and transforming* to *a stationary OLG economy* E^* if and only if

(i) for all $t = 1, 2, \ldots,$ $\underline{Y}^t \subseteq \underline{Y}^{t+1}$ and $\omega_b^t \leqq \omega_b^{t+1}$ hold, and
(ii) $\lim_{t\to\infty} \underline{Y}^t = \underline{Y}^* \supset \underline{Y}^1$, $\lim_{t\to\infty} \omega_b^t = \omega_b^*$.

Here, the growing process from a period economy to the next period economy within a general OLG economy is characterized by condition (i), which represents the dynamic process of technical progress due to the success of technological innovation. This technological innovation can take place both in $\underline{Y}^t$ and Y_b^t, where $\omega_b^t \leqq \omega_b^{t+1}$ represents a technological innovation in the latter production possibility set. Morever, the condition (ii) requires that the growing economy converges to a period economy E^*. This implies that the growing OLG economy E^∞ transforms to a stationary OLG economy in the limit, where *the stationary OLG economy is characterized by the infinite repetition of a period economy* E^*.

In this formulation of the growing and converging transition, a sequence of each period's production possibility sets is given to meet the presumption of the *consistent application of Pareto efficient policies* defined by Corden (1984). Setting aside the issue of how the R&D investment for technological innovation can be incentivized, as discussed in Section 1, we will focus on the issue whether and how such a dynamic transition of period-economies can improve individual and social welfare in the long run as captured by the idea of the Hicksian Optimism discussed in Section 1.

'The improvements' that Hicks had in mind are visualized as the improvements of the 'convergent stationary OLG economy' resulting from a sequence of growing period economies over the initial period economy. Our notion of a growing and transforming economy requires more than what Hicks had in mind: the expansion of production possibility sets occurs for any two periods, and is therefore much stronger than Hicks' requirement for the improvements of the economy. Now, comparing this eventual stationary OLG economy with the initial period economy, we can formulate the following principle requiring that none of the young and the old be made worse off:

Hicksian Optimism (HO): For any general OLG economy $E^\infty = \langle Y^t; \omega_l; u \rangle_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 \in \mathbb{R}_+^n$ such that E^∞ is *growing and transforming* to a stationary OLG economy E^*, if $\left(x_a^0; \left((x_b^t, x_a^t), l^t, \omega_a^t\right)_{t \in \mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t \in \mathcal{T}}\right)$ $\in \varphi(E^\infty)$ has $\lim_{t\to\infty} \left(x_a^{t-1}; \left((x_b^t, x_a^t), \omega_a^t\right); \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)\right) =$ $\left(x_a^*; \left((x_b^*, x_a^*), l^*, \omega_a^*\right); \left(-y_l^*, -\underline{y}^*, \overline{y}^*\right)\right) \in F(E^*; \omega_a^*)$, then

$$v_b(x_b^*) \geqq v_b\left(x_b^1\right) \text{ and } v_a(x_a^*) \geqq v_a\left(x_a^0\right).$$

According to our notion of Hicksian Optimism, the improvements based on 'growth policies' should leave no one behind–all young and old generations will be made at least as well-off as their ancestors who existed before implementing such policies.[9] Note that it is not required that everyone should (weakly) benefit at each period from the growth policies. It merely requires that, eventually and this could be a long time requiring 'a great deal of human patience, more patience than is characteristic of the twentieth century, even of the economists

[9]It may be of interest to note that our idea that 'no one is left behind' exposed in the Hicksian Optimism is similar to that advocated and used by the UNDP: "People get left behind when they lack the choices and opportunities to participate in and benefit from development progress." (UNDP, 2018)

of the twentieth century; more patience, perhaps, than we ought to ask' as observed by Hicks (1941) himself, no one is going to get hurt. Since our notion of a growing and transforming economy demands more than what Hicks requires for the growing of the economy, our formulation of the Hicksian optimism is therefore much weaker than what Hicks had in mind, making our version of the Hicksian optimism even more appealing.

Our notion of Hicksian Optimism formally may be viewed as similar to various monotonicity properties proposed in the related literatures of axiomatic bargaining (Kalai (1977)) and of resource allocations (see, for example, Chambers and Hayashi (2020), Chun and Thompson (1988), Moulin and Roemer (1989), Moulin and Thompson (1988), and Roemer (1986)). We note, however, that there are several substantial differences between those monotonicity properties and our notion of Hicksian Optimism. The monotonicity properties introduced in those literatures deal with two different static economies and thus require monotonicity across different economic environments, and the underlying allocation rules are static as well. In our case, the axiom of HO is applied to each simple OLG economy so that it operates within a given economic environment, and the allocation rules are intertemporal. These differences will make our main impossibility result (see the next section) intrinsically different from the impossibility results obtained in the literature of resource allocations.

4 Any Efficient and Individually Rational Allocation Rule Is Not Hicksian Optimistic

In this section, we examine the fate of the Hicksian optimism in the context of certain mechanisms of allocating resources. For this purpose, we introduce the following two familiar axioms for intertemporal allocation rules that are to be embedded in the class of resource allocation mechanisms that we study.

Weak Pareto Efficiency (WPE): For any general OLG economy $E^{\infty} = \langle Y^t; \omega_l; u\rangle_{t=1}^{\infty} \in \mathcal{E}^{\infty}$ associated with ω_a^0, and every intertemporal allocation $\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right)$ in $\varphi(E^{\infty})$, there is no other $\left(x_a'^0; \left(\left(x_b'^t, x_a'^t\right), l'^t, \omega_a'^t\right)_{t\in\mathcal{T}}; \left(-y_l'^t, -\underline{y}'^t, \overline{y}'^t\right)_{t\in\mathcal{T}}\right)$ in $F(E^{\infty}; \omega_a^0)$ such that:

(i) $(x_b'^t, x_a'^t) = (x_b^t, x_a^t)$ except for a finite number of t, and

(ii) $v_a\left(x_a'^0\right) \geqq v_a\left(x_a^0\right)$, $u\left(x_b'^t, x_a'^t\right) \geqq u\left(x_b^t, x_a^t\right)$ for all $t \in \mathcal{T}$ and either $v_a\left(x_a'^0\right) > v_a\left(x_a^0\right)$ or $u\left(x_b'^t, x_a'^t\right) > u\left(x_b^t, x_a^t\right)$ for some $t \in \mathcal{T}$.

Individual Rationality (IR): For any general OLG economy $E^{\infty} = \langle Y^t; \omega_l; u\rangle_{t\in\mathcal{T}}$ in $\mathcal{E}^{\infty}$ associated with ω_a^0, and every $\left(x_a^0; \left(\left(x_b^t, x_a^t\right), l^t, \omega_a^t\right)_{t\in\mathcal{T}}; \left(-y_l^t, -\underline{y}^t, \overline{y}^t\right)_{t\in\mathcal{T}}\right)$ in $\varphi(E^{\infty})$, the following conditions hold:

$$\begin{aligned} v_b\left(x_b^1\right) &\geqq v_b\left(\omega_b^1\right) \text{ and } v_a\left(x_a^0\right) \geqq v_a\left(\omega_a^0\right) \text{ in period } t = 1; \\ v_b\left(x_b^t\right) &\geqq v_b\left(\omega_b^t\right) \text{ and } v_a\left(x_a^{t-1}\right) \geqq v_a\left(\omega_a^{t-1}\right) \text{ in period } t = 2, \ldots, \ . \end{aligned}$$

Weak Pareto Efficiency (**WPE**) is introduced by Balasko and Shell (1980) in the context of OLG models. It is weaker than the standard Pareto efficiency condition: the Pareto domination test is applied only to certain types of feasible intertemporal allocations in which the consumption bundles are identical except for a finite number of generations. Following the works of Balasko and Shell (1980, Proposition 4.4) and of Tvede (2010, Corollary 7.9 and Corollary 7.10),[10] it can be shown that, in our OLG model, a Walrasian competitive equilibrium allocation satisfies **WPE**, and any feasible intertemporal allocation satisfying **WPE** can be a Walrasian competitive equilibrium allocation. It may be noted that, since our result is an impossibility (see Theorem 1 below), it is sufficient to formulate this weaker version of the Pareto efficiency condition.

Individual Rationality (**IR**) requires that an intertemporal allocation rule be such that, for every period $t \in \mathcal{T}$, every agent in the population of this period has an incentive to participate in social economic activities of the period economy E_t: by participating in social economic activities of the period economy, the agent would be at least as well-off as by choosing to engage in her own autarkic economy. Note that in every period $t \in \mathcal{T}$, generation $t-1$ at the old age can choose an autarkic economic action by consuming her savings vector ω_a^{t-1}, while generation t at the young age can produce ω_b^t by operating Y_b^t with her one unit labor supply and consuming ω_b^t in her autarkic economy.[11]

The two axioms capture a broader class of intertemporal allocation rules. Can any such intertemporal allocation rules deliver the Hicksian optimism? The following result provides an answer to this question and its proof can be found in the Appendix.

Theorem 1 *Suppose* $\omega_a^0 > \mathbf{0}$ *and* $\omega_b^1 > \mathbf{0}$. *Then, there exists no intertemporal allocation rule satisfying Hicksian Optimism, Weak Pareto Efficiency, and Individual Rationality simultaneously.*

Therefore, Theorem 1 suggests that any resource allocation mechanism characterized by an efficient and individually rational intertemporal allocation rule is bound to leave some individual behind even if the economy is growing and expanding due to technological innovation and even after a long period of time.

We note that, as shown in the proof of Theorem 1 developed in the Appendix, the fundamental incompatibility of **HO** with **WPE** and **IR** is rooted in the possibility of at least two types of growing and transforming OLG economies emerging from the same initial-period economy. Indeed, if there is only one type

[10] While Balasko and Shell (1980) focus on pure exchange OLG model, Tvede (2010) also discusses OLG economies with production.

[11] Note that we could have introduced a stronger individual rationalilty condition than this version of **IR** by incorporating information concerning individuals' ownerships of production technologies in $\underline{\boldsymbol{Y}}^t$. Since our result is an impossibility result (see the theorem below), it becomes unnecessary to formulate a stronger individual rationality condition.

of growing and transforming OLG economy available from one initial period economy, it is not difficult to construct a monotone increasing path of each period's individually rational and efficient utility allocation corresponding to the growing and transforming path of the period economies for each and every initial-period economy. This implies that an efficient and individually rational intertemporal allocation rule delivers the Hicksian optimism for this kind of growing and transforming OLG economy.

On the other hand, that only a unique type of growing and transforming OLG economies emerging from the initial-period economy is highly questionable and implausible. It is more natural and plausible to see that there will be more than one type of growing and transforming OLG economies starting from the initial-period economy,[12] and, there is no good reason to select just one type of general OLG economy as the domain of the Hicksian optimism, since these alternative paths of period-economies can respectively be regarded as the consequences of the consistent application of Pareto efficient policies.

As shown in Balasko and Shell (1980, Proposition 4.4) and Tvede (2010, Corollary 7.9), in OLG models, the Walrasian intertemporal allocation rule is weakly Pareto efficient. It is clear that the Walrasian intertemporal allocation rule is also individually rational. Then, by Theorem 1, we obtain the following result.

Corollary 2 *Suppose $\omega_a^0 > \mathbf{0}$ and $\omega_b^1 > \mathbf{0}$. Then, the Walrasian intertemporal allocation rule does not satisfy Hicksian Optimism.*

Thus, the market mechanism does not deliver the Hicksian Optimism: in a market system, economic expansions can leave some individuals behind–some individuals would be made worse off than the pre-expansion economy.

5 Conclusion

In a simple OLG economy, we have shown that any resource allocation mechanism characterized by a weakly efficient and individually rational intertemporal allocation rule faces a dilemma: with continued technological innovation and progress, some individual is bound to be left behind even after a long period of time. Since the market mechanism characterized by the Walrasian allocation rule belongs to the class of weakly efficient and individually rational intertemporal allocation rules, the market mechanism faces the same dilemma: it cannot deliver the Hicksian optimism–an ideal as envisioned by Hicks (1941) and, more recently, as advocated by the UNDP (2015) in its 2030 Agenda for Sustainable Development.

[12] For instance, multiple types of production possibility sets could potentially emerge in successive periods as the outcomes derived from even one type of policy to promote R&D investments applied in the initial period.

Pareto efficiency is deeply rooted in economics and is highly appealing for resource allocation. Given the tremendous technological innovation made in recent human history and the great economic progress made in recent decades, the Hicksian optimism seems a very attractive property to be demanded for resource allocation mechanisms. If we follow these lines of reasoning, the property of individual rationality needs a careful and possibly critical examination.

To begin with, we note that, as our result in Section 4 shows, the unfettered market mechanism characterized by the Walrasian allocation rule cannot deliver Hicksian optimism. Part of the reasons is that the Walrasian intertemporal allocation rule is both weakly efficient and individually rational. Suppose we abandon individual rationality but still want to use the market mechanism to allocate resources. One possibility is to use the market mechanism coupled with a redistributive scheme (perhaps similar to that required for the second welfare theorem) to ensure that, eventually, every generation, being young or old, is at least as well-off as his/her ancestor who existed at the very beginning. Resource-wise, this is feasible since the economy is expanding and everyone's eventual initial endowment is bigger than the initial endowment of his/her ancestor in period 0 or period 1. Of course, the Walrasian intertemporal allocation rule coupled with a redistributive scheme may violate individual rationality introduced in Section 4, the reason being that, some individuals may be made worse off than their initial endowments in that period after redistributions.[13]

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[13] If, on the other hand, one wants to go beyond the market mechanism, there are several studies on certain egalitarian type allocation rules for specific static economies involving one input and one output, where it has been shown that the proposed allocation rules are Pareto efficient and satisfy some technological monotonicities. See the contributions by Moulin (1987, 1990), Moulin and Roemer (1989), and Roemer and Silvestre (1987). An important reason why positive results can be obtained in those settings is that their static economies involve one input and one output and, for such specific economies, the property of individual rationality almost has no bite. For our economy, individual rationality is non-trivial and is rather stringent.

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Appendix

Proof of Theorem 1. Suppose that there are only two commodities. Given $\varepsilon \in \left(0, \frac{1}{2}\right)$, let $\alpha \in \mathbb{R}_+$ be such that $\alpha > \frac{1-\varepsilon}{\varepsilon}$, which implies that $\alpha > 1$. Let $\bar{\delta}$ and δ be sufficiently small with $\bar{\delta} > \delta > 0$. Let $u = U(v_b, v_a)$ satisfy the following properties:[14] for any $(x_1, x_2) \in \mathbb{R}^2_+$,

$$v_b(x_1, x_2) \equiv \min\left\{\frac{x_1}{\alpha}(1+\delta), x_2\right\} \text{ and } v_a(x_1, x_2) \equiv \min\left\{x_1, \frac{x_2}{\alpha}(1+\delta)\right\},$$

and U is strictly increasing in $\mathbb{R}^2$.

Let a production possibility set Y^t at $t \in \mathcal{T}$ be represented by a von Neumann production technology (A^t, B^t, L^t), where A^t is a matrix of material input coefficients; B^t is a matrix of gross output coefficients; and L^t is a vector of direct labor input coefficients. In particular, let $\{(A^t, B^t, L^t)\}_{t=1, \ldots,}$ be specified as follows:

$$\begin{aligned} A^1 &\equiv \begin{bmatrix} \varepsilon & 0 \\ \varepsilon & 0 \end{bmatrix}, \ B^1 \equiv \begin{bmatrix} F_1(t)+\varepsilon & \varepsilon \\ F_2(t)+\varepsilon & \varepsilon \end{bmatrix}, \ L^1 \equiv (1,1), \text{ and} \\ A^t &\equiv \begin{bmatrix} \omega^t_{b1} & 0 \\ \omega^t_{b2} & 0 \end{bmatrix}, \ B^t \equiv \begin{bmatrix} F_1(t)+\omega^{t+1}_{b1} & \omega^t_{b1} \\ F_2(t)+\omega^{t+1}_{b2} & \omega^t_{b2} \end{bmatrix}, \ L^t \equiv (1,1) \ \ (\forall t = 2, \ldots,), \end{aligned}$$

where $F_1(t)$ is continuous and increasing at every $t = 1, \ldots,$ such that $F_1(1) = 1$

[14] The specific one-period utility functions are used here to provide a simplest economy to establish our result. We may note that, at the expense of increasing complexity, the proof goes through with a more standard type of one-period utility functions, where the utility functions are strongly monotonic and are not the Leontief preferences of the two goods.

and $\lim_{t\to\infty} F_1(t) = \alpha$; and $F_2(t)$ is continuous and increasing at every $t = 1, \ldots,$ such that $F_2(1) = 1$ and $\lim_{t\to\infty} F_2(t) = 1+\delta$; and moreover, let

$$\begin{aligned} \omega_b^t &= \left(\omega_{b1}^t, \omega_{b2}^t\right) = \varepsilon \cdot \left[(\alpha+\bar{\delta})^{\frac{t-2}{t+2\alpha}}, (1+\bar{\delta})^{\frac{t-2}{t+2\alpha}}\right] \text{ for every } t = 2, \ldots, \text{ and} \\ \omega_a^0 &= \left(\omega_{a1}^0, \omega_{a2}^0\right) = \varepsilon(1,1). \end{aligned}$$

Note that $\lim_{t\to\infty}(A^t, B^t, L^t) = (A^*, B^*, L^*)$ holds, where

$$A^* = \begin{bmatrix} \omega_{b1}^* & 0 \\ \omega_{b2}^* & 0 \end{bmatrix},\ B^* = \begin{bmatrix} \alpha+\omega_{b1}^* & \omega_{b1}^* \\ (1+\delta)+\omega_{b2}^* & \omega_{b2}^* \end{bmatrix},\ L^* = (1,1),$$

in which $\omega_b^* = (\omega_{b1}^*, \omega_{b2}^*) = \varepsilon(\alpha+\bar{\delta}, 1+\bar{\delta})$ holds. Also, note that $A^2 = A^1$, as $\omega_{b1}^2 = \varepsilon = \omega_{b2}^2$ holds by construction. For each (A^t, B^t, L^t), let

$$Y_{(A^t,B^t,L^t)} \equiv \left\{y = \left(-y_l, -\underline{y}, \overline{y}\right) \in \mathbb{R}_- \times \mathbb{R}_-^2 \times \mathbb{R}_+^2 \mid \exists z \in \mathbb{R}_+^2 : y \leqq \left(-L^t z, -A^t z, B^t z\right)\right\},$$

and set $Y^t \equiv Y_{(A^t,B^t,L^t)}$. It can be checked that Y^t satisfies **A1**, **A2**, **A3**. In particular, Y_b^t can be derived from the 2nd column production process of (A^t, B^t, L^t) while $\underline{Y}^t$ can be derived from the 1st column production process of (A^t, B^t, L^t). Moreover, for any $t = 1, \ldots,$, $\underline{Y}^t \subseteq \underline{Y}^{t+1}$ and $\omega_b^t \leqq \omega_b^{t+1}$ hold. Since $\bar{\delta} > \delta$, it follows that $\varepsilon(a, 1+\delta) < \omega_b^*$.

Let an intertemporal allocation rule φ satisfy **WPE**, **IR**, and **HO**. Note that the above defined general OLG economy $E^\infty = (E_t = \langle Y^t; \omega_l; u\rangle)_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 = (\varepsilon, \varepsilon)$ is growing and transforming to the stationary OLG economy $E^* = \langle Y^*; \omega_l; u\rangle$, where $Y^* \equiv Y_{(A^*,B^*,L^*)}$. Let $\mathbf{e}_1 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $\left(x_a^0; (x^t, l^t, \omega_a^t)_{t=1,\ldots}; (y^t)_{t=1,\ldots}\right) \in \varphi(E^\infty)$ be such that, for each period $t \in \mathcal{T}$,

$$l^t = 1;\ \omega_a^t = \omega_b^{t+1};\ y^t = \left(-L^t\mathbf{e}_1, -A^t\mathbf{e}_1, B^t\mathbf{e}_1\right);\ \text{and}\ x_a^{t-1} + x_b^t + \omega_a^t = \left(B^t - A^t\right)\mathbf{e}_1 + \omega_a^{t-1},$$

and $\lim_{t\to\infty}\left(x_a^{t-1}; (x^t, l^t, \omega_a^t); y^t\right) = (x_a^*; (x^*, l^*, \omega_a^*); y^*) \in F(E^*; \omega_a^*)$, where $\omega_a^* = \omega_b^*$. Then, by **IR**, $v_b(x_b^*) \geqq v_b(\varepsilon(\alpha, 1+\delta))$ and $v_a(x_a^*) \geqq v_a(\varepsilon(\alpha, 1+\delta))$ as $\bar{\delta} > \delta$.[15] Thus, $v_b(x_b^*) \geqq \min\left\{\frac{\varepsilon\alpha}{\alpha}(1+\delta), \varepsilon(1+\delta)\right\} = \varepsilon(1+\delta)$ and $v_a(x_a^*) \geqq \min\left\{\varepsilon\alpha, \frac{\varepsilon(1+\delta)^2}{\alpha}\right\} = \frac{\varepsilon}{\alpha}(1+\delta)^2$. Note that, as $\varepsilon\alpha - \frac{\varepsilon}{\alpha} > 0$, $\varepsilon\alpha - \frac{\varepsilon(1+\delta)^2}{\alpha} > 0$ holds for the sufficiently small $\delta > 0$. Then, $v_a(x_a^*) \leqq \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$ holds. To see it, suppose that $v_a(x_a^*) > \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$. This implies that $x_{a1}^* > \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$, $x_{a2}^* > (1+\delta)(1-\varepsilon)$. Then, $x_{b1}^* < \alpha - \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$ and $x_{b2}^* < \varepsilon(1+\delta)$. Therefore,

$$v_b(x_b^*) < \min\left\{(1+\delta)\left[1 - \frac{(1+\delta)^2(1-\varepsilon)}{\alpha^2}\right], \varepsilon(1+\delta)\right\} = \varepsilon(1+\delta).$$

[15] The last property follows from the fact that the individually rational correspondence, that is, the allocation rule selecting all individually rational allocations at each and every economy, is upper hemi-continuous.

Note that $1-\frac{(1+\delta)^2(1-\varepsilon)}{\alpha^2} > \varepsilon$ holds, as $\alpha^2-(1+\delta)^2(1-\varepsilon)-\alpha^2\varepsilon = \alpha^2(1-\varepsilon)-(1+\delta)^2(1-\varepsilon) = \left(\alpha^2-(1+\delta)^2\right)(1-\varepsilon) = (\alpha+(1+\delta))(\alpha-(1+\delta))(1-\varepsilon) > 0$ by $\alpha > 1$ and δ being sufficiently small. Then, since $v_b(x_b^*) < \varepsilon(1+\delta) \leqq v_b(x_b^*)$, we have a contradiction. Therefore, $v_a(x_a^*) \leqq \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$ holds. By **HO**, we thus obtain

(B.1) $v_a\left(x_a^0\right) \leqq \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$.

We now consider another growing and transforming OLG economies starting from the same initial-period economy. For this purpose, we define $Y'^t \equiv Y_{(A'^t,B'^t,L'^t)}$, where

$$\begin{aligned} A'^1 &\equiv \begin{bmatrix} \varepsilon & 0 \\ \varepsilon & 0 \end{bmatrix},\ B'^1 \equiv \begin{bmatrix} F_2(t)+\varepsilon & \varepsilon \\ F_1(t)+\varepsilon & \varepsilon \end{bmatrix},\ L'^1 \equiv (1,1), \text{ and} \\ A'^t &\equiv \begin{bmatrix} \omega_{b2}^t & 0 \\ \omega_{b1}^t & 0 \end{bmatrix},\ B'^t \equiv \begin{bmatrix} F_2(t)+\omega_{b2}^{t+1} & \omega_{b2}^t \\ F_1(t)+\omega_{b1}^{t+1} & \omega_{b1}^t \end{bmatrix},\ L'^t \equiv (1,1)\ \ (\forall t=2,\ \ldots,). \end{aligned}$$

Then, we have $A'^2 = A'^1$ and $\lim_{t\to\infty} Y'^t = Y'^* \equiv Y_{(A'^*,B'^*,L'^*)}$, where

$$A'^* \equiv \begin{bmatrix} \omega_{b2}^* & 0 \\ \omega_{b1}^* & 0 \end{bmatrix},\ B'^* \equiv \begin{bmatrix} (1+\delta)+\omega_{b2}^* & \omega_{b2}^* \\ \alpha+\omega_{b1}^* & \omega_{b1}^* \end{bmatrix},\ L'^t \equiv (1,1).$$

Note that $Y'^1 = Y_{(A'^1,B'^1,L'^1)} = Y_{(A^1,B^1,L^1)} = Y^1$ holds by construction.

Consider $E'^\infty = (E'_t = \langle Y'^t; \omega_l; u\rangle)_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a'^0 = \omega_a^0$. Let $\left(x_a'^0; (x'^t, l'^t, \omega_a'^t)_{t=1,\ldots}; (y'^t)_{t=1,\ldots}\right) \in \varphi(E'^\infty)$ be such that

$$\left(x_a'^0; \left(x_b'^1, x_a'^1\right), l'^1, \omega_a'^1; y'^1\right) = \left(x_a^0; \left(x_b^1, x_a^1\right), 1, \omega_a^1; y^1\right),$$

which is warranted by $\varphi(E'_1) = \varphi(E_1)$ due to $E'_1 = E_1$, and

$$\lim_{t\to\infty} \left(x_a'^{t-1}; \left(x'^t, l'^t, \omega_a'^t\right); y'^t\right) = \left(x_a'^*; \left(x'^*, l'^*, \omega_a'^*\right); y'^*\right) \in F(E'^*; \omega_a'^*).$$

Then, by a symmetric argument, we obtain $v_b\left(x_b'^*\right) \leqq \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$. Thus, by **HO**, we have

(B.2) $v_b\left(x_b'^1\right) = v_b\left(x_b^1\right) \leqq \frac{(1+\delta)^2(1-\varepsilon)}{\alpha}$.

To complete our proof, we now construct an intertemporal feasible allocation $\left(x_a''^0; (x''^t, l''^t, \omega_a''^t)_{t=1,\ldots}; (y''^t)_{t=1,\ldots}\right) \in F(E^\infty; \omega_a^0)$ such that

$$\begin{aligned} \left(x_a''^0; \left(x_b''^1, x_a''^1\right), l''^1, \omega_a''^1; y''^1\right) &= \left(x_a''^0; \left(x_b''^1, x_a^1\right), 1, \omega_a^1; y^1\right) \\ \text{with } \left(x_b''^1, x_a''^0\right) &= \left(\left(\frac{\alpha}{\alpha+1}, \frac{1}{\alpha+1}\right), \left(\frac{1}{\alpha+1}, \frac{\alpha}{\alpha+1}\right)\right) \text{ at } E_1; \\ \text{and } \left(x_a''^{t-1}; \left(x_b''^t, x_a''^t\right), l''^t, \omega_a''^t; y''^t\right) &= \left(x_a^{t-1}; \left(x_b^t, x_a^t\right), 1, \omega_a^t; y^t\right) \text{ at } E_t \quad (\forall t=2,\ \ldots,). \end{aligned}$$

Note that $\left(x_a''^0; \left(x_b''^1, x_a''^1\right), l''^1, \omega_a''^1; y''^1\right)$ is a temporary feasible for E_1 with ω_a^0, since $x_a''^0 + x_b''^1 + \omega_a''^1 = (1,1) + \omega_a^1 = (1,1) + (\varepsilon, \varepsilon) = \left(B^1 - A^1\right)\mathbf{e}_1 + \omega_a^0$ holds, where $\omega_a^1 = (\varepsilon, \varepsilon)$ comes from $\omega_a^1 = \omega_b^2$ by definition of $\left(x_a^0; \left(x^t, l^t, \omega_a^t\right)_{t=1,\ldots}; \left(y^t\right)_{t=1,\ldots}\right)$ and from $\omega_b^2 = (\varepsilon, \varepsilon)$ by construction of B^t. Thus, the intertemporal allocation $\left(x_a''^0; \left(x''^t, l''^t, \omega_a''^t\right)_{t=1,\ldots}; \left(y''^t\right)_{t=1,\ldots}\right)$ is indeed feasible for E^∞ with ω_a^0, since it is identical to the φ-optimal allocation $\left(x_a^0; \left(x^t, l^t, \omega_a^t\right)_{t=1,\ldots}; \left(y^t\right)_{t=1,\ldots}\right)$ except for the period $t = 1$.

Then, $v_b\left(x_b''^1\right) = v_a\left(x_a''^0\right) = \min\left\{\frac{\alpha}{\alpha+1}\frac{(1+\delta)}{\alpha}, \frac{1}{\alpha+1}\right\} = \frac{1}{\alpha+1}$. However,

$$\frac{1}{\alpha+1} - \frac{(1+\delta)^2(1-\varepsilon)}{\alpha} > 0 \text{ for the sufficiently small } \delta > 0,$$

as $\frac{1}{\alpha+1} - \frac{(1-\varepsilon)}{\alpha} = \frac{\alpha\varepsilon+\varepsilon-1}{\alpha(\alpha+1)} > 0$ by $\alpha > \frac{1-\varepsilon}{\varepsilon}$. By (B.1) and (B.2), we must have

(B.3) $v_a\left(x_a''^0\right) > v_a\left(x_a^0\right)$ and $v_b\left(x_b''^1\right) > v_b\left(x_b^1\right)$.

Since $u(\cdot,\cdot)$ is strictly inreasing, we must also have

(B.4) $u\left(x_b''^1, x_a''^1\right) > u\left(x_b^1, x_a^1\right)$.

Then, the φ-optimal allocation $\left(x_a^0; \left(x^t, l^t, \omega_a^t\right)_{t=1,\ldots}; \left(y^t\right)_{t=1,\ldots}\right)$ cannot be weakly Pareto efficient, a contradiction. ■