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Bubbles and Economic Fluctuations

Pablo A. Guerrón Quintana
(Department of Economics, Boston College)
Ryo Jinnai
(Institute of Economic Research, Hitotsubashi University)
and
Yohei Yamamoto
(Faculty of Economics, Hitotsubashi University)

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Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603 Japan

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Pablo A. Guerrón Quintana[*] Ryo Jinnai[†] Yohei Yamamoto[‡]

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Abstract

This chapter studies the relationship between asset price bubbles and macroeconomic fluctuations through both empirical analysis and theoretical modeling. We begin by applying the right-tailed unit root tests of Phillips et al. (2015a,b) to real stock and housing price indices in G-7 economies. These tests identify explosive dynamics in asset prices, and our findings show that such bubbly episodes frequently align with periods of economic expansion, suggesting a strong empirical link between asset booms and business cycle upswings. To investigate the mechanisms behind this co-movement, we modify the canonical bubble models of Tirole (1985) and Martin and Ventura (2012) by incorporating endogenous labor supply. However, in both cases, the emergence of a bubble fails to generate a robust macroeconomic expansion. Output and investment either decline or respond sluggishly, while labor hours fall in response to bubble formation. We then turn to the model of Guerron-Quintana et al. (2023), which embeds a variable capacity utilization mechanism into a dynamic general equilibrium framework. This amplification channel allows the model to produce simultaneous increases in output, consumption, investment, and labor during bubbly periods, consistent with empirical patterns. We also discuss the quantitative implementation challenges faced by this approach, highlighting the trade-offs involved in quantitatively modeling bubble-driven fluctuations.

Keywords: asset price bubble; business cycles

[*]Department of Economics, Boston College. Email: pguerron@gmail.com.
[†]Institute of Economic Research, Hitotsubashi University. Email: ryo.jinnai@r.hit-u.ac.jp.
[‡]Faculty of Economics, Hitotsubashi University. Email: yohei.yamamoto@r.hit-u.ac.jp.

1 Introduction

This chapter investigates the interplay between asset price bubbles and macroeconomic fluctuations. As emphasized by Kindleberger and Aliber (2017, p.18–19), "Manias almost always have occurred during the expansion phase of the business cycle." While this observation aligns with conventional wisdom and may appear self-evident even without reference to economic historians, we adopt a more agnostic and empirically grounded approach in the initial part of the chapter to assess the validity of this hypothesis.

We first look at the data using a formal econometric approach. Testing for speculative bubbles in asset prices is a long-standing challenge, and numerous econometric techniques have been developed for this purpose; see Gürkaynak (2008) and Homm and Breitung (2012) for a review. We specifically apply the method of Phillips et al. (2015a,b), as their seminal work represents one of the most recent and widely used approaches in the empirical bubble-detection literature. They define a bubble as a subperiod during which the univariate time series of real asset prices exhibits an explosive process—that is, an autoregressive process with a root greater than unity. See Appendix A for details.

We apply the method to real stock and housing price indices from G7 countries and find that the periods identified as bubbles using this relatively simple approach almost always coincide with economic booms documented by independent sources. This finding provides strong empirical support for the connection between asset price bubbles and macroeconomic expansions.[1]

We then examine the economic mechanism behind the link. In this regard, Kindleberger and Aliber (2017, p.18–19) observed a causal relationship from the asset price bubble to the real economy, stating that "During the mania the increases in the prices of real estate or stocks or in one or several commodities contribute to increases in consumption spending and in investment spending that in turn lead to a quickening of economic growth." We examine this thesis using seminal works in the literature—Tirole (1985) and Martin and Ventura (2012)—as analytical laboratories. However, labor supply is inelastic in these models, which makes them restrictive for studying business cycles. Hence, we make minimal modifications to endogenize labor supply. We then simulate the emergence of a

[1] Recent empirical studies have examined the co-movement of bubbles across multiple regions and countries (Horie and Yamamoto, 2024; Pavlidis et al., 2016). Although the macroeconomic models presented in the later sections do not explicitly incorporate such cross-country dynamics, this empirical pattern is consistent with existing evidence on business cycle synchronization (Canova et al., 2007; Cesa-Bianchi et al., 2019; Davis, 2014).

bubble in these models and examine whether it generates an economic boom.

We find that an economic boom does not accompany the bubble in these classical models. In the Tirole (1985) model with endogenous labor, investment decreases when the bubble emerges. In the Martin and Ventura (2012) model with endogenous labor, output, consumption, and investment do not increase swiftly but instead lag behind the asset price boom. Labor hours respond immediately, but they decrease when a bubble emerges. These exercises show that the causal mechanism from an asset price bubble to an economic boom is not straightforward.

We then explain the Guerron-Quintana et al. (2023) model. With a quantitative application in mind, the authors construct an infinite-horizon macroeconomic model in which an asset price bubble causes an economic boom. Importantly, they introduce a variable capacity utilization rate as a shock amplification mechanism; that is, the owner of capital can choose how intensively to use it, but at the cost of faster depreciation. In the model, economic agents use capital more intensively during bubbly periods than during bubbleless periods because the opportunity cost is lower. As a result, output, consumption, investment, and labor hours increase simultaneously when a bubble emerges.

We also discuss a quantitative application of the model, highlighting key methodological challenges the authors face, as well as their innovations. They identify bubbles using a structural model. We examine both the similarities and differences between bubble periods identified by different methods.

2 Empirical Evidence

In this section, we present empirical evidence linking asset price bubbles to business cycles. To this end, we use monthly data on the real stock price index and quarterly data on the real house price index from the G7 countries. The sample period spans January 1960 to December 2020 for stock prices, and 1970Q1 to 2020Q4 for house prices. We identify bubble periods by applying the sequential right-tailed unit root tests proposed by Phillips et al. (2015a,b) to these data on a country-by-country basis. We obtain business cycle dates from the Economic Cycle Research Institute (ECRI). Although business cycle reference dates are determined either officially or unofficially by each country using different methodologies, we use the expansion and recession dates provided by the ECRI because it applies a National Bureau of Economic Research (NBER)-type procedure across countries. These dates are widely used in research involving international comparisons (e.g., Canova et al. (2007) and Koutsoumanis and Castro (2023)).

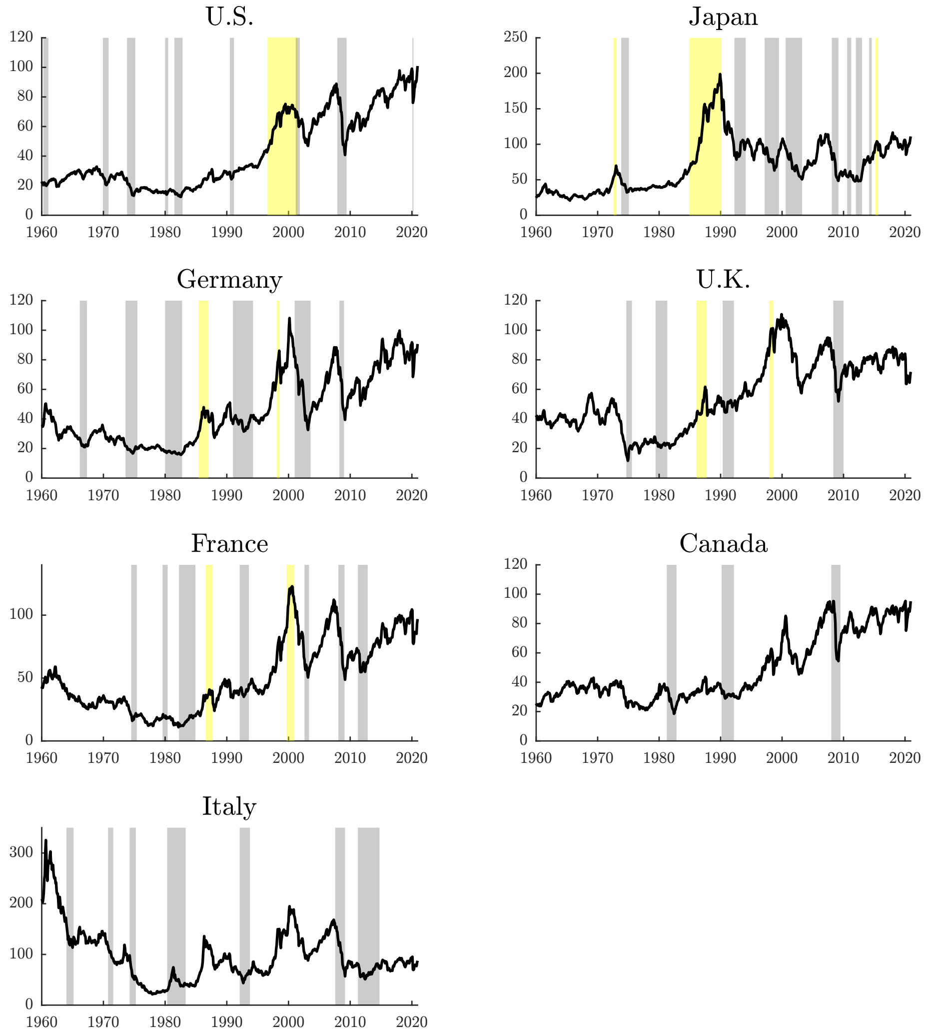

Figure 1: Stock Price Bubbles and Business Cycles
Note: The yellow shadow highlights the identified bubble period, while the gray shadow represents the recession period.
Source: OECD and ECRI.

Figure 1 presents the results for the stock market. Bubble periods are highlighted in yellow and include both the dot-com bubble in the United States in the late 1990s and the asset price bubble in Japan in the 1980s. We do not identify any stock market bubbles in Canada or Italy in our sample. The periods highlighted in gray represent recessions, while the others indicate expansions.

All identified bubble episodes occur during expansionary phases of the business cycle. Moreover, these episodes are frequently followed by sharp contractions, as evidenced by the post-bubble recessions in the United States in the early 2000s and in Japan in the early 1990s. These patterns point to a robust empirical association between stock market bubbles, economic expansions, and subsequent downturns.

Figure 2 highlights the identified housing bubble periods in yellow and the recession periods in gray. At least one housing bubble is identified in each country. Housing bubbles in the U.K. and the U.S. are relatively longer in duration compared to stock price bubbles. With the exception of three short recessions (from March 2001 to November 2001 and from February 2020 to April 2020 in the U.S., and from January 2008 to July 2009 in Canada), all housing bubbles occurred during expansionary phases of the business cycle. In addition, housing bubbles are often followed by recessions, as seen in the late 2000s in the U.S., the early 1990s in Japan, and both the early 1990s and the late 2000s in the U.K. Similar to stock market bubbles, these observations suggest a strong link between housing bubbles and economic expansions.

3 Bubbles and Economic Fluctuations in Classic Models

The preceding section established that asset price bubbles consistently coincide with expansionary phases of the business cycle. In this section, we investigate the underlying economic mechanism behind this link. To that end, we adopt the canonical theoretical frameworks of Tirole (1985) and Martin and Ventura (2012) as computational laboratories: we simulate the emergence of speculative bubbles in each model and assess whether their formation induces a concomitant boom in aggregate economic activity.

Following the literature, we define an economic boom as an expansionary deviation from trend during which output, consumption, investment, and hours worked move in unison.[2] However, both Tirole (1985) and Martin and Ventura (2012) assume an inelastic labor supply—a restriction that limits our analysis. We therefore introduce a key modification to each framework, allowing agents to choose hours worked optimally.

3.1 The Tirole (1985) Model with Endogenous Labor

3.1.1 Agents

We rely on an overlapping generations model. A new cohort of size 1 is born in each period and lives for two periods. Agents born in period $t \geq 0$ maximize the

[2] In an influential book chapter, Cooley and Prescott (1995, p. 26) argue that "the business cycle should be thought of as apparent deviations from a trend in which variables move together," and highlight hours worked as one of the key series their real business cycle model seeks to explain.

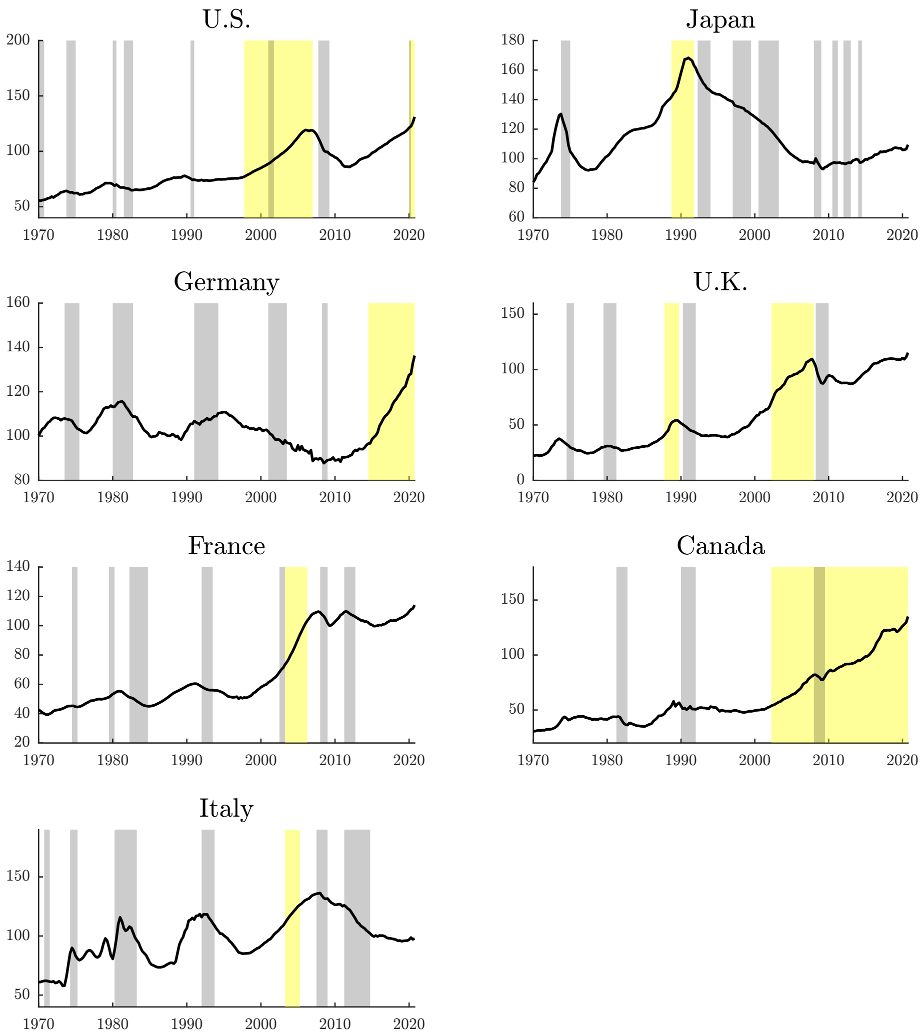

Figure 2: Housing Price Bubbles and Business Cycles
Note: The yellow shadow highlights the identified bubble period, while the gray shadow represents the recession period.
Source: OECD and ECRI.

utility function given by

$$\eta \log \left(\underbrace{1 - l_t}_{\text{leisure}} \right) + \log \left(\underbrace{c_{t+1}}_{\text{consumption}} \right). \tag{3.1}$$

They derive utility from leisure $(1 - l_t)$ and consumption (c_{t+1}). The parameter η governs the relative importance of leisure compared to consumption. This utility function is of the King-Plosser-Rebelo (KPR) type, as proposed by King et al. (1988). KPR utility is widely used in the business cycle literature, for reasons discussed below.

In equation (3.1), l_t denotes hours worked, while the total time endowment is normalized to one. Households sell labor services at a wage rate w_t, and allocate their income between investment in capital for the next period k_{t+1} and the *bubbly asset* $\tilde{m}_t$, which is intrinsically useless in the sense that it yields neither utility nor goods. The budget constraint faced by young agents is

$$k_{t+1} + \tilde{p}_t \tilde{m}_t = w_t l_t, \tag{3.2}$$

where $\tilde{p}_t$ is the market price of bubbly assets in terms of goods. Agents are not allowed to choose negative investment $(k_{t+1} \geq 0)$ or to short-sell bubbly assets $(\tilde{m}_t \geq 0)$.

When agents become old, they rent out capital at a rental rate r_{t+1} and sell all of their bubbly asset holdings. Capital fully depreciates, while bubbly assets do not depreciate at all. The consumption level in old age is given by

$$c_{t+1} = r_{t+1} k_{t+1} + \tilde{p}_{t+1} \tilde{m}_t. \tag{3.3}$$

At date 0, there is an initial cohort of old agents. They hold K_0 units of capital and M units of bubbly assets. Their consumption level is given by $c_0 = r_0 K_0 + \tilde{p}_0 M$.

3.1.2 Firms

Competitive firms rent capital services K_t and labor services L_t in order to maximize profits, which are given by

$$Y_t - r_t K_t - w_t L_t, \tag{3.4}$$

where output is produced using a Cobb-Douglas production technology:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \tag{3.5}$$

and A_t denotes the exogenous level of technology. The first-order conditions for profit maximization are

$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{3.6}$$

and

$$w_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}. \tag{3.7}$$

Finally, we assume exogenous technological growth:

$$A_t = \left(g^{1-\alpha}\right)^t, \tag{3.8}$$

where $g > 1$ is a parameter.

3.1.3 Competitive Equilibrium

The market clearing conditions for capital, labor, and bubbly assets are

$$K_{t+1} = k_{t+1}, \tag{3.9}$$

$$L_t = l_t, \tag{3.10}$$

$$\tilde{m}_t = M, \tag{3.11}$$

respectively. We impose these conditions and define the competitive equilibrium as follows. Given the initial capital stock K_0, a competitive equilibrium in the model consists of a sequence of prices $\{r_t, w_t, \tilde{p}_t\}$ and allocations $\{l_t, K_{t+1}\}$ such that the following conditions hold:

(i) $\{l_t, K_{t+1}, M\}$ solves the agent's utility maximization problem;

(ii) $\{l_t, K_t\}$ solves the firm's profit maximization problem.

3.1.4 Fundamental Equilibrium

We first discuss an equilibrium in which the bubbly asset is traded at its fundamental value, i.e., $\tilde{p}_t = 0$ for all $t \geq 0$. In this case, agents have no choice but to save all of their wage income in capital. Hence, both $k_{t+1} = w_t l_t$ and $c_{t+1} = r_{t+1} w_t l_t$ hold. Agents choose labor hours optimally, which implies that the following first-order condition is satisfied:

$$\frac{\eta}{1-l_t} = \left(\frac{1}{c_{t+1}}\right) r_{t+1} w_t \tag{3.12}$$

The left-hand side represents the marginal disutility of work, while the right-hand side represents the marginal benefit of work, expressed in terms of utility.

At this point, we can highlight an advantage of the KPR utility. The real wage w_t on the right-hand side exhibits trend growth, consistent with empirical data. All else equal, a higher wage increases the marginal benefit of work. However, in the data, hours worked remain relatively stable over the long run. This creates the need for a mechanism that stabilizes labor supply despite rising real wages.

KPR utility is designed to provide exactly such a mechanism. In the model, as the real wage increases, consumption also rises. The resulting decrease in the marginal utility of consumption reduces the incentive to earn additional wage income. These two opposing forces offset each other, preventing labor hours from increasing indefinitely. This is a desirable and widely appreciated feature of KPR utility.

Substituting $c_{t+1} = r_{t+1} w_t l_t$ into equation (3.12), we obtain:

$$l_t = \bar{l} := \frac{1}{1+\eta} \tag{3.13}$$

We thus observe that labor supply remains constant even as w_t grows. Because all wage income is invested, we derive:

$$\hat{K}_{t+1} g = \underbrace{(1-\alpha)(\bar{l})^{1-\alpha} \hat{K}_t^{\alpha}}_{\text{wage income}}, \tag{3.14}$$

where $\hat{K}_t$ is the detrended level of capital, defined by $\hat{K}_t := K_t / g^t$. Given an initial value of $\hat{K}_0$, this equation determines the equilibrium path of capital.

3.1.5 Bubbly Equilibrium

We now discuss a *bubbly equilibrium*, in which $\tilde{p}_t > 0$ for all $t \geq 0$. An asset price bubble exists because the fundamental value of the bubbly asset is zero.

The following equation must hold in a bubbly equilibrium

$$r_{t+1} = \frac{\tilde{p}_{t+1}}{\tilde{p}_t} \tag{3.15}$$

This is the no-arbitrage condition: the left-hand side represents the return to investing in capital, while the right-hand side represents the return to holding bubbly assets. If this condition does not hold, either the bubbly asset market or the capital rental market fails to clear.

Combining equations (3.3) and (3.15), we again obtain $c_{t+1} = r_{t+1} w_t l_t$. The optimal labor supply condition is therefore

$$l_t = \bar{l} \coloneqq \frac{1}{1+\eta} \tag{3.16}$$

Although labor supply is endogenized in the model, agents choose the same labor hours in both the fundamental equilibrium and the bubbly equilibrium. As we shall see momentarily, this is not the case in Guerron-Quintana et al. (2023) where employment is regime specific.

Using equation (3.2) and an equilibrium condition $w_t \bar{l} = (1-\alpha) Y_t$, we derive the law of motion for capital

$$\hat{K}_{t+1} g = (1-\alpha)(\bar{l})^{1-\alpha} \hat{K}_t^{\alpha} - \underbrace{m_t}_{\text{crowding out}}, \tag{3.17}$$

where m_t denotes the bubble size relative to the growth trend, defined as $m_t \coloneqq (\tilde{p}_t M)/g^t$. Capital is crowded out when $m_t > 0$ because young agents purchase bubbly assets from old agents, and the proceeds are not invested in physical capital but consumed.

The law of motion for the bubble size is derived from equation (3.15):

$$\frac{m_{t+1}}{m_t} g = \alpha (\bar{l})^{1-\alpha} \hat{K}_{t+1}^{\alpha-1}. \tag{3.18}$$

Given the initial conditions $\hat{K}_0$ and m_0, equations (3.17) and (3.18) jointly determine the equilibrium paths of capital $\hat{K}_t$ and bubble size m_t.

3.1.6 Comovement Problem

We simulate the dynamics of a bubble as follows. We assume that the economy is initially in the fundamental equilibrium described in Section 3.1.4. In addition, the detrended capital stock $\hat{K}_t$ is at its steady-state level under that equilibrium. In period $t = 0$, a new vintage of bubbly assets is injected into the economy suddenly and unexpectedly. Specifically, the old agents in period 0 receive M units of brand new bubbly assets as a lump-sum transfer. From then on, the economy transitions to the bubbly equilibrium described in Section 3.1.5. The initial value of $\hat{K}_0$ is predetermined, as it is determined in period $t = -1$, but not m_0. We choose m_0 (equivalently, $\tilde{p}_0$) such that the sequences $\{\hat{K}_t, m_t\}$ converge to a steady state in which m_t remains strictly positive.

Figure 3 presents a numerical simulation.[3] The bubble size jumps from zero to a positive value in period $t = 0$, and subsequently converges to a positive steady state. The remaining four panels show how output (Y_t), consumption (c_t), investment ($I_t := k_{t+1}$), and labor hours (l_t) respond to this event.

The model does not predict an economic boom in period $t = 0$. Investment falls, output initially remains unchanged and then declines, and labor hours do not move. These variables must rise together in a boom, but they clearly fail to do so in this simulation.

Only consumption increases in period $t = 0$. This is because the old agents receive a bubble and increase their consumption by selling it to the young. However, after this trade, the young agents have fewer resources to invest. As a result, investment is crowded out by the bubble. The rise in consumption is the flip side of the fall in investment.

3.2 The Martin and Ventura (2012) Model with Endogenous Labor

Martin and Ventura (2012) introduce financial frictions into an environment similar to that of the Tirole (1985) model. They assume that some agents are more productive than others. If credit markets function well, productive agents can borrow funds from unproductive agents, and capital is produced only by productive agents. However, if there are frictions in the credit markets, capital creation may be inefficiently low. In such cases, bubbles may be associated with higher investment and output if they help to relax these constraints.

3.2.1 Model

We consider an overlapping generations model in which each agent lives for two periods. A new cohort of size 1 is born in each period. Agents maximize the utility function

$$\eta \log(1 - l_t) + \log(c_{t+1}), \tag{3.19}$$

[3] We set $\alpha = 0.4$, $g = 1.1$, and $\eta = 1$. The qualitative results are robust to variations in parameter values.

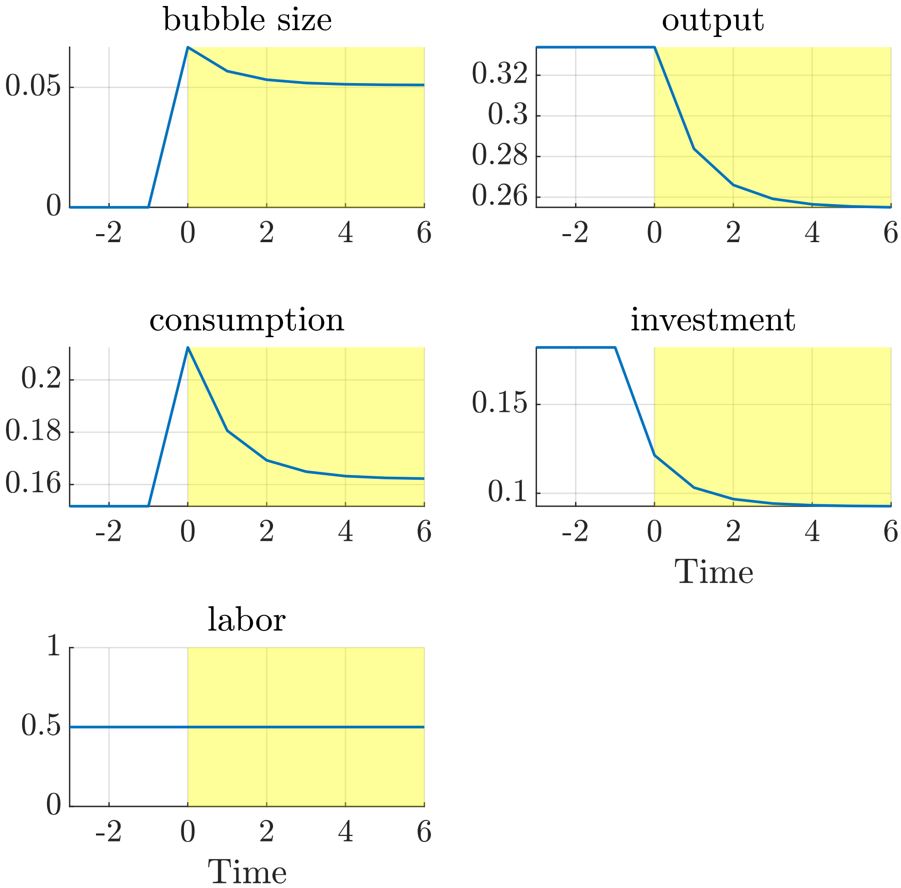

Figure 3: Bubble and Economic Fluctuation in the Tirole (1985) Model with Endogenous Labor
Note: Bubble size, output, consumption, and investment are divided by an exogenous trend component g^t. The bubble period is highlighted in yellow.

where l_t denotes labor hours and c_{t+1} denotes consumption.

In what follows, the superscripts P and U denote the productive agents and unproductive agents, respectively. A fraction π of each cohort is productive. Productive agents are endowed with their own supply of intrinsically worthless assets when young. Each vintage of bubbly assets is distinct, and different vintages are traded at different prices. Let $\tilde{p}_{s|t}$ denote the price in period t of the vintage originally endowed to the cohort born in period $s \in \{0, \cdots, t\}$.[4] Productive agents also supply labor and allocate their income between capital investment and bubbly

[4]Ryo Jinnai learned a great deal about the Martin and Ventura (2012) model from an unpublished book manuscript by Barlevy (2025), including the explicit treatment of vintage-specific prices.

assets. Their budget constraint is:

$$k_{t+1}^P + \sum_{s=0}^{t} \tilde{p}_{s|t} \tilde{m}_{s|t}^P = w_t l_t^P + \tilde{p}_{t|t} M, \tag{3.20}$$

where k_{t+1}^P is the amount of capital produced, $\tilde{m}_{s|t}^P$ is the net holding of the asset vintage endowed to those born in period s, and M is the amount of bubbly assets endowed to productive agents born in period t. Their consumption is given by:

$$c_{t+1}^P = r_{t+1} k_{t+1}^P + \sum_{s=0}^{t} \tilde{p}_{s|t+1} \tilde{m}_{s|t}^P. \tag{3.21}$$

We assume that they cannot choose negative investment ($k_{t+1}^P \geq 0$) or short-sell bubbly assets ($\tilde{m}_{s|t}^P \geq 0$ for all $s \in \{0, \cdots, t\}$).

The remaining fraction of each cohort consists of unproductive agents. Their budget constraint is:

$$\frac{k_{t+1}^U}{\delta} + \sum_{s=0}^{t} \tilde{p}_{s|t} \tilde{m}_{s|t}^U = w_t l_t^U, \tag{3.22}$$

where $\delta < 1$ indicates the inefficiency of unproductive agents: they can produce only δ units of capital per unit of output. We assume unproductive young agents are not endowed with bubbly assets.[5] Their consumption is:

$$c_{t+1}^U = r_{t+1} k_{t+1}^U + \sum_{s=0}^{t} \tilde{p}_{s|t+1} \tilde{m}_{s|t}^U. \tag{3.23}$$

We also assume that they cannot choose negative investment or short-sell bubbly assets.

There is also a cohort of agents born old in period 0, who are collectively endowed with K_0 units of capital. Their consumption is $c_0 = r_0 K_0$. The firm's problem is the same as in Section 3.1.2.

Given initial capital K_0, a competitive equilibrium of the model consists of a sequence of prices r_t, w_t, and $\{\tilde{p}_{0|t}, \cdots, \tilde{p}_{t|t}\}$, and a sequence of quantities L_t, K_{t+1}, l_t^P, k_{t+1}^P, $\{\tilde{m}_{0|t}^P, \cdots, \tilde{m}_{t|t}^P\}$, l_t^U, k_{t+1}^U, and $\{\tilde{m}_{0|t}^U, \cdots, \tilde{m}_{t|t}^U\}$ such that:

(i) Young agents solve their utility maximization problems;

(ii) Firms employ labor and capital optimally;

[5] This assumption can be relaxed without altering the qualitative results.

(iii) The markets for capital, labor, and bubbly assets clear:

$$K_{t+1} = \pi k^P_{t+1} + (1-\pi)k^U_{t+1}, \tag{3.24}$$

$$L_t = \pi l^P_t + (1-\pi)l^U_t, \tag{3.25}$$

$$\pi \tilde{m}^P_{s|t} + (1-\pi)\tilde{m}^U_{s|t} = \pi M \tag{3.26}$$

for all $s \in \{0, \cdots, t\}$ and $t \geq 0$.

3.2.2 Fundamental Equilibrium

We first consider the fundamental equilibrium, in which bubbly assets are always worthless; that is, $\tilde{p}_{s|t} = 0$ holds for all $s \in \{0, \cdots, t\}$ and $t \geq 0$. Both productive and unproductive agents must invest all wage income in capital. Their capital accumulation is given by $k^P_{t+1} = w_t l^P_t$ and $k^U_{t+1} = \delta w_t l^U_t$. Their consumption levels are $c^P_{t+1} = r_{t+1} w_t l^P_t$ and $c^U_{t+1} = r_{t+1}\delta w_t l^U_t$. Labor hours are

$$l^P_t = l^U_t = \bar{l} := \frac{1}{1+\eta}. \tag{3.27}$$

The total capital stock produced is:

$$K_{t+1} = [\pi + (1-\pi)\delta]\, w_t \bar{l}. \tag{3.28}$$

Using the equilibrium condition $w_t \bar{l} = (1-\alpha)Y_t = (1-\alpha)A_t K^\alpha_t \bar{l}^{1-\alpha}$, we can determine the equilibrium path of capital given the initial capital stock K_0.

3.2.3 Bubbly Equilibrium

We now consider a bubbly equilibrium in which $\tilde{p}_{s|t} > 0$ for all $s \in \{0, \cdots, t\}$ and $t \geq 0$. Specifically, we focus on an equilibrium in which unproductive young agents are indifferent between purchasing bubbly assets and producing capital; that is,

$$r_{t+1}\delta = \frac{\tilde{p}_{s|t+1}}{\tilde{p}_{s|t}} \tag{3.29}$$

holds for all $s \in \{0, \cdots, t\}$ and $t \geq 0$. Given (3.29), the consumption level of unproductive agents is $c^U_{t+1} = r_{t+1}(\delta w_t l^U_t)$, and their optimal labor supply is $l^U_t = \bar{l}$. These agents are willing to purchase bubbly assets in the amount

$$\tilde{m}^U_{s|t} = \frac{\pi}{1-\pi} M \tag{3.30}$$

for all $s \in \{0, \cdots, t\}$, and invest the remainder of their income:

$$\frac{k^U_{t+1}}{\delta} = w_t \bar{l} - \sum_{s=0}^{t} \tilde{p}_{s|t} \frac{\pi}{1-\pi} M. \tag{3.31}$$

The value of k^U_{t+1} remains non-negative as long as the bubble size is sufficiently small, which we assume.[6]

Given (3.29), productive agents find capital production more profitable than purchasing bubbly assets. They therefore produce capital according to:

$$k^P_{t+1} = w_t l^P_t + \tilde{p}_{t|t} M. \tag{3.33}$$

Their consumption is given by $c^P_{t+1} = r_{t+1} k^P_{t+1} = r_{t+1}(w_t l^P_t + \tilde{p}_{t|t} M)$. Their optimal labor supply is:

$$l^P_t = \bar{l}\left(1 - \eta \frac{\tilde{p}_{t|t} M}{w_t}\right). \tag{3.34}$$

Combining (3.31) and (3.33), we obtain the total capital produced in the economy:

$$K_{t+1} = \left(\pi l^P_t + (1-\pi)\delta \bar{l}\right) w_t - \underbrace{\delta B^O_t}_{\text{crowding out}} + \underbrace{(1-\delta) B^N_t}_{\text{crowding in}}, \tag{3.35}$$

where B^O_t and B^N_t denote the old and new bubbles in period t, defined by $B^O_t := \sum_{s=0}^{t-1} \tilde{p}_{s|t} \pi M$ and $B^N_t := \tilde{p}_{t|t} \pi M$, respectively. As shown in the first term, capital increases if young agents earn more wage income. However, capital decreases when $B^O_t > 0$ because young agents use part of their income to purchase bubbly assets from old agents, reducing their ability to invest in capital. This is essentially the same crowding-out effect seen in the Tirole (1985) model.

In contrast, the third term shows a *crowding-in effect*: capital increases when $B^N_t > 0$. This mechanism is novel in the Martin and Ventura (2012) model. Productive young agents sell new bubbly assets endowed to them to unproductive agents. This transaction reallocates resources from less productive to more productive agents, thereby enhancing aggregate investment efficiency.

The law of motion for bubble size is derived from the no-arbitrage condition (3.29). Multiplying both sides by πM and summing over all $s \in \{0, \cdots, t\}$, we

[6] We assume the condition

$$\sum_{s=0}^{t} \tilde{p}_{s|t}(\pi M) < (1-\pi) w_t \bar{l} \tag{3.32}$$

holds for all $t \geq 0$.

obtain:

$$B_{t+1}^O = r_{t+1}\delta \underbrace{\left(B_t^O + B_t^N\right)}_{\text{total bubble in } t}. \tag{3.36}$$

Because unproductive agents are indifferent between holding bubbly assets and producing capital, the bubble grows at the same rate as the return on capital production for unproductive agents.

Finally, following Martin and Ventura (2012, p.3050), we assume that the size of the new vintage is determined by endogenous variables:

$$B_t^N = \iota_0 w_t + \iota_1 B_t^O, \tag{3.37}$$

where both ι_0 and ι_1 are positive parameters.

Given initial capital K_0, equations (3.35), (3.36), and (3.37) jointly determine the equilibrium paths of capital K_t and bubble sizes $\{B_t^N, B_t^O\}$.[7]

3.2.4 Comovement Problem

We conduct an exercise similar to that in Section 3.1.6. Suppose the economy is initially in the fundamental equilibrium described in Section 3.2.2, and the

[7] We solve the model as follows. First, rewrite (3.36) as:

$$x_{t+1}^O = \frac{r_{t+1}}{w_{t+1}}\delta(x_t^O + x_t^N)w_t, \tag{3.38}$$

where $x_t^O := \frac{B_t^O}{w_t}$ and $x_t^N := \frac{B_t^N}{w_t}$. Since $r_{t+1}K_{t+1} = \alpha Y_{t+1}$ and $w_{t+1}L_{t+1} = (1-\alpha)Y_{t+1}$, it follows that

$$\frac{r_{t+1}}{w_{t+1}} = \frac{\alpha}{1-\alpha}\cdot\frac{L_{t+1}}{K_{t+1}}. \tag{3.39}$$

Substituting this into the expression for x_{t+1}^O, we get:

$$x_{t+1}^O = \delta\frac{\alpha}{1-\alpha}(x_t^O + x_t^N)\left(\frac{w_t}{K_{t+1}}\right)[\pi l_{t+1}^P + (1-\pi)\bar{l}]. \tag{3.40}$$

Rewriting (3.35), we obtain:

$$\frac{K_{t+1}}{w_t} = \pi l_t^P + (1-\pi)\delta\bar{l} + (1-\delta)x_t^N - \delta x_t^O. \tag{3.41}$$

Substituting (3.41) into (3.40) and using $l_t^P = \bar{l}\left(1 - \frac{\eta}{\pi}x_t^N\right)$, we arrive at:

$$x_{t+1}^O = \frac{\delta\frac{\alpha}{1-\alpha}(x_t^O + x_t^N)\left[\bar{l}(1-\eta x_{t+1}^N)\right]}{[\pi + (1-\pi)\delta]\bar{l} + (\bar{l}-\delta)x_t^N - \delta x_t^O}. \tag{3.42}$$

Finally, rewrite (3.37) as:

$$x_t^N = \iota_0 + \iota_1 x_t^O. \tag{3.43}$$

Equations (3.42) and (3.43) jointly determine the equilibrium paths of x_t^O and x_t^N, from which we can derive the paths of other endogenous variables including K_t.

detrended capital stock $\hat{K}_t \coloneqq K_t/g^t$ is at its steady-state level under that equilibrium. Starting from period $t = 0$, the bubbly equilibrium described in Section 3.2.3 is selected.

Figure 4 presents a numerical simulation.[8] The top-left panel shows the total bubble size $(B_t^O + B_t^N)$ relative to the hourly wage w_t. It jumps from zero to a positive value in period $t = 0$ and then converges to a steady-state level. This marks the emergence of the bubble. The other four panels show how output, consumption, investment, and labor hours respond to this event.

Unlike in the Tirole (1985) model, consumption does not increase in period $t = 0$ because old agents are not endowed with bubbly assets.[9] Investment does not rise either, even though productive young agents are endowed with a new vintage of bubbly assets.[10] This is because the new vintage is traded among the young and does not expand the total funds available for investment.

Output increases in period $t = 1$ due to the crowding-in effect discussed in the previous section. As output rises, both consumption and investment also begin to increase.

Labor hours decline in period $t = 0$ and remain low thereafter. This is due to the income effect.[11] Productive young agents increase their consumption using the proceeds from the newly endowed bubbly assets. This windfall allows them to enjoy more leisure. As a result, output falls in period $t = 0$ due to reduced labor input. Consumption also falls in period $t = 0$ because the return on capital declines. Investment declines as well because labor compensation falls.

Overall, output, consumption, and investment do not rise immediately after the emergence of the bubble. Labor input declines immediately and remains per-

[8]We set $\alpha = 0.4$, $\delta = 0.1$, $\pi = 0.06$, $\iota_0 = 0.03$, $\iota_1 = 0.036$, $g = 1.1$, and $\eta = 1$. The qualitative results are robust to variations in parameter values.

[9]Aggregate consumption is defined as $C_t \coloneqq \pi c_t^P + (1-\pi)c_t^U$. In equilibrium,

$$C_t = \alpha Y_t + B_t^O. \tag{3.44}$$

[10]Aggregate investment is defined as $I_t \coloneqq \pi k_{t+1}^P + (1-\pi)(k_{t+1}^U/\delta)$. In equilibrium,

$$I_t = \left(\pi l_t^P + (1-\pi)\bar{l}\right) w_t - B_t^O. \tag{3.45}$$

[11]Aggregate labor supply is given by $L_t = \pi l_t^P + (1-\pi)l_t^U$. In equilibrium,

$$L_t = \bar{l}\left(1 - \underbrace{\frac{\eta}{\pi}\frac{B_t^N}{w_t}}_{\text{income effect}}\right). \tag{3.46}$$

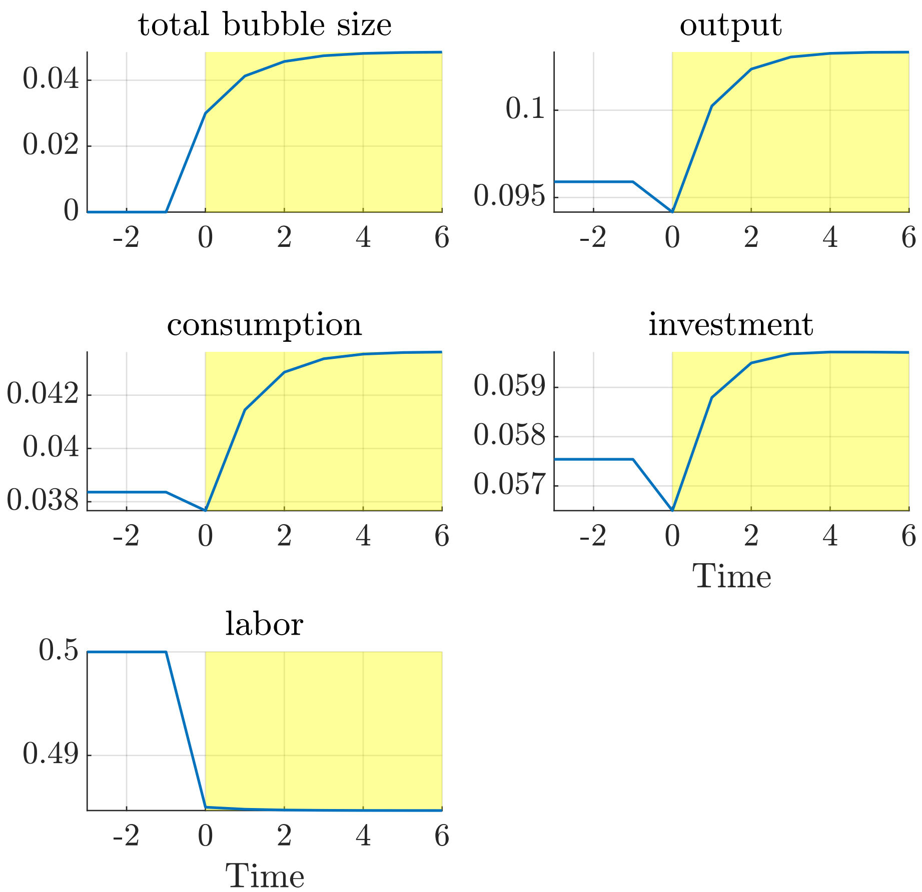

Figure 4: Bubble and Economic Fluctuation in the Martin and Ventura (2012) Model with Endogenous Labor
Note: Total bubble size is divided by the hourly wage w_t. Output, consumption, and investment are divided by the exogenous trend component g^t. The bubble period is highlighted in yellow.

sistently low. While output, consumption, and investment start to recover with a lag, the model does not predict an economic boom at the time the bubble emerges.

4 Bubbles and Economic Fluctuations in the Guerron-Quintana et al. (2023) Model

We study the Guerron-Quintana et al. (2023) model because it successfully generates an economic boom when a bubble emerges. It incorporates a crowding-in effect similar to that in the Martin and Ventura (2012) model. In addition, it features variable capacity utilization: the owner of capital can choose how intensively to use it, at the cost of faster depreciation. This mechanism is a common source of shock amplification in the business cycle literature. Aiming for quantitative applications, Guerron-Quintana et al. (2023) introduce bubbles into an infinite-horizon macroeconomic framework.

4.1 Households

The economy is populated by a continuum of households, with measure one. All households behave identically. Each household has a unit measure of members who are identical at the beginning of each period. During the period, members are separated from each other, and each member receives a shock that determines her role in the period. A member will be an investor with probability $\pi \in [0, 1]$ and will be a saver/worker with probability $1 - \pi$. These shocks are i.i.d. among members and across time.

A period is divided into four stages: household's decisions, production, investment, and consumption. In the household's decision stage, all members of a household are together and pool their assets: n_t units of capital and $\tilde{m}_t$ units of bubbly assets. The head of the household decides the capacity utilization rate u_t. Because all members of the household are identical in this stage, the household head evenly divides the assets among the members. The household head also gives contingency plans to each member as follows. If the member becomes an investor, he or she spends i_t units of final goods to invest, and brings home the following items before the consumption stage: x_t^i units of final goods, n_{t+1}^i units of capital, and $\tilde{m}_{t+1}^i$ units of bubbly assets. Likewise, if the member becomes a saver, he or she supplies l_t units of labor, and brings home the following items before the consumption stage: x_t^s units of final goods, n_{t+1}^s units of capital, and $\tilde{m}_{t+1}^s$ units of bubbly assets. After receiving these instructions, members go to the market and remain separated from each other until the consumption stage. The household maximizes the utility function given by

$$\sum_{t=0}^{\infty} \beta^t \left(\pi \log\left(c_t^i\right) + (1 - \pi)\left[\log\left(c_t^s\right) + \eta \log\left(1 - l_t\right)\right]\right) \tag{4.1}$$

At the beginning of the second stage, each member receives the shock that determines her role in the period. Markets open and competitive firms produce final goods. Compensation for productive factors is paid to their owners. A fraction of capital depreciates. Following Greenwood et al. (1988), we assume that a higher utilization rate causes a faster depreciation of the capital stock.[12]

[12]This is either because wear and tear increase with use or because less time can be devoted to maintenance.

Specifically, the depreciation rate $\delta(u_t)$ is given by

$$\delta(u_t) = \delta_0 + \frac{\delta_1}{1+\zeta} u_t^{1+\zeta} \tag{4.2}$$

where $\zeta > 0$.

Investors seek finance to undertake investment projects in the third stage. Financing comes through different channels: own resources, selling of new and existing capital, and bubbly assets. Investors have access to a linear technology that transforms any amount of final goods into the same amount of new capital. Asset markets close at the end of this stage. Members of the household meet again in the consumption stage. An investor consumes c_t^i units of final goods and a saver consumes c_t^s units of final goods. After consumption, members' identities are lost. They start a new period as identical members. Capital and bubbly assets are aggregated at the household level before starting a new period;

$$n_{t+1} = \pi n_{t+1}^i + (1-\pi) n_{t+1}^s \tag{4.3}$$

and

$$\tilde{m}_{t+1} = \pi \tilde{m}_{t+1}^i + (1-\pi) \tilde{m}_{t+1}^s \tag{4.4}$$

The instructions must meet a set of constraints. First, they have to satisfy an intratemporal budget constraint. For an investor, it is

$$x_t^i + i_t + \underbrace{q_t\left(n_{t+1}^i - i_t - (1-\delta(u_t))\, n_t\right)}_{\text{net capital purchase}} + \underbrace{\tilde{p}_t\left(\tilde{m}_{t+1}^i - \tilde{m}_t\right)}_{\text{net bubble purchase}} = r_t \underbrace{u_t n_t}_{\text{capital service}} \tag{4.5}$$

and for a saver, it is

$$x_t^s + q_t\left(n_{t+1}^s - (1-\delta(u_t))\, n_t\right) + \tilde{p}_t\left(\tilde{m}_{t+1}^s - \tilde{m}_t\right) = r_t u_t n_t + w_t l_t \tag{4.6}$$

Here, q_t and $\tilde{p}_t$ denote prices of capital and bubbly assets, respectively. The household can earn larger rental income now by increasing the utilization rate at the expense of faster depreciation of capital. In addition, the instructions must satisfy a feasibility constraint in the consumption stage given by

$$\pi x_t^i + (1-\pi) x_t^s = \pi c_t^i + (1-\pi) c_t^s \tag{4.7}$$

Following Kiyotaki and Moore (2019), we assume that an investor can issue new equity on, at most, a fraction ϕ of investment. In addition, she can sell, at most, a fraction ϕ of existing capital in the market too. Effectively, these constraints introduce a lower bound to the capital holdings at the end of the period:

$$n_{t+1}^i \geq (1-\phi)\left(i_t + (1-\delta(u_t))\, n_t\right) \tag{4.8}$$

A similar constraint applies to n_{t+1}^s, but we omit it because it does not bind in equilibrium. We also omit non-negativity constraints for u_t, c_t^i, i_t, n_{t+1}^i, x_t^s, c_t^s, l_t, n_{t+1}^s, and $\tilde{m}_{t+1}^s$ for the same reason. However, there are two exceptions

$$\tilde{m}_{t+1}^i \geq 0 \tag{4.9}$$

and

$$x_t^i \geq 0 \tag{4.10}$$

which mean that the investor can't short sell bubbly assets and must bring a non-negative amount of consumption back to the household.

We assume that ϕ is sufficiently small. The price of capital q_t exceeds one in this case. Because capital creation is profitable, investors will increase i_t as much as possible, implying that the following feasibility constraint for investment holds in equilibrium:

$$\underbrace{(1-\phi q_t)\, i_t}_{\text{minimum cost to conduct } i_t} = \underbrace{u_t r_t n_t + \phi q_t\left(1-\delta(u_t)\right) n_t + \tilde{p}_t \tilde{m}_t}_{\text{maximum liquidity an investor can attain}} \tag{4.11}$$

The left-hand side is the minimum cost investors have to finance in order to conduct i_t, which is smaller than i_t because a part of the costs can be covered by selling newly created capital. The right-hand side is the maximum liquidity an investor can attain.

Combining (4.3), (4.5), (4.6), and (4.7), we obtain the budget constraint at the household level:

$$\begin{aligned} &\pi c_t^i + (1-\pi)\, c_t^s + \pi i_t + q_t\left[n_{t+1} - (1-\delta(u_t))\, n_t\right] + \tilde{p}_t\left[(1-\pi)\,\tilde{m}_{t+1}^s - \tilde{m}_t\right] \\ &= u_t r_t n_t + \pi q_t i_t + (1-\pi)\, w_t l_t \end{aligned} \tag{4.12}$$

Substituting (4.11) into (4.12), we obtain

$$\begin{aligned}&\pi c_t^i + (1-\pi)\, c_t^s + q_t n_{t+1} + \tilde{p}_t (1-\pi)\, \tilde{m}_{t+1}^s \\ &= u_t r_t n_t + (1-\pi)\, w_t l_t + (1-\delta(u_t))\, q_t n_t + \tilde{p}_t \tilde{m}_t \\ &+ \lambda_t \pi \underbrace{[u_t r_t n_t + \phi q_t (1-\delta(u_t))\, n_t + \tilde{p}_t \tilde{m}_t]}_{\text{maximum liquidity an investor can attain}}\end{aligned} \tag{4.13}$$

where

$$\lambda_t \equiv \frac{q_t - 1}{1 - \phi q_t} \tag{4.14}$$

The left-hand side is gross spending, consisting of consumption and gross asset purchases. The first line in the right-hand side is gross income, consisting of dividends, labor income, and the market value of the portfolio. The second line in the right-hand side is the total profit from capital creation. The reason is the following. An investor can create $1/(1-\phi q_t)$ units of capital from a unit of liquidity. A fraction ϕ of the investment is sold, and the rest is added to the investor's portfolio, which is worth $(1-\phi)\, q_t/(1-\phi q_t)$. Finally, subtracting the costs of the investment from it, we find

$$\frac{(1-\phi)\, q_t}{1-\phi q_t} - 1 = \frac{q_t - 1}{1-\phi q_t} = \lambda_t \tag{4.15}$$

Hence, λ_t measures how much value an investor can create from a unit of liquidity. Finally, because investors as a group have $\pi [u_t r_t n_t + \phi q_t (1-\delta(u_t))\, n_t + \tilde{p}_t \tilde{m}_t]$ units of liquidity, the second line in the right-hand side is the total profit from capital creation at the household level.

The household's problem can be reformulated in a simpler way as follows. It chooses a sequence of u_t, c_t^i, c_t^s, l_t, n_{t+1}, and $\tilde{m}_{t+1}^s$ to maximize the utility (4.1) subject to the budget constraint (4.13) and the law of motion of bubbly assets $\tilde{m}_{t+1} = (1-\pi)\, \tilde{m}_{t+1}^s$. The first-order conditions are

$$c_t^i = c_t^s \tag{4.16}$$

$$\eta \frac{c_t^s}{1-l_t} = w_t \tag{4.17}$$

$$r_t - \delta'(u_t)\, q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0 \tag{4.18}$$

$$q_t = \beta \left(\frac{c_t^s}{c_{t+1}^s}\right) (u_{t+1} r_{t+1} + (1-\delta(u_{t+1}))\, q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1-\delta(u_{t+1})))) \tag{4.19}$$

and

$$\tilde{p}_t = \beta \left(\frac{c_t^s}{c_{t+1}^s} \right) (1 + \pi \lambda_{t+1}) \, \tilde{p}_{t+1} \tag{4.20}$$

The first equation states that the marginal utility from consumption has to be equalized across members of the household. The second equation states that the marginal rate of substitution between leisure and consumption has to be equal to the wage. The third equation states that the marginal benefit of raising the capacity utilization rate (the rental rate of capital) has to be equal to its opportunity cost (the value of the depreciated capital at the margin). The fourth equation is the Euler equation for capital, in which λ_t appears because capital is not only a production factor but also a means of providing liquidity to investors. The fifth equation is the Euler equation for the bubbly assets. The left-hand side of this equation is strictly positive only if there is a chance that the bubbly assets in period t will be traded at a strictly positive price in the next period. In other words, it is the resalability of bubbly assets that justifies their positive prices, which is the same in Tirole (1985) and Martin and Ventura (2012).

4.2 Firms

Competitive firms produce output from capital and labor services denoted by KS_t and L_t, respectively. The production function is

$$Y_t = A_t \left(KS_t\right)^{\alpha} \left(L_t\right)^{1-\alpha}, \tag{4.21}$$

where A_t is the technology level that agents in the economy take as given. Firms maximize profits defined as $Y_t - r_t KS_t - w_t L_t$ by choosing KS_t and L_t, where r_t is the rental price of capital and w_t is the wage rate. The production technology is freely available to potential entrants. Firms make zero profits in equilibrium.

We assume that, while all the economic agents in the model take it as given, the technology level A_t is actually endogenous:

$$A_t = \bar{A} \left(K_t\right)^{1-\alpha}. \tag{4.22}$$

$\bar{A}$ is a scale parameter. Following Arrow (1962), Sheshinski (1967), and Romer (1986), we interpret the dependency of A_t on K_t as learning-by-doing; namely, knowledge is a by-product of investment, and in addition, it is a public good that anyone can access at zero cost. With this assumption, the long-run tendency for capital to experience diminishing returns is eliminated. Long-run growth is sustained by both capital and knowledge accumulation. Moreover, the growth rate is endogenous and influenced by not only the state of the economy but also actions taken by economic agents.

The AK structure is important for tractability. As we discuss in the following section, once we de-trend the model by K_t, the equilibrium conditions depend only on the exogenous state variables. The resulting system is computationally tractable. Importantly, we can find the regime-dependent steady states and estimate the model in reasonable time.[13]

4.3 Competitive Equilibrium

Competitive equilibrium is defined in a standard way; all agents optimize given prices, and the market clearing conditions are satisfied; i.e.,

$$n_{t+1} = K_{t+1} \tag{4.23}$$

$$L_t = (1 - \pi)\, l_t \tag{4.24}$$

$$KS_t = u_t K_t \tag{4.25}$$

$$\pi c_t^i + (1 - \pi)\, c_t^s + \pi i_t = Y_t \tag{4.26}$$

$$\pi \tilde{m}_{t+1}^i + (1 - \pi)\, \tilde{m}_{t+1}^s = M \tag{4.27}$$

The law of motion for aggregate capital stock is

$$K_{t+1} = (1 - \delta(u_t))\, K_t + \pi i_t \tag{4.28}$$

which automatically holds by Walras' law.

[13] The analysis is way more complicated if the aggregate production function has diminishing returns to scale in capital. This is because not only capital becomes a state variable but also the number of possible cases for the level of capital grows over time for the following reason. First, the stock of capital today depends on the one yesterday, but capital yesterday could take different values depending on the regime yesterday. The same reasoning applies for the stock of capital on the day before yesterday. This means that we need to track the entire history of regime switches up to time t in order to solve the model. The possible levels of capital explode as time evolves, making the solution effectively unfeasible.

4.4 Crowding-In and Crowding-Out Effects of a Realized Bubble

Rewriting equation (4.11) and imposing the market clearing condition for bubbly assets, we obtain

$$\underbrace{\pi i_t}_{\text{gross investment}} = \pi \underbrace{\frac{1}{1-\phi q_t}}_{\text{financial leverage}} \left(u_t r_t n_t + \phi q_t \left(1 - \delta(u_t)\right) n_t + \underbrace{\tilde{p}_t M}_{\text{crowding-in}} \right). \tag{4.29}$$

All else equal, gross investment is higher in an equilibrium with a bubble ($\tilde{p}_t > 0$) than in one without ($\tilde{p}_t = 0$). This is because productive members (investors) sell bubbly assets to less productive members (savers) and use the proceeds to expand investment. Although, unlike in Martin and Ventura (2012), investors are not endowed with a new vintage of bubbly assets each period, a similar crowding-in effect arises because the assets are distributed across members before it is known who will become investors.

A crowding-out effect also exists and manifests as a pecuniary externality. Other things being equal, the price of capital q_t is lower in a bubbly equilibrium ($\tilde{p}_t > 0$) than in a bubbleless one ($\tilde{p}_t = 0$). The reason is that, as shown in equation (4.19), capital is valued not only for its dividends but also for the liquidity it provides to investors. In a bubbly equilibrium, this liquidity value is diluted, since the bubble itself provides liquidity. A lower q_t reduces gross investment πi_t by increasing the down payment investors must make—appearing in equation (4.29) as a reduction in financial leverage.

4.5 Comovement

We conduct a similar exercise as in Sections 3.1.6 and 3.2.4. We assume that the economy is initially in the fundamental equilibrium, where bubbly assets are valueless. In period $t = 0$, a new vintage of bubbly assets is injected into the economy suddenly and unexpectedly; specifically, M units of brand-new bubbly assets are distributed to households in a lump-sum manner. Starting in period $t = 0$, a bubbly equilibrium is selected. The economy then follows a transition path that converges to a steady state in which the ratio of bubble size to capital stock ($\tilde{p}_t M/K_t$) remains positive.

Figure 5 presents a numerical simulation.[14] As shown in the top-left panel, the

[14]Following Guerron-Quintana et al. (2023), we set $\beta = 0.99$, $\pi = 0.06$, $\phi = 0.19$, $\alpha = 0.33$,

bubble size jumps from zero to a positive value in period $t = 0$. The subsequent panels show that output, consumption, investment, and labor hours all increase simultaneously upon the emergence of the bubble. An economic boom occurs immediately, rather than with a delay.

The capacity utilization rate plays a central role in this dynamic. As shown in equation (4.18), the household chooses the utilization rate by weighing the marginal benefit (the rental rate of capital) against the opportunity cost (the marginal value of depreciated capital). When the bubble emerges, the opportunity cost falls because the price of capital declines, as discussed in Section 4.4. In response, the household raises the utilization rate, as depicted in the bottom-right panel.

Because capital is now used more intensively, demand for labor services increases due to the complementarity between capital and labor in production. Consequently, both wages and labor supply rise, contributing to higher output. Consumption increases as households become wealthier following the bubble's emergence. Investment also rises, as the crowding-in effect of the bubble outweighs the crowding-out effect, both discussed in Section 4.4.

4.6 Stochastic Model

Before closing this section, we introduce aggregate shocks into the model. This stochastic extension is used in the quantitative analysis presented in the next section.

We begin by introducing regime switches. Let $z_t \in \{b, f\}$ denote the regime at time t, where b and f indicate the bubbly and fundamental regimes, respectively. The regimes differ in whether bubbly assets exist. In the fundamental regime, there are no bubbly assets in the economy. When the regime switches to the bubbly state, M units of a new vintage of bubbly assets are created and distributed to households in a lump-sum manner. No bubbly assets are created under other circumstances. Once created, bubbly assets persist without depreciation as long as the economy remains in the bubbly regime, but they vanish immediately when the regime reverts to the fundamental state. z_t follows a Markov process.

The law of motion for bubbly asset holdings, previously specified in equation (4.4), is modified to:

$$\tilde{m}_{t+1} = \pi \tilde{m}^i_{t+1} + (1-\pi)\tilde{m}^s_{t+1} + \mathbf{1}_{\{z_t = f,\, z_{t+1} = b\}} M \tag{4.30}$$

$\delta_0 = 0.001$, $\delta_1 = 0.065$, $\zeta = 0.33$, $A = 0.49$, and $\eta = 2.67$.

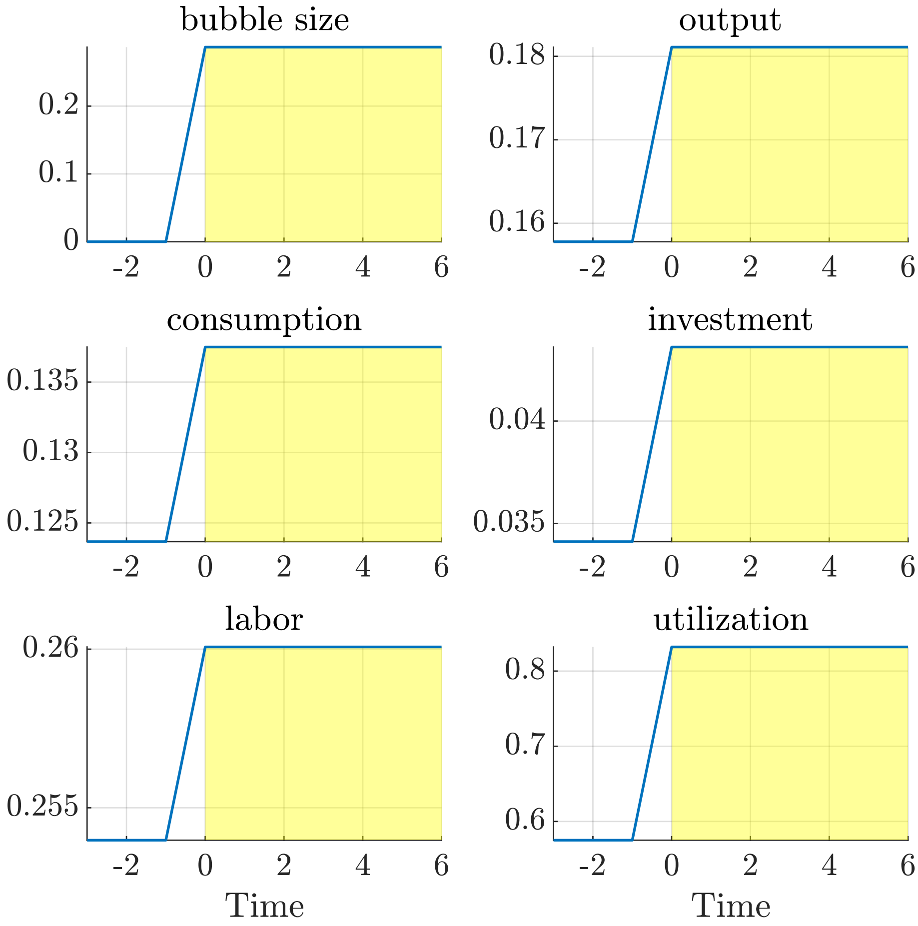

Figure 5: Bubble and Economic Fluctuation in the Guerron-Quintana et al. (2023) Model
Note: Bubble size, output, consumption, and investment are divided by capital stock K_t. The bubble period is highlighted in yellow.

where **1** is an indicator function. The last term reflects the issuance of a new vintage of bubbly assets when the regime shifts from fundamental to bubbly.

In the fundamental regime, there are no spot or forward markets for bubbly assets, and future bubbles cannot be used as collateral. We implement this restriction by modifying the budget constraints in equations (4.5) and (4.6) as follows:

$$x_t^i + i_t + q_t\left(n_{t+1}^i - i_t - (1-\delta(u_t))\,n_t\right) + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\left(\tilde{m}_{t+1}^i - \tilde{m}_t\right) = u_t r_t n_t \quad (4.31)$$

$$x_t^s + q_t\left(n_{t+1}^s - (1-\delta(u_t))\,n_t\right) + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\left(\tilde{m}_{t+1}^s - \tilde{m}_t\right) = u_t r_t n_t + w_t l_t \quad (4.32)$$

We also impose:

$$\mathbf{1}_{\{z_t=f\}}\tilde{m}_{t+1}^i = \mathbf{1}_{\{z_t=f\}}\tilde{m}_{t+1}^s = 0 \quad (4.33)$$

These restrictions ensure that bubbly assets cannot be traded in the fundamental regime. The market-clearing condition for bubbly assets, replacing equation (4.27), becomes:

$$\pi\tilde{m}^i_{t+1} + (1-\pi)\tilde{m}^s_{t+1} = \mathbf{1}_{\{z_t=b\}}M \tag{4.34}$$

Both sides are zero in the fundamental regime.

Next, we introduce a productivity shock. We modify the technology level, originally defined in equation (4.22), as:

$$A_t = \bar{A}K_t^{1-\alpha}e^{a_t} \tag{4.35}$$

where a_t is an exogenous AR(1) process. This shock captures stylized supply-side fluctuations.

Finally, we introduce a preference shock that temporarily affects household impatience. The utility function in equation (4.1) is modified as follows:

$$\mathbf{E}_0\left[\sum_{t=0}^{\infty}\frac{\beta^t}{e^{d_t}}\left(\pi\log(c^i_t) + (1-\pi)\left[\log(c^s_t) + \eta\log(1-l_t)\right]\right)\right] \tag{4.36}$$

Here, d_t is an exogenous AR(1) process representing a stylized demand-side shock. This preference shock shares many characteristics with financial shocks frequently analyzed in the literature; see Fisher (2014) and Smets and Wouters (2007).

4.7 Crowding-Out Effect of Future Bubbles

Guerron-Quintana et al. (2023) study a *recurrent-bubble equilibrium* in which bubbly assets are traded at positive prices during the bubbly regime but do not exist in the fundamental regime. They identify a crowding-out effect of *future bubbles*, which is related to—but distinct from—the crowding-out effect of realized bubbles discussed in Section 4.4.

To illustrate this effect graphically, we simulate the emergence of a bubble in two different scenarios. In the first, the bubble is entirely unanticipated prior to period $t = 0$ and emerges suddenly. In the second, agents in periods $t < 0$ anticipate that a bubble may emerge in the future with some probability, and it indeed materializes in period $t = 0$ as expected, albeit with uncertainty. For simplicity, we assume in both scenarios that the bubble, once formed, persists indefinitely.

The top panel of Figure 6 displays the path of investment under these two scenarios.[15] A notable difference arises prior to the bubble's emergence: investment

[15] We set the transition probability from $z_t = f$ to $z_t = b$ at 1.5%.

is lower in the economy where agents anticipate a future bubble (solid blue line) compared to the economy where the bubble is completely unexpected (red dotted line). Since no bubble exists in periods $t < 0$, this difference cannot be attributed to the crowding-out effect of realized bubbles. Rather, it reflects the crowding-out effect of *anticipated future bubbles*.

This effect also operates through a pecuniary externality. When agents expect future bubbles, the current price of capital q_t declines because they foresee that realized bubbles will later depress capital prices, as discussed in Section 4.4. A lower capital price reduces investors' financial leverage, which adversely affects gross investment.

The bottom panel of Figure 6 plots the ratio of stock market value to annualized GDP. We define stock market value as:

$$stock_t := \phi q_t K_{t+1} + \underbrace{\mathbf{1}_{\{z_t=b\}} \tilde{p}_t M}_{\text{bubble}} \tag{4.37}$$

The bubble component is included in the stock market valuation because we assume that bubbly assets are tied to the equities of final goods firms, which are intrinsically worthless. Additionally, we assume that a fraction ϕ of the capital stock is publicly traded, following empirical evidence provided in Guerron-Quintana et al. (2023, p. 349).

When the bubble emerges, we observe a stock market boom. Although the capital price q_t declines, the bubble term offsets this drop and elevates overall market valuation.

5 Quantitative Investigation

Section V of Guerron-Quintana et al. (2023) presents an attempt to bridge theoretical insights about asset price bubbles with empirical evidence. The analysis here examines the key methodological challenges and innovations in their approach, with a particular focus on solution/estimation complexities.

5.1 Solving the model

As explained above, our model contains two features that complicate and alleviate its numerical computation. First, the economy can be in either the fundamental regime or the bubbly regime. Since the regimes follow a Markov chain of order 1, the model inherits the Markovian characteristic. As shown by Farmer et al. (2011),

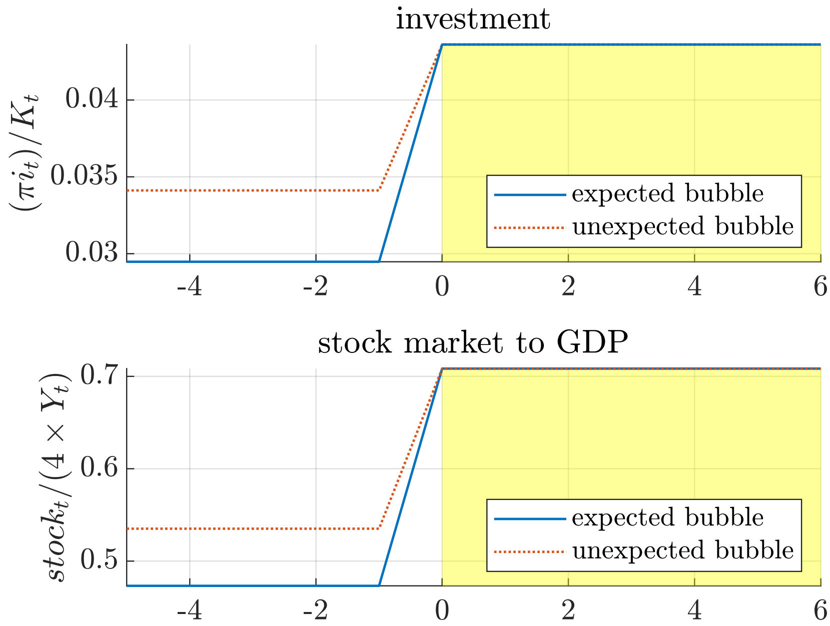

Figure 6: Crowding-Out Effect of Future Bubbles in the Guerron-Quintana et al. (2023) Model
Note: Investment is divided by the capital stock K_t. The bubble period is highlighted in yellow.

solving this class of models is intrinsically difficult and is prone to suffering from indeterminacy. Moreover, the switches imply that the state of the economy at a given point in time depends on the entire history of the bubbles' rise and collapse.

Second, there is growth in the economy, making the model non-stationary. This means that we need to make the model stationary before trying to apply the methods in the toolkit of modern macroeconomics (Fernández-Villaverde et al. (2016)).[16] In the Guerron-Quintana et al. (2023) model, however, the only endogenous state variable is capital. So the model is de-trended by dividing growing variables such as output Y_t by capital K_t. The resulting model is entirely forward-looking, which removes the model's dependence on the past history of bubbles. This property is crucial for tractability.

To see these two forces in action, let us consider some of the equilibrium conditions in Guerron-Quintana et al. (2023). A simplified version of the production function is

$$Y_{j,t} = K_t((1-\pi)l_{j,t})^{1-\alpha}.$$

Here, the subindex j takes the value f if the economy is in the fundamental regime or b if the economy is in the bubbly regime. Capital stock K_t does not

[16] The approach outlined in Kulish and Pagan (2017) is an alternative and promising way of dealing with the non-stationarity and Markov switches.

have the subindex j because it is a predetermined variable and hence does not depend on the realization of the current regime z_t. If we define de-trended output as $\hat{y}_{j,t} \equiv Y_{j,t}/K_t$, the production function depends exclusively on labor today. Importantly, capital is no longer a state variable. The only states of the economy are the bubble regime and the demand/productivity shocks. Because they follow exogenous stochastic processes, their realization at time t is known when solving the model.

The optimality condition of the household for capital is

$$q_{j,t} = \mathbf{E}_t \Bigg[\beta \left(\frac{\hat{c}^s_{j,t}}{\hat{c}^s_{t+1}} \frac{1}{g_{j,t}} \right) (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} \\ + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \Bigg]$$

where $g_{j,t}$ is capital growth defined by $g_t \coloneqq K_{j,t+1}/K_t$.[17] Here, the subindex j indicates the regime at time t and the expectation operator is with respect to the probability of switching regimes at $t+1$. The reader should keep in mind that the variables in the next period are not indexed by j because, from the perspective of today, they can be fundamental or bubbly. Similarly to the production function approach, we de-trend consumption by capital to render the optimality condition stationary.

If we proceed with this approach over all the equilibrium conditions of the model, we obtain a system that is stationary and forward-looking. The final step in the solution process is to recognize that to solve the model today, say in the bubbly regime, we need the controls and states in the bubbly regime tomorrow as well those variables in the fundamental regime tomorrow. This can be achieved by solving an expanded version of the model that states the equilibrium conditions of the fundamental and blubbly regimes one on top of the other. To help visualize the structure of the expanded model, the block corresponding to the production function will consist of these equations

$$\hat{y}_{f,t} = ((1-\pi) l_{f,t})^{1-\alpha},$$
$$\hat{y}_{b,t} = ((1-\pi) l_{b,t})^{1-\alpha}$$

Next, we provide a more detailed description of the solution and estimation of the model. It is based on the technical appendix in Guerron-Quintana et al.

[17] Capital tomorrow K_{t+1} depends on the current regime, because it is determined at t.

(2023)

First, we de-trend the equilibrium conditions of the model by scaling them with the capital stock. This transformation ensures that the model becomes stationary, which is a necessary step before any perturbation or estimation technique is applied. In the stationary version of the model, the dynamics are governed entirely by expectations about the future, given the realization of current structural shocks and the prevailing regime. As a result, the system is forward-looking and its behavior is characterized by analogous equations in Section 4.

We introduce the notation used to differentiate between the two regimes in the model. Let X_t^f and Y_t^f denote the vectors of endogenous state and control variables, respectively, when the economy is in the fundamental regime. Analogously, X_t^b and Y_t^b represent the state and control vectors in the bubbly regime. The de-trended model can then be written as a system of conditional equilibrium equations:

$$\mathbf{E}_t \Gamma_f(X_t^f, Y_t^f, X_{t+1}^f, Y_{t+1}^f, X_{t+1}^b, Y_{t+1}^b) = 0,$$
$$\mathbf{E}_t \Gamma_b(X_t^b, Y_t^b, X_{t+1}^f, Y_{t+1}^f, X_{t+1}^b, Y_{t+1}^b) = 0.$$

These two systems represent the equilibrium dynamics of the model conditional on the regime being fundamental or bubbly. Importantly, the formulation captures the possibility of regime switching in the future, as indicated by the dependence of each regime's equilibrium on both sets of future state and control variables. The functions $\Gamma_f(\cdot)$ and $\Gamma_b(\cdot)$ encapsulate the equilibrium conditions for each regime.

The next step involves computing the steady states for each regime, which we do by solving the system in the absence of structural shocks while allowing for regime switching. Specifically, we search for fixed points (X^f, Y^f, X^b, Y^b) that satisfy the following conditions:

$$\Gamma_f(X^f, Y^f, X^f, Y^f, X^b, Y^b) = 0,$$
$$\Gamma_b(X^b, Y^b, X^f, Y^f, X^b, Y^b) = 0.$$

This approach ensures that the computed steady states account for the potential transitions between regimes and thus reflect a more realistic long-run characterization of the economy. It captures a non-linear impact of the emergence of a bubble as a transition from (X^f, Y^f) to (X^b, Y^b), as well as a non-linear impact of the collapse as a transition from (X^b, Y^b) to (X^f, Y^f). It captures effects of the household's anticipation of regime switches, including the crowding-out effect of future bubbles discussed in Section 4.7.

Once the steady states are determined, we linearize the model around them using a first-order perturbation method.[18] This yields a linear approximation of the model's law of motion for endogenous variables. The linearized system captures the dynamic response of the model to small deviations from the steady states, including the effects of structural shocks. For this purpose, we adopt the first-order perturbation technique described in Schmitt-Grohe and Uribe (2004).

The linearized dynamics of the system can be compactly expressed as:

$$\mathbb{X}_t = \Lambda_x \mathbb{X}_{t-1} + \Omega_x \Xi_{x,t},$$

where $\mathbb{X}_t = [X_t^f, Y_t^f, X_t^b, Y_t^b]'$ is the vector of all endogenous variables in both regimes, and $\Xi_{x,t}$ denotes the vector of structural innovations at time t.

To connect the model with observed macroeconomic data, we specify a measurement equation:

$$\mathbb{Y}_t = \Lambda_y \mathbb{X}_t + \Omega \Xi_{y,t}.$$

In this expression, $\mathbb{Y}_t$ represents the vector of observed data, which are constructed from the variables of the model through the mapping matrix Λ_y. The term $\Xi_{y,t}$ captures classical measurement errors that account for the discrepancies between the model and the actual data.

For estimation, we compute the likelihood of the model using a non-linear filtering method capable of handling regime switching. Specifically, we apply the filtering approach described in Chapter 5 of Kim and Nelson (1999), which is designed to handle Markov-switching state-space models.

Finally, we estimate the model using a Bayesian approach, following the framework developed in Fernandez-Villaverde et al. (2016). This method involves specifying prior distributions for the model parameters and updating them with the observed data to obtain the posterior distributions. The Bayesian approach provides a coherent framework for incorporating both parameter uncertainty and regime uncertainty into the estimation.

[18] In our solution algorithm, there is no restriction on the type of approximation technique around each of the steady states. We could use higher order perturbation or projection methods. However, this will come at the expense of complicating the estimation stage, which is already complex due to the regime switches.

5.2 Identifying Bubbles in the Data

Guerron-Quintana et al. (2023) identify the presence of asset price bubbles by leveraging their model's ability to predict that both GDP growth and the stock-market-to-GDP ratio are concurrently elevated during bubbly periods. Figure 7 shows their estimate of probability of the U.S. economy being in the bubbly regime. Their sample period is from 1984Q1 to 2017Q4.

It strongly suggests the existence of at least three distinct bubbly episodes. The first happened before the stock market crash (Black Monday) of October 1987. The second likely occurred between roughly 1997 and 2001, while the third spanned the years leading up to the Great Recession, beginning around 2006. During both intervals, robust performance in the asset markets coincided with vigorous economic growth, conditions that our model interprets as symptomatic of bubble formation.

However, not every episode of economic expansion qualifies as a bubble in their framework. For instance, the pronounced GDP growth observed in the mid-1990s is attributed to positive productivity shocks rather than speculative dynamics, as the stock-market-to-GDP ratio remained insufficiently elevated to signal a bubble. Similarly, a surge in equity markets alone does not automatically imply the presence of a bubble. A notable example of this is the year 2014, when stock prices were soaring but GDP growth remained comparatively modest. This divergence illustrates the importance of considering both indicators jointly; a boom is classified as a bubbly boom only if high asset valuations are accompanied by correspondingly strong real economic growth. This is an important difference from the econometric approach discussed in Section 2, in which a unit root test is applied to a univariate time series.

The most recent two bubbles identified by Guerron-Quintana et al. (2023) are followed by immediate recessions, which are highlighted in gray in Figure 7.[19] The Guerron-Quintana et al. (2023) model is consistent with this pattern too; their model predicts that a recession should be observed when the bubble collapses. The mechanism is analogous to the one we discussed in Section 4.5.

In the appendix of Guerron-Quintana et al. (2023), the authors extend their analysis by using quarterly U.S. data on GDP growth and the credit-to-GDP ratio to identify the presence of bubbles. Much like the stock-market-to-GDP ratio, the model predicts that the credit-to-GDP ratio rises during bubbly regimes relative to fundamental ones. According to our findings, the early 2000s begin in

[19] Recession dates from the ECRI.

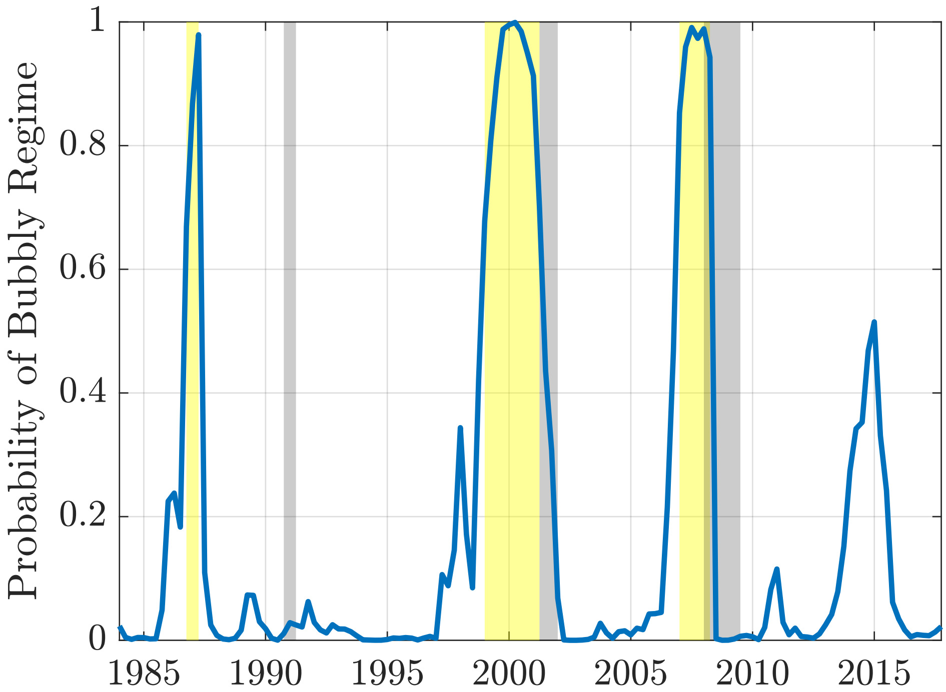

Figure 7: Estimated Probability of Bubbly Regime in the Guerron-Quintana et al. (2023) Model
Note: The yellow shadow highlights periods in which the posterior probability exceeds 60%, while the gray shadow represents the recession period.

a fundamental regime, but as credit expands rapidly over the decade, the likelihood of entering a bubbly regime increases. By mid-2005, the model assigns a smoothed probability exceeding 50% to the bubbly state, signaling its growing importance. From 2007 to early 2009, the bubble was fully developed, with growth during this interval largely driven by speculative forces—marking a clear contrast with the productivity-led expansion of the 1990s. At its height, the observed level of credit in the data can be attributed to a combination of speculative bubble dynamics and positive productivity shocks. However, the bubble dissipates in the early 2010s. During the initial phase of the Great Recession, credit levels begin to adjust downward, although they remain elevated compared to the 1990s. Our model interprets this early crisis period as being driven by a sudden, exogenous decline in investment demand—captured by a negative shock to preferences—rather than a reversion to fundamentals. As credit contraction persists and GDP growth remains subdued, the economy gradually transitions back to the fundamental regime, which becomes the dominant state from 2011 onward.

6 Conclusion

This chapter explored the relationship between asset price bubbles and macroeconomic booms. Empirically, we found that bubbles—identified using formal econometric methods—almost always coincide with periods of economic expansion. However, this empirical regularity is not easily explained by classical models of rational bubbles, even when they incorporate endogenous labor supply.

A more recent model proposed by Guerron-Quintana et al. (2023) offers a plausible explanation for bubble-driven expansions. Their model is both tractable and suitable for quantitative analysis, making it a valuable tool for identifying and studying bubbles in macroeconomic data.

Although quantitative approaches have gained prominence in economics, the application of rational bubble models in empirical research remains relatively limited. We hope this chapter serves as a useful reference for researchers seeking to further investigate the role of asset price bubbles in macroeconomic fluctuations.

References

Arrow, K. J. (1962). "The Economic Implications of Learning by Doing". *The Review of Economic Studies* 29.3, 155–173. ISSN: 00346527, 1467937X. URL: http://www.jstor.org/stable/2295952.

Barlevy, G. (2025). *Asset Bubbles and Macroeconomic Policy (The Tel Aviv Lecture Series)*. The MIT Press.

Canova, F., M. Ciccarelli, and E. Ortega (2007). "Similarities and convergence in G-7 cycles". *Journal of Monetary Economics* 54.3, 850–878. ISSN: 0304-3932. DOI: https://doi.org/10.1016/j.jmoneco.2005.10.022.

Cesa-Bianchi, A., J. Imbs, and J. Saleheen (2019). "Finance and synchronization". *Journal of International Economics* 116, 74–87. ISSN: 0022-1996. DOI: https://doi.org/10.1016/j.jinteco.2018.08.007.

Cooley, T. F. and E. C. Prescott (1995). "Economic Growth and Business Cycles". In: *Frontiers of Business Cycle Research*. Princeton University Press, 1–38.

Davis, J. S. (2014). "Financial integration and international business cycle comovement". *Journal of Monetary Economics* 64, 99–111. ISSN: 0304-3932. DOI: https://doi.org/10.1016/j.jmoneco.2014.01.007.

Farmer, R. E., D. F. Waggoner, and T. Zha (2011). "Minimal state variable solutions to Markov-switching rational expectations models". *Journal of Economic Dynamics and Control* 35.12, 2150–2166.

Fernandez-Villaverde, J., J. F. Rubio-Ramirez, and F. Schorfheide (2016). "Solution and Estimation Methods for DSGE Models". In: *Handbook of Macroeconomics*. Ed. by J. B. Taylor and H. Uhlig. Vol. 2. Elsevier, 527–724. URL: http://www.sciencedirect.com/science/article/pii/S1574004816000070.

Fernández-Villaverde, J., J. Rubio-Ramírez, and F. Schorfheide (2016). "Chapter 9 - Solution and Estimation Methods for DSGE Models". In: ed. by J. B. Taylor and H. Uhlig. Vol. 2. Handbook of Macroeconomics. Elsevier, 527–724. DOI: https://doi.org/10.1016/bs.hesmac.2016.03.006. URL: https://www.sciencedirect.com/science/article/pii/S1574004816000070.

Fisher, J. D. M. (2014). *On the Structural Interpretation of the Smets-Wouters "Risk Premium" Shock*. Tech. rep. Federal Reserve Bank of Chicago. URL: https://www.chicagofed.org/publications/working-papers/2014/wp-08.

Greenwood, J., Z. Hercowitz, and G. W. Huffman (1988). "Investment, Capacity Utilization, and the Real Business Cycle". *The American Economic Review* 78.3, 402–417. ISSN: 00028282. URL: http://www.jstor.org/stable/1809141.

Guerron-Quintana, P. A., T. Hirano, and R. Jinnai (2023). "Bubbles, Crashes, and Economic Growth: Theory and Evidence". *American Economic Journal: Macroeconomics* 15.2, 333–71. DOI: 10.1257/mac.20220015.

Gürkaynak, R. S. (2008). "ECONOMETRIC TESTS OF ASSET PRICE BUBBLES: TAKING STOCK". *Journal of Economic Surveys* 22.1, 166–186. DOI: https://doi.org/10.1111/j.1467-6419.2007.00530.x.

Homm, U. and J. Breitung (2012). "Testing for Speculative Bubbles in Stock Markets: A Comparison of Alternative Methods". *Journal of Financial Econometrics* 10.1, 198–231. ISSN: 1479-8409. DOI: 10.1093/jjfinec/nbr009. eprint: https://academic.oup.com/jfec/article-pdf/10/1/198/2280083/nbr009.pdf. URL: https://doi.org/10.1093/jjfinec/nbr009.

Horie, T. and Y. Yamamoto (2024). "Identifying Common and Idiosyncratic Explosive Behaviors in the Large Dimensional Factor Model with an Application to U.S. State-Level House Prices". *Journal of Econometric Methods* 13.1, 1–27. DOI: doi:10.1515/jem-2022-0017.

Kim, C.-J. and C. R. Nelson (1999). *State-Space Models with Regime Switching*. The MIT Press.

Kindleberger, C. P. and R. Z. Aliber (2017). *Manias, Panics, and Crashes: A History of Financial Crises*. 7th ed. Kindle Edition. London: Palgrave Macmillan.

King, R. G., C. I. Plosser, and S. T. Rebelo (1988). "Production, growth and business cycles: I. The basic neoclassical model". *Journal of Monetary Economics* 21.2, 195–232. ISSN: 0304-3932. DOI: https://doi.org/10.1016/0304-3932(88)90030-X.

Kiyotaki, N. and J. Moore (2019). "Liquidity, Business Cycles, and Monetary Policy". *Journal of Political Economy* 127.6, 2926–2966. URL: https://doi.org/10.1086/701891.

Koutsoumanis, G. and V. Castro (2023). "The duration of acceleration cycle downturns: duration dependence, international dynamics and synchronisation". *Empirical Economics* 64, 1667–1698. DOI: 10.1007/s00181-022-02299-1.

Kulish, M. and A. Pagan (2017). "Estimation and Solution of Models with Expectations and Structural Changes". *Journal of Applied Econometrics* 32.2, 255–274. DOI: https://doi.org/10.1002/jae.2527. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/jae.2527. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/jae.2527.

Martin, A. and J. Ventura (2012). "Economic Growth with Bubbles". *American Economic Review* 102.6, 3033–3058. URL: https://ideas.repec.org/a/aea/aecrev/v102y2012i6p3033-58.html.

Ng, S. and P. Perron (2001). "LAG Length Selection and the Construction of Unit Root Tests with Good Size and Power". *Econometrica* 69.6, 1519–1554. DOI: https://doi.org/10.1111/1468-0262.00256.

Pavlidis, E. et al. (2016). "Episodes of Exuberance in Housing Markets: In Search of the Smoking Gun". *The Journal of Real Estate Finance and Economics* 53.4, 419–449. DOI: 10.1007/s11146-015-9531-2.

Phillips, P. C. B., S. Shi, and J. Yu (2015a). "TESTING FOR MULTIPLE BUBBLES: HISTORICAL EPISODES OF EXUBERANCE AND COLLAPSE IN THE S&P 500". *International Economic Review* 56.4, 1043–1078. DOI: https://doi.org/10.1111/iere.12132.

Phillips, P. C. B., S. Shi, and J. Yu (2015b). "TESTING FOR MULTIPLE BUBBLES: LIMIT THEORY OF REAL-TIME DETECTORS". *International Economic Review* 56.4, 1079–1134. DOI: https://doi.org/10.1111/iere.12131.

Romer, P. M. (1986). "Increasing Returns and Long-Run Growth". *Journal of Political Economy* 94.5, 1002–1037. URL: http://www.jstor.org/stable/1833190.

Schmitt-Grohe, S. and M. Uribe (2004). "Solving dynamic general equilibrium models using a second-order approximation to the policy function". *Jour-*

nal of Economic Dynamics and Control 28.4, 755–775. URL: http://www.sciencedirect.com/science/article/pii/S0165188903000435.
Sheshinski, E. (1967). "Optimal accumulation with learning by doing". In: *Essays on the Theory of Optimal Economic Growth*. Ed. by K. Shell. MIT Press, 31–52.
Smets, F. and R. Wouters (2007). "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach". *American Economic Review* 97.3, 586–606.
Tirole, J. (1985). "Asset Bubbles and Overlapping Generations". *Econometrica* 53.6, 1499–1528. URL: http://www.jstor.org/stable/1913232.

A Date-stamping Strategy of Asset Price Bubbles

This appendix outlines the identification strategy for housing and stock market bubbles proposed by Phillips et al. (2015a,b), which we employed in Section 2 country-by-country. The method is based on the maximum value of the right-tailed unit root test. Phillips et al. (2015a,b) is particularly useful in our context, as our sample of advanced countries likely includes multiple bubbles after the 1960s or 1970s. The method mitigates the well-known power loss in detecting subsequent bubbles, which occurs when earlier bubbles are included in the sample. To address this, we define the backward supremum ADF (BSADF) test for the time series of real asset prices p_t for $t = 1, ..., T$ conducted at time $t = k$ as

$$BSADF^k = \max_{s \in [1, k-T_0+1]} ADF^{[s,k]} \text{ for } k = T_0, ..., T,$$

where $ADF^{[s,k]}$ is a t-test statistic using the sample from $t = s+1, ..., k$ for the coefficient δ in the regression

$$\Delta p_t = \mu + \delta p_{t-1} + \sum\nolimits_{l=1}^{q} \rho_l \Delta p_{t-l} + error$$

for the null hypothesis of $H_0 : \delta = 0$ against an alternative hypothesis of $H_1 : \delta > 0$ and the lag order q is determined by Ng and Perron (2001) modified information criteria.

The origination and termination dates of the jth bubble, $\hat{T}_e^j$ and $\hat{T}_f^j$, for $j =$

$1, 2, 3, ...$ are identified by

$$\begin{aligned}\hat{T}_e^j &= \min_{t \in [\hat{T}_f^{j-1}+d, T]} \left\{ t : BSADF^t > cv_t \right\}, \\ \hat{T}_f^j &= \min_{t \in [\hat{T}_e^{j}+d, T]} \left\{ t : BSADF^t < cv_t \right\},\end{aligned}$$

where cv_t is a sequence of critical values determined by bootstrap with 999 repetitions at the 1% significance level. The minimum interval between two adjacent bubbles is set to $d = 6$. When estimating the origination of the first bubble, we specifically set $\hat{T}_f^0 = 0$ and $d = T_0$. T_0 is the minimum interval to conduct the above ADF test and particularly set to $T_0 = \left\lfloor \left(0.01 + \frac{1.8}{\sqrt{T}}\right) T \right\rfloor$ with $\lfloor \cdot \rfloor$ indicating the integer part of the argument. However, these particular choices do not qualitatively affect the final results.