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Generic Indeterminacy of Steady-State Competitive Equilibria in Walras-von Neumann Production Economies

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Generic Indeterminacy of Steady-State Competitive Equilibria in Walras-von Neumann Production Economies*

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Abstract

This paper studies the structure of the set of steady-state competitive equilibria defined in a quite generalized von Neumann economic model. First, it is shown that in any von Neumann production economy, there exists an admissible domain of non-negative interest rates such that for every interest rate within the domain, there exists an associated steady state equilibrium. Second, for almost all interest rates within the domain, the associated steady state equilibrium is *indeterminate*. Thus, in summary, for any von Neumann production economy, there is a dense subset of the admissible domain over which the set of steady state equilibria consists of a finite number of one-dimensional continuums of those equilibria. This property is observed regardless of whether the underlying economy is *regular* or not, which contrasts sharply with the finite and discrete properties of the other types of Walrasian equilibria in both static and intertemporal regular economies. These main results suggest, as a new, future research agenda, the need to study an appropriate *equilibrium selection mechanism* that should be applied prior to market competition.

JEL Classification Code: B51, D33, D50.

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1 Introduction

Since Adam Smith, understanding the functioning of the capitalist economy as a whole has been a central concern of economists, and general equilibrium theory has played a central role in economics by providing a framework for working on this issue. Indeed, perfect competition and the traditional theory of homo economicus are seen as describing the enduring tendencies of economic activity, or as benchmarks against which real-world deviations can be measured.

As a benchmark theory, general equilibrium theory has been successful in providing a coherent solution to economic resource allocation problems. In addition, contemporary general equilibrium theory has extended the classical solution concept of Walrasian competitive equilibrium to a number of more general economies, such as the perfectly foresighted equilibrium in intertemporal economies with a finite set of infinitely lived agents as well as with an infinite set of finitely lived agents, and the sequential equilibrium in multi-stage sequential trading.

The literature on general equilibrium theory also includes work on von Neumann equilibrium (von Neumann, 1945), which is interpreted here as the *balanced growth* (or *steady-state*) *competitive equilibrium in the generalized von Neumann production economies*, where aggregate consumption demand can vary due to changes in prices.[1]

The von Neumann equilibrium is a solution concept in intertemporal economies, but it is different from the standard perfectly foresighted equilibrium as well as the sequential equilibrium solutions. In fact, unlike these intertemporal equilibrium solutions, it can be defined by equilibrium conditions for the resource allocations of one period, since its stationary features appear in all periods. However, there are some specific problems in establishing its existence that are not encountered in static production economies, as suggested by Malinvaud (1972; ch. 10).

The von Neumann equilibrium is usually referred to in the context of the *turnpike theorem* as a *golden rule steady state* that attracts the intertemporal competitive equilibrium paths starting from any initial position. Thus, as Mandler (2002) points out, it can be recognized as a cogent representation of the *long-period equilibrium* that serves as a 'center of gravity' for economic activity under capitalist competition, to which short-run prices would quickly return.

Nevertheless, there has been a rather limited number of studies, such as Morishima (1960) and Bidard and Hosoda (1987), on the (refined) von Neumann equilibrium. As a result, the properties of this solution concept remain relatively unexplored, including the general existence problem of it. In particular, there is no full-fledged analysis of the *equilibrium manifold of von Neumann equilibria*.

In this paper, we will develop a full-fledged analysis of the *structure of the set of von Neumann equilibria*. Recall that general equilibrium theory has developed many full-fledged analyses of the structure of the set of Walrasian competitive equilibria since

[1]That is, we discuss here a *refined* version of the original definition of von Neumann equilibrium discussed in von Neumann (1945) and Gale (1960). The original definition has often been criticized as a model of "slave economies" because it does not properly incorporate the optimal consumption choices of individuals through market exchange. In the face of such criticism, Morishima (1960) proposed a refined version with *generalized von Neumann models*, introducing specific types of Marshallian demand functions for the capitalist and working classes, as well as an exogenously given saving rate for the capitalist class. As discussed below, the model of *generalized von Neumann production economies* in this paper is much more general than even the Morishima (1960) model.

Debreu (1970), which were motivated to warrant the explanatory power of this theory. Indeed, the theory of *regular economies* initiated by Debreu (1970) shows that for almost all exchange economies, the set of Walrasian competitive equilibria is *finite* and *discrete*. According to Debreu (1976), such a structure of the solution set is highly desirable because it guarantees the *generic determinacy* of equilibrium allocations via market competition, which further guarantees the predictability and stability of such an economic system.

As Mandler (1999a,b, 2002) has argued forcefully, there is another important reason to explore the question of generic (in)determinacy of market equilibria. Given the extreme levels of wealth and income inequality in present-day capitalist economies, as reported by Oxfam (2024), the task of exploring the mechanisms that generate the growing inequality would be one of the central concerns. In economics, however, there is no common view on the mechanism that determines the functional distribution of income across different schools of economics. For example, the classical and Marxian schools have often argued that functional income distribution is determined by various factors, including not only market competition but also some historical and institutional conditions of capitalist society. In contrast, the early neoclassical school had originally argued that functional income distribution in competitive market economies is determined by the principle of marginal productivity, although, as Hahn (1982) pointed out, this principle is no longer considered indispensable for this determination.

Nowadays, this fundamental debate can be discussed from the perspective of the generic (in)determinacy of market equilibria, as emphasized by Mandler (1999a, b, 2002). The classical and Marxian views on the functional distribution of income have been formally studied by Sraffa's (1960) system of price equations, which is conceived as a representation of the *long-period equilibrium under free competition* and is known as underdetermined: in the system, the number of unknown variables is greater than the number of equations, which implies that one of the rates of wages and interest should be the parameter of the market mechanism that must be determined outside market competition in order to close the system of equations. Such a structure of price determination under free competition is compatible with the classical and Marxian view that the wage rate is determined by historical and institutional factors, rather than by the matching mechanism of supply and demand in labor markets.

In contrast, Debreu's (1970) work on regular economies suggests that there is no room for factors other than market competition in determining the functional distribution of income, since Walrasian equilibrium prices and allocations change smoothly as a function of parameters representing the economic environment. Although Debreu (1970) focused only on exchange economies, Mas-Colell (1975) and Kehoe (1980, 1982, 1985) showed that the set of Walrasian competitive equilibria is finite and discrete in (static) production economies with constant returns to scale technologies. Thus, the generic determinacy of Walrasian competitive equilibria casts doubt on the *one-degree-of-freedom* view of the Sraffian system of price equations.[2]

In response to these doubts, Yoshihara and Kwak (2023, 2024) recently established the generic indeterminacy of steady state equilibria by showing that in a simple model of overlapping generations (OLG, hereafter) production economies with a Leontief technique, there exists a steady state equilibrium that is regular and indeterminate in the sense that

[2]In fact, the Sraffian school simply defines the long-period equilibrium by the Sraffian system of price equations, which lacks the conditions for the matching of supply and demand in the commodity and factor markets. This point has been criticized by Mandler (1999a).

in its neighborhood there exists a *continuum of other steady state equilibria* that converge to it.

The present paper significantly generalizes the latter contribution by considering von Neumann, rather than simple Leontief, production economies. Moreover, the main results of this paper provide a more full-fledged characterization of the equilibrium manifold of von Neumann equilibria than the results of Yoshihara and Kwak (2023, 2024). The model considered in this paper is also significantly more general than the economies analyzed by Morishima (1960) and Bidard and Hosoda (1987), in that we drop the assumptions that the aggregate demand functions are derived from homothetic preferences over the consumption space and that the saving rate of the capitalist class is exogenously given.

To be precise, we introduce a von Neumann production economy with a simple OLG structure, which is represented as a profile of one-period data of production economies: a list of von Neumann production technology, a given size of population as labor endowment data, and two Marshallian demand functions. In this economy, the von Neumann equilibrium solution is reduced to a *steady-state* competitive equilibrium.

We show that for each economy there is an admissible range of non-negative interest rates such that for each interest rate in this range, there exists a von Neumann equilibrium associated with that interest rate. Moreover, in each economy, for almost all interest rates within the admissible domain, the corresponding von Neumann equilibria are *indeterminate*, in the sense that within a neighborhood of each of these equilibria there exists a continuum of other von Neumann equilibria converging to it.

More precisely, for each economy there exists a non-empty set of von Neumann equilibria, which is described as a closed graph of an upper hemi-continuous correspondence between the von Neumann equilibria and their associated interest rates. Moreover, there exists a dense subset of the admissible domain over which such a closed graph consists of a finite number of continuous curves, each representing a one-dimensional continuum of equilibria. These results hold whether or not the underlying economy is regular.[3] Thus, the one-dimensional indeterminacy of von Neumann equilibria is *generic*.

These main results do *not* depend on the OLG structure of the model. In fact, Addendum A of this paper analyzes an even more general version of the model, in which the population grows at a constant, exogenous rate $g \geqq 0$, and the economic environment in any given period is specified by a list of von Neumann production technologies, the size of the population, and a Marshallian aggregate demand function. Accordingly, a more general solution concept is defined as a *von Neumann balanced growth equilibrium* associated with g as the warranted rate of capital accumulation. For this solution concept, essentially the same two main results as those mentioned above are obtained.

1.1 Related Literature Review

In addition to the classical contributions of Debreu (1970), Mas-Colell (1975), and Kehoe (1980, 1982, 1985), the generic determinacy of Walrasian equilibria also holds in some intertemporal economic models. As Kehoe and Levine (1985) argued, in intertemporal models with a finite number of infinitely lived agents, regular economies are of full measure and each of them has a finite number of isolated, perfectly foresighted equilibria. In

[3] In this paper, as discussed below, an *economy* is defined to be *regular* if all the von Neumann equilibria in that economy are regular, and a *von Neumann equilibrium* is defined to be *regular* if its corresponding Jacobian has full row rank.

intertemporal exchange models with an infinite number of finitely living OLGs, the regular economies are again generic and each of them has a finite number of steady state equilibria, as shown by Kehoe and Levine (1984).

A number of contributions have also proved the *local indeterminacy* of such steady states in some intertemporal economic models. A steady state is said to be locally indeterminate if there is a continuum of nearby steady state equilibrium paths, all of which converge to the steady state. Kehoe and Levine (1984, 1985) and Calvo (1978) show that each of the steady states in each intertemporal OLG economy with a pure exchange or simple production technology is locally indeterminate.[4]

Mandler (1995, 1999a) has analyzed the *generic indeterminacy of sequential equilibrium* in intertemporal production economies with a finite number of finitely lived agents. Using Radner's (1972) method of decomposing an intertemporal (Arrow-Debreu) equilibrium into a sequential equilibrium under two-period sequential trade, it can be shown, under some conditions, that the second-period continuation equilibrium is indeterminate for almost any induced second-period economy.[5]

In summary, the existing literature on generic (in)determinacy can be summarized as follows:

Insert Table 1 aroud here.

As argued by Debreu (1970), Mas-Colell (1975), Kehoe (1980, 1982), and Kehoe and Levine (1985), the above-mentioned finite and discrete properties of Walrasian competitive equilibria in static economies and perfectly foresighted equilibria in intertemporal economies are no longer warranted if these underlying economies are not regular. Hence, the verification of the full measure of regularity in these economies is crucial to establish the generic determinacy of these equilibria. In contrast, the study of regular economies is less important in this paper, since the generic indeterminacy of von Neumann equilibria can be established essentially without reference to the regularity of the economies, as discussed in detail below.

The generic indeterminacy of von Neumann equilibria also contrasts sharply with the finite and discrete features of steady states in the standard literature on OLG economies, such as Kehoe and Levine (1984) and Calvo (1978). Such finiteness and discreteness in the standard literature would arise from some specific features of their OLG economic models. For example, steady states in OLG pure exchange economies have a structure quite similar to Walrasian equilibria in static pure exchange economies, as argued in Kehoe and Levine (1984). In the case of OLG production economies, Calvo's (1978) neoclassical two-sector model is so specific that the stationary level of the capital stock and the corresponding stationary production activities can be solved completely independently of the price system, which is the source of the finiteness of steady states in his model.[6]

[4]Even in infinite-horizon intertemporal economies with a finite number of infinitely lived individuals, there is some literature, such as Benhabib and Farmer (1994) and Benhabib and Nishimura (1998), on the *local indeterminacy of equilibrium paths* converging to a steady state, which is shown to exist in economies with some degree of *market imperfections*. See also Nishimura and Venditti (2006) for a useful survey of this work.

However, this line of research will not be discussed further in this paper, as we focus on Walrasian equilibria under perfectly competitive economies.

[5]However, this indeterminacy arises only in economies with linear, non-differentiable production technology, see Mandler (1997).

[6]See Yoshihara and Kwak (2024) for more detailed comments on the source of Calvo's (1978) steady

Finally, it may be worth noting that Kehoe (1985; Section VI; Theorem 7) considered the class of production technologies as that of alternative Leontief production techniques, and then showed that each of such economies is regular and has a *unique* (steady-state) equilibrium by applying the nonsubstitution theorem.[7] This result may seem incompatible with the main result (the generic indeterminacy of von Neumann equilibria) of this paper, but it is not. For what Kehoe (1985; section VI) focused on is the case of steady state equilibrium associated with *zero interest rate* alone, while we also consider the cases of steady state equilibria associated with *positive interest rates*.

The remainder of the paper is organized as follows. Section 2 presents a von Neumann production economic model with a simple OLG structure. Given this class of economies, Section 3 discusses the general existence of von Neumann steady-state equilibria. In addition, Section 4 verifies the general indeterminacy of such equilibria. Section 5 contains some concluding remarks. All the proofs of the main theorems in Sections 3 and 4 are relegated to Appendix. Addendum A contains another variant of von Neumann production economies and its corresponding solution concept, as mentioned above.

2 An overlapping generation economy with von Neumann production technology

Consider an OLG model in which each generation $t = 1, 2, \ldots,$ is a single individual living in two periods, working only in her young age and retiring in her old age to purchase consumption goods from her wealth due to her past savings. Let ω_l be the labor endowment of a generation, which is assumed to be fixed over time.[8]

There are $n \geqq 2$ goods produced in this economy that are used as consumption goods and/or capital goods. There are m ($\geqq n$) alternative production processes that can be used. Consider a von Neumann production technology (A, B, L), where A is an $n \times m$ nonnegative matrix of material input coefficients, B is an $n \times m$ nonnegative matrix of gross output coefficients, and L is a $1 \times m$ positive vector of direct labor coefficients. At the end of each period t, let $p_t \in \mathbb{R}^n_+$ be a vector of prevailing *prices* of n commodities; $w_t \in \mathbb{R}_+$ be a prevailing *wage rate*; and $r_t \in \mathbb{R}$ be a prevailing *interest rate*.

Let $z_b : \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+$ (*resp.* $z_a : \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+$) be a *Marshallian demand function* of each generation t in their youth (*resp.* in their old age) such that, for any commodity price vectors $p_t, p_{t+1} \in \mathbb{R}^n_+$, wage rates $w_t, w_{t+1} \in \mathbb{R}_+$, and an interest factor $1 + r_{t+1} \in \mathbb{R}_+$, $z_k (p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) \in \mathbb{R}^n_+$ is a consumption vector purchasable by each generation at age $k = b, a$. As is standard in OLG models with production, at least since Calvo (1978), the Marshallian demand of each agent t is assumed to satisfy the following budget constraint:

$$p_t z^t_b + \frac{p_{t+1} z^t_a}{1 + r_{t+1}} = w_t \omega_l,$$

state results.

[7]Note that the class of economies in Kehoe (1985; section VI) need not have the OLG structure, but this point is not essential to the following argument.

[8]For any positive integer q, $\mathbb{R}^q$ (*resp.* $\mathbb{R}^q_+$, $\mathbb{R}^q_{++}$ and $\mathbb{R}^q_-$) denotes the q-fold Cartesian product of $\mathbb{R} = (-\infty, +\infty)$ (*resp.* $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$). For any $x, x' \in \mathbb{R}^q$, $x \geqq x'$ denotes $[x_1 \geqq x'_1, \cdots, x_q \geqq x'_q]$, $x \geq x'$ denotes $[x \geqq x'$ and $x \neq x']$, and $x > x'$ denotes $[x_1 > x'_1, \cdots, x_q > x'_q]$.

where agents supply inelastically all of their labor endowment ω_l and, given their earnings $w_t\omega_l$, they spend to purchase consumption goods z_b^t after deducting $\frac{p_{t+1}z_a^t}{1+r_{t+1}}$ for saving, which is to finance consumption during retirement. Savings are used to finance productive investment during old age.

The demand function z_k is assumed to be *continuously differentiable* and to satisfy *homogeneity*: for $k = b, a$,

$$\begin{aligned} z_k\left(p_t, w_t, p_{t+1}, w_{t+1}, 1+r_{t+1}\right) &= z_k\left(\lambda p_t, \lambda w_t, \lambda p_{t+1}, \lambda w_{t+1}, 1+r_{t+1}\right) \\ &= z_k\left(p_t, w_t, \lambda p_{t+1}, \lambda w_{t+1}, \lambda\left(1+r_{t+1}\right)\right) \end{aligned}$$

for every $\lambda > 0$ and every $(p_t, w_t, p_{t+1}, w_{t+1}, 1+r_{t+1}) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+$; and *Walras' law*. In addition, assume that each demand function z_k is derived from a strongly monotonic preference on the consumption space in each period. When z_k is evaluated at stationary prices, $(p_t, w_t) = (p_{t+1}, w_{t+1}) = (p, w)$ for each t, we will use the notation $z_k(p, w, r)$ for $k = a, b$. Let $z(p, w, r) \equiv z_b(p, w, r) + z_a(p, w, r)$ be the aggregate demand function at each period t when the market prices are stationary.

An *overlapping generation economy* is given by a profile $E = \langle (A, B, L); \omega_l; z\rangle$. As in the literature on von Neumann production models, we are interested in studying a specific long-period feature of economic resource allocation through market competition, where prices are stationary and all investment activity is simply of the replacements. Thus, a steady state of the Walrasian competitive equilibrium is defined, following Mandler (1999a; section 6), as follows:

Definition 1: A *steady-state equilibrium* ($\boldsymbol{SE}$) in the economy $E = \langle (A, B, L); \omega_l; z\rangle$ is a pair of a price vector $(p, w, r) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times [-1, \infty)$ and a production activity vector $y \in \mathbb{R}_+^m$, such that:

$$pB \leqq (1+r)\,pA + wL, \tag{a}$$

$$By \geqq z(p, w, r) + Ay, \text{ and} \tag{b}$$

$$Ly \leqq \omega_l. \tag{c}$$

In Definition 1, $1 + r \geqq 0$. This implies that non-negativity of equilibrium interest rates is not necessarily required. However, steady-state equilibria with strictly negative interest rates would be a fluke and would not be of interest. Therefore, in the following argument we will legitimately focus on a specific case of $\boldsymbol{SE}$ in which equilibrium interest rates are non-negative.[9] Such a refined $\boldsymbol{SE}$ is defined as follows.

Definition 2: A *steady-state equilibrium* $((p, w, r), y)$ for the economy $E = \langle (A, B, L); \omega_l; z\rangle$ is called *Walras-von Neumann* (in short, **W-N** $\boldsymbol{SE}$) if and only if $(p, w, 1 + r) > \mathbf{0}$ with $r \geqq 0$ and condition (c) in Definition 1 holds as an equality.

Note that in Definition 2, $y \geq \mathbf{0}$ is implied by $Ly = \omega_l$. Thus, a **W-N** $\boldsymbol{SE}$ satisfies the nonzero condition of equilibrium production activities, which is a necessary condition for the von Neumann equilibrium in the standard literature such as Morishima (1960).

[9] In fact, as argued in Yoshihara and Kwak (2024, p. 384), there is a reasonable microeconomic model of an individual optimization program to rationalize the aggregate demand function z of an economy E, which makes $\boldsymbol{SE}$ with strictly negative interest rates impossible whenever the production activity vector is required to be nonzero, $y \geq \mathbf{0}$, in equilibrium. See also Addendum B of this paper for details.

In any economy $E = \langle (A, B, L); \omega_l; z \rangle$, Walras' law at any stationary price vector (p', w', r') is given by:

$$p'[z_b + z_a] - r'W - w'\omega_l = 0 \qquad (*),$$

where $W \equiv w'\omega_l - p'z_b$. Note that $(*)$ is simply the aggregate of the young generation's budget equation, $p'z_b + W - w'\omega_l = 0$, and the old generation's budget equation, $p'z_a - (1 + r')W = 0$. Since W is used to finance productive investment, there exists a productive activity vector $y' \in \mathbb{R}^m_+$ that satisfies $p'Ay' = W$. Therefore, $(*)$ is replaced by:[10]

$$p'[z_b + z_a] - r'p'Ay' - w'\omega_l = 0 \qquad (*)'.$$

Furthermore, if the price vector (p', w', r') satisfies condition (a) of Definition 1 and the vector y' is a profit-maximizing activity at those prices, then $(*)'$ can be reduced to:

$$w'[Ly' - \omega_l] + p'[z_b + z_a + Ay' - By'] = 0 \qquad (*)''.$$

Here, $((p', w', r'), y')$ is not necessarily an equilibrium in $(*)''$ since it does not necessarily satisfy condition (b) of Definition 1.

3 Existence

In this section, we will study the existence of **W-N *SE***. As a preliminary step, let $\boldsymbol{e}_i \in \mathbb{R}^n_+$ be a unit row vector such that its i-th component is unity and its any other components are zero, where $i = 1, \ldots, n$. Likewise, let $\boldsymbol{e}_j \in \mathbb{R}^m_+$ be a unit column vector such that its j-th component is unity and its any other components are zero, where $j = 1, \ldots, m$. Then, we introduce the following assumptions.

Assumption 1 (A1): For any non-negative vector $d \in \mathbb{R}^n_+$, there exists $x \in \mathbb{R}^m_+$ such that $[B - A]x \geqq d$.

Assumption 2 (A2): Every commodity needs to be produced in some production process: $\boldsymbol{e}_i B \geq \mathbf{0}$ for every commodity $i \in \{1, 2, \ldots, n\}$. Every process needs to use some commodity inputs: $A\boldsymbol{e}_j \geq \mathbf{0}$ for every process $j \in \{1, 2, \ldots, m\}$.

Assumption 3 (A3): For every process that produces commodity $i \in \{1, 2, \ldots, n\}$ jointly with commodity $i^* \in \{1, 2, \ldots, n\}$ $(i^* \neq i)$, there is another process with the same outputs and the same inputs and labor, except that commodity i is not produced.

Assumption 4 (A4): If a sequence of prices $\{(p^q, w^q, r^q)\} \subseteq \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+$ converges to $(p, w, r) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ with $p \in \mathbb{R}^n_+ \backslash \left(\mathbb{R}^n_{++} \cup \{\mathbf{0}\}\right)$, then $\|z(p^q, w^q, r^q)\|$ converges to infinity.

These four assumptions are standard. A4 is imposed in the literature on *regular economies*, and it is a natural requirement for economies with strictly monotonic preferences over consumption. From A4, if a **W-N *SE*** exists, then its corresponding equilibrium commodity price vector should be positive. A2 is a natural requirement in the

[10] Even if a part of W can be used to finance some non-productive investment, as specified in Addendum B, we can verify that Walras' law for any given stationary price vector is still given by $(*)'$, as shown in Yoshihara and Kwak (2024; Proof of Theorem 1) and in Addendum B of this paper.

literature on von Neumann production models. A1 is a natural condition ensuring productiveness. Finally, A3 is a condition of free disposal.

By A2 it follows from the generalized Perron-Frobenius Theorem (Mangasarian, 1971, Theorem 4.1, p. 91) that there exists a semi-positive vector $p^R \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ with a positive $(1+R) > 0$ such that $p^R B = (1+R)\, p^R A$ holds. This positive eigenvalue $\frac{1}{1+R}$ is maximal, in the sense that there is no $\lambda > \frac{1}{1+R}$ such that $\lambda p B = pA$ for some $p \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, according to Mangasarian (1971, Theorem 4.1-(i), p. 91). In addition, by A1, $R > 0$ holds from Fujimoto and Krause (1988, Theorem 2, p. 192).

The existence of $R > 0$ in our model ensures the existence of the maximum interest rate associated with the zero wage rate. Then, for any $r \in [0, R)$, while there is no $p^r \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $p^r [B - (1+r) A] = \mathbf{0}$, as argued above, it also follows from $L > \mathbf{0}$ that there exists $p^r \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $p^r [B - (1+r) A] \leqq L$. Therefore, $[0, R)$ should serve as the domain of interest rates over which the existence of the associated **W-N** $\boldsymbol{SE}$ can be examined. Indeed, by setting $w = 1$ (using labor as the numeraire), the existence of a **W-N** $\boldsymbol{SE}$ is established for each and every interest rate in $[0, R)$, as shown in the following theorem:

Theorem 1: For any $E = \langle (A, B, L)\, ; \omega_l; z \rangle$ and *each interest rate* $r \in [0, R)$, there exists a *Walras-von Neumann steady state equilibrium* $((p^*, 1, r)\, , y^*)$.

The proof of Theorem 1 is relegated to Appendix. Here, we discuss the basic scenario of the proof. Let $((p^*, 1, r)\, , y^*)$ be a **W-N** $\boldsymbol{SE}$ for $E = \langle (A, B, L)\, ; \omega_l; z \rangle$. Then, conditions (a) and (b) of Definition 1 are satisfied by $((p^*, 1, r)\, , y^*)$.

First, post-multiplying condition (a) of Definition 1 by y^*, we obtain:

$$p^* B y^* = (1+r)\, p^* A y^* + L y^*.$$

Pre-multiplying condition (b) of Definition 1 by p^*, we obtain:

$$p^* B y^* = p^* z\, (p^*, 1, r) + p^* A y^*.$$

Therefore, in equilibrium, we have:

$$p^* z\, (p^*, 1, r) = p^* (B - A)\, y^* = (r p^* A + L)\, y^*. \qquad \text{(d)}$$

Second, consider the following linear programming problems: given r and p^* of the equilibrium $((p^*, 1, r)\, , y^*)$,

$$(MP_1^*) \quad \max_{p' \in \mathbb{R}^n_+} p' \cdot z\, (p^*, 1, r) \ \text{ subject to } p' [B - A] \leqq r p^* A + L.$$

and

$$(MP_2^*) \min_{y \in \mathbb{R}^m_+} (r p^* A + L)\, y \text{ subject to } [B - A]\, y \geqq z\, (p^*, 1, r)\, .$$

Then, since $p^* \in \mathbb{R}^n_+$ is feasible in (MP_1^*) while $y^* \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ is feasible in (MP_2^*), it follows from condition (d) and the duality theorem that p^* is an optimal solution for (MP_1^*) while y^* is an optimal solution for (MP_2^*).

In other words, a crucial step in proving the existence of a **W-N** $\boldsymbol{SE}$ is to find an *appropriate* price vector p^* that is an optimal solution to a linear programming maximization problem, such as (MP_1^*), defined by that price vector itself. Denote the set of solutions to a linear programming maximization problem defined by means of each price vector $p \in \mathbb{R}^n_+$ by $\psi^r(p)$. Then, $p \in \psi^r(p)$ holds if and only if this p is an optimal solution to the problem. Thus, the crucial step is to formulate a fixed point problem of such a mapping ψ^r.

One difficulty is that the linear programming maximization problem, like (MP_1^*), is defined over the *universal domain* $\mathbb{R}^n_+$, whereas the fixed point problem requires that the domain and the range of ψ^r be identical and a *proper subset of* $\mathbb{R}^n_+$. In the proof of Theorem 1 developed in Appendix, we solve this problem by appropriately defining the domain and the range of ψ^r in such a way that the fixed point of such a mapping is indeed optimal over the *universal domain* $\mathbb{R}^n_+$.

Unlike the existence theorems in the main literature on von Neumann equilibrium, Theorem 1 explicitly highlights the parametric nature of interest rates in the determination of **W-N** $\boldsymbol{SE}$s: it shows the existence of equilibria corresponding to each and every interest rate, and there is no condition for determining an equilibrium interest rate, even though it is a variable that determines the aggregate demand of all goods and the optimal production plans.

Recall that in the Sraffian literature, degree one freedom is often emphasized in the determination of a steady state equilibrium, in the sense that a steady state equilibrium can be determined only after either an interest rate or a wage rate has been fixed outside of market competition.[11] In such arguments, however, neither the Marshallian demand function nor the labor market equilibrium condition is seriously discussed in the determination of equilibria.[12] Therefore, it is not obvious whether the Sraffian view of degree one freedom can be robust if these factors are introduced into the model and the excess demand conditions for all markets are explicitly examined. Theorem 1 shows that the answer to this question is yes in von Neumann production economies with the OLG structure. Indeed, as discussed below, Theorem 1 implies the existence of a closed graph relationship between **W-N** $\boldsymbol{SE}$s and interest rates in any economy.[13]

Let $\triangle \equiv \left\{(p, w) \in \mathbb{R}^{n+1}_+ \mid \sum_{i=1}^n p_i + w = 1\right\}$ be the set of vectors of commodity prices and wage rates. For the sake of analytical tractability, let us change the normalization rule to represent equilibrium price vectors. For the equilibrium $((p^*, 1, r), y^*)$, consider normalization of $(p^*, 1)$ as

$$\left(\frac{p_1^*}{1+\sum_{j=1}^n p_j^*}, \ldots, \frac{p_n^*}{1+\sum_{j=1}^n p_j^*}, \frac{1}{1+\sum_{j=1}^n p_j^*}\right).$$

Then, this normalized price vector belongs to $\triangle$. Thus, each equilibrium price vector is also represented within $\triangle \times [0, R)$.

Note that, based on Therorem 1, for each economy $E = \langle (A, B, L); \omega_l; z\rangle$, we can construct a mapping $\Psi_E : [0, R) \twoheadrightarrow \triangle \times [0, R) \times \mathbb{R}^m_+$ as follows: for any $r \in [0, R)$, $\Psi_E(r)$ is the set of **W-N** $\boldsymbol{SE}$s associated with r in the economy E. That is, for every $r \in [0, R)$,

[11] For instance, see Kurz and Salvadori (1995, Chapter 8) and Bidard (2004, Chapter 11).

[12] An exception may be found in Bidard (2004, Chapter 22, Theorem 1, p. 256), which introduces a Marshallian aggregate demand function and then shows the existence of a balanced growth equilibrium for any given profit rate. However, the latter result is based on an inappropriate formulation of Walras' law: the form of Walras' law (A2) in Bidard (2004, p. 256) represents an equilibrium condition rather than the Walras identity.

[13] This property cannot be highlighted by the existence theorems of contributions such as Morishima (1960) and Bidard and Hosoda (1987). For a detailed discussion, see Addendum A.

every $(p(r), w(r), r; y(r)) \in \Psi_E(r)$ is a **W-N $\boldsymbol{SE}$** in the economy E. Let this mapping Ψ_E be called the *equilibrium correspondence of the economy E*. This mapping Ψ_E is non-empty, according to Theorem 1. Furthermore, it can be shown that Ψ_E is *upper hemi-continuous at every $r \in [0, R)$*.

Proposition 1: For any economy $E = \langle (A, B, L); \omega_l; z \rangle$, the equilibrium correspondence Ψ_E of that economy is *upper hemi-continuous.*

The proof of Proposition 1 is relegated to Appendix.

Thus, according to Proposition 1, the set of **W-N $\boldsymbol{SE}$**s for the economy E is described as a *closed graph in* $[0, R) \times \left(\triangle \times [0, R) \times \mathbb{R}_+^m\right)$ of Ψ_E.

However, this result *per se* does not warrant the indeterminacy of the equilibria, because the lower hemi-continuity of Ψ_E is not yet warranted. Indeed, if Ψ_E were not to be lower hemi-continuous, there could be a **W-N $\boldsymbol{SE}$** $(p(r'), w(r'), r'; y(r')) \in \Psi^E(r')$ which is *locally unique* in the sense that for a sufficiently small open neighborhood $\mathcal{O}(r') \subseteq (0, R) \times \Psi_E([0, R))$ of $(p(r'), w(r'), r'; y(r'))$,

$$\mathcal{O}(r') \setminus \{(p(r'), w(r'), r'; y(r'))\} = \varnothing$$

holds. In other words, the upper hemi-continuity *alone* does not generally exclude the possibility of the existence of a *locally isolated equilibrium* within $\Psi_E([0, R))$. Therefore, Proposition 1 *per se* cannot exclude the possibility of determinate equilibria.[14]

4 Generic Indeterminacy

In this section, we will show that each **W-N $\boldsymbol{SE}$** is indeterminate whenever it is regular, where the concept of regular equilibrium is defined in Definition 4 below. Moreover, we will also show that for every economy and for almost all positive interest rates, the associated **W-N $\boldsymbol{SE}$**s are *regular*: for every economy, any non-regular equilibrium is *non-generic.*

Let us first introduce the definition of indeterminacy, which we owe to Mandler (1999a).

Definition 3: For an economy $E = \langle (A, B, L); \omega_l; z \rangle$, a Walras-von Neumann steady state equilibrium $((p, w, r); y)$ is *indeterminate* if for any $\varepsilon > 0$, there exists another Walras-von Neumann steady state equilibrium $((p', w', r'); y')$ in this economy such that $(p', w', r') \neq (p, w, r)$ and $\|(p', w', r') - (p, w, r)\| < \varepsilon$.

For any **W-N $\boldsymbol{SE}$** $((p, w, r); y)$, we should have $By > \mathbf{0}$, since $z(p, w, r) > \mathbf{0}$ holds by **A4**. In this case, the corresponding system of equilibrium inequalities is given by:

$$\begin{aligned} z(p, w, r) - [B - A] y &= \mathbf{0}; \ (1) \\ p[B - (1 + r) A] - wL &\leqq \mathbf{0}, \ (2) \end{aligned}$$

where there are $n + m$ inequalities while there are $n + 1 + m$ unknown variables, assuming that one commodity is used as the numeraire.[15]

[14] In this paper, the determinacy of equilibria is defined as the complement of the cases of indetermiancy given in Definition 3 below.

[15] There is no equation here that satisfies condition (c) of Definition 1. This is because if there is a solution to the system of inequalities (1) and (2), then it also satisfies condition (c), given that Walras' law $(*)$ holds.

To analyze generic indeterminacy, we may need to reduce the inequalities (2) to a system of equations. Let $B(p,w,r)$ and $A(p,w,r)$ be $n \times k$ matrices, and $L(p,w,r)$ be a $1 \times k$ row vector, where k $(\leqq m)$ is the number of processes actually operated at prices (p,w,r). Each of the processes actually operated at (p,w,r) achieves profit maximization at the prices. Then, by definition, (2) can be reduced to:

$$p\left[B(p,w,r) - (1+r)A(p,w,r)\right] - wL(p,w,r) = \mathbf{0}. \quad (2)^*$$

Now, we can get a mapping F as follows:

$$F(p,w,r,y) \equiv \begin{cases} z(p,w,r) - [B-A]\,y \\ \left(pB(p,w,r) - (1+r)\,pA(p,w,r) - wL(p,w,r)\right)^T \end{cases},$$

where the superscript T stands for *transpose*. The map F has a regular value if $F(p,w,r,y) = \mathbf{0}$ for some $(p,w,r;y) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times (0,R) \times \mathbb{R}^m_{+}$.

Without loss of generality, we assume in the following that whenever $F(p,w,r,y) = \mathbf{0}$ holds and k is the number of actually operated processes in $y \geq \mathbf{0}$ at prices (p,w,r), then for each process $j = 1,\dots,m$, $y_j = 0$ if and only if $\left[p\left(B - (1+r)A\right) - wL\right]\boldsymbol{e}_j < 0$, and the number k is *minimal*, in the sense that no other vector $y' \geq \mathbf{0}$ with less than k positive components can be an equilibrium activity vector at prices (p,w,r).

The system of equations $F(p,w,r,y) = \mathbf{0}$ reduces to $F(\bar{p},w,r,\overline{y}) = \mathbf{0}$, where $\bar{p} \equiv (\frac{p_1}{p_n},\dots,\frac{p_{n-1}}{p_n},1)$ and $\overline{y} \in \mathbb{R}^k_{++}$ is obtained by removing the $m-k$ zero components of the vector $y \in \mathbb{R}^m_{+}$. In this reduced system there are $n+k$ equations, while there are $n+1+k$ unknowns.

Formally:

$$F(\bar{p},w,r,\overline{y}) \equiv \begin{bmatrix} z(\bar{p},w,r) - \left[B(\bar{p},w,r) - A(\bar{p},w,r)\right]\overline{y} \\ \left(\bar{p}B(\bar{p},w,r) - (1+r)\bar{p}A(\bar{p},w,r) - wL(\bar{p},w,r)\right)^T \end{bmatrix},$$

which is continuously differentiable at least in a small neighborhood of each equilibrium $(\overline{p},w,r,\overline{y})$. Here, the continuous differentiability of the excess demand functions

$$z(\overline{p},w,r) - \left[B(\overline{p},w,r) - A(\overline{p},w,r)\right]\overline{y}$$

is guaranteed at least in a small neighborhood of each equilibrium $(\overline{p},w,r,\overline{y})$, as argued in Kehoe (1980, 1982). Indeed, the matrix $\left[B(\overline{p},w,r) - A(\overline{p},w,r)\right]$ is invariant with respect to a small change of $(\overline{p},w,r)$ at each equilibrium, since a sufficiently small change of each of $\overline{p},w,r$ induces a sufficiently small change of $\overline{y}$ such that all of the k positive components in $\overline{y}$ are still positive, while the remaining $m-k$ processes are still inactive, since any process that has not reached the zero-profit condition will not do so after such a small change in prices. Therefore, for such a small change in prices, the k processes actually operating remain constant, which implies that $\left[B(\overline{p},w,r) - A(\overline{p},w,r)\right]$ is invariant with respect to a small change in prices. Then, the derivatives of the functions $z(\overline{p},w,r) - \left[B(\overline{p},w,r) - A(\overline{p},w,r)\right]\overline{y}$ with respect to prices are identical to those of the demand functions $z(\overline{p},w,r)$ with respect to each of $\overline{p},w,r$.

By the definition of **W-N** $\boldsymbol{SE}$, $F(\bar{p},w,r,\overline{y}) = \mathbf{0}$ holds if and only if $((p,w,r);y)$ is a **W-N** $\boldsymbol{SE}$. We then introduce the notion of regular equilibria.

Definition 4: A Walras-von Neumann steady state equilibrium $((p,w,r),y)$ for an economy $E = \langle (A,B,L);\omega_l;z\rangle$ is *regular* if the Jacobian of $F(\bar{p},w,r,\overline{y}) = \mathbf{0}$ has full row rank.

Now, we are ready to argue the indeterminacy of **W-N** $\boldsymbol{SE}$s:

Theorem 2: For any economy $E = \langle (A, B, L); \omega_l; z \rangle$, and for almost all $r \in (0, R)$, the associated Walras-von Neumann steady state equilibrium $((p, w, r), y)$ is *regular* and *indeterminate*.

The proof of Theorem 2 is relegated to Appendix.

The next numerical example illustrates the indeterminacy of **W-N** $\boldsymbol{SE}$s proved by Theorem 2.

Example 1 *Assume that $n = 2$, $m = 3$, $\omega_l = 1$, and the aggregate Marshallian demand function is derived from the following form of the utility function: for any $(z_b, z_a) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$,*

$$u(z_b, z_a) \equiv \left[(z_{b1})^{\alpha} \cdot (z_{b2})^{1-\alpha}\right] \cdot \left[(z_{a1})^{\alpha} \cdot (z_{a2})^{1-\alpha}\right]$$

where $\alpha \in (0, 1)$. Let a von Neumann production technology be given by:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}, \text{ and } L = (1, 1, 1).$$

Thus, an economy is specified by $E = \langle (A, B, L); 1; \alpha \rangle$, where α represents the aggregate Marshallian demand function derived from the utility function associated with this parameter value.

In this economy, the von Neumann production technology (A, B, L) has the maximum eigenvalue $\frac{1}{1+R} = \frac{1}{3}$, and thus $R = 2$ is the maximum interest rate. The aggregate Marshallian demand function is given by:

$$z(p, w, r) = \begin{pmatrix} \alpha \frac{w + r(p_1+p_2)(y_1+y_2+y_3)}{p_1} \\ (1-\alpha) \frac{w + r(p_1+p_2)(y_1+y_2+y_3)}{p_2} \end{pmatrix}.$$

Then, fixing $w = 1$, the equilibrium system of inequalities (1) and (2) in this economy is specified as follows: for each $r < R = 2$,

$$\begin{bmatrix} 2 & 4 & -1 \\ 2 & -1 & 4 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \alpha \frac{1 + r(p_1+p_2)(y_1+y_2+y_3)}{p_1} \\ (1-\alpha) \frac{1 + r(p_1+p_2)(y_1+y_2+y_3)}{p_2} \end{pmatrix}; \tag{1}$$

$$(p_1, p_2) \begin{bmatrix} 2-r & 4-r & -(1+r) \\ 2-r & -(1+r) & 4-r \end{bmatrix} \leqq (1, 1, 1). \tag{2}$$

By computing the solutions to this system, the set of **W-N** $\boldsymbol{SE}$*s is specified as follows:*

(I) for any economy E with $\alpha \in \left(0, \frac{2}{5}\right)$,

$$\left\{ ((p, w, r), y) \in \triangle \times [0, 2) \times \mathbb{R}^3_+ \mid (p, w) = \left(\left(\frac{0.4}{3-r}, \frac{0.6}{3-r} \right), \frac{2-r}{3-r} \right), \ y = \left(\frac{1+5\alpha}{3}, 0, \frac{2-5\alpha}{3} \right)^T \right\};$$

(II) for any economy E with $\alpha \in \left[\frac{2}{5}, \frac{3}{5}\right]$,

$$\left\{ ((p, w, r), y) \in \triangle \times [0, 2) \times \mathbb{R}^3_+ \mid (p, w) = \left(\left(\frac{\alpha}{3-r}, \frac{1-\alpha}{3-r} \right), \frac{2-r}{3-r} \right), \ y = (1, 0, 0)^T \right\}; \text{ and}$$

(III) for any economy E with $\alpha \in \left(\frac{3}{5}, 1\right)$,

$$\left\{ \left((p, w, r), y \right) \in \triangle \times [0, 2) \times \mathbb{R}^3_+ \mid (p, w) = \left(\left(\frac{0.6}{3-r}, \frac{0.4}{3-r} \right), \frac{2-r}{3-r} \right), \ y = \left(\frac{6-5\alpha}{3}, \frac{5\alpha - 3}{3}, 0 \right)^T \right\}.$$

■

In this example, each **W-N *SE*** is a continuous function of interest rates within $[0, 2)$, and thus the set of **W-N *SE***s in each economy constitutes a one-dimensional continuum.

Theorem 2 is quite appealing because it claims that for each and every economy, almost all **W-N *SE***s are indeterminate and each of them can be represented as the image of a single-valued continuous mapping of interest rates. This indeterminacy is observed generically without imposing any stringent assumption on the class of economies.

Together with Proposition 1, Theorem 2 implies that for each economy E, the associated map Ψ_E is not only upper hemi-continuous, but also lower hemi-continuous. More precisely, for almost all $r \in (0, R)$, $\Psi_E(r)$ consists of a finite number of associated **W-N *SE***s, each of which is locally isolated within $\Psi_E(r)$. Thus, Ψ_E essentially consists of a finite number of distinct single-valued continuous mappings. Moreover, although the entire set of **W-N *SE***s in each economy E constitutes a closed graph $\Psi_E((0, R)) = \{(p(r), w(r), r, y(r)) \in \Psi_E(r) \mid r \in (0, R)\}$, according to Proposition 1, Theorem 2 implies that this closed graph of Ψ_E consists of a finite number of continuous curves at almost all r in $(0, R)$.

While the previous paragraph describes the properties of the set of equilibria of a given economy E, let $\mathcal{E}$ denote the set of economies satisfying the assumptions in this paper. Then, define a correspondence $\Psi^{WN} : \mathcal{E} \twoheadrightarrow \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}^m_{++}$ such that for each $E \in \mathcal{E}$, $\Psi^{WN}(E) = \Psi_E((0, R_E))$, where R_E comes from the maximum Frobenius eigenvalue $\frac{1}{1+R_E}$ associated with E. We call this mapping the *Walras-von Neumann correspondence* (W-N correspondence).

With a suitable topology on $\mathcal{E}$, it can be shown that the W-N correspondence is upper hemi-continuous, as in the case of the *Walrasian correspondence* in static economies.[16] However, it is well known that the image of the Walrasian correspondence in any given (regular) economy consists of a finite number of distinct and locally unique Walrasian equilibria. In contrast, the image $\Psi^{WN}(E)$ of the W-N correspondence at any given economy E *generically* consists of a finite number of distinct *curves* (one-dimensional continuums) of the **W-N *SE***s. We say "generically" because there can be at most a finite number of interest rates such that $\Psi_E(r)$ contains *non-regular* **W-N *SE***s.

Insert Figures 1a and 1b aroud here.

As in the standard literature on regular economies, we say that an economy $E \in \mathcal{E}$ is *regular* if and only if its corresponding image $\Psi^{WN}(E)$ contains only regular **W-N *SE***s. Then, as discussed in Section 4.1 below, we can see that the set of regular economies is of full measure, so almost all economies are regular. Furthermore, Theorem 2 suggests

[16] The Walrasian correspondence is a mapping that associates with each economy the set of competitive equilibria.

that *even within a non-regular economy*, the *set of non-regular* **W-N** $\boldsymbol{SE}$s is *of Lebesgue measure zero*.[17]

Finally, a special remark is worth making about the structure of Ψ^{WN} in a special case of economies with no possibility of joint production. An economy $E = \langle (A, B, L); \omega_l; z \rangle$ has *no joint production* if each and every process $j = 1, \ldots, m$ in the associated von Neumann production technology (A, B, L) can produce *only one type of commodity* as gross output: for each production process $j = 1, \ldots, m$, there exists a commodity $i = 1, \ldots, n$ such that $B\boldsymbol{e}_j = \boldsymbol{e}_i^T$ holds. In an economy without joint production, the von Neumann production technology can be reorganized as a profile of alternative Leontief production techniques $\{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\ldots,\Upsilon}$, where A^σ is an $n \times n$ non-negative square matrix of material input coefficients; I^σ is an $n \times n$ identity matrix; and L^σ is a $1 \times n$ positive vector of labor input coefficients, and σ is an index number to label each alternative Leontief production technique. Then, assuming that each Leontief production technique $(A^\sigma, I^\sigma, L^\sigma)$, $\sigma = 1, \ldots, \Upsilon$, is productive and indecomposable, it can be shown that for any economy $E = \langle (A, B, L); \omega_l; z \rangle$ with no joint production, each **W-N** $\boldsymbol{SE}$ is regular and $\Psi^{WN}(E)$ consists solely of a *single* one-dimensional continuum of all the **W-N** $\boldsymbol{SE}$s. (See Corollary 1 in Appendix for details.) That is, the set of steady-state equilibrium prices in an economy with no joint production is represented by a *single continuous curve*.

4.1 On Genericity of Regular Economies

In this subsection, we define a parameter set of economies, and then define *regular economies* within such a parameter set, and then examine the openness and genericity of such regular economies when the solution concept is of **W-N** $\boldsymbol{SE}$.

Given the demand function of two generations z^a, z^b, and for a vector of perturbations $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$, define a perturbed demand function, as in Mandler (1999a), by

$$z_i(h) \equiv z_i^b(h) + z_i^a(h)$$

where

$$z_i^b(h) \equiv z_i^b(p, w, r) + \frac{w}{p_i} h_i, \quad z_i^a(h) \equiv z_i^a(p, w, r) + \frac{w}{p_i} h^o$$

for each $i = 1, 2, ..., n$. To preserve Walras' law and homogeneity, the labor endowment ω_ℓ is also perturbed as follows: $\omega_l(h) \equiv \omega_l + \sum_{i=1}^n h_i + \frac{nh^o}{1+r}$.

Thus, given an economy $E = \langle (A, B, L); \omega_l; z \rangle$, a *perturbed economy* can be defined by $E(h) = \langle (A, B, L); \omega_l(h); z(h) \rangle$ for each $h \in \mathbb{R}^{n+1}$. Considering that a von Neumann

[17] In this context, a brief remark can be made about the problem of the index theorem. Because of the generic indeterminacy result here, it is meaningless to count the number of all **W-N** $\boldsymbol{SE}$s even in a regular economy. However, it may be relevant to count the finite number of continuous curves of these equilibria in the regular economy. For this question, given a regular economy E, for each interest rate $r \in (0, R_E)$, we may apply an 'index theorem' similar to Proposition 6.4.1 in Mas-Colell (1985; p.250) to $\Psi_E(r)$ to verify that the number of equilibria within $\Psi_E(r)$ is odd. This may lead us to conclude that the number of continuous curves of **W-N** $\boldsymbol{SE}$s in a regular economy is odd.

Perhaps, Bidard and Erreygers (1998) are relevant to this issue: they consider a restricted version of the von Neumann production model, in that it has a fixed composition of final demand and a fixed uniform rate of profit. They then show that the total number of equilibrium growth equilibria associated with the fixed rate of profit is odd. However, since their analysis does not address the effect of consumer behavior on the determination of market equilibria, we cannot apply the main theorem of Bidard and Erreygers (1998) to the indexation problem discussed here.

production technology (A, B, L) can also be perturbed, we denote a (perturbed) economy by (A, B, L, h) without loss of generality.

Define a function $\mathcal{F}$ on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1}, 1)$, k quantity variables $\overline{y} \in \mathbb{R}^k_{++}$, and (perturbed) economy (A, B, L, h) into $\mathbb{R}^{n+k}$, *i.e.*

$$\mathcal{F} : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^{k}_{++} \times \mathbb{R}^{nm}_{+} \times \mathbb{R}^{nm}_{+} \times \mathbb{R}^{m}_{++} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+k}$$

such that

$$\mathcal{F}(\bar{p}, w, r, \overline{y}, A, B, L, h) = \begin{bmatrix} z(h) - [B(\overline{p}, w, r) - A(\overline{p}, w, r)]\,\overline{y} \\ (\bar{p}B(\overline{p}, w, r) - (1+r)\bar{p}A(\overline{p}, w, r) - wL(\overline{p}, w, r))^T \end{bmatrix}.$$

A (perturbed) economy (A, B, L, h) is *regular* if every **W-N** $\boldsymbol{SE}$ $((p, w, r), y)$ in that economy is regular, i.e., the Jacobian $\boldsymbol{D}\mathcal{F}$ has full rank at $(\bar{p}, w, r, \overline{y})$. Denote the set of (perturbed) economies as P and the set of regular (perturbed) economies as P_R. Then, through some routine work, the following theorem is obtained.

Theorem 3: P_R is open and has full measure in P.

The proof of Theorem 3 can be obtained in a similar way to the proof of Theorem 2 in Yoshihara and Kwak (2024).

5 Concluding Remarks

In the above sections, we have considered a class of von Neumann production economies with a simple OLG structure to study Walras-von Neumann steady state equilibria. We then established the following distinctive features of the set of Walras-von Neumann steady state equilibria. That is, for every production economy E there exists a non-empty set of such equilibria which can be described as a closed graph of the equilibrium correspondence Ψ_E defined over the admissible domain, $[0, R_E)$, of non-negative interest rates. Moreover, for almost every interest rate in the domain $[0, R_E)$, the corresponding Walras-von Neumann steady state equilibrium is regular and indeterminate. Therefore, except for a negligible subset of the domain $[0, R_E)$, the closed graph of the correspondence Ψ_E consists of a finite number of one-dimensional continuous curves of regular Walras-von Neumann steady state equilibria. This property is observed regardless of whether the economy E is regular or not, although a full measure of the regularity of economies is also warranted. As noted in Section 1 and developed in Addendum A of this paper, these main results hold even if the economy has population growth and/or does not have an OLG structure.

What lessons can we learn from the generic indeterminacy of Walras-von Neumann equilibria? Recall that Debreu (1970, 1976) argued that equilibrium indeterminacy is an undesirable feature to warrant the explanatory power of the theory. However, such an interpretation may be inappropriate.

Rather, it may indicate that the classical and Marxian views that the functional distribution of income is determined, at least in part, by some historical, institutional, and socio-political schemes are indeed compatible with standard general equilibrium reasoning. In other words, in determining the long-period equilibrium position, it may suggest the need for an *equilibrium selection mechanism* that is *non-market competitive* in nature and is applied prior to the implementation of the competitive market mechanism. That is, a *two-stage comprehensive resource allocation mechanism* should be established, in which the first stage consists of a *non-market scheme* to determine a functional income distribution, that is, to select either an interest rate or a wage rate. Then, the second stage is the competitive market mechanism, which determines a **W**-**N** $\boldsymbol{SE}$ associated with the selected interest rate (or wage rate).

Although the question of what kind of non-market scheme would be relevant is beyond the scope of this paper, it may even involve a *democratic decision* on an *appropriate social welfare function* that can specify an optimal equilibrium selection among infinitely many **W**-**N** $\boldsymbol{SE}$s. Such an equilibrium selection might be relevant to the central bank's choice of monetary policy to influence the long-term interest rate. Or, given that we can choose wage rates rather than interest rates as the parameter of market competition in Theorems 1 and 2, the centralized collective bargaining system, as in the Nordic countries, may be considered as such an equilibrium selection mechanism. Recently, Oxfam (2024) proposes, as prescriptions for fighting inequality, to "ensure no share dividend payments before living wages" as well as to "limit top pay to no more than 20 times that of the average (median) worker", which could also serve as examples of the first-stage equilibrium selection mechanism.

Thus, the main results of this paper would open new windows for some new research agendas. For example, the design problem of an optimal income (re)distribution policy can be formulated as that of the first-stage scheme within an appropriate comprehensive two-stage resource allocation mechanism. Perhaps, the design problem of such a non-market scheme would require an appropriate view of which of infinitely many steady-state equilibria the intertemporal equilibrium paths from the first-stage selection would approach. Thus, it would be interesting to examine how the so-called turnpike theorem might be robust or require revision given the existence of a finite number of one-dimensional continuums of Walras-von Neumann steady-state equilibria. We leave this for future research.

6 References

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7 Appendix: Proofs of Main Theorems

7.1 Proof of Theorem 1

Let us fix the wage rate to *unity*: $w = 1$. Then, by condition (a) of Definition 1, any equilibrium commodity price vector associated with an interest rate $r \in [0, R)$ must belong to the following set:

$$\triangle^r \equiv \left\{p \in \mathbb{R}^n_+ \mid p\left[B - (1+r)\,A\right] \leqq L\right\}.$$

Since $\boldsymbol{e}_i B \geq \mathbf{0}$ $(\forall i)$ by A2 and $L > \mathbf{0}$, $\triangle^r$ is non-empty, compact, and convex. Then, for each $p \in \triangle^r$, write

$$\triangle^{(p,r)} \equiv \left\{p' \in \mathbb{R}^n_+ \mid p'\left[B - A\right] \leqq rpA + L\right\}.$$

Since $p \in \triangle^{(p,r)}$ follows from $p \in \triangle^r$, $\triangle^{(p,r)}$ is not empty. It is also compact and convex. Define $\triangle^{(\triangle^r,r)} \equiv \cup_{p\in\triangle^r}\triangle^{(p,r)}$.

Define the domain of commodity price vectors by the following set:

$$\triangle^K \equiv \left\{p \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} p_i \leqq K\right\},$$

where $K > 0$ is sufficiently large. Let $\partial\triangle^K \equiv \left\{p \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i = K\right\}$, which is the upper boundary of $\triangle^K$. Then, since K is sufficiently large, for each $p \in \triangle^{(\triangle^r,r)}$ there exists $t \geqq 1$ such that $tp \in \partial\triangle^K$. This implies that $\triangle^K \supset \triangle^{(\triangle^r,r)}$. It also follows that for any $p \in \partial\triangle^K$, there exists $\lambda \in (0,1)$ such that $\lambda p \in \triangle^{(\triangle^r,r)}$.

Given $r \in [0,R)$ and for each $p \in \triangle^K$, the Marshallian demand vectors $z_b(p,1,r)$ and $z_a(p,1,r)$ are uniquely identified at prices $(p,1,r)$. Then, given such fixed $z_b(p,1,r)$ and $z_a(p,1,r)$, define the following problem:

$$(MP_1(p)) \quad \max_{p' \in \triangle^K} p' \cdot [z_b(p,1,r) + z_a(p,1,r)] \text{ subject to } p'[B-A] \leqq rpA + L.$$

Let us denote the set of optimal solutions to $(MP_1(p))$ by $\psi^r(p)$. Then:

Lemma 1: The correspondence $\psi^r : \triangle^K \twoheadrightarrow \triangle^K$ has a fixed point.

Proof. It is easy to see that $\psi^r(p)$ is non-empty, compact, and convex for any $p \in \triangle^K$. Moreover, by Berge's maximum theorem, ψ^r is upper hemi-continuous. Therefore, by the Kakutani fixed point theorem, there exists $p^* \in \triangle^K$ such that $p^* \in \psi^r(p^*)$. ■

Lemma 1 implies that there exists $p^* \in \triangle^K$ such that

$$p^* \in \arg \max_{p' \in \triangle^K;\ p'[B-A] \leqq rp^*A + L} p' \cdot z(p^*,1,r).$$

Note that $z_b(p^*,1,r) + z_a(p^*,1,r) \geq \mathbf{0}$ follows from $w = 1$. Moreover, $p^* > \mathbf{0}$ holds, since if $p^* \geq \mathbf{0}$ and $p^* \not> \mathbf{0}$, then it cannot be a solution of $(MP_1(p^*))$ by A4.

Given this $p^* \in \triangle^K$, we define the following *linear programming problems*:

$$(MP_1^*) \quad \max_{p' \in \mathbb{R}^n_+} p' \cdot z(p^*,1,r) \text{ subject to } p'[B-A] \leqq rp^*A + L.$$

and

$$(MP_2^*) \min_{y \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}} (rp^*A + L)\,y \text{ subject to } [B-A]\,y \geqq z(p^*,1,r).$$

Then:

Lemma 2: $p^* \in \triangle^K$ is an optimal solution to (MP_1^*).

Proof. To show the claim, it suffices to verify that for any $p' \in \mathbb{R}^n_+$, if $p'[B-A] \leqq rp^*A + L$, then $p' \in \triangle^K$. Let

$$\triangle^{(p^*,r)} \equiv \left\{p \in \mathbb{R}^n_+ \mid p[B-A] \leqq rp^*A + L\right\}.$$

Then, by definition, $\triangle^{(p^*,r)} \subseteq \triangle^{(\triangle^r,r)} \subset \triangle^K$. Therefore, for any $p \in \mathbb{R}^n_+ \backslash \triangle^K$, $p \notin \triangle^{(p^*,r)}$ holds, so $p[B-A] \nleqq rp^*A + L$. Thus, for $p \in \mathbb{R}^n_+ \backslash \triangle^K$, p cannot be a solution of (MP_1^*). Since $p^* \in \triangle^K$ is a solution to $(MP_1(p^*))$ by Lemma 1, it is also a solution to (MP_1^*). ■

Next, consider (MP_2^*). By A1, there exists a feasible solution $y \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}$ such that $[B-A]\,y \geqq z(p^*,1,r) \geq \mathbf{0}$. Therefore there exists a solution $y^* \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}$ to (MP_2^*).

So we have

$$p^* \cdot z(p^*, 1, r) \leqq p^* [B - A] y^* \leqq (rp^* A + L) y^*.$$

Then:

Lemma 3: $p^* \cdot z(p^*, 1, r) = p^* [B - A] y^* = (rp^* A + L) y^*$.

Proof. This follows from the duality theorem and Lemma 2. ■

Proof of Theorem 1: We can see that $((p^*, 1, r); y^*)$ satisfies conditions (a) and (b) of Definition 1. Note that, as argued in Section 2, Walras' law $(*)$ reduces to $(*)''$ in $((p^*, 1, r), y^*)$. Then it can be shown by means of Walras' law $(*)''$ and Lemma 3 that $((p^*, 1, r); y^*)$ also satisfies condition (c) of Definition 1 with equality. Thus, $((p^*, 1, r); y^*)$ is a **W-N *SE*** associated with the non-negative interest rate $r \geqq 0$ in the economy $E = \langle (A, B, L); \omega_l; u \rangle$. ■

7.2 Proof of Proposition 1

Proof of Proposition 1: For each economy $E = \langle (A, B, L); \omega_l; z \rangle$, define a correspondence $\gamma : [0, R) \times \triangle^K \twoheadrightarrow \triangle^K$ such that for each $(r, p) \in [0, R) \times \triangle^K$, $\gamma(r, p) = \psi^r(p)$, where ψ^r is defined in the proof of Theorem 1 and it is non-empty, compact-valued, and upper hemi-continuous. Therefore, γ is closed. Next, define $\Gamma : [0, R) \twoheadrightarrow \triangle^K$ by $\Gamma(r) \equiv \{p^* \in \triangle^K \mid p^* \in \gamma(r, p^*)\}$ for each $r \in [0, R)$. Then, by Proposition 12.9 of Border (1985, p.65), Γ is upper hemi-continuous. Define $\eta : [0, R) \times \Gamma([0, R)) \twoheadrightarrow \mathbb{R}^m_+$ as: $y^* \in \eta(r, p^*)$ if and only if y^* is a solution of (MP_2^*) at (r, p^*) with $p^* \in \Gamma(r)$. Then, by Proposition 11.23 of Border (1985, p.60), η is upper hemi-continuous. Now, define $\varphi_E : [0, R) \twoheadrightarrow \mathbb{R}^m_+ \times \Gamma([0, R))$ such that for any $r \in [0, R)$, $(y^*, p^*) \in \varphi_E(r)$ if and only if $p^* \in \Gamma(r)$ and $y^* \in \eta(r, p^*)$ hold. By the proof of Theorem 1, this (y^*, p^*) constitutes a **W-N *SE*** associated with r. Since η is upper hemi-continuous, φ_E is also upper hemi-continuous by definition. Finally, define $\Psi_E : [0, R) \twoheadrightarrow \triangle \times [0, R) \times \mathbb{R}^m_+$ such that for each $r \in [0, R)$,

$$\left(\left(\frac{p_1^*}{1 + \sum_{j=1}^n p_j^*}, \ldots, \frac{p_n^*}{1 + \sum_{j=1}^n p_j^*}, \frac{1}{1 + \sum_{j=1}^n p_j^*} \right), r, y^* \right) \in \Psi_E(r)$$

if and only if $(y^*, p^*) \in \varphi_E(r)$ holds. Since φ_E is upper hemi-continuous, Ψ_E is also upper hemi-continuous by definition. ■

7.3 Proof of Theorem 2

Given an economy E, for each $r \in (0, R)$, let $((p, w, r), y) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times (0, R) \times \mathbb{R}^m_+$ be an associated **W-N *SE***, whose existence is ensured by Theorem 1. Then $F(\bar{p}, w, r, \overline{y}) = \mathbf{0}$ holds, where $\bar{p} \equiv (\frac{p_1}{p_n}, \ldots, \frac{p_{n-1}}{p_n}, 1)$ and $\overline{y} \in \mathbb{R}^k_{++}$ is obtained by removing the $m - k$ zero components of the $m \times 1$ column vector $y \in \mathbb{R}^m_+$. In the following argument, we will show that for any given economy E and *for almost all* $r \in (0, R)$, the Jacobian of F at the equilibrium associated with this r has full row rank.

The Jacobian of F at the equilibrium $((p, w, r), y)$ is given by

$$\begin{aligned} &\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\bar{p}, w, r, \overline{y})) \\ = &\begin{bmatrix} [A - B](\overline{p}, w, r) & \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r) & \boldsymbol{D}_w z(\overline{p}, w, r) & \boldsymbol{D}_r z(\overline{p}, w, r) \\ \mathbf{0} & [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T(\overline{p}, w, r) & -L(\overline{p}, w, r)^T & -(\overline{p}A(\overline{p}, w, r))^T \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} [A - B](\overline{p}, w, r) &\equiv [A(\overline{p}, w, r) - B(\overline{p}, w, r)], \\ [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T(\overline{p}, w, r) &\equiv [\boldsymbol{b}_{-n}(\overline{p}, w, r) - (1+r)\boldsymbol{a}_{-n}(\overline{p}, w, r)]^T, \end{aligned}$$

and $[\boldsymbol{b}_{-n}(\overline{p}, w, r) - (1+r)\boldsymbol{a}_{-n}(\overline{p}, w, r)]^T$ is the $k \times (n-1)$ matrix obtained by deleting the n-th column of the matrix $[B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T$. We will verify that $rank\left[\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\bar{p}, w, r, \overline{y}))\right] = n + k$ holds for almost all r within $(0, R)$.

First, we will verify the following claim:

Lemma 4: $rank\begin{bmatrix} [A - B](\overline{p}, w, r) & \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r) & \boldsymbol{D}_w z(\overline{p}, w, r) & \boldsymbol{D}_r z(\overline{p}, w, r) \end{bmatrix} = n.$

Proof. The claim follows from

$$rank\begin{bmatrix} \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r) & \boldsymbol{D}_w z(\overline{p}, w, r) \end{bmatrix} = n,$$

since the $n \times n$ matrix $\begin{bmatrix} \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r) & \boldsymbol{D}_w z(\overline{p}, w, r) \end{bmatrix}$ is invertible (see 1.11 on page 6 in Balasko (2009)). ■

Second, we need to check that

$$rank\begin{bmatrix} \mathbf{0} & [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T(\overline{p}, w, r) & -L(\overline{p}, w, r)^T & -(\overline{p}A(\overline{p}, w, r))^T \end{bmatrix} = k.$$

This statement holds whenever the row vectors in $[B - (1+r)A]^T(\overline{p}, w, r)$ are linearly independent. To verify it, we need to do some preliminary analysis:

Lemma 5: The columns of $[B(\overline{p}, w, r) - A(\overline{p}, w, r)]$ are linearly independent and $k \leqq n$.

Proof. Suppose not. Then, there exists $\overline{x} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $[B(\overline{p}, w, r) - A(\overline{p}, w, r)]\overline{x} = \mathbf{0}$. Therefore, there exists another equilibrium activity vector $\overline{y} + \lambda\overline{x} \in \mathbb{R}^k_+$ for some $\lambda \in \mathbb{R}$ such that $[B(\overline{p}, w, r) - A(\overline{p}, w, r)](\overline{y} + \lambda\overline{x}) = z(\bar{p}, w, r)$ holds. Then, by a proper choice of λ, the number of positive components of $(\overline{y} + \lambda\overline{x})$ can be less than k. However, this contradicts that k is the minimum number of actually operated processes at $(\overline{p}, w, r)$. Therefore, the columns of $[B(\overline{p}, w, r) - A(\overline{p}, w, r)]$ are linearly independent and $k \leqq n$ must hold. ■

By Lemma 5 there exists a $k \times k$ submatrix of $[B(\overline{p}, w, r) - A(\overline{p}, w, r)]$ which is invertible. Denote this submatrix by

$$(B(\overline{p}, w, r) - A(\overline{p}, w, r))^{(k)}$$

and its determinant by

$$\det\left(B\left(\overline{p},w,r\right)-A\left(\overline{p},w,r\right)\right)^{(k)}.$$

Similarly, define a $k\times k$ submatrix of $[B(\overline{p},w,r)-(1+r)A(\overline{p},w,r)]$ by

$$\left(B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right)^{(k)}\equiv\left(B\left(\overline{p},w,r\right)-A\left(\overline{p},w,r\right)\right)^{(k)}-rA\left(\overline{p},w,r\right)^{(k)},$$

where $A(\overline{p},w,r)^{(k)}$ is the $k\times k$ submatrix of $A(\overline{p},w,r)$ that appears in $\left(B(\overline{p},w,r)-A(\overline{p},w,r)\right)^{(k)}$. Then:

Lemma 6: $\det\left(B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right)^{(k)}\neq 0$ if and only if

$$rank\left[\begin{array}{cccc}\mathbf{0} & \left[\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}\right]^{T}\left(\overline{p},w,r\right) & -L\left(\overline{p},w,r\right)^{T} & -\left(\overline{p}A\left(\overline{p},w,r\right)\right)^{T}\end{array}\right]=k.$$

Proof. There are two possibilities: (i) $k<n$; and (ii) $k=n$.

Consider case (i): $k<n$. Then, $\left(\left[B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)}$ is a submatrix of $\left[\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}\right]^{T}\left(\overline{p},w,r\right)$, so the claimed equivalence relation holds.

Consider case (ii): $k=n$. First, the equilibrium condition (2)* in Section 4 can be rewritten as:

$$\sum_{i=1}^{n-1}\frac{\overline{p}_{i}}{w}\left(\boldsymbol{b}_{i}\left(\overline{p},w,r\right)-(1+r)\boldsymbol{a}_{i}\left(\overline{p},w,r\right)\right)^{T}+\frac{1}{w}\left(\boldsymbol{b}_{n}\left(\overline{p},w,r\right)-(1+r)\boldsymbol{a}_{n}\left(\overline{p},w,r\right)\right)^{T}=L\left(\overline{p},w,r\right)^{T},$$

where $\left(\boldsymbol{b}_{i}\left(\overline{p},w,r\right)-(1+r)\boldsymbol{a}_{i}\left(\overline{p},w,r\right)\right)^{T}$ (resp. $\left(\boldsymbol{b}_{n}\left(\overline{p},w,r\right)-(1+r)\boldsymbol{a}_{n}\left(\overline{p},w,r\right)\right)^{T}$) is the i-th column vector (resp. the n-th column vector) of $\left(\left[B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)}$. Therefore, we have the following property:

$$\begin{aligned}&\det\left[\begin{array}{cc}\left(\left[\boldsymbol{b}_{-n}\left(\overline{p},w,r\right)-\left(1+r\right)\boldsymbol{a}_{-n}\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)}, & -L\left(\overline{p},w,r\right)^{T}\end{array}\right]\\ = & -\frac{1}{w}\det\left(\left[B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)}\\ & +\sum_{i=1}^{n-1}\det\left[\left(\left[\boldsymbol{b}_{-n}\left(\overline{p},w,r\right)-\left(1+r\right)\boldsymbol{a}_{-n}\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)},-\frac{\overline{p}_{i}}{w}\left(\boldsymbol{b}_{i}\left(\overline{p},w,r\right)-(1+r)\boldsymbol{a}_{i}\left(\overline{p},w,r\right)\right)^{T}\right]\\ = & -\frac{1}{w}\det\left(\left[B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)}.\end{aligned}$$

So we have:

$$\det\left[\begin{array}{cc}\left(\left[\boldsymbol{b}_{-n}\left(\overline{p},w,r\right)-\left(1+r\right)\boldsymbol{a}_{-n}\left(\overline{p},w,r\right)\right]^{T}\right)^{(k)} & -L\left(\overline{p},w,r\right)^{T}\end{array}\right]\neq 0$$

if and only if

$$\det\left(B\left(\overline{p},w,r\right)-\left(1+r\right)A\left(\overline{p},w,r\right)\right)^{(k)}\neq 0.$$

So the claimed equivalence relation holds. ■

By Lemma 6, it suffices to show that $\det\left(B\left(\overline{p},w,r\right)-(1+r)\,A\left(\overline{p},w,r\right)\right)^{(k)}\neq 0$ for almost all r in $(0,R)$. To show this, we define

$$\mathcal{R}\equiv\left\{r\in[0,R)\mid\left[B\left(\overline{p},w,r\right)-(1+r)A\left(\overline{p},w,r\right)\right]^{T}\text{ is not a full row rank}\right\}.$$

We will verify that the set $\mathcal{R}$ is of Lebesgue measure zero in $[0,R)$. To show this, we assume $r\in\mathcal{R}$ and then show that for a small open neighborhood $\mathcal{N}\left(r\right)\subset[0,R)$ of r, $\mathcal{N}\left(r\right)\cap\mathcal{R}=\{r\}$.

As a preliminary step, let $\mathcal{N}^{(\overline{p},w,r,\overline{y})}\subset\mathbb{R}^{n}_{++}\times\mathbb{R}_{++}\times(0,R)\times\mathbb{R}^{k}_{++}$ be a sufficiently small open neighborhood of $(\bar{p},w,r,\overline{y})$ such that for any $(\overline{p}',w',r',\overline{y}')\in\mathcal{N}^{(\overline{p},w,r,\overline{y})}$, $B\left(\overline{p}',w',r'\right)=B\left(\overline{p},w,r\right)$, $A\left(\overline{p}',w',r'\right)=A\left(\overline{p},w,r\right)$, and $L\left(\overline{p}',w',r'\right)=L\left(\overline{p},w,r\right)$. The existence of such a small neighborhood is guaranteed since $\left[B\left(\overline{p},w,r\right)-(1+r)\,A\left(\overline{p},w,r\right)\right]$ is invariant with respect to a small change in prices, as argued when discussing the continuous differentiability of $F\left(\overline{p},w,r,\overline{y}\right)$ in Section 4. Furthermore, let $\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)\subset(0,R)$ be the projection of $\mathcal{N}^{(\overline{p},w,r,\overline{y})}$ onto $(0,R)$. Then, for any $r'\in\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ and any **W-N *SE*** $\left(\overline{p}\left(r'\right),w\left(r'\right),r',\overline{y}\left(r'\right)\right)\in\mathcal{N}^{(\overline{p},w,r,\overline{y})}$, it follows that $B\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)=B\left(\overline{p},w,r\right)$, $A\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)=A\left(\overline{p},w,r\right)$, and $L\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)=L\left(\overline{p},w,r\right)$.

Define a real-valued function $\Xi^{(\overline{p},w,r)}:\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)\to\mathbb{R}$ by: for each $r'\in\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$,

$$\Xi^{(\overline{p},w,r)}\left(r'\right)\equiv\det\left(B\left(\overline{p},w,r\right)-(1+r')\,A\left(\overline{p},w,r\right)\right)^{(k)}.$$

Then, we have

Lemma 7: Let $r\in\mathcal{R}$, and $(\bar{p},w,r,\overline{y})$ be a **W-N *SE*** associated with this r in E. Then, $\Xi^{(\overline{p},w,r)}\left(r\right)=0$ and there exists a small open neighborhood $\mathcal{N}\left(r\right)\subseteq\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ of r such that for any $r'\in\mathcal{N}\left(r\right)\setminus\{r\}$, $\Xi^{(\overline{p},w,r)}\left(r'\right)\neq 0$.

Proof. Since $r\in\mathcal{R}$, $rank\left[B\left(\overline{p},w,r\right)-(1+r)A\left(\overline{p},w,r\right)\right]^{T}<k$ holds, so there exists a $k\times k$ submatrix $\left(B\left(\overline{p},w,r\right)-(1+r)A\left(\overline{p},w,r\right)\right)^{(k)}$ of $B\left(\overline{p},w,r\right)-(1+r)A\left(\overline{p},w,r\right)$ such that $\Xi^{(\overline{p},w,r)}\left(r\right)=\det\left(B\left(\overline{p},w,r\right)-(1+r)A\left(\overline{p},w,r\right)\right)^{(k)}=0$. Let $\left(\Xi^{(\overline{p},w,r)}\right)^{-1}:\mathbb{R}\twoheadrightarrow\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ be the inverse mapping of $\Xi^{(\overline{p},w,r)}$. Since $\Xi^{(\overline{p},w,r)}$ is a polynomial function defined over $\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ with at most k degree, the set $\left(\Xi^{(\overline{p},w,r)}\right)^{-1}(0)$ is at most finite. Therefore, by choosing a sufficiently small open neighborhood $\mathcal{N}\left(r\right)\subseteq\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ of r, it follows that for any $r'\in\mathcal{N}\left(r\right)\setminus\{r\}$, $\Xi^{(\overline{p},w,r)}\left(r'\right)\neq 0$. ■

Then:

Lemma 8: Let $r\in\mathcal{R}$, and $(\bar{p},w,r,\overline{y})$ be a **W-N *SE*** associated with this r in E. Then, there exists a small open neighborhood $\mathcal{N}\left(r\right)\subseteq\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ of r such that for any $r'\in\mathcal{N}\left(r\right)\setminus\{r\}$ and any **W-N *SE*** $\left(\overline{p}\left(r'\right),w\left(r'\right),r',\overline{y}\left(r'\right)\right)\in\mathcal{N}^{(\overline{p},w,r,\overline{y})}$, $\left[B\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)-(1+r')\,A\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)\right]^{T}$ has full row rank.

Proof. By Lemma 7, there exists a small open neighborhood $\mathcal{N}\left(r\right)\subseteq\mathcal{N}^{(\overline{p},w,r,\overline{y})}\left(r\right)$ of r such that, for any $r'\in\mathcal{N}\left(r\right)\setminus\{r\}$, $\Xi^{(\overline{p},w,r)}\left(r'\right)\neq 0$. Then, by construction of $\mathcal{N}^{(\overline{p},w,r,\overline{y})}$, for any $r'\in\mathcal{N}\left(r\right)\setminus\{r\}$ and any **W-N *SE*** $\left(\overline{p}\left(r'\right),w\left(r'\right),r',\overline{y}\left(r'\right)\right)\in\mathcal{N}^{(\overline{p},w,r,\overline{y})}$,

$$\det\left(B\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)-(1+r')\,A\left(\overline{p}\left(r'\right),w\left(r'\right),r'\right)\right)^{(k)}=\Xi^{(\overline{p},w,r)}\left(r'\right)\neq 0.$$

Thus $[B(\overline{p}(r'), w(r'), r') - (1+r') A(\overline{p}(r'), w(r'), r')]^T$ has full row rank. ■

With these preliminary analyses, we can conclude that:

Lemma 9: $\mathcal{R}$ is of Lebesgue measure zero in $[0, R)$.

Proof. By Lemma 8, for any $r \in \mathcal{R}$, there exists a sufficiently small open neighborhood $\mathcal{N}(r) \subset (0, R)$ of r such that $\mathcal{N}(r) \cap \mathcal{R} = \{r\}$. So the set $\mathcal{R}$ is discrete. Since $\mathcal{R}$ is a subset of a compact set $[0, R]$, it is at most finite. Thus, the set $\mathcal{R}$ is of Lebesgue measure zero within $[0, R)$. ■

Now, we can complete the proof of Theorem 2.

Proof of Theorem 2: By Lemma 9, $rank[B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T = k$ holds for almost all $r \in [0, R)$. Thus, for almost all $r \in (0, R)$, the corresponding **W-N *SE*** $(\overline{p}, w, r, \overline{y})$ in this economy is *regular*.

Then, by the implicit function theorem, for almost all $r \in (0, R)$, there are an open neighborhood $\mathcal{O}(r) \subset (0, R)$ of r and also an open neighborhood $\mathcal{O}(\overline{p}, w, \overline{y}) \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}^k_{++}$ of $(\overline{p}, w, \overline{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}^k_{++}$ such that there exists a continuous single-valued mapping $\eta : \mathcal{O}(r) \to \mathcal{O}(\overline{p}, w, \overline{y})$ such that, for any $r' \in \mathcal{O}(r)$, there exists $(\overline{p}', w', \overline{y}') = \eta(r')$ with $F(\overline{p}', w', r', \overline{y}') = \mathbf{0}$.

By the definition of the mapping F, $F(\overline{p}', w', r', \overline{y}') = \mathbf{0}$ implies that $z(\overline{p}', w', r') = [B(\overline{p}', w', r') - A(\overline{p}', w', r')]\overline{y}'$ and $\overline{p}' B(\overline{p}', w', r') = (1+r')\overline{p}' A(\overline{p}', w', r') + w' L(\overline{p}', w', r')$. Then it follows by Walras' law that $L(\overline{p}', w', r')\overline{y}' = \omega_l$. Thus, $(\overline{p}', w', r', \overline{y}')$ is a **W-N *SE*** associated with $r' \in \mathcal{O}(r)$. ■

7.4 Ψ^{WN} in the Case of No Joint Production

Consider an economy $E = \langle (A, B, L); \omega_l; z \rangle$ with *no joint production.* As briefly explained in Section 4, such an economy can be represented by a profile $E = \left\langle \{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\dots,\Upsilon}; \omega_l; z \right\rangle$, where $\{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\dots,\Upsilon}$ is a profile of alternative Leontief production techniques derived from this von Neumann production technology (A, B, L) in the following way. As a first step, we introduce n production sectors, where sector $i = 1, \dots, n$ is the collection of processes that produce commodity i alone. Then each process j can be classified as a *Leontief process of sector* i if and only if $B\boldsymbol{e}_j = \boldsymbol{e}_i^T$ holds. After such classification, let us pick up one process from each sector $i = 1, \dots, n$, then we can constitute a *Leontief production technique* $(A^\sigma, I^\sigma, L^\sigma)$, where A^σ is an $n \times n$ non-negative square matrix of material input coefficients, I^σ is an $n \times n$ identity matrix, and L^σ is a $1 \times n$ positive vector of labor input coefficients. Denote the number of all such Leontief production techniques derived from (A, B, L) by Υ.

Assume that each such Leontief production technique $(A^\sigma, I^\sigma, L^\sigma)$, $\sigma = 1, \dots, \Upsilon$, derived from (A, B, L) is productive and indecomposable. Then, each available Leontief production technique $\sigma \in \{1, \dots, \Upsilon\}$ has its associated maximum eigenvalue $\frac{1}{1+R^\sigma} < 1$. Among them, let $\sigma^* \in \{1, \dots, \Upsilon\}$ be the technique whose associated maximum eigenvalue is minimal. Then let $R_E \equiv R^{\sigma^*}$. Obviously, for such an economy $E =$

$\left\langle \{(A^{\sigma}, I^{\sigma}, L^{\sigma})\}_{\sigma=1,\ldots,\Upsilon}; \omega_l; z \right\rangle$, Theorems 1 and 2 and Proposition 1 can hold. In addition, a sharper characterization of the set of **W-N** $\boldsymbol{SE}$s can be observed:

Corollary 1: In any economy with no joint production, $E = \left\langle \{(A^{\sigma}, I^{\sigma}, L^{\sigma})\}_{\sigma=1,\ldots,\Upsilon}; \omega_l; z \right\rangle$, $\Psi_E(r)$ is a singleton for each $r \in [0, R_E)$.

Proof. It is known that for each $r \in [0, R_E)$, there exists a unique price vector $(p(r), w(r), r)$ associated with the cost-minimizing Leontief production technique (A^r, I^r, L^r) at that prices (see Kurz and Salvadori (1995, p. 131, Theorem 5.1)): that is,

$$\begin{aligned} & (p(r), w(r), r) \\ \equiv & \left(\frac{L^r [I^r - (1+r) A^r]^{-1}}{\sum_{i=1}^{n} L^r \left([I^r - (1+r) A^r]^{-1} \right)_i + 1}, \frac{1}{\sum_{i=1}^{n} L^r \left([I^r - (1+r) A^r]^{-1} \right)_i + 1}, r \right) \end{aligned}$$

where $\left([I^r - (1+r) A^r]^{-1} \right)_i$ is the i-th column vector of the matrix $[I^r - (1+r) A^r]^{-1}$.

By Theorem 1, there should be a **W-N** $\boldsymbol{SE}$ associated with this r. In such an equilibrium, its equilibrium price vector must be the unique price vector $(p(r), w(r), r)$ defined above.

Summarizing the above arguments, it follows that $\Psi_E(r)$ is a singleton for each $r \in [0, R_E)$. ■

Thus, by Corollary 1, $\Psi^{WN}(E)$ consists solely of a *single* one-dimensional continuum of all the **W-N** $\boldsymbol{SE}$s. Finally, for each $r \in [0, R_E)$, $(p(r), w(r), r, y(r)) \in \Psi_E(r)$ is regular, which can be shown as in Yoshihara and Kwak (2024).

<table>
<tr><th colspan="2">Economic Domains</th><th>Generic Determinacy</th><th>Generic Indeterminacy</th></tr>
<tr><td>Static</td><td>Pure Exchange: Debreu (1970)
Production with CRS Technologies:
Mas-Collel (1975), Kehoe (1980,1982).</td><td>regular economies: full measure.
regular equilibria: finite, locally unique Walrasian CEs.</td><td></td></tr>
<tr><td rowspan="6">Intertemporal</td><td>a finite number of infinitely lived agents:
Kehoe and Levine (1985).</td><td>regular economies: full measure.
regular equilibria: finite, locally unique perfectly farsighted equilibria.</td><td></td></tr>
<tr><td>OLG Pure Exchange: Kehoe and Levine (1985).</td><td rowspan="2">regular economies: full measure.
regular equilibria: finite, locally unique steady states.</td><td rowspan="3">Local Indeterminacy:
the existence of a continuum of nearby steady-state equilibrium paths converging to the same steady-state.</td></tr>
<tr><td>OLG two-sector neoclassical production:
Calvo (1978)</td></tr>
<tr><td>a finite number of infinitely lived agents with market imperfection:
Nishimura and Venditti (2016), etc.</td><td>finite, locally unique steady states.</td></tr>
<tr><td>a finite number of finitely lived agents:
Mandler (1995, 1999).</td><td></td><td>Generic Indeterminacy of Sequential Equilibria:
the second-period continuation equilibria are indeterminate for almost every induced second-period economy.</td></tr>
<tr><td>OLG with simple Leontief production:
Yoshihara and Kwak (2023, 2024).</td><td></td><td>Generic Indeterminacy of non-trivial steady state equilibria.</td></tr>
</table>

Table 1: Literature on Generic (In)determinacy

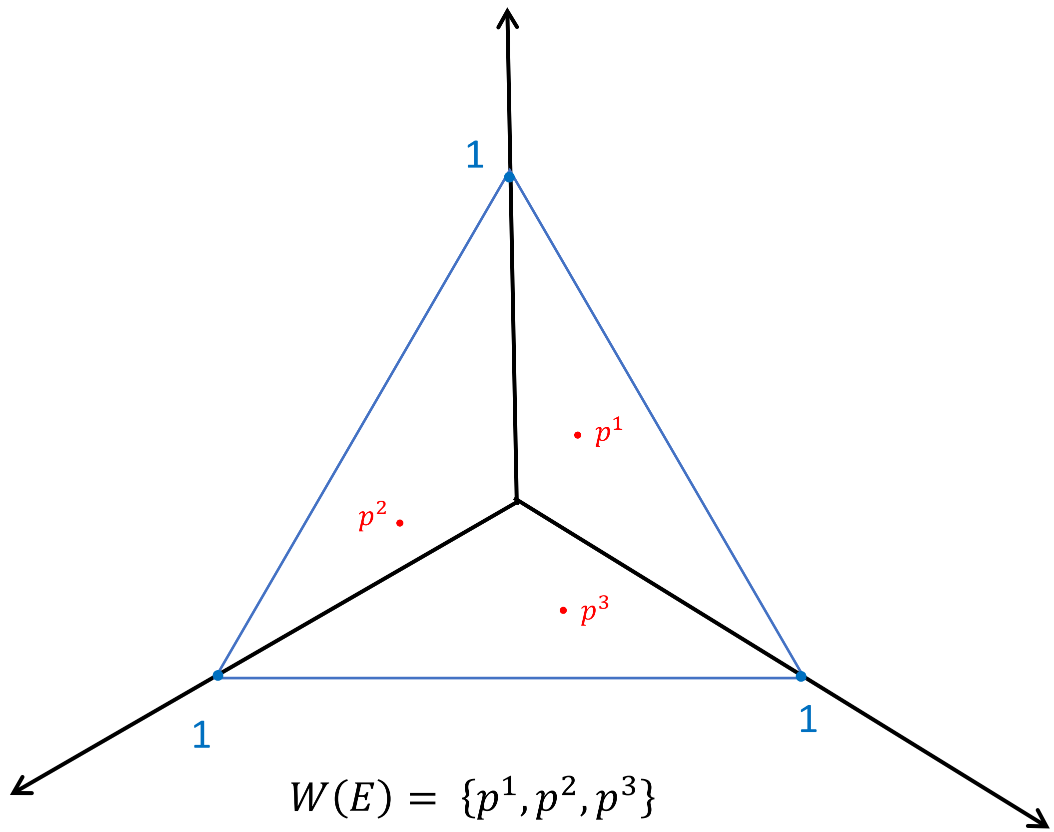

Figure 1a: the set of the standard Walrasian competitive equilibria in a regular economy *E*

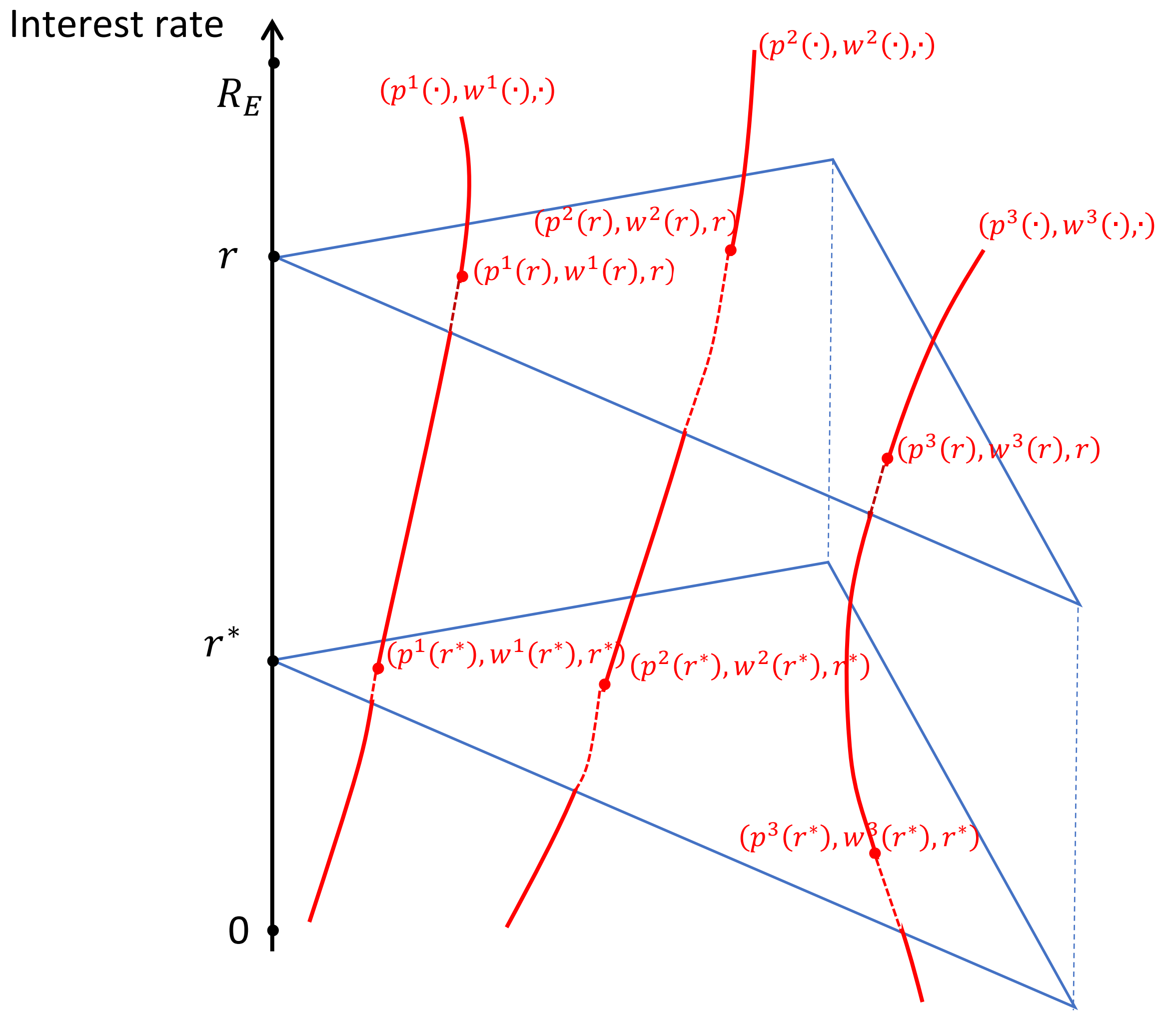

Figure 1b: the set of *W*-*N* equilibria in a regular economy *E*

$$\Psi^{WN}(E)$$
$$= \{(p^1(r'), w^1(r'), r'), (p^2(r'), w^2(r'), r'), (p^3(r'), w^3(r'), r') | r' \in (0, R_E)\}$$

Addendum for *Generic Indeterminacy of Steady-State Competitive Equilibria in Walras-von Neumann Production Economies*[*]

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May 1, 2025

Abstract

This is the addendum to the main text of the paper I wrote entitled *Generic Indeterminacy of Steady-State Competitive Equilibria in Walras-von Neumann Production Economies*. First, Addendum A shows that the main results of the paper do not depend on the OLG structure of the model. More precisely, Addendum A analyzes a more general version of the model in which the population grows at a constant, exogenous rate $g \geqq 0$, and the economic environment in any given period is specified by a list of von Neumann production technologies, the size of the population, and a Marshallian aggregate demand function. Accordingly, a more general solution concept is defined as a *von Neumann balanced growth equilibrium* associated with g as the warranted rate of capital accumulation. For this solution concept, it is shown that essentially the same two main results as those developed in the main text of the paper are obtained. Second, in Addendum B, it is shown that there exists a reasonable microeconomic model of an individual optimization program for rationalizing the aggregate demand function z of an economy E defined in the main text of the paper. It is also shown that in such a microeconomic model, a steady-state equilibrium with a strictly negative interest rate is impossible to exist whenever the production activity vector is required to be nonzero in equilibrium.

JEL Classification Code: B51, D33, D50.

Keywords: generic indeterminacy of Walras-von Neumann steady state equilibria; von Neumann production economies; a finite number of one-dimensional continuums of Walras-von Neumann steady state equilibria;

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1 Addendum A: An Extension: Generic Indeterminacy of Balanced Growth Equilibria

In this section, we consider a general von Neumann economic model with an exogenous population growth rate. Because of the growing population, a von Neumann equilibrium must be defined as a balanced growth equilibrium associated with an equilibrium rate of capital accumulation being equal to the population growth rate. Here, an economic environment is specified by a list $E = \langle (A, B, L); \omega_l, g; z \rangle$, where $g \geqq 0$ represents a population growth rate, ω_l the current population size, and z the aggregate Marshallian demand function. Since ω_l is the population size of the present period, it becomes $(1 + g)\,\omega_l$ in the next period of production.

Unlike the model in Section 2 of the main text of the paper, we will leave the underlying microeconomic structure of consumers' behavior unspecified, except that z must satisfy the *aggregate budget constraint*. However, as discussed below, the latter point invites more complexity in the analysis of von Neumann equilibria under such a general economic model than under the model with an OLG structure.

1.1 Walras-von Neumann Balanced Growth Equilibria with a General Aggregate Demand Function

Let the Marshallian demand function $z(p, w, r, I)$ represent the aggregate demand of the whole population, where I represents the aggregate income distributed to the whole population after firms deduct the investment funds for capital accumulation, so that the *aggregate budget constraint* $p \cdot z(p, w, r, I) = I$ must hold.[1] Note that the aggregate income I is a continuous function of the price system (p, w, r), the aggregate production plan $y \in \mathbb{R}^m_+$, and the aggregate labor endowment ω_l : $I = I(p, w, r, y, \omega_l)$. Thus, the Marshallian demand function is reduced to:

$$\begin{aligned} z(p, w, r, I) &= z(p, w, r, I(p, w, r, y, \omega_l)) \\ &= z(p, w, r, I(p, w, r, y)), \text{ since } \omega_l \text{ is fixed throughout the analysis,} \\ &= z(p, w, r, y). \end{aligned}$$

In the following, we will denote the Marshallian demand function by $z(p, w, r, y)$. Note that $p \cdot z(p, w, r, y) = (r - g)\,pAy + \omega_l$ holds by the definition of this function.

This definition differs from that of the Marshallian demand function in Section 2 of the main text of the paper, because z in Section 2 is a function of prices only. Recall that the underlying budget constraint in Section 2 was given by the wage income $w\omega_l$, apart from the price information (p, w, r). Therefore, Since ω_l is fixed throughout the entire periods, the Marshallian aggregate demand in Section 2 of the main text of the paper can be defined as a function of price information *only*. Such treatment is no longer possible in this addendum section, so the Marshallian aggregate demand should also depend on the information of the production plan y.

[1] A typical example of the aggregate income function is $I(p, w, r, y) = (r - g)\,pAy + w\omega_l$. Here, $(r - g)\,pAy$ represents the aggregate net profit income distributed to all households, which is equal to the residual of aggregate profit income $rpAy$ after firms deduct investment funds for capital accumulation $gpAx$, given that a production plan $y \in \mathbb{R}^m_+$ is activated in a present period.

In this case, for each $r \in (0, R)$, a *Walras-von Neumann balanced growth equilibrium* (in short, **W-N *BGE***) is a profile $((p, w, r); y)$ that is a solution to the following system of inequalities:

$$\begin{aligned} pB &\leqq (1+r)\, pA + wL; \text{ (a)} \\ By &\geqq (1+g)\, Ay + z\,(p, w, r, y)\,, \text{ (b)} \\ Ly &\leqq \omega_l. \text{ (c)} \end{aligned}$$

Here, Walras' law is represented correspondingly by:

$$p \cdot [By - (1+g)\, Ay - z\,(p, w, r, y)] + w\,(\omega_l - Ly) = 0.$$

Multiplying y from the right in (a), we have:

$$pBy = (1+r)\, pAy + wLy. \text{ (a*)}$$

Then, Walras' law together with (a*) can be reduced to:

$$pz\,(p, w, r, y) - (r-g)\, pAy - w\omega_l = 0.$$

Next, by multiplying p from the left in (b), we have:

$$pBy = (1+g)\, pAy + pz\,(p, w, r, y)\,. \text{ (b*)}$$

From (a*) and (b*), we have $pz\,(p, w, r, y) = wLy + (r-g)\, pAy$. Then, together with the reduced form of Walras' law, we get

$$Ly = \omega_l.$$

Thus we can get rid of condition (c).

1.2 Existence of a Closed Graph Relationship between Balanced Growth Equilibria and Interest Rates

As discussed above, a Marshallian aggregate demand vector cannot be determined by price information alone, but also by the information of the (ex ante) production plan. However, as observed below, an equilibrium production plan will be determined corresponding to the given aggregate demand vector. This equilibrium production plan should also stipulate a balanced growth equilibrium path of capital accumulation. Therefore, under a balanced growth equilibrium, the Marshallian aggregate demand vector must be fixed by the information of the balanced growth equilibrium production plan, but the latter is a variable determined corresponding to the former. Thus, the existence problem of a balanced growth equilibrium in this section should involve an additional complication in that the 'ex-ante' production plan must coincide with the 'ex-post' equilibrium production plan.

With this last point in mind, consider the existence of a balanced growth equilibrium for each $r \in [0, R)$. Assume again that $w = 1$. Define $X \equiv \left\{x \in \mathbb{R}^m_+ \mid Lx = \omega_l\right\}$. Let

$$p^m A x^m \equiv \arg \max_{p^m \in \triangle^K;\ x \in X} pAx,$$

and then let

$$I^m \equiv rp^m Ax^m + \omega_l.$$

Finally, define

$$X^m \equiv \left\{x \in \mathbb{R}^m_+ \mid \exists p \in \triangle^K : \ rpAx + Lx \leqq I^m\right\}.$$

Note that X^m is compact and convex. The latter property holds because $\triangle^K$ is convex.

Choose any $x \in X^m$ to represent an ax ante plan for capital accumulation. In other words, gAx represents an ax ante demand for capital goods for new investment.

For each $r \in [0, R)$ and each $p \in \triangle^K$, the aggregate demand vector is determined by $z(p, 1, r, x)$. Then, for such fixed $z(p, 1, r, x)$ and $x \in X^m$, define the following program:

$$(MP_1) \quad \max_{p' \in \triangle^K} p' \cdot [z(p, 1, r, x) + gAx] \text{ subject to } p'[B - A] \leqq rpA + L.$$

As already shown in the proof of Theorem 1, there exists $p^*(x) \in \triangle^K$ such that

$$p^*(x) \in \arg \max_{p' \in \triangle^K;\ p'[B-A] \leqq rp^*(x)A + L} p' \cdot [z(p^*(x), 1, r, x) + gAx].$$

Then, given this datum $p^*(x) > \mathbf{0}$, define the following problems:

$$(MP_1^*) \quad \max_{p' \in \mathbb{R}^n_+} p' \cdot [z(p^*(x), 1, r, x) + gAx] \text{ subject to } p'[B - A] \leqq rp^*(x)A + L,$$

and

$$(MP_2) \min_{y \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}} [rp^*(x)A + L]y \text{ subject to } [B - A]y \geqq z(p^*(x), 1, r, x) + gAx.$$

As we already observed above, $p^*(x) > \mathbf{0}$ is a solution to (MP_1^*). Let $y^*(x) \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be a solution to this (MP_2). Note that the existence of such a solution is guaranteed by **A1**.

In this case we have

$$p^*(x) \cdot [z(p^*(x), 1, r, x) + gAx] \leqq p^*(x)[B - A]y^*(x) \leqq [rp^*(x)A + L]y^*(x).$$

By the duality theorem, it follows that

$$p^*(x) \cdot [z(p^*(x), 1, r, x) + gAx] = p^*(x)[B - A]y^*(x) = [rp^*(x)A + L]y^*(x). \quad (**)$$

By the way, the definition of $z(p^*(x), 1, r, x)$ implies that

$$\begin{aligned} p^*(x)[z(p^*(x), 1, r, x) + gAx] &= gp^*(x)Ax + I(p^*(x), 1, r, x) \\ &= rp^*(x)Ax + \omega_l \leqq \max_{p" \in \triangle^K} rp"Ax + \omega_l \leqq I^m. \end{aligned}$$

Thus, since $p^*(x) \cdot [z(p^*(x), 1, r, x) + gAx] = [rp^*(x)A + L]y^*(x)$, we have

$$[rp^*(x)A + L]y^*(x) \leqq I^m,$$

which implies $y^*(x) \in X^m$.

Define a correspondence $\phi : X^m \twoheadrightarrow X^m$ such that

$$\phi(x) \equiv \left\{y^*(x) \in X^m \mid y^*(x) \in \arg \min_{[B-A]y \geqq z(p^*(x), 1, r, x) + gAx} [rp^*(x)A + L]y\right\}.$$

Note that this correspondence is upper hemi-continuous and convex-valued. Since X^m is compact and convex, by the Kakutani fixed point theorem there exists a fixed point $y^* \in X^m$ such that $y^* \in \phi(y^*)$. Then let $p^* \equiv p^*(y^*)$. By definition,

$$\begin{aligned} p^* &\in \arg \max_{p' \in \mathbb{R}^n_+;\ p'[B-A] \leqq rp^*A+L} p' \cdot [z(p^*, 1, r, y^*) + gAy^*] \ ; \\ y^* &\in \arg \min_{y \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\};\ [B-A]y \geqq z(p^*,1,r,y^*)+gAy^*} [rp^*A + L]\, y \ . \end{aligned}$$

As argued above, it follows that

$$p^* \cdot [z(p^*, 1, r, y^*) + gAy^*] = p^* [B - A]\, y^* = [rp^*A + L]\, y^* \quad (***)$$

by the duality theorem. Finally, since $p^* \cdot [z(p^*, 1, r, y^*) + gAy^*] = rp^*Ay^* + \omega_l$ holds by definition of the demand function, we have $Ly^* = \omega_l$ from (***). Similarly, by Walras' law, it follows that

$$(p^* \cdot [z(p^*, 1, r, y^*) + gAy^*] - p^* [B - A]\, y^*) + (Ly^* - \omega_l) = 0.$$

Thus, again by (***), we have $Ly^* = \omega_l$.

Therefore, we have a **W-N *BGE*** $((p^*, 1, r), y^*)$ associated with $r \in [0, R)$ for the economy $\langle (A, B, L); \omega_l; g; z \rangle$.

Theorem 4: Let $E = \langle (A, B, L); \omega_l; g; z \rangle$ be an economy as defined above. Then, for *each interest rate* $r \in [0, R)$, there exists a *Walras-von Neumann balanced growth equilibrium* $((p^*, 1, r), y^*)$ under this economy.

1.3 Generic Indeterminacy of Balanced Growth Equilibria

The generic indeterminacy of balanced growth equilibria can be shown as argued in Section 4 of the main text of the paper. First, a continuously differentiable mapping F is given in this section, as follows:[2]

$$F(\overline{p}, w, r, \overline{y}) \equiv \begin{cases} [B(\overline{p}, w, r) - (1+g)\, A(\overline{p}, w, r)]\, \overline{y} - z(\overline{p}, w, r, \overline{y}) \\ p\, [B(\overline{p}, w, r) - (1+r)\, A(\overline{p}, w, r)] - wL(\overline{p}, w, r) \end{cases} .$$

Here, the definitions of $\overline{p}$ and $\overline{y}$ are the same as in Section 4 of the main text of the paper. Thus, $\overline{y} \in \mathbb{R}^k_{++}$ is obtained by subtracting the $m-k$ zero components of the $m \times 1$ column vector $y \in \mathbb{R}^m_+$, where k is the minimum number of processes that can satisfy the equilibrium condition (b) in equality.

Then, the corresponding Jacobian of $F(\overline{p}, w, r, \overline{y}) = 0$ in a **W-N *BGE*** $(\overline{p}, w, r, \overline{y})$ is given by:

$$\begin{aligned} &\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}, w, r, \overline{y})) \\ =& \begin{bmatrix} JF_{n\times k} & \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_w z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_r z(\overline{p}, w, r, \overline{y}) \\ \mathbf{0} & [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T (\overline{p}, w, r) & -L(\overline{p}, w, r)^T & -(\overline{p}A(\overline{p}, w, r))^T \end{bmatrix}, \end{aligned}$$

[2] As in the case of steady state equilibria discussed in Section 4, the continuous differentiability of the excess demand functions

$$[B(\overline{p}, w, r) - (1+g)\, A(\overline{p}, w, r)]\, \overline{y} - z(\overline{p}, w, r, \overline{y})$$

is guaranteed at least in a small neighborhood of each equilibrium $(\overline{p}, w, r, \overline{y})$.

where

$$JF_{n\times k}\equiv\left[(1+g)\,A-B\right](\overline{p},w,r)+\boldsymbol{D}_{\overline{y}}z\,(\overline{p},w,r,\overline{y})\,.$$

As shown in the proof of Theorem 2 of the main text of the paper, the first n-row vectors

$$\left[\begin{array}{cccc}\left[(1+g)\,A-B\right](\overline{p},w,r)+\boldsymbol{D}_{\overline{y}}z\,(\overline{p},w,r,\overline{y}) & \boldsymbol{D}_{\overline{p}}z\,(\overline{p},w,r,\overline{y}) & \boldsymbol{D}_{w}z\,(\overline{p},w,r,\overline{y}) & \boldsymbol{D}_{r}z\,(\overline{p},w,r,\overline{y})\end{array}\right]$$

are linearly independent since $\left[\begin{array}{cc}\boldsymbol{D}_{\overline{p}}z\,(\overline{p},w,r,\overline{y}) & \boldsymbol{D}_{w}z\,(\overline{p},w,r,\overline{y})\end{array}\right]$ is invertible. Furthermore, the next k row vectors

$$\left[\begin{array}{cccc}\boldsymbol{0} & \left[\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}\right]^{T}(\overline{p},w,r) & -L\,(\overline{p},w,r)^{T} & -\left(\overline{p}A\,(\overline{p},w,r)\right)^{T}\end{array}\right]$$

are shown to be linearly independent for almost all $r\in(0,R)$. First, as shown in the proof of Theorem 2 of the main text of the paper, we can see that

$$rank[B-(1+g)\,A]^{T}\,(\overline{p},w,r)=k.$$

Then, as shown in the proof of Theorem 2 of the main text of the paper, we can see that for almost all $r\in(0,R)$,

$$rank\left[\begin{array}{cc}\left[\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}\right]^{T}(\overline{p},w,r) & -L\,(\overline{p},w,r)^{T}\end{array}\right]=k.$$

In summary, we get the following result:

Theorem 5: Let $E=\langle(A,B,L)\,;\omega_{l};g;z\rangle$ be a general von Neumann production economy specified as above. Then, for almost all $r\in(0,R)$, the associated Walras-von Neumann balanced growth equilibrium $((p,w,r)\,,y)$ under this economy is *indeterminate*.

1.4 Remark on the existence theorems of the balanced growth equilibrium in the present literature

Morishima (1960, 1969) also considered von Neumann production economies with Mashallian demand functions, and then Salvadori (1980) and Bidard and Hosoda (1988) examined the existence of a balanced growth equilibrium in that model. While assuming that wages are paid in advance of production, Morishima (1960) introduced the exogenous saving rate of the capitalist class as s_c with $0<s_c<1$. Then, a balanced growth equilibrium is defined by the following system of equilibrium inequalities:

$$\begin{aligned}pB &\leqq (1+r)\,[pA+wL]\,;\ \text{(a)}\\ By &\geqq (1+g)\,[Ay+wLy\cdot d^{w}\,(p,w,r)]+r\,[pAy+wLy]\cdot d^{c}\,(p,w,r)\,,\ \text{(b)}\\ g &= rs_c,\ \text{(c)}\end{aligned}$$

where $d^{c}\,(p,w,r)$ represents the Marshallian consumption demand of the capitalist class per $1-s_c$ unit of expenditure: For any $p\in\mathbb{R}^{n}_{+}$, $p\cdot d^{c}\,(p,w,r)=1-s_c$. Likewise, $d^{w}\,(p,w,r)$ represents the Marshallian consumption demand of the working class per unit of expenditure: for any $p\in\mathbb{R}^{n}_{+}$, $p\cdot d^{w}\,(p,w,r)=1$.[3]

[3] Morishima (1969) also provided a more generalized von Neumann model than Morishima (1960) by introducing an exogenous saving rate for the working class. Since then, Salvadori (1980) and Bidard and Hosoda (1988) have investigated the existence of a balanced growth equilibrium in the Morishima (1969) model.

However, to highlight the determinacy of the equilibrium in von Neumann models of the Morishima type, it is sufficient to examine the original model of Morishima (1960).

Here, condition (c) represents the Cambridge equation, where Morishima (1960) assumed that population growth $g > 0$ is exogenously given. Then, the system of inequalities consists of $n + k + 1 + 1$ equations (n equations of excess demand conditions for commodity markets (b); k equations of zero profit conditions for the operated processes (a); the Cambridge equation (c); and the commodity price normalization equation, $\sum_{i=1}^{n} p_i = 1$. In contrast, the unknowns are p, y, w, and r. Here, Walras' law is represented by:

$$\begin{aligned} pBy &= (1+g)\, p\left[Ay + wLy \cdot d^{w}(p, w, r)\right] + r\left[pAy + wLy\right] \cdot pd^{c}(p, w, r) \\ &= (1+g)\left[pAy + wLy\right] + r\left[pAy + wLy\right](1 - s_c). \end{aligned}$$

Then, by multiplying y in (a) from the right, we have:

$$pBy = (1+r)\left[pA + wL\right] y. \quad \text{(a*)}$$

Then, by means of Walras' law and (a*), the Cambridge equation (c) is derived. So we can get rid of (c). However, r can also be removed from the list of the unknowns by the derived Cambridge equation, since the population growth rate g is given. So there are $n + k + 1$ independent equations while there are $n + k + 1$ unknowns. Thus, we cannot observe the degree one freedom feature in Morishima's (1960) system of equilibrium inequalities. Thus, a balanced growth equilibrium in the Morishima (1960) model is determinate. In fact, while r can be determined by the derived Cambridge equation, the equilibrium wage rate w is determined by the intersection of the growth rate curves of labor demand and of labor supply, where the former is derived from the wage-interest-rate frontier of this economy and is downward sloping with respect to wage rates, and the latter is a flat curve drawn at point g.

In contrast, the model discussed in this Addendum A of this paper also introduces an exogenous population growth rate, but does not have a fixed saving rate as a parameter. However, the model in this Addendum A introduces the labor market equilibrium condition, which does not appear in Morishima's (1960) system of equilibrium inequalities. As we have observed, the labor equilibrium condition is also removed by the application of Walras' law, so that there are also $n + k$ independent equations in Addendum A, while the $n + k + 1$ unknown variables cannot be further reduced. As a consequence, given the population growth rate, the system of equilibrium inequalities in Addendum A cannot determine the equilibrium interest rate, and thus the continuum of Neumann equilibria can be observed for the exogenously given population growth rate.

2 Addendum B: A Simple Microeconomic Model of OLG Economies

This appendix presents a simple microeconomic model of OLG economies, where an individual optimization problem for their economic decisions is specified to provide an underlying structure of economic data E, as developed in Yoshihara and Kwak (2023, 2024).

Given the basic information about the OLG structure of each generation with a labor endowment ω_l and a profile of von Neumann production technology (A, B, L) specified in Section 2 of the main text of the paper, let $u : \mathbb{R}_{+}^{n} \times \mathbb{R}_{+}^{n} \to \mathbb{R}$ be a welfare function of lifetime consumption activities common to all generations. As usual, u is assumed to be continuous and strongly monotonic. Thus, an *overlapping generation economy* is given by a profile $E = \langle (A, B, L); \omega_l; u \rangle$.

Assume also that for each generation t, $l^t \in \mathbb{R}_+$ represents the labor supplied by t at the beginning of the young age; $\omega^{t+1} \in \mathbb{R}^n_+$ represents a bundle of goods for the purpose of saving money $p_t\omega^{t+1}$, chosen by generation t at the end of the young age and used in the old age; $\delta^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle purchased by generation t at the beginning of the old age for the purpose of speculative activities; $y^{t+1} \in \mathbb{R}^m_+$ represents a production activity vector chosen by generation t at the beginning of the old age; z^t_b is the consumption bundle consumed by generation t at young age; and z^t_a is the consumption bundle consumed by generation t at old age.

Each generation t at young age is confronted with the following optimization program MP^t: for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$,

$$\max_{l^t,\omega^{t+1},\delta^{t+1},y^{t+1},z^t_b,z^t_a} u\left(z^t_b, z^t_a\right)$$

subject to

$$\begin{aligned} p_t z^t_b + p_t\omega^{t+1} &\leqq w_t l^t, \\ l^t &\leqq \omega^t_l, \\ p_t\delta^{t+1} + p_t A y^{t+1} &= p_t\omega^{t+1}, \text{ and} \\ p_{t+1} z^t_a &\leqq p_{t+1}\delta^{t+1} + p_{t+1} B y^{t+1} - w_{t+1} L y^{t+1}. \end{aligned}$$

That is, each generation t can supply l^t amount of labor at young age as a worker employed by generation $t-1$. From the wage income $w_t l^t$ earned at the end of the young age, she can save $p_t\omega^{t+1}$ amount of money and purchase a consumption bundle z^t_b. With the saved money $p_t\omega^{t+1}$, the generation t at the beginning of the old age can purchase δ^{t+1} for the speculative purpose and can purchase a vector of capital goods Ay^{t+1} for the productive investment. As an industrial capitalist, she can employ Ly^{t+1} amount of labor of generation $t+1$. Then, at the end of the old age, she can earn $p_{t+1}\delta^{t+1}$ as the income of the speculative investment and $p_{t+1}By^{t+1} - w_{t+1}Ly^{t+1}$ as the income of the productive investment. With this income she can purchase a consumption bundle z^t_a.

Let $\left(l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z^t_b, z^t_a\right)$ be a solution of the optimization program MP^t for each generation t. In the optimum, all weak inequalities in the above constraints should be equal, given the assumption of u. That is,

$$\begin{aligned} p_t z^t_b + p_t\omega^{t+1} &= w_t l^t, \\ l^t &= \omega^t_l, \text{ and} \\ p_{t+1} z^t_a &= p_{t+1}\delta^{t+1} + p_{t+1} B y^{t+1} - w_{t+1} L y^{t+1}. \end{aligned}$$

Note that the production activity vector y^t planned by generation $t-1$ at the beginning of the old age should satisfy the profit maximization condition. Since market prices in equilibrium should satisfy the zero profit condition, the following condition holds for each period t:

$$p_t B \leqq (1 + r_t)\, p_{t-1} A + w_t L.$$

Therefore, the equilibrium profit maximization condition for each period t is given by:

$$p_t B y^t = (1 + r_t)\, p_{t-1} A y^t + w_t L y^t.$$

Thus, the revenue constraint $p_{t+1}z_a^t = p_{t+1}\delta^{t+1} + p_{t+1}By^{t+1} - w_{t+1}Ly^{t+1}$ of generation t at the end of the old age can be reduced to

$$p_{t+1}z_a^t = p_{t+1}\delta^{t+1} + (1+r_{t+1})\,p_t A y^{t+1}.$$

Given a pair of sequence of price vectors $(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) \equiv \{(p_t,w_t,r_t)\}_{t\geq 0}$, let $(z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}), z_a^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}))$ be a solution of the generation $t = 1,2,\ldots,$ to the above MP^t utility maximization problem under budget constraint. Then, a competitive equilibrium can be formulated as follows.

Definition A1: A *competitive equilibrium* under the overlapping generation economy $E = \langle (A,B,L); \omega_l; u\rangle$ is a pair of sequence of price vectors $(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) \equiv \{(p_t,w_t,r_t)\}_{t\geq 0}$ and sequence of optimal actions of each generation

$$\left\{\left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}), z_a^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r})\right)\right\}_{t\geq 0}$$

satisfying the following conditions:

$$p_t B \leqq (1+r_t)\,p_{t-1}A + w_t L \ (\forall t); \quad \text{(A1.1)}$$

$$\delta^t + By^t \geqq z^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) + \omega^{t+1} \ (\forall t); \quad \text{(A1.2)}$$

where $z^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) \equiv z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r})$ is the aggregate consumption demand at each t;

$$\delta^t + Ay^t \leqq \omega^t \ (\forall t); \quad \text{(A1.3)}$$

$$\text{and } Ly^t \leqq \omega_l^t \ (\forall t). \quad \text{(A1.4)}$$

In the above definition, the excess demand condition in commodity markets is given by (A1.2). In each period t, the aggregate consumption demand vector is given by $z^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) = z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r})$. It may contain some zero components. For a commodity i such that $z_i^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) = 0$, it follows that in equilibrium, $\delta_i^t + y_i^t \geqq \omega_i^{t+1}$. In the above inequality of excess demand condition (A1.2), y^t is the gross output vector which is planned by generation $t-1$ at the beginning of period t and is harvested at the end of this period, while δ^t is the commodity bundle purchased by generation $t-1$ at the beginning of period t and is sold by generation $t-1$ at the end of period t.

In each period t, the capital market equilibrium condition is given by (A1.3) of Definition A1. Note that the choice between speculative investment δ^t and productive investment Ay^t is given by generation $t-1$ at the beginning of the old age. Moreover, the saving of the commodity bundle ω^t is given by generation $t-1$ at the end of the young age.

In each period t, the labor market equilibrium condition is given by (A1.4) of Definition A1. Note that the aggregate labor demand Ly^t is chosen by generation $t-1$ at the old age, while the aggregate labor supply ω_l^t is given by generation t at the young age.

A specific long-period feature of competitive equilibrium is given as a steady state equilibrium, in which all investment activities are simply of the replacement type. In such a case, given a pair of sequence of price vectors $(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) \equiv \{(p_t,w_t,r_t)\}_{t\geq 0}$ such that $(p_t,w_t,r_t) = (p,w,r)$ for every period t, the solution $(z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}), z_a^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}))$ of the generation $t = 1,2,\ldots,$ to the optimization problem MP^t can be represented by $z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) = z_b(p,w,r)$ and $z_a^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) = z_a(p,w,r)$ for each generation t. Correspondingly, the aggregate demand function $z^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) = z_b^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r})$ for each period t can be represented by $z^t(\boldsymbol{p},\boldsymbol{w},\boldsymbol{r}) = z(p,w,r)$. Such an equilibrium is given as follows.

Definition A2: A *steady state equilibrium* under the overlapping economy $E = \langle (A, B, L); \omega_l; u \rangle$ is a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ associated with

$$\left\{ \left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \right) \right\}_{t \geq 0},$$

such that there exists a profile of a stationary price vector (p, w, r), a gross output vector $y \geq \mathbf{0}$, and a speculative activity vector $\delta \geqq \mathbf{0}$, satisfying $(p_t, w_t, r_t) = (p, w, r)$, $y^{t+1} = y$, $\delta^{t+1} = \delta$, $\omega^{t+1} = Ay + \delta$, $z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b(p, w, r)$, and $z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a(p, w, r)$ for each t, and the following conditions hold:

$$\begin{aligned} pB &\leqq (1 + r) pA + wL; \text{ (A.a)} \\ By + \delta &\geqq z(p, w, r) + \omega, \text{ (A.b)} \\ \text{where } z(p, w, r) &= z_b(p, w, r) + z_a(p, w, r); \text{ and} \\ Ly &\leqq \omega_l, \text{ (A.c)} \\ Ay + \delta &= \omega, \text{ (A.d).} \end{aligned}$$

In the above definitions of two types of equilibria, the choice between speculative investment δ^{t+1} and productive investment Ay^{t+1} is the consequence of each generation's optimal action in the program MP^t. Therefore, $\delta = \mathbf{0}$ can be optimal under the steady state equilibrium whenever the equilibrium interest rate r is non-negative.

To see the last point, let us consider under what conditions the market equilibrium without speculative activity, $\delta^t = \mathbf{0}$ $(\forall t)$, holds in general. Note that if the entire monetary wealth $p_{t-1}\omega^t$ of generation $t - 1$ is used for the productive investment, it would earn $(1 + r_t) p_{t-1}\omega^t$, while if it is used for the speculative investment, it would earn $p_t\omega^t$. Therefore, the productive investment of all monetary wealth is an optimal action for generation $t - 1$ at the beginning of old age if and only if $(1 + r_t) p_{t-1}\omega^t \geqq p_t\omega^t$. In general, if

$$(1 + r_t) p_{t-1} \geqq p_t$$

holds for each period $t = 1, \ldots$, then $\delta^t = \mathbf{0}$ is an optimal action for each generation $t - 1$ at the beginning of old age. Thus, under the steady state equilibrium, this inequality condition holds automatically, since $(1 + r) p \geqq p$ holds whenever $r \geqq 0$.

In contrast, under a stationary price system associated with $r < 0$, each generation would devote all of their wealth to speculative investment. Then, no production takes place in any period, and therefore no consumption good can be supplied in any period. Thus, if a steady state equilibrium is associated with $r < 0$, it would only be a trivial one. Since we are interested in the non-trivial case of equilibria, we focus on the case with $r \geqq 0$ as well as $\delta = \mathbf{0}$ throughout the analysis of this paper.

Thus, we can introduce a special case of a steady state equilibrium, which is defined as follows.

Definition A3: A *steady state equilibrium* $((p, w, r), y, \omega)$ under the overlapping economy $E = \langle (A, B, L); \omega_l; u \rangle$ is called *Walras-von Neumann* (**W-N**) if and only if $(p, w, 1 + r) > \mathbf{0}$ with $r \geqq 0$ and condition (A.c) in Definition A2 holds in equality.

Thus, given a steady state price vector $(p, w, r) \in \mathbb{R}_+^n \times \mathbb{R}_{++} \times \mathbb{R}_+$, each and every generation t at young age is faced with the following optimization program MP:

$$\max_{\omega, z_b, z_a} u(z_b, z_a)$$

subject to

$$\begin{aligned} pz_b + p\omega &= w\omega_l, \text{ and} \\ pz_a &= (1+r)\, p\omega. \end{aligned}$$

Let $z_b(p, w, r)$ and $z_a(p, w, r)$ be the Marshallian demand functions defined as optimal solutions to the above problem MP at the price system (p, w, r).

Finally, **Definition A3** can be reduced to **Definition 2** in Section 2 of the main text of the paper by removing ω from the list of optimal actions at the steady-state price vector, since the equilibrium choice of ω is automatically fixed whenever the equilibrium activity vector y is determined, according to condition (A.d) of **Definition A2**.

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