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# Sraffian indeterminacy of steady-state equilibria in the Walrasian general equilibrium framework

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# Sraffian indeterminacy of steady-state equilibria in the Walrasian general equilibrium framework*

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**Abstract**

In contrast to Mandler's (1999a; Theorem 6) generic determinacy of the steady-state equilibrium, we first show that any regular Sraffian steady-state equilibrium is *indeterminate* in terms of Sraffa (1960) under a simple overlapping generation economy with a fixed Leontief technique. We also check that this indeterminacy is *generic*. These results are obtained by explicitly introducing a simple model of every generation's utility function and individual optimization program to the overlapping generation economy, which also explains the main source of the difference between our results and Mandler (1999a; section 6). We also argue the distinctiveness of our results in comparison with the standard literature, like Calvo (1978), of overlapping generations indeterminacy.

*JEL Classification Code*: B51, D33, D50.

*Keywords*: Sraffian indeterminacy; factor income distribution; general equilibrium framework

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# 1 Introduction

It is well-known that Sraffa's (1960) system of equilibrium price equations contains one more unknown than equation, which leads to the indeterminacy of the steady-state equilibrium. This Sraffian indeterminacy has been regarded as a basis to argue that some non-market-competitive force is indispensable to determine the factor income distribution between capital and labor.

Mandler (1999a) critically examined Sraffian indeterminacy by embedding the Sraffian system of price equations in a Walrasian general equilibrium framework. First, Mandler (1999a, section 3) confirms the generic indeterminacy of non-stationary equilibria within a sequential equilibrium framework with fixed production coefficients and the price inelastic supply of endowments. Second, Mandler (1999a, section 6) shows that steady-state equilibria are generically determinate, given overlapping generations economies with a fixed labor endowment and endogenous supplies of capital goods.[1] He then claims that his sequential equilibrium approach to Sraffian indeterminacy, developed in Mandler (1999a, section 3), is the sole possible way of defending Sraffa's idea of explaining income distribution by social and institutional conditions.

Fratini and Levrero (2011) criticize Mandler's sequential equilibrium approach to Sraffian indeterminacy, and argue that Mandler's analysis would bring us back to methodological questions about the Walrasian intertemporal general equilibrium theories and eventually justify Sraffa's rediscovery of the surplus approach of the classical economists and Marx to value and distribution. The surplus approach views distribution as the result of social conditions that are more fundamental than those determining relative prices, rather than explaining it on the basis of the substitution principle among factors and goods.

In this paper, putting such a methodological debate aside, we address Mandler's way of defending Sraffa's view of income distribution by re-examining Mandler's (1999a, section 6) generic determinacy of steady-state equilibria. That is, by constructing a simple model of overlapping generations economies and introducing the same definition of steady-state equilibrium as Mandler (1999; section 6, p. 705), we show that a steady-state equilibrium is generically indeterminate, unlike the result of Mandler (1999a; section 6). It may suggest that Sraffa's view of income distribution can be deemed valid even when it is embedded in the Walrasian intertemporal general equilibrium framework.

In the rest of this paper, section 2 provides a brief review of the literature on indeterminacy in Walrasian general equilibrium theory. Section 3 introduces a simple model of overlapping generation economies and defines the steady-state equilibrium. Then, section 4 argues for the generic indeterminacy of such an equilibrium, contrary to Mandler's (1999a; section 6) conclusion. Finally, section 5 provides concluding remarks. The general existence theorem of a steady-state

[1] As Mandler (1999a, p. 699) himself points out, the Walrasian system of general equilibrium has an inherent problem of overdetermination: when the endowment of reproducible means of production is arbitrarily given, the system of equations is overdetermined under the uniform rate of profit. For further details of the implications of this issue, see Eatwell (1999), Garegnani (1990), and Petri (2004).

equilibrium is provided in the Appendix.

# 2 A brief literature review

## 2.1 Generic determinacy of Walrasian equilibria in static economic models

In the historic work by Debreu (1970),[2] it was proven that exchange economies have only a finite number of Walrasian equilibria. This means that the Walrasian equilibrium prices and allocations change smoothly as a function of the parameters representing economic environments, so that agents in large economies can have only a negligible effect on equilibrium prices; i.e. there is no longer an incentive for market manipulation. For production economies, Mas-Colell (1975) and Kehoe (1980, 1982) established generic determinacy for constant returns to scale technologies and for linear activity analysis. This implies that determinacy is now generic with almost any type of technology regardless of inelasticity in factor supply. For the model of an incomplete market with a nominal asset, generic determinacy is established in Geanakoplos and Plemarchakis (1987) and Balasko and Cass (1989).

## 2.2 Generic indeterminacy of Walrasian equilibria in intertemporal economic models

However, generic indeterminacy of Walrasian equilibria has also been verified in some intertemporal economic frameworks. At least two streams of research on such indeterminacy would be relevant to this paper. One stream of research is to consider indeterminacy of sequential Walrasian equilibria in finite-horizon intertemporal economies with finite number of finitely lived individuals. Another stream is to consider infinite-horizon intertemporal economies with infinite number of finitely lived and overlapping generations (OLGs hereafter).

Note that, even in infinite-horizon intertemporal economies with finite number of infinitely lived individuals, there is some literature on local indeterminacy of equilibrium paths converging to a steady state, which is shown to exist under economies with some degree of market imperfections.[3 4] Here, a further investigation will not be devoted along this line of research as we focus on Walrasian equilibria under perfectly competitive economies.

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[2] For further details, see Mandler (1999b)

[3] For instance, Benhabib and Farmer (1994) and Behhabib and Nishimura (1998). See also Nishimura and Venditti (2006) for a useful survey of these works.

[4] Instead, whenever such economies are under perfect competition, a Walrasian equilibrium path is generically determinate as argued in Kehoe and Levine (1985, section 2).

### 2.2.1 Generic indeterminacy of non-stationary prices in sequential equilibrium

Mandler (1995) shows the genericity of sequential indeterminacy. Using Radner's (1972) method to decompose an intertemporal equilibrium into a sequential one, the second period production activities can be fixed in the second period continuation equilibrium by the vector of factors endowed and produced in the first period. Therefore, the continuation equilibrium condition consists of the second-period equilibrium price equations and the equations of the second-period excess demand condition for consumption goods, where the only unknown variables are the second-period prices of consumption goods and factors. Under this structure, it is shown that if an intertemporal equilibrium has fewer activities using positively priced second period factors than the number of those factors (implicitly degenerated), then there is a generic set of economies such that the continuation equilibrium of almost every induced second-period economy is indeterminate; and if it is not implicitly degenerated, then the continuation equilibrium of almost every induced second-period economy is regular. In Mandler (1997), the determinacy of both the intertemporal equilibria and the endogenously generated second period equilibria is verified under differentiable production technology.

This conclusion of generic sequential indeterminacy results from the assumptions of linear activities, the production of a fixed quantity, and the investment of part of the first period products into the second period production. Given the same features, Mandler (1999a) investigated Sraffa's indeterminacy claim for an equilibrium with a non-stationary price vector. This equilibrium is defined by the zero-profit condition for a non-stationary price vector and the excess demand conditions for commodities and factors by reflecting Hahn's (1982) criticism of stationary prices and the lack of the demand side in the original model of Sraffa (1960). It has a similar structure to the above-mentioned sequential second-period equilibrium, in that the equilibrium production activity vector is non-arbitrarily fixed by the endowment vector of factors given at the beginning of this period.[5] The only unknown variables in the system of equilibrium equations for the zero-profit and excess demand conditions are the prices of commodities, a wage rate, and an interest rate. In such an equilibrium, the Sraffian indeterminacy is observed whenever the total number of commodity inputs, labor, and financial capital with positive prices is greater than the number of activities used in production. In particular, the former is $n+1$ while the latter is $n$ in a simple Leontief production model with no alternative technique nor joint production, so that one-dimensional indeterminacy is generically observed, as shown by Mandler (1999a, section 3).

[5] Fratini and Levrero (2011) raise a methodological question about the Walrasian intertemporal general equilibrium approach to Sraffian indeterminacy, by pointing out that the sequential indeterminacy occurs due to the non-arbitrary determination of capital endowments via an *ad hoc* use of the future price expectations: such expectations were formed in the first period on the ground of an Arrow-Debreu complete market equilibrium, but will generally be wrong as the equilibrium prices of the second period will be indeterminate, and thus generally different from the expected ones.

### 2.2.2 Generic indeterminacy of non-stationary equilibrium price paths in overlapping generation economies

Generic indeterminacy of Walrasian equilibrium has been also observed for overlapping generation (OLG hereafter) economies, in that the set of locally stable equilibrium price paths converging to a steady-state equilibrium price vector constitutes a continuum, though the steady-state is locally unique. Here, such indeterminacy is associated with the arbitrariness of some initial conditions in equilibrium paths. A typical example of such conditions is the existence of fiat money or the existence of a nonzero stock of nominal debt in the initial period. See Kehoe and Levine (1985, 1990) for a more detailed discussion.

Among many works for such economies, there are a few that may have some links with the analysis in this paper. First, Geanakoplos and Plemarchakis (1986, Proposition 2) consider an one-sector OLG production economy, where not only two-periods consumption plans but also real money balances enter into each generation's utility function, and then show that one-dimensional indeterminacy of Walrasian equilibria occurs in such economies. While the indeterminacy is attributed to an arbitrary choice of commodity price at the initial period, the equilibria entail holding positive monetary balances due to such a utility function: all of total savings are not necesarily devoted to finance productive investments, even if the return by productive investment is higher than that by holding money. Such a feature could be relevant to Mandler's (1999a, section 6) work regarding the generic determinacy of steady-state equilibria in a general OLG production economy, as discussed in more detailed in the following section 3.[6]

Calvo (1978) shows that generic indeterminacy of Walrasian equilibrium is also observed even for a non-monetary OLG production economy with the standard type of preference for two-period consumption plans. In his paper, a two-sector production model is defined, in which one sector produces a consumption good and the other produces a capital good, and it is shown that there is a unique stationary capital stock that is determined independently of any other variables. Then, corresponding to the stationary capital stock, a unique stationary production activity vector can also be specified, given a fixed labor endowment. Assuming that the economy is on the path of the stationary capital stock, Calvo (1978) shows that there exist a stationary equilibrium price vector and one-dimensional continuum of equilibrium price sequences, each of which converges to this stationary price vector. Here, the indeterminacy of the equilibrium price path is indexed by the price of the capital good at the initial period.

This result shows that the indeterminacy in OLG production economies arises by a completely different mechanism than the case of sequential indeterminacy in finite-horizon intertemporal economies. This is because, unlike the case of sequential indeterminacy, Calvo's (1978) model exhibits indeterminacy even if the total number of positively priced productive factors (one commodity

[6] Indeed, the possibility of holding positive monetary balances could be a basis of the claim of Mandler (1999a, p. 705) that Walras' law cannot make one equilibrium equation redundant.

input and labor) is not greater than the number of sectors activated in equilibria.

With this result for the model treating capital as a homogeneous reproducible good, Calvo (1978) insists that the indeterminacy of equilibrium prices is independent of the existence of reswitching techniques that are typically observed under economies with capital as a bundle of heterogenous reproducible goods. However, this claim does not refute the Sraffian one-dimensional indeterminacy observed in economies with capital as a bundle of heterogenous reproducible goods, since the Sraffian indeterminacy theory is of steady-state equilibrium prices whereas the one-dimensional indeterminacy in Calvo's (1978) model is observed only for (sequences of) non-stationary equilibrium prices. Moreover, the reswitching techniques would be irrelevant even for the Sraffian indeterminacy of the steady-state equilibrium prices, as they are observed in equilibrium transition of steady-state prices due to technical changes.

### 2.3 Generic determinacy of steady-state equilibrium prices in overlapping generation economies by Mandler (1999a, section 6)

Mandler (1999a, section 6) shows that generic determinacy is observed for steady-state equilibria in an overlapping generation economy with a simple Leontief production model. In this case, no analogical reasoning developed in the above argument of sequential equilibrium can be applied, as the equilibrium production activity vector should be endogenously determined while the price vector of commodity inputs is equal to that of commodity outputs. He then argued that in a long-run OLG setting the number of equilibrium equations and that of unknown variables are identical because none of the market-clearing equations are redundant by means of Walras' law (Mandler, 1999a; section 6; p. 704). Note that he begins with an abstract Marshallian demand function of every generation as the primitive data for the OLG economy, and no explicit information about the underlying economy such as each generation's utility function and her optimization program is provided. Therefore, any further information about each generation's decision-making of investments on productive and non-productive assets is absent except that it would be financed from past savings. As discussed in more detail in the next section, this last point leads Mandler (1999a, section 6) to conclude the generic determinacy of steady-state equilibria.

In contrast to Mandler (1999a, section 6), in the following sections, we will show that one-dimensional Sraffian indeterminacy is generically observed even for the steady-state equilibria in the same OLG setting, by specifying an explicit model of each generation's utilify function and individual optimization program. Our model, augmented with an individual optimization program, does not allow for agents' holding positive monetary balances in equilibrium whenever the return by productive investment is higher than that by holding money, which leads us to derive a (reduced form of) Walras' law that can make one equilibrium equation redundant.

# 3 An overlapping generation economy

A simple overlapping generation model is constructed, in which each generation $t = 1, 2, \dots,$ lives for two periods. Each generation can earn only from labor supply in his youth but in his old age can earn both from labor supply and productive investment of his past saving. Let $\omega_l^b > 0$ (resp. $\omega_l^a \geqq 0$) be the labor endowment of one generation when he is young (resp. when he is old), and so $\omega_l \equiv \omega_l^b + \omega_l^a > 0$. Assume in the following that every generation has a common preference over his lifetime consumption activities, and labor is supplied inelastically for every generation in all of his ages.

There are $n \geqq 2$ commodities which are produced in this economy and used as consumption goods or capital goods, respectively. Let $(A, L)$ be a *Leontief production technique* prevailing in this economy, where $A$ is a $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients and $L$ is a $1 \times n$ positive row vector of direct labor coefficients.

Let $z_b : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^n$ (*resp.* $z_a : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^n$) be a *Marshallian demand function* of every generation $t$ in his youth (*resp.* in his old age) such that for each commodity price vectors $p_t, p_{t+1} \in \mathbb{R}_+^n$, each wage rates $w_t, w_{t+1} \in \mathbb{R}_+$, and a profit factor $1 + r_{t+1} \in \mathbb{R}_+$, $z_k(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) \in \mathbb{R}_+^n$ is a consumption vector purchasable for every generation when his age is $k = b, a$. The demand function $z_k$ is assumed to be *continuously differentiable* and satisfies *homogeneity*: for $k = b, a$,

$$\begin{aligned} z_k(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) &= z_k(\lambda p_t, \lambda w_t, \lambda p_{t+1}, \lambda w_{t+1}, 1 + r_{t+1}) \\ &= z_k(p_t, w_t, \lambda p_{t+1}, \lambda w_{t+1}, \lambda(1 + r_{t+1})) \end{aligned}$$

for any $\lambda > 0$ and every $(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+$; and *Walras' law*. Note that it is not obvious how to define Walras' law without reference to any specific structure of the underlying economy. Here, let us introduce the definition of Walras' law given by Mandler (1999; section 6, p. 704):

$$(1 + r_{t+1})\left(p_t z_b - w_t \omega_l^b\right) + p_{t+1} z_a - w_{t+1} \omega_l^a = 0.$$

When $z_k$ is evaluated at stationary market prices $(p_t, w_t) = (p_{t+1}, w_{t+1}) = (p, w)$ for every $t$, we will use the notation $z_k(p, w, 1 + r)$. Thus, Mandler's (1999; section 6, p. 704) definition of Walras' law can be reduced to the following form:

$$(1 + r)\left(p z_b - w \omega_l^b\right) + p z_a - w \omega_l^a = 0. \quad (\alpha)$$

Let $z(p, w, 1 + r) \equiv z_b(p, w, 1 + r) + z_a(p, w, 1 + r)$ be the aggregate demand function at every period $t$ when the market prices are stationary.

In this way, an *overlapping generation economy* is given by a profile $\left\langle (A, L); \left(\omega_l^b, \omega_l^a\right); (z_b, z_a) \right\rangle$. Let us introduce a steady-state equilibrium in the following way:

**Definition 1 [Mandler (1999, section 6; Definition D6.2)]:** A *steady-state equilibrium* under the overlapping generation economy $\left\langle (A, L); \left(\omega_l^b, \omega_l^a\right); (z_b, z_a) \right\rangle$ is a pair of a stationary price vector $(p, w, 1 + r) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+$ and a gross output vector $y \in \mathbb{R}_+^n$, such that the following conditions hold:

$$
\begin{aligned}
p &\leqq (1+r)\,pA + wL; \text{ (a)} \\
y &\geqq z\,(p, w, 1+r) + Ay, \text{ (b); and} \\
Ly &\leqq \omega_l. \text{ (c)}
\end{aligned}
$$

In particular, a steady-state equilibrium $((p, w, 1+r), y)$ is called *Sraffian* if $r \geqq 0$ and all of the above (a), (b), and (c) hold with equality.

Given that $r \geqq 0$ for a Sraffian steady-state equilibrium, we use, from now on, the notation $((p, w, r), y)$ to represent a Sraffian steady-state equilibrium.

### 3.1 Source of Mandler's (1999a, Theorem 6) generic determinacy of the Sraffian steady-state equilibrium

In the above system of equilibrium equations (a)-(c) with equality, there are $2n+1$ equations while $2n+1$ unknowns, given that one of the $n$ commodities can be chosen as the *numeràire*. Mandler's (1999a, Theorem 6) generic determinacy of the Sraffian steady-state equilibrium is based on the view that none of the $2n+1$ equations can be redundant by means of Mandler's (1999; section 6, p. 704) Walras' law ($\alpha$).

To see this point, let $((p, w, r), y)$ be a solution of the equations (a) and (b) with equality. Let us multiply both sides of the equation (a) from the right by $y \geqq \mathbf{0}$, which implies (a') $py = (1+r)\,pAy + wLy$, while let us multiply both sides of the equation (b) from the left by $p \geqq \mathbf{0}$, which implies (b') $py = pz\,(p, w, 1+r) + pAy$. Then, from (a') and (b'), we have (b") $pz\,(p, w, 1+r) = rpAy + wLy$. On the other hand, Mandler's (1999; section 6, p. 704) Walras' law ($\alpha$) can be rewritten as $(pz\,(p, w, 1+r) - w\omega_l) + r\left(pz_b - w\omega_l^b\right) = 0$, which is further reduced to $(wLy - w\omega_l) + \left(rpAy + r\left(pz_b - w\omega_l^b\right)\right) = 0$ by applying (b"). From the last equation, the equation (c) $wLy - w\omega_l = 0$ automatically follows whenever all the savings of the young are financed to productive investments in this solution: $w\omega_l^b - pz_b = pAy$.

However, as mentioned in section 2.3, the data of the Marshallian demand functions $(z_b, z_a)$ in the underlying economy $\left\langle (A, L); \left(\omega_l^b, \omega_l^a\right); (z_b, z_a)\right\rangle$ would be too abstract to identify whether an underlying individual optimization can achieve $w\omega_l^b - pz_b = pAy$ or not. It may allow us to consider even a case that a portion of the savings would be devoted to non-productive investments: $w\omega_l^b - pz_b > pAy$. Indeed, as in Genekoplos and Polemarchakis (2006), it would be generally possible that the return by holding money is indifferent to the return by holding capital for productive investments in (no steady-state) equilibrium.[7] Thus, Mandler (1999a, section 6) concludes that Walras' law ($\alpha$) (Mandler, 1999a; section 6; p. 704) can make none of the $2n+1$ equations redundant, which leads to generic determinacy of the Sraffian steady-state equilibrium.

[7] However, it is questionable whether such indifference relation could actually follow under a steady-state equilibrium with $r > 0$, given a deterministic framework with no uncertainty and no contingent markets, like this paper's model.

## 3.2 An explicit model of all generations' utility function and individual optimization program

In contrast to Mandler's (1999a, section 6) approach, we will introduce, in the following arugment, an explicit model of all generations' utility function and individual optimization program, where individual optimal actions would not finance non-productive investments whenever the return by productive investments is higher than that by non-productive investments.

Let $u : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function of lifetime consumption activities, which is common to all generations. As usual, $u$ is assumed to be continuous and strongly monotonic. In the whole of the following discussions, let $\omega^a_l = 0$ and so $\omega_l = \omega^b_l$ to refuse unessential complication. Thus, an *overlapping generation economy* is given by a profile $\langle (A, L); \omega_l; u \rangle$.

For each period $t$, let $p_t \in \mathbb{R}^n_+$ represent a vector of *prices* of $n$ commodities prevailing at the end of this period; $w_t \in \mathbb{R}_+$ represent a *wage rate* prevailing at the end of this period; and $r_t \in \mathbb{R}_+$ represent an *interest rate* prevailing at the end of this period. Assume also, for each generation $t$, that $l^t \in \mathbb{R}_+$ represents $t$'s labor supplied at the beginning of their youth; $\omega^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle for the purpose of saving monetary value $p_t \omega^{t+1}$, which will be chosen by generation $t$ at the end of their youth and will be used in their old age; $\delta^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle purchased for the purpose of speculative activities by generation $t$ at the beginning of their old age; $y^{t+1} \in \mathbb{R}^n_+$ represents a production activity vector decided by generation $t$ at the beginning of their old age; $z^t_b$ is the consumption bundle consumed by the generation $t$ in their youth; and $z^t_a$ is the consumption bundle consumed by generation $t$ in their old age.

Each generation $t$ in their youth is faced with the following optimization program $MP^t$: for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$,

$$\max_{l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z^t_b, z^t_a} u\left(z^t_b, z^t_a\right)$$

subject to

$$\begin{aligned} p_t z^t_b + p_t \omega^{t+1} &\leqq w_t l^t, \\ l^t &\leqq \omega_l, \\ p_t \delta^{t+1} + p_t A y^{t+1} &= p_t \omega^{t+1}, \text{ and} \\ p_{t+1} z^t_b &\leqq p_{t+1} \delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}. \end{aligned}$$

That is, each generation $t$ can supply $l^t$ amount of labor in her youth as a worker employed by generation $t-1$. From the wage income $w_t l^t$ earned at the end of her youth, she can save $p_t \omega^{t+1}$ amount of money and can purchase a consumption bundle $z^t_b$. By using the saved money $p_t \omega^{t+1}$, generation $t$ at the beginning of her old age can purchase $\delta^{t+1}$ for speculative purposes and can purchase a vector of capital goods $A y^{t+1}$ as a productive investment. As an industrial capitalist, she can employ $L y^{t+1}$ amount of generation $t+1$'s labor. Then, at the end of her old age, she can earn $p_{t+1} \delta^{t+1}$ as the revenue of the speculative investment and can earn $p_{t+1}y^{t+1} - w_{t+1}Ly^{t+1}$ as the return on the productive investment. From these revenues, she can purchase a consumption bundle $z_a^t$.

Let $\left(l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z_b^t, z_a^t\right)$ be a solution to the optimization program $MP^t$ for each generation $t$. At the optimum, all of the weak inequalities in the above constraints should hold with equality, given the assumption of $u$. That is,

$$\begin{aligned} p_t z_b^t + p_t \omega^{t+1} &= w_t l^t, \\ l^t &= \omega_l, \text{ and} \\ p_{t+1} z_a^t &= p_{t+1}\delta^{t+1} + p_{t+1}y^{t+1} - w_{t+1}Ly^{t+1}. \end{aligned}$$

Note that the production activity vector $y^{t+1}$, planned by generation $t$ at the beginning of old age, should satisfy the profit maximization condition. As market prices should satisfy the zero-profit condition in equilibrium, the following condition holds for every period $t+1$, where $t \geq 0$:

$$p_{t+1} \leqq (1 + r_{t+1})\, p_t A + w_{t+1} L.$$

Therefore, the profit maximization condition in equilibrium for every period $t+1$ is represented by:

$$p_{t+1} y^{t+1} = (1 + r_{t+1})\, p_t A y^{t+1} + w_{t+1} L y^{t+1}.$$

Thus, the revenue constraint $p_{t+1} z_a^t = p_{t+1}\delta^{t+1} + p_{t+1}y^{t+1} - w_{t+1}Ly^{t+1}$ of generation $t$ at the end of the old age can be reduced to

$$p_{t+1} z_a^t = p_{t+1}\delta^{t+1} + (1 + r_{t+1})\, p_t A y^{t+1}.$$

Given a sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$, let $\left(z_b^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right), z_a^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right)\right)$ be a solution of the generations $t = 1, 2, \ldots,$ to the problem $MP^t$ of utility maximization under the budget constraint. Then, a competitive equilibrium can be formulated as follows.

**Definition 2:** A *competitive equilibrium* under the overlapping generation economy $\langle (A, L); \omega_l; u \rangle$ is a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$ and sequence of each generation's optimal actions $\left\{\left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right), z_a^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right)\right)\right\}_{t \geq 1}$ satisfying the following conditions:

$$\begin{aligned} p_t &\leqq (1 + r_t)\, p_{t-1} A + w_t L \;\; (\forall t)\,; \;\; (1.1) \\ \delta^t + y^t &\geqq z^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right) + \omega^{t+1} \;\; (\forall t)\,; \;\; (1.2) \\ \text{where } z^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right) &\equiv z_b^t\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right) + z_a^{t-1}\left(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}\right) \text{ is the aggregate consumption demands at each } t; \\ \delta^t + Ay^t &\leqq \omega^t \;\; (\forall t)\,; \;\; (1.3) \\ \text{and } Ly^t &\leqq \omega_l^t \;\; (\forall t)\,. \;\; (1.4) \end{aligned}$$

In the above definition, the excess demand condition in commodity markets is given by (1.2). In each period $t$, the aggregate consumption demand vector is given by $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$. It may contain some zero components. For commodity $i$ such that $z_i^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = 0$, it follows that in equilibrium, $\delta_i^t + y_i^t \geqq \omega_i^{t+1}$. In the inequality of excess demand condition (1.2) above, $y^t$ is the gross output vector which is planned by generation $t-1$ at the beginning of period $t$ and is harvested at the end of this period, while $\delta^t$ is the commodity bundle purchased by generation $t-1$ at the beginning of period $t$ and is sold by generation $t-1$ at the end of period $t$.

In each period $t$, the capital market equilibrium condition is given by (1.3) of Definition 2. Note that the choice between the speculative investment $\delta^t$ and the productive investment $Ay^t$ is made by generation $t-1$ at the beginning of old age. Moreover, the bundle of saving commodities $\omega^t$ is chosen by generation $t-1$ at the end of the young age.

In each period $t$, the labor market equilibrium condition is given by (1.4) of Definition 2. Note that the aggregate labor demand $Ly^t$ is chosen by generation $t-1$ in their old age, while the aggregate labor supply $\omega_l^t$ is given by generation $t$ at the young age.

Now, it can be seen that a steady-state equilibrium presented in Definition 1 is a specific case of competitive equilibria given in Definition 2.

**Definition 1*:** A *steady-state equilibrium* under the overlapping economy $\langle (A, L); \omega_l; u \rangle$ is a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ associated with

$$\left\{ \left( \omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \right) \right\}_{t \geq 1},$$

such that there exists a profile of a stationary price vector $(p, w, r)$, a gross output vector $y \geq \mathbf{0}$, and a speculative activity vector $\delta \geqq \mathbf{0}$, satisfying $(p_t, w_t, r_t) = (p, w, r)$, $y^{t+1} = y$, $\delta^{t+1} = \delta$, $\omega^{t+1} = Ay + \delta$, $z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b(p, w, r)$, and $z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a(p, w, r)$ for every $t$, and the inequalities (a), (b), and (c) in Definition 1 hold true.

In particular, a steady-state equilibrium is called *Sraffian* if all of (a), (b), and (c) in Definition 1 hold in equality.

In the above definitions, we simply assume that the individual choice between speculative investment $\delta^{t+1}$ and productive investment $Ay^{t+1}$ is the consequence of each generation's optimal action in the program $MP^t$. We impose no restriction on that choice. Therefore, even for Definition 1*, $\delta > \mathbf{0}$ seems to be compatible with such equilibrium notions. However, we will see that $\delta = \mathbf{0}$ should hold under the steady-state equilibrium whenever the equilibrium interest rate $r$ is positive.

To see this last point, let us consider under what conditions in general the market equilibrium holds with no speculative activity, $\delta^{t+1} = \mathbf{0}$ $(\forall t)$. Note that if the whole monetary wealth $p_t \omega^{t+1}$ of generation $t$ is used for productive investment, she would earn $(1 + r_{t+1}) p_t \omega^{t+1}$, while if it is used for speculative investment, she would earn $p_{t+1} \omega^{t+1}$. Therefore, allocating her whole monetary wealth to productive investment is an optimal action for generation $t$ at the beginning of her old age if and only if $(1 + r_{t+1}) p_t \omega^{t+1} \geqq p_{t+1} \omega^{t+1}$. In general, if

$$(1 + r_{t+1}) p_t \geqq p_{t+1}$$

holds for every period $t \geq 0$, then $\delta^{t+1} = \mathbf{0}$ is an optimal action for every generation $t$ at the beginning of the old age. Thus, under the steady-state equilibrium, this inequality condition holds automatically, as $(1 + r) p \geqq p$ holds whenever $r \geqq 0$. However, if $r = 0$, then the generation is indifferent between speculative investment and productive investment, and so $\delta \geq \mathbf{0}$ may constitute a steady-state equilibrium associated with $r = 0$. In contrast, if $r > 0$, then the productive investment is strictly preferred to the speculative investment for every generation $t$ under the steady-state equilibrium. Thus, $\delta = \mathbf{0}$ should hold under the steady-state equilibrium whenever $r > 0$.

# 4 Indeterminacy of the Sraffian steady-state equilibrium

In this section, we show that a Sraffian steady-state equilibrium is generically indeterminate. Firstly, again following Mandler (1999a), let us formulate the notion of indeterminacy in this model.

**Definition 3** (Mandler (1999a)): Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above. Then, a Sraffian steady-state equilibrium $((p, w, r), y)$ under this economy is *indeterminate* if for any $\varepsilon > 0$, there is a Sraffian steady-state equilibrium $((p', w', r'), y')$ such that $(p', w', r') \neq (p, w, r)$ and $\|(p', w', r') - (p, w, r)\| < \varepsilon$.

It should be worth emphasizing that indeterminacy of a Sraffian steady-state equilibrium requires a continuum of Sraffian steady-state equilibria including this particular one. Such a continuum could be represented by (a part of) the *wage-interest rate curve* derived from the Leontief technique $(A, L)$, which constitutes a well-known linear line connecting between the point of zero interest rate-the maximal wage rate and the point of the maximal interest rate-zero wage rate if the *numeràire* is defined by the *standard commodity* of Sraffa (1960).

Definition 3 is quite distinctive from the standard OLG indeterminacy like one discussed in Calvo (1978). In the standard notion, a steady-state equilibrium is deemed indeterminate if there is a continuum of equilibrium paths of non-stationary prices, each of which converges to this steady-state. Thus, a steady-state equilibrium studied in Calvo (1978) is deemed indeterminate in the standard sense, but it is determinate in terms of Definition 3.

Let the profile $((p, w, r), y)$ be a Sraffian steady-state equilibrium. It can be shown that it is indeterminate. To see this point, let us examine the system of equations that characterizes the Sraffian steady-state equilibrium, which is given as follows:

$$\begin{aligned} p &= (1+r)\, pA + wL;\ (1) \\ y &= z\,(p,w,r) + Ay;\ (2) \text{ and} \\ Ly &= \omega_l.\ (3) \end{aligned}$$

Note that (1) has $n$ equations, (2) has $n$ equations, and (3) has one equation. In contrast, there are $n$ unknown variables regarding the vector $y$ and there are $(n-1)+2$ unknown variables regarding $(p,w,r)$, assuming hereafter that commodity $n$ is selected as the *numeràire*. Therefore, there are $2n+1$ unknown variables in the system of $2n+1$ equations. However, we can decrease the number of equations using Walras' law. Based on this fact, we can show the indeterminacy of the Sraffian steady-state equilibrium in terms of Definition 3.

Given a Sraffian steady-state equilibrium $((p,w,r)\,,y)$, define $\bar{p} \equiv (\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, 1)$ and the associated system of equilibrium equations as follows:

$$F(\bar{p},w,r,y) \equiv \begin{bmatrix} z(p,w,r) - [I-A]\,y \\ (\bar{p} - (1+r)\bar{p}A - wL)^T \end{bmatrix}.$$

By the definition of Sraffian steady state-equilibrium, $F(\bar{p},w,r,y) = 0$ holds. Note that the mapping $F$ does not contain the counterpart of equation (3). This is because the equation (3) is shown to be redundant, as discussed below in the proof of Theorem 1. Therefore, let us introduce the notion of regular equilibria by means of this $F$.

**Definition 4** (Mandler (1999a)): Let $\langle (A,L)\,; \omega_l; u\rangle$ be an overlapping generation economy as specified above. Then, a Sraffian steady-state equilibrium $((p,w,r)\,,y)$ under this economy is *regular* if the Jacobian of $F(\bar{p},w,r,y) = 0$ has full row rank.

Now, we are ready to argue the indeterminacy of Sraffian steady-state equilibria, which is summarized as follows:

**Theorem 1:** Let $\langle (A,L)\,; \omega_l; u\rangle$ be an overlapping generation economy as specified above, and let $((p,w,r)\,,y)$ be a Sraffian steady-state equilibrium under this economy. Then, it is *indeterminate* whenever it is regular.

**Proof.** First, let us show that the equation (3) is redundant by means of Walras' law. In the overlapping generation economy, Walras' law is generally given by the following equation:

$$\left[p_t\left(z_b^t + z_a^{t-1}\right) + p_t\omega^{t+1}\right] - \left[p_t\delta^t + (1+r_t)\,p_{t-1}Ay_t + w_t\omega_l^t\right] = 0,\ \ (4)$$

which is derived from the aggregation of $p_t z_b^t + p_t\omega^{t+1} - w_t\omega_l^t = 0$ and $p_t z_a^{t-1} - p_t\delta^t - (1+r_t)\,p_{t-1}Ay^t = 0$. Moreover, (4) can be reduced to the following form under stationary prices:

$$\left[p\left(z_b^t + z_a^{t-1}\right) + p\omega^{t+1}\right] - \left[p\delta^t + (1+r)\,pAy_t + w\omega_l^t\right] = 0.\ \ (4a)$$

Note that (4a) can be rewritten to the following form:

$$\left[p\left(z_b^t + z_a^{t-1}\right) + pAy_{t+1} + p\delta^{t+1}\right] - \left[p\delta^t + (1+r)\,pAy_t + w\omega_l^t\right] = 0. \quad (4b)$$

As $z_b^t = z_b$, $z_a^{t-1} = z_a$, and $y_{t+1} = y_t = y$ hold for every $t$ under the steady-state, (4b) can be reduced to

$$\left[p\left(z_b + z_a\right) + p\delta^{t+1}\right] - \left[p\delta^t + rpAy + w\omega_l\right] = 0. \quad (4b^*)$$

Furthermore, $\delta_{t+1} = \delta_t = \delta$ also holds for every $t$ under the steady-state. Indeed, $\omega^{t+1} = \omega^t = \omega$ holds in the steady-state. Thus, as $\delta_t + Ay_t = \omega$ holds for every $t$ whenever $p > \mathbf{0}$, $y_{t+1} = y_t = y$ implies $\delta_{t+1} = \delta_t = \delta$. Finally, $p > \mathbf{0}$ follows from the definition of Sraffian steady-state equilibrium prices (1), given the assumption of productive and indecomposable $A$ and the positivity of $L$. Thus, (4b*) can be reduced to

$$p\left(z_b + z_a\right) - \left[rpAy + w\omega_l\right] = 0. \quad (4c)$$

Let us take a profile $((p, w, r), y)$ satisfying the system of equations (1) and (2). From (2), we have

$$\begin{aligned} py &= pz\left(p, w, r\right) + pAy \quad (5) \\ \text{where } z\left(p, w, r\right) &= z_b\left(p, w, r\right) + z_a\left(p, w, r\right). \end{aligned}$$

By combining (1), (5) can be written as:

$$pz\left(p, w, r\right) = p\left(I - A\right)y = rpAy + wLy. \quad (5a)$$

Note that the profile $((p, w, r), y)$ meets Walras' law (4c), which implies that

$$pz\left(p, w, r\right) = rpAy + w\omega_l. \quad (6)$$

From (5a) and (6), we obtain the equation (3):

$$Ly = \omega_l.$$

Thus, the system of $2n+1$ equations (1), (2), and (3) characterizing the Sraffian steady-state equilibrium $((p, w, r), y)$ can be reduced to the system of $2n$ equations (1) and (2), given the reduced form of Walras' law (4c). Then, since the system of $2n$ equations has $2n+1$ unknown variables, it has freedom of degree one.

If the equilibrium $((p, w, r), y)$ is regular, then the Jacobian matrix of the system of equations (1) and (2) at $((\overline{p}, w, r), y)$ has rank $2n$. Therefore, we can show the indeterminacy of the Sraffian steady-state equilibrium by applying the implicit function theorem (A detailed proof is given in Theorem A2 of Appendix). ■

As mentioned in section 3.1, given the same definition of steady-state equilibrium as Definition 1* in this paper, Mandler (1999a; section 6) argues that such an equilibrium is determinate, which is incompatible with Theorem 1. He reaches this conclusion by the following reasoning: "Due to the way in which $1+r$ appears in Walras' law, the standard argument that one of the equilibrium conditions is redundant is not valid in the present model" (Mandler, 1999a; section 6; p. 705). Indeed, if $r = 0$, then Mandler's (1999; section 6, p. 704) definition of Walras' law ($\alpha$) is reduced to

$$pz\,(p, w, r) - w\omega_l = 0. \quad (\alpha^*)$$

In this case, equation (3) is obtained from equations (1) and (2) and this reduced form ($\alpha$*) of Walras' law, and so can it be redundant.

However, the above proof of Theorem 1 verifies that even if $r > 0$, equation (3) can be redundant. Indeed, remember that $p\delta = 0$ holds under a stationary price vector with $r > 0$. Therefore, noting that $\omega_l = \omega_l^b$ by $\omega_l^a = 0$, the first component $(1 + r)\,(pz_b - w\omega_l)$ of the left hand side of ($\alpha$) implies that all of the residual of the wage revenue after purchasing the young generation's consumption bundle, $w\omega_l - pz_b$, is invested in production activity. Thus, the revenue of the old generation $(1 + r)\,(w\omega_l - pz_b)$ represents the gross return of the productive investment $w\omega_l - pz_b$ with the return rate $r > 0$, that is expended for the consumption of the old generation, $pz_a$. This implies that there exists a production activity $y \geq \mathbf{0}$ such that $pAy = w\omega_l - pz_b$. Therefore, the equation ($\alpha$), which can be rewritten as follows:

$$pz_b + pz_a - w\omega_l + r\,(pz_b - w\omega_l) = 0,$$

is equivalent to

$$pz_b + pz_a - w\omega_l - rpAy = 0,$$

that is (4c). Thus, Mandler's conclusion of generic determinacy is no longer applicable whenever an individual optimization program as in section 3.2 of this paper is explicitly introduced.

## 4.1 Openness and genericity

Next, we examine the openness and genericity of parameter set of economies in which every steady-state equilibrium is regular. The openness and genericity are related to the stability and coverage of indeterminacy in the perturbation of parameters characterizing the set of economies.

For the demand function of two generations $z_b$, $z_a$, labor endowment $\omega_\ell$ and for $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$, define a perturbed demand function with similar form to Mandler (1999a) as

$$z_i(h) \ \equiv z_i^b(h) \ + z_i^a(h)$$

where

$$z_i^b(h) \ \equiv z_{bi}(p, w, r) + \frac{w}{p_i}h_i, \ \ z_i^a(h) \ \equiv z_{ai}(p, w, r) + \frac{w}{p_i}h^o$$

for each $i = 1, 2, ..., n$.

In order to preserve Walras' law and homogeneity, the perturbation of labor endowment is given as $\omega_l(h) \equiv \omega_\ell + \sum_{i=1}^n h_i + \frac{nh^o}{1+r}$.

Now define a function $F$ on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1}, 1)$, $n$ quantity variables $(y_1, y_2, ...y_n)$, and adding the parameter set $(A, L, h)$ to $\mathbb{R}^{2n}$, *i.e.*

$$F : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n^2}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n}$$

such that

$$F(\bar{p}, w, r, y, A, L, h) = \begin{bmatrix} z(h) - [I - A]\, y \\ (\bar{p} - (1+r)\bar{p}A - wL)^T \end{bmatrix}.$$

**Definition 6:** An *economy* is a profile of $(A, L, h)$ where $(A, L)$ is a Leontief production technique, in which $A$ is $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients, $L$ is $1 \times n$ positive row vector of direct labor coefficients, and $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$ is for perturbation.

An economy $(A, L, h)$ is *regular* if every Sraffian steady-state equilibrium $((p, w, r), y)$ is regular, that is, the Jacobian $\boldsymbol{D}F$ has full-rank at $(\bar{p}, w, r, y)$. Denote the set of economies as $P$ and the set of regular economies as $P_R$.

**Theorem 2**: $P_R$ is open and has full measure in $P$.

**Proof.** Before examining whether $P_R$ has full measure, let's first check whether the Jacobian $\boldsymbol{D}F$ has full rank with respect to $p_1, ...p_{n-1}, w, r,\ y_1, ..., y_n$ in order to check the regularity of an equilibrium whenever the economy $(A, L, h)$ has the property that $L$ cannot be the Frobenius eigenvector of $A$. The system of equations above has $2n$ equations and $n+1$ price variables $(p_1, ..., p_{n-1}, w, r)$. Hence, the quantity variables $(y_1, ..., y_n)$ are to be determined simultaneously in the Jacobian. Including perturbed parameters, for any $(A, L, h)$, $\boldsymbol{D}_{(y,\bar{p},w,r)}(F_{A,L,h}(\bar{p}, w, r, y))$ is given by:

$$\begin{bmatrix} [A - I] & \boldsymbol{D}_{\bar{p}} z(h) & \boldsymbol{D}_w z(h) & \boldsymbol{D}_r z(p, w, r) \\ & & & \\ \mathbf{0} & I^*_{n-1} - (1+r)A^T_{-n} & -L^T & -(\bar{p}A)^T \\ & & & \end{bmatrix}$$

where

$$\boldsymbol{D}_{\bar{p}} z(h) = \boldsymbol{D}_{\bar{p}} z(\bar{p}, w, r) - \begin{bmatrix} \frac{w}{p_1^2}(h_1 + h^o) & \mathbf{0} & & \cdots & \\ \mathbf{0} & \frac{w}{p_2^2}(h_2 + h^o) & \mathbf{0} & \cdots & \\ & & \ddots & & \\ \mathbf{0} & \cdots & & \mathbf{0} & \frac{w}{p_{n-1}^2}(h_{n-1} + h^o) \\ \mathbf{0} & \cdots & & & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{D}_w z(h) = \boldsymbol{D}_w z(\bar{p}, w, r) + [\frac{1}{p_1}(h_1 + h^o), \frac{1}{p_2}(h_2 + h^o), ..., \frac{1}{p_{n-1}}(h_{n-1} + h^o), (h_n + h^o)]^T ,$$

$A^T$ is the transpose of $A$ and $A^T_{-n}$ is the $n \times (n-1)$ matrix obtained by deleting the $n$-th column of $A^T$, and

$$I^*_{n-1} = \begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix}.$$

Here, note that the last row of $\boldsymbol{D}_{\bar{p}} z(h)$ is nonzero because the last row of $\boldsymbol{D}_{\bar{p}} z(p, w, r)$ is non-zero. As we observed in the calculation result above, the Jacobian has full rank of $2n$ unless the vectors $\begin{bmatrix} \mathbf{0} & I^*_{n-1} - (1+r)A^T_{-n} & -L^T & -(\bar{p}A)^T \end{bmatrix}$ are linearly dependent. Note that the linear dependence of the vectors is observed only in the exceptional case that $L$ becomes the Frobenius eigenvector of $A$.

The full-measure claim of $P_R$ is proven by the transversality theorem. Let's consider the perturbation of parameters $(A, L, h)$ in $\mathbb{R}^{n^2}_{+} \times \mathbb{R}^n_{++} \times \mathbb{R}^{n+1}$. If 0 is a regular value of $F$ at $(\bar{p}, w, r, y)$ and $\boldsymbol{D}F$ has full rank $2n$ with respect to $(A, L, h)$ in $\mathbb{R}^{n^2}_{+} \times \mathbb{R}^n_{++} \times \mathbb{R}^{n+1}$, then except for a set of $(A', L', h') \in \mathbb{R}^{n^2}_{+} \times \mathbb{R}^n_{++} \times \mathbb{R}^n$ of measure zero, $F_{A,L,h}(\bar{p}, w, r, y) : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^n \to \mathbb{R}^{2n}$ has 0 as a regular value.

Define the Jacobian $\boldsymbol{D}F$ with respect to $(A, L, h)$, which is denoted by $\boldsymbol{D}_{A,L,h}F$, as below:

$$\boldsymbol{D}_{A,L,h}F = \begin{bmatrix} \frac{w}{p_1} & & & 0 & \frac{w}{p_1} & y^T & & & \\ & \ddots & & & & & & & \mathbf{0} \\ & & \frac{w}{p_{n-1}} & 0 & \frac{w}{p_{n-1}} & & \ddots & & \\ & 0 & & w & w & & & y^T & \\ & & & & \mathbf{0} & & (*) & & -wI_n \end{bmatrix}$$

where the row vector $y^T$ is the transpose of $y$, $I_n$ is the $n \times n$ identity matrix, $(*) = -(1+r)[p_1 I_n \; ... \; p_{n-1} I_n \; I_n]$ is $n \times n^2$ matrix. Here, each $p_i I_n$ is an $n \times n$ matrix:

$$p_i I_n = \begin{bmatrix} p_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p_i \end{bmatrix}.$$

The first $n+1$ columns are for $(h_1, ..., h_n, h^o)$, the next $n^2$ columns are for the components of $A$ and the last $n$ columns are for the components of $L$. We can see that the above matrix has full-rank.

As for openness, consider the contrary case. Suppose $P_R$ is not open. Then there exists a sequence $\{(A, L, h)_k\}$ of non-regular economies converging to a regular economy $(A, L, h)_* \in P_R$. Correspondingly, there exists a sequence of non-regular equilibria $\{(\bar{p}, r, w, y)_k\}$ which converges to a regular equilibrium $(\bar{p}, r, w, y)_*$ at $(A, L, h)_*$. Then the corresponding Jacobian matrices $\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k$ of $2n$ rows and $2n+1$ columns exist, which have less than full rank. For a Jacobian matrix, we can pick $2n+1$ separate square submatrices of order $2n$. The determinants of square submatrices of order $2n$ are all zero. Now we can define a continuous function, say $c$, from the set of Jacobian matrices to the set of $2n+1$-dimensional vectors whose components are determinants of square submatrices derived from the Jacobian $\boldsymbol{D}F_{A,L,h}$. Since $c(\boldsymbol{D}F_{A,L,h}) = (0, ..., 0) \in \mathbb{R}^{2n+1}$ for any $\boldsymbol{D}F_{A,L,h}$ of less than full rank, $c(\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k) = (0, ..., 0)_k \to (0, ..., 0) \in \mathbb{R}^{2n+1}$ as $k \to \infty$.

Since $\{(0, ..., 0)_k\}$ converging to $(0, ..., 0)$ is closed in $\mathbb{R}^{2n+1}$ and $c$ is continuous, the inverse image $c^{-1}(\{(0, ..., 0)_k\}) = \{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\}$ is closed. Its elements are Jacobian matrices from $P \backslash P_R$ of less than full rank. Since $\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\}$ is closed, $\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y)_*$ is contained in $\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\}$.

Note that $c(\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y)_*) = (0, ..., 0) \in \mathbb{R}^{2n+1}$. This implies that the converging point of the sequence $\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\}$, each element of which is correspondingly defined from $(A, L, h)_k \in P \backslash P_R$, must also have less than full rank. In other words, the convergent point of the sequence of non-regular economies must also be non-regular. This contradicts our initial assumption. Therefore, the set of regular economies $P_R$ is open. ■

# 5 Concluding Remarks

In the above argument, we have shown that under an overlapping generation production economy with a fixed Leontief technique, generic indeterminacy arises in Sraffian steady-state equilibria. Remember that a Sraffian steady-state equilibrium is indeterminate if there is a continuum of nearby Sraffian steady-state equilibria (Definition 3). This conclusion is strong and remarkable because the nearby equilibria are Sraffian steady states whereas the main literature on overlapping-generations indeterminacy typically finds determinate steady states[8] (and it finds a continuum of equilibrium nearby a steady-state but those nearby equilibria are not steady states), as discussed in sections 2.2.2 and 2.3. This conclusion has been obtained by the following two features of our

[8] Note that Nishimura and Shimomura (2002, 2006) show the existence of a continuum of steady-state equilibria in dynamic Heckshar-Ohlin international economies. However, the generation of this continuum is due to the infinitely many allocations across two countries of a uniquely determined aggregate capital stock associated with a unique steady-state equilibrium price vector, which corresponds to the unique steady-state equilibrium in our terminology.

model: first, providing a reasonable individual optimization program, and second, introducing capital as a bundle of heterogenous reproducible commodities.

As the consequence of the first feature, we have explicitly derived Walras' law from the individual optimization program that can make one of the equilibrium equations redundant, and eventually leads us to the opposite conclusion from Mandler's (1999a; section 6) generic determinacy. This suggests that Marshallian demand functions and a specific form ($\alpha$) of Walras' law in Mandler's (1999a; section 6) approach would be insufficient as the primitive data of the economy for characterizing the OLG equilibria.

The second feature can be explained by comparing our conclusion with Calvo (1978). As discussed in section 2.2.2, Calvo (1978) defines capital as a homogenous reproducible good in a two-sector production model, which makes the system of equilibrium equations completely 'decomposed' into two sub-systems. Then, one of the sub-systems yields the stationary level of capital stock and the corresponding stationary production activities, entirely independent of the price system. With the solution of these variables, the remaining sub-system can be solved for the remaining unknowns (the prices), as the numbers of the equations and of the unknowns in the sub-system are identical. However, if capital is defined as a vector of two reproducible goods, then the system of steady-state equilibrium equations cannot be 'decomposed', and thus the stationary levels of capital goods and the corresponding production activities cannot be solved independently of the price system and the excess demand functions. This would be the source of the opposite conclusions between ours and Calvo (1978).

Given the generic indeterminacy of steady-state equilibria in the simple Leontief production model, a natural next question would be whether this indeterminacy is robust in more general models. There are at least two interesting more general models: a production model with alternative Leontief techniques to represent economies with the possibility of technical changes; and the von Neumann production model of economies with joint production. Note that the discussion developed in section 5 of Mandler (1999a), referring to both of these models, is irrelevant to this robustness question, as it refers only to the sequential equilibria with non-stationary prices, as in section 3 of Mandler (1999a).

For the model with alternative Leontief techniques, it can be verified that the generic feature of one-dimensional indeterminacy of steady-state equilibria is still observed. Moreover, this conclusion still holds even if the number of alternative Leontief production techniques is infinite or uncountable. Therefore, unlike the case of sequential equilibria in Mandler (1997), the differentiability of overall production techniques cannot affect the generic feature of the indeterminacy for the case of steady-state equilibria.

For the von Neumann model, we conjecture that the generic one-dimensional indeterminacy of steady-state equilibria would be still observed in economies with joint production. At the present stage, we leave it for future research.

Finally, as Mandler's (2002) reference to Morishima (1961) indicates, it would also be interesting to investigate and characterize equilibrium paths in infinite-horison intertemporal economies as argued in the turnpike theorems, given that the continuum of Sraffian steady state equilibria exists.

# 6 References


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# 7 Appendix: The Existence of Sraffian Steady-State Equilibrium

In this Appendix, we show that, given an economy $\langle (A, L)\,; \omega_l; u\rangle$, there exists an open subset of available non-negative interest rates such that for every interest rate in this subset, an associated steady-state equilibrium exists. By such an existence theorem, it is ensured that the generic indeterminacy discussed in Theorems 1 and 2 is not an empty claim.

Note that if speculative investment were allowed to be non-zero and non-negative under a steady-state equilibrium, then the commodity market clearing condition (b) in Definition 2 would be given by the following form:

$$y + \delta \geqq z(p, w, r) + Ay + \delta,$$

which is also the reduced form of condition (1.2) in Definition 1.

Finally, given that the utility function is strongly monotonic, $\delta \geq \mathbf{0}$ would appear under the steady-state equilibrium only when the equilibrium interest rate is zero. However, even when the equilibrium interest rate is zero, $\delta = \mathbf{0}$ is still an optimal action. Therefore, without loss of generality, we may focus on the case of no speculative investment when we discuss the indeterminacy of the Sraffian steady-state equilibrium.

With Definition 2, we can obtain the following existence theorem of the Sraffian steady-state equilibrium in this overlapping economy.

**Theorem A1**: Let $\langle (A, L); \omega_l; u \rangle$ be an economy as specified above. Then, there exists a *Sraffian steady-state equilibrium* $((p, w, r), y(p, w, r))$ under this economy.

**Proof.** Let us define $\triangle \equiv \left\{(p, w) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^n p_i + w = 1\right\}$ and $\mathring{\triangle} \equiv \{(p, w) \in \triangle \mid (p, w) > \mathbf{0}\}$. For each $(p, w) \in \triangle$, consider the following opmizaition problem:

$$\max_{(z_b, z_a, y)} u(z_b, z_a)$$

subject to

$$\begin{aligned} pz_b + W &\leqq w\omega_l, \\ pAy &= W, \text{ and} \\ pz_a &\leqq \max\{py - wLy, W\}. \end{aligned}$$

Denote the set of solutions to this optimization problem by $\mathcal{O}(p, w)$.

Take $(z_b(p, w), z_a(p, w), y(p, w)) \in \mathcal{O}(p, w)$. Then,

$$y(p, w) \in \arg\max \left\{ \max_{y \geq \mathbf{0};\ pAy = W} py - pAy - wLy, 0 \right\}$$

holds. It is also shown that the correspondence $\mathcal{O} : \mathring{\triangle} \twoheadrightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ is non-empty, compact and convex-valued, and upper hemicontinuous.

Let us define the excess demand correspondence $\mathcal{D} : \mathring{\triangle} \twoheadrightarrow \mathbb{R}^n$ by

$$\begin{aligned} &\mathcal{D}(p, w) \\ \equiv\ &\{(z(p, w) - (I - A) y(p, w), Ly(p, w) - \omega_l) \mid (z_b(p, w), z_a(p, w), y(p, w)) \in \mathcal{O}(p, w)\}. \end{aligned}$$

It can be shown that this correspondence is non-empty, compact and convex-valued, and upper hemicontinuous. By the strong monotonicity of $u$, the following form of Walras' law holds: for any $(p, w) \in \mathring{\triangle}$ and any $d(p, w) \in \mathcal{D}(p, w)$, $(p, w) \cdot d(p, w) = 0$.

Let us take any price sequence $\{(p^k, w^k)\} \subset \overset{\circ}{\triangle}$ such that $(p^k, w^k) \rightarrow (\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$. Take $d(p^k, w^k) \in \mathcal{D}(p^k, w^k)$ for each $(p^k, w^k)$.

Suppose that $(\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$ with $\overline{w} > 0$. Then, there exists a commodity $i$ such that $\overline{p}_i = 0$. Then, for sufficiently large $k$, $p_i^k$ is sufficiently close to zero. Then, $z_i(p^k, w^k)$ is sufficiently large by the strong monotonicity of $u$. In contrast, $y(p^k, w^k)$ is bounded by the condition $p^k Ay(p^k, w^k) < w^k \omega_l$. Therefore, for sufficiently large $k$, $z_i(p^k, w^k) - y_i(p^k, w^k) + A_i y(p^k, w^k) > 0$ should hold, where $A_i$ is the $i$-th row vector of $A$. Now, let us define $(p', w') \in \overset{\circ}{\triangle}$ such that $(p', w') \equiv \frac{1}{\lambda}(p^k, w^k) - \frac{1-\lambda}{\lambda}(\overline{p}, \overline{w})$ for some sufficiently small $\lambda \in (0, 1)$. Then, $(p', w') \cdot d(p^k, w^k) > 0$ holds as $p_i' [z_i(p^k, w^k) - y_i(p^k, w^k) + A_i y(p^k, w^k)] > 0$ is sufficiently greater.

Suppose that $(\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$ with $\overline{w} = 0$. Then, for sufficiently large $k$, $w^k$ is sufficiently close to zero. Then, $y(p^k, w^k)$ must be sufficiently close to zero vector as $p^k Ay(p^k, w^k) < w^k \omega_l$. Thus, for sufficiently large $k$, $Ly(p^k, w^k) < \omega_l$ should hold. Now, let us define $(p', w') \in \overset{\circ}{\triangle}$ such that $(p', w') \equiv \left(p^k\left(1 + \frac{\varepsilon}{1-w^k}\right), w^k - \varepsilon\right)$ for some sufficiently small $\varepsilon > 0$. Then,

$$\begin{aligned}
& (p', w') \cdot d(p^k, w^k) \\
= & \left(p^k\left(1 + \frac{\varepsilon}{1-w^k}\right), w^k - \varepsilon\right) \cdot \left(z(p^k, w^k) - (I - A)y(p^k, w^k), Ly(p^k, w^k) - \omega_l\right) \\
= & \frac{\varepsilon}{1-w^k} p^k \cdot \left[z(p^k, w^k) - (I - A)y(p^k, w^k)\right] - \varepsilon\left(Ly(p^k, w^k) - \omega_l\right) \\
= & \frac{w^k}{1-w^k} \varepsilon\left(\omega_l - Ly(p^k, w^k)\right) - \varepsilon\left(Ly(p^k, w^k) - \omega_l\right) > 0.
\end{aligned}$$

In summary, we have shown that for any price sequence $\{(p^k, w^k)\} \subset \overset{\circ}{\triangle}$ such that $(p^k, w^k) \rightarrow (\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$, and for any $d(p^k, w^k) \in \mathcal{D}(p^k, w^k)$, there exists $(p', w') \in \overset{\circ}{\triangle}$ such that $(p', w') \cdot d(p^k, w^k) > 0$ for infinitely many $k$. Then, by Grandmont (1977, Lemma 1), there exists $(p^*, w^*) \in \overset{\circ}{\triangle}$ such that $z(p^*, w^*) - (I - A)y(p^*, w^*) = \mathbf{0}$ and $Ly(p^*, w^*) - \omega_l = 0$. Thus, $y(p^*, w^*) = (I - A)^{-1} z(p^*, w^*)$, and so $y(p^*, w^*) > \mathbf{0}$ by the indecomposability of $A$, unless $z(p^*, w^*) = \mathbf{0}$. Since $p^* > \mathbf{0}$ and $w^* > 0$, $z(p^*, w^*) \geq \mathbf{0}$ follows from the strong monotonicity of $u$. Thus, $y(p^*, w^*) > \mathbf{0}$. Then, for $r^* \equiv \frac{p^* y(p^*, w^*) - Ly(p^*, w^*)}{p^* Ay(p^*, w^*)} - 1$, $r^* \geqq 0$ holds from $y(p^*, w^*) \in \arg\max\{\max_{y \geq \mathbf{0};\ p^* Ay = W} p^* y - p^* Ay - w^* Ly, 0\}$. Moreover, it should follow from the optimal behavior and $y(p^*, w^*) > \mathbf{0}$ that

$$p^* = (1 + r^*)\, p^* A + w^* L.$$

Thus, there exists a Sraffian steady-state equilibrium $((p^*, w^*, r^*), y(p^*, w^*, r^*))$ with $y(p^*, w^*, r^*) = y(p^*, w^*)$. ■

Denote the Frobenius eigenvalue of the matrix $A$ by $(1+R)^{-1} \in (0,1)$. Then, by Theorem A1 and Theorem 1, we have the following existence theorem.

**Theorem A2**: Let $\langle (A, L); \omega_l; u \rangle$ be an economy as specified above. Let $((p^*, w^*, r^*), y(p^*, w^*, r^*))$ be a Sraffian steady-state equilibrium, which is regular. Then, there exists an open neighborhood $\mathcal{N}(r^*) \subseteq [0, R)$ of $r^*$ such that there exists a Sraffian steady-state equilibrium

$$((p(r), w(r), r), y(p(r), w(r), r))$$

for every $r \in \mathcal{N}(r^*)$.

**Proof.** Let us define a continuously differentiable function $F : \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times [0, R) \times \mathbb{R}_+^n \to \mathbb{R}^{2n}$ as:

$$F(\overline{p}, w, r, y) = \begin{bmatrix} z(p, w, r) - [I - A]\, y \\ \overline{p}_{-n} - (1+r)\,\overline{p}A_{-n} - wL_{-n} \\ Ly - \omega_l \end{bmatrix}.$$

Let $(p^*, w^*, r^*, y^*)$ be a Sraffian steady-state equilibrium, whose existence is ensured by Theorem A1. Assume it is a regular equilibrium. Then, the Jacobian of $F$ at $(p^*, w^*, r^*, y^*)$ is given by $\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*))$ is given by:

$$\begin{aligned} &\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*)) \\ = &\begin{bmatrix} [A - I] & \boldsymbol{D}_{\overline{p}} z(\overline{p}^*, w^*, r^*) & \boldsymbol{D}_w z(\overline{p}^*, w^*, r^*) & \boldsymbol{D}_r z(\overline{p}^*, w^*, r^*) \\ \mathbf{0} & I_{n-1} - (1+r)A_{-n}^T & -L_{-n}^T & -(\bar{p}A_{-n})^T \\ L & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

As $(p^*, w^*, r^*, y^*)$ is regular, it follows that $rank\left[\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*))\right] = 2n$.[9]

Then, by the implicit function theorem, there exist an open neighborhood $\mathcal{N}(r^*) \subset [0, R)$ of $r^*$ and also an open neighborhood $\mathcal{M}(\overline{p}^*, w^*, y^*) \subset \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+^n$ of $(\overline{p}^*, w^*, y^*)$ such that there exists a continuous single-valued mapping $\eta : \mathcal{N}(r^*) \to \mathcal{M}(\overline{p}^*, w^*, y^*)$ such that for any $r' \in \mathcal{N}(r^*)$, there exists $(\overline{p}', w', y') = \eta(r')$ with $F(\overline{p}', w', r', y') = \mathbf{0}$. By the definition of the mapping $F$, $F(\overline{p}', w', r', y') = \mathbf{0}$ implies that $\overline{p}' \cdot (z(\overline{p}', w', r') - [I - A]\, y') + w' \cdot (Ly' - \omega_l) = 0$. As $\overline{p}'_{-n} = (1+r')\,\overline{p}'A_{-n} + w'L_{-n}$, it also follows that $1 = (1+r')\,\overline{p}'A_n + w'L_n$. Thus, $\overline{p}' = (1+r')\,\overline{p}'A + w'L$ holds, which implies that $(\overline{p}', w', r', y')$ is a Sraffian steady-state equilibrium associated with $r' \in \mathcal{N}(r^*)$. ■

In this way, we can show that for each non-negative interest rate within a subset of $[0, R)$, there exists a Sraffian steady-state equilibrium associated with this interest rate.

[9] As shown in the proof of Theorem 2, the regularity of the equilibrium $(p^*, w^*, r^*, y^*)$ is indeed verified except for a non-generic case that the vector $L$ becomes the Frobenius eigenvector of $A$.