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Natural Implementation with Partially Honest Agents in Economic Environments

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Abstract

In this paper, we introduce the weak and the strong notions of partially honest agents (Dutta and Sen, 2012), and then study implementation by natural price-quantity mechanisms (Saijo et al., 1996, 1999) in pure exchange economies with three or more agents in which pure-consequentialistically rational agents and partially honest agents coexist. Firstly, assuming that there exists at least one partially honest agent in either the weak notion or the strong notion, the class of efficient social choice correspondences which are Nash-implementable by such mechanisms is characterized. Secondly, the (unconstrained) Walrasian correspondence is shown to be implementable by such a mechanism when there is at least one partially honest agent of the strong type, which may provide a behavioral foundation for decentralized implementation of the Walrasian equilibrium. Finally, in this set-up, the effects of honesty on the implementation of more equitable Pareto optimal allocations can be viewed as negligible.

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1 Introduction

A goal of implementation theory is to characterize the class of social choice correspondences (*SCC*s) which could be implemented via some decentralized decision making processes, given various information structures. Maskin (1999) provides a general framework of implementation theory and, given the complete information structure and the (Nash) equilibrium solution concept, discusses a necessary condition and a sufficient condition for (Nash) implementation.[1] This canonical theory of implementation, however, rests on two basic assumptions: 1) There is no restriction on mechanisms for decentralized decision making, and 2) agents act in a pure-consequentialistically rational manner.

The first assumption can be justified whenever the main concern is to draw a demarcation line between what *SCC* is or is not implementable. However, if the theory of implementation is to have any practical meaning, implementable *SCC*s must rely on mechanisms having unquestionable features whose suitability depends greatly on the type of implementation problems at hand (Jackson, 1992). For instance, in the case of resource allocation problems in a market structure, a prominent and natural restriction on mechanisms is represented by *price-quantity mechanisms*, because only information reported by agents in the form of prices and their own quantity demands is required for decision making.

Indeed, a central concern in resource allocation problems of classical economic environments regards the types of resource allocations implemented by the so-called *competitive market*, in which each agent exchanges the information of *prices* and their own *demand quantities* to determine an allocation. This interest has triggered fundamental theoretical contributions which have sharpened the understanding of market mechanisms in two different ways. On the one hand, when individuals in the economy are *sincere*, the Walrasian equilibrium allocations are implemented as being Pareto efficient via a competitive market, as suggested by the general equilibrium theory and the fundamental theorems of welfare economics. Conversely, when agents *strategize*, as Hurwicz (1978), Otani and Sicilian (1982) and Thomson (1984) discuss, the market mechanism is manipulated by such agents so that the equilibrium allocations are identical to the lens-shaped area delimited by the agents' true offer curves, which are not truthfully Walrasian. This latter strand of literature suggests that the existence of a scheme for punishing those manipulating agents is indispensable to the successful operation of the market mechanism. Seminal studies such as Hurwicz (1979), Schmeidler (1980), Postlewaite and Wettstein (1989), and Tian (1992, 2000) design, for implementation of the (constrained) Walrasian equilibrium allocations, a desirable price-quantity mechanism which could be regarded as a formulation of a comprehensive market mechanism consisting of a trading rule (a scheme for the exchange of the information of prices and quantities) and a punishment scheme.

[1] Many studies of Nash implementation have followed, such as Saijo (1988), Moore and Repullo (1990), Dutta and Sen (1991) and Lombardi and Yoshihara (2012a). Hereafter unless otherwise specified "equilibrium" should be understood as Nash equilibrium, and "implementation" as Nash implementation.

Following the criticism of Jackson (1992) and the above mentioned works in economic environments, Dutta et al. (1995), Sjöström (1996), and Saijo et al. (1996, 1999) initiate the theory of natural implementation, which defines the class of natural mechanisms as being appropriate for the allocation problems in economic environments and then characterizes the class of *SCC*s that are implementable by natural mechanisms. A typical example is of a price-quantity mechanism satisfying *individual feasibility*, *balancedness*, *forthrightness*, and the *best response property*; henceforth, *natural* price-quantity mechanism.[2]

The second assumption is that agents behave in a pure-consequentialistically rational manner. In other words, the maximization of material gains is the only intention of agents' actions. This is frequently criticized for excluding honesty as a powerful motivator.[3] In fact, simple reasoning and everyday observation suggest that a concern about honesty is an important determinant of behavior. Furthermore, actual behavior is often the outcome of a compromise between what honesty prescribes and what the pursuit of material gains dictates. Experimental evidence confirms these impressions (see Green et al., 2009).

Acknowledging these criticisms, Dutta and Sen (2012) address the question of implementation in the abstract social choice framework where some of the agents share the virtue of honesty in their decentralized decision making.[4] Formulating an agent having a weak sense of the virtue of honesty as a *partially honest* one, who prefers to be truthful in her preference profile report when a lie does not better serve her material interests, and assuming that there exists at least one such partially honest agent in the society and the mechanism designer does not know her identity, Dutta and Sen (2012) show that in a society with three or more agents, any *SCC* satisfying the no-veto power condition is implementable.[5]

This line of research is particularly relevant if the task at hand is to achieve societal goals via market like-mechanisms. Indeed, "Cutthroat competitiveness in the market can go together with strict adherence to norms of honesty" as Elster (1989, p. 102) states, but the aforementioned literature has neglected the virtue of honesty as a self-imposed standard of conduct, and so has failed to appreciate its influence on the design of market-like mechanisms. There is, therefore, a need to develop decentralized decision making processes that can take into account more sophisticated behavior than has been analyzed up until this point, in order to answer questions such as: What are the economic effects of honesty in the design of market like-mechanisms? Which implementation problems can be resolved through honestly motivated behaviour and which limitations cannot be overcome? This

[2]The definition of natural price-quantity mechanism with these four conditions was firstly given by Saijo et al. (1996).

[3]For instructive discussions on the role of emotions and norms in economics, see Bowles and Gintis (2000), Camerer (2003), Elster (1998), Kreps (1997), Sen (1997) and Suzumura and Xu (2001).

[4]The first seminal work related to this subject is of Matsushima (2008a, 2008b). Moreover, there are seminal related works such as Glazer and Rubinstein (1998), Eliaz (2002), Corchón and Herrero (2004) and Kartik and Tercieux (2012).

[5]A complete characterization is provided in Lombardi and Yoshihara (2013).

paper focuses on classical exchange economies in which pure-consequentialistically rational agents and partially honest agents coexist, limits its analysis to natural price-quantity mechanisms, and devises two different types of honestly motivated behavior.

The first type is a weak notion in that only the announcement of demand quantity is relevant for agents' honest behavior. Thus, a partially honest agent in this weak sense strictly prefers to report her true demand quantity rather than misreport it whenever the consequences of both actions are indifferent in her preference over outcomes, but she is not concerned about honesty in her price announcements. This is a very limited deviation from the standard natural implementation set-up, since the degree of honesty injected into implementation problems is minimized, as in the approach of Dutta and Sen (2012). However, the scope of naturally implementable *SCC*s in such a setting is greatly enlarged in comparison with the standard set-up, as will be argued later. The second type of partial honesty is of a slightly stronger notion than the first, in that the announcement of both price and demand quantity is relevant. Although this notion is not a minimal injection of the virtue of honesty, causal introspection indicates that it is not unrealistic for an agent to have strict preferences for reports that disclose all the true information in her hands when a lie does not lead to any better outcome.

For each of the two notions of partial honesty, this paper provides a full characterization of Nash implementable efficient *SCC*s by natural price-quantity mechanisms when there is at least one partially honest agent in a society with more than two persons. Note that in this paper, as in Saijo et al. (1999), no efficient *SCC* is restricted to interior points. It is therefore possible that *boundary SCC*-optimal allocations exist in some economies. Moreover, since no differentiability of utility functions is assumed, there are *multiple* efficiency price vectors corresponding to an *SCC*-optimal allocation, as in Saijo et al. (1999). Unlike the case of Dutta and Sen (2012) and Lombardi and Yoshihara (2012b),[6] it is not true that every efficient *SCC* is implementable by a natural price-quantity mechanism, even if there is at least one partially honest agent. Indeed, neither of the two representative fair allocation rules such as the *No-envy and efficient correspondence*[7] and the *efficient egalitarian-equivalent correspondence* is implementable by such comprehensive market mechanisms.

Note that the *Walrasian correspondence* does not satisfy Maskin monotonicity (Maskin, 1999), and it is, therefore, not implementable, as shown by Hurwicz et al. (1995). Thus, works on implementation of the Walrasian correspondence such as Postlewaite and Wettstein (1989) and Tian (1992, 2000)[8] replace the Wal-

[6] Lombardi and Yoshihara (2012b; Theorem 1) show that every efficient *SCC* is implementable by a natural price-allocation mechanism whenever there is at least one partial honest agent in the weak sense.

[7] However, this correspondence is Nash-implementable by some simple mechanism, as in Thomson (2005).

[8] Postlewaite and Wettstein (1989) and Tian (1992, 2000) consider the domain restriction in which every Walrasian equilibrium allocation is interior in every admissible economy. Since the Walrasian correspondence and the constrained Walrasian correspondence are identical in a

rasian correspondence with the *constrained Walrasian correspondence* (Hurwicz et al. 1995) as the target of implementation problems. Otherwise, designed mechanisms for implementation of the Walrasian correspondence, such as Hurwicz (1979) and Schemeidler (1980), must be individually *in*feasible,[9] though individual feasibility and balancedness seem to be indispensable for characterizing market-like mechanisms.

However, one of the most striking implications of this paper is that the Walrasian correspondence is implementable by a natural price-quantity mechanism whenever there is at least one partially honest agent. To be more precise, given that there is at least one partially honest agent in the weak sense, the Walrasian correspondence is implementable by such a mechanism in economies with two commodities, though it is not implementable in economies with more than two. In contrast, given that there is at least one partially honest agent in the strong sense, the Walrasian correspondence is implementable by such a mechanism in *all* pure exchange economies.

Combined with the existing works, the above result may provide a behavioral foundation for the theory of the Walrasian equilibrium. Indeed, our main result may indicate that given the presumption of strategic agents, a careful and thoughtful design of comprehensive market mechanisms is insufficient, and that the presence of the virtue of honesty is indispensable for decentralized implementation of the Walrasian equilibrium allocations.

The results of the paper are summarized in Table 1 of section 4 and compared with those found in the conventional framework.

The remainder of the paper is structured as follows: Section 2 describes the formal environment. Section 3 provides the characterization results and explores their consequences. Section 4 contains concluding remarks. The appendix includes proofs omitted from the text.

2 The model

2.1 *Preliminaries*

There are $n \geq 3$ agents in $N \equiv \{1, ..., n\}$ and $\ell \geq 2$ distinct commodities in $L \equiv \{1, ..., \ell\}$. Unless otherwise specified, we assume that the cardinality of L is $\ell \geq 2$. $\mathbb{R}$ is the set of all real numbers; $\mathbb{R}_+$ ($\mathbb{R}_{++}$) denotes the set of all non-negative (positive) real numbers; $\mathbb{R}^\ell$ is the Cartesian product of ordered ℓ-tuples of real numbers, whereas $\mathbb{R}^\ell_+$ ($\mathbb{R}^\ell_{++}$) denotes its non-negative (positive) orthant. Vector inequalities are defined as follows: For all $x, y \in \mathbb{R}^\ell$, $x \geq y$ if $x_\ell \geq y_\ell$ for each $\ell \in L$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_\ell > y_\ell$ for each $\ell \in L$.

such domain, it is not inappropriate to state that they consider Nash implementation of the constrained Walrasian correspondence.

[9] Note that Maskin (1999)'s original framework of implementation problems presumes that every mechanism has an outcome function whose range is the set of feasible social alternatives, which implies that in the case of economic environments, every mechanism must be individually feasible.

Each $i(\in N)$ is characterized by a consumption space $\mathbb{R}_{+}^{\ell}$ (where $x_i = (x_{i1}, ..., x_{i\ell}) \in \mathbb{R}_{+}^{\ell}$ is the i's commodity bundle), by an endowment vector $\omega_i \in \mathbb{R}_{+}^{\ell}$ and by a preference relation defined over $\mathbb{R}_{+}^{\ell}$.[10] We assume that i's preferences have a utility representation $u_i : \mathbb{R}_{+}^{\ell} \to \mathbb{R}$ which is continuous and quasi-concave on $\mathbb{R}_{+}^{\ell}$, and *either* strictly monotonic on $\mathbb{R}_{+}^{\ell}$ *or* strictly monotonic on $\mathbb{R}_{++}^{\ell}$ where the utility of every interior consumption bundle is strictly higher than the utility of any consumption bundle on the boundary. U is the class of all such utility functions, whereas U_i is the class of admissible utility functions for i. Given a profile of endowment vectors, we denote $\sum_{i \in N} \omega_i \equiv \Omega \in \mathbb{R}_{++}^{\ell}$ as the aggregate endowment. It is assumed that the distribution of endowments is known and fixed.

For i, $u_i \in U_i$, and $x_i \in \mathbb{R}_{+}^{\ell}$, $L(x_i, u_i) \equiv \left\{x_i' \in \mathbb{R}_{+}^{\ell} \mid x_i' \leq \Omega \text{ and } u_i(x_i) \geq u_i(x_i')\right\}$ denotes the *weak lower contour set of agent i for u_i at x_i*. An economy is specified by a list $u = (u_i)_{i \in N} \in U_N \equiv \times_{i \in N} U_i$. An allocation is a list of bundles $x = (x_1, ..., x_n) \in \mathbb{R}_{+}^{n\ell}$, whereas a *feasible allocation* is an allocation $x = (x_1, ..., x_n) \in \mathbb{R}_{+}^{n\ell}$ such that $\sum_{i \in N} x_i = \Omega$. The set of all feasible allocations is denoted by A.[11]

A *social choice correspondence* (*SCC*) is a multi-valued mapping $F : U_N \twoheadrightarrow A$ such that for each $u \in U_N$, $F(u)$ is a non-empty subset of feasible allocations, that is, $\varnothing \neq F(u) \subseteq A$.[12] Unless specified otherwise, we do not assume that for all $u \in U_N$ and all $x \in F(u)$, $x_i \gg 0$ for all i. The set of (*Pareto*) *efficient allocations* for the economy $u \in U_N$, denoted $P(u)$, is $P(u) \equiv \{x \in A \mid \text{there is no } y \in A : u_i(y_i) > u_i(x_i) \text{ for all } i\}$. An *SCC* F defined on U_N is *efficient* if for any $u \in U_N$, $F(u) \subseteq P(u)$ holds. $\mathcal{F}$ is the class of all efficient *SCC*s defined on U_N. Among many elements of $\mathcal{F}$, we shall be concerned with the following well-known *SCC*s:

Pareto correspondence P: $P(u) = \{x \in A \mid \text{there is no } y \in A : u_i(y_i) > u_i(x_i) \text{ for all } i\}$.

No-envy and efficient correspondence NP: $NP(u) \equiv \{x \in A \mid u_i(x_i) \geq u_i(x_j) \text{ for all } i \text{ and } j\} \cap P(u)$.

Walrasian correspondence W: $W(u) \equiv \{x \in A \mid \text{there is } p \in \mathbb{R}_{+}^{\ell} \text{ s.t. for all } i,\ p \cdot x_i = p \cdot \omega_i \text{ and for all } y_i \in \mathbb{R}_{+}^{\ell},\ u_i(x_i) \geq u_i(y_i) \text{ if } p \cdot y_i \leq p \cdot \omega_i\}$.

Constrained Walrasian correspondence W_c: $W_c(u) \equiv \{x \in A \mid \text{there is } p \in \mathbb{R}_{+}^{\ell} \text{ s.t. for all } i,\ p \cdot x_i = p \cdot \omega_i \text{ and for all } y_i \in \mathbb{R}_{+}^{\ell},\ u_i(x_i) \geq u_i(y_i) \text{ if } y_i \leq \Omega \text{ and } p \cdot y_i \leq p \cdot \omega_i\}$.

Efficient egalitarian-equivalent correspondence EE: $EE(u) \equiv \{x \in A \mid \text{there is a unique maximal number } \lambda \in (0,1) \text{ s.t. } u_i(x_i) = u_i(\lambda\Omega) \text{ for all } i\} \cap P(u)$.

For any $(u_i, x_i) \in U_i \times \mathbb{R}_{+}^{\ell}$, $V_i(x_i, u_i) \equiv \{y_i \in \mathbb{R}_{+}^{\ell} | y_i \leq \Omega \text{ and } u_i(x_i) \leq u_i(y_i)\}$ denotes the *weak upper contour set of agent i for u_i at x_i*. Given $(u_i, x_i) \in U_i \times \mathbb{R}_{+}^{\ell}$,

[10] Hereafter unless otherwise specified "i" should be understood as agent $i \in N$.

[11] Note that because of this definition of feasible allocations, any admissible domain U_N in this paper cannot be *separable* in the sense of Dutta and Sen (2012).

[12] The weak set inclusion is denoted by $\subseteq$, while the strict set inclusion is denoted by $\subset$.

a price vector p belonging to the unit simplex Δ, that is, $p \in \Delta$, is said to be a *sub-gradient of u_i at x_i* if $p \cdot x'_i \geq p \cdot x_i$ for all $x'_i \in V_i(x_i, u_i)$. The set of all sub-gradients of u_i at x_i, that is, $\delta u_i(x_i) \equiv \{p \in \Delta | p \cdot x'_i \geq p \cdot x_i \text{ for all } x'_i \in V_i(x_i, u_i)\}$, is called the *sub-differential of u_i at x_i*. For any $(x, u) \in \mathbb{R}^{n\ell}_+ \times U_N$, let $\Pi(x, u) \equiv \cap_{i \in N} \delta u_i(x_i)$. Notice that $x \in P(u)$ if $\Pi(x, u)$ is non-empty. In words, $\Pi(x, u)$ consists of prices p each of which is normal to a hyperplane separating the weak upper contour sets of all agents with u at x. Any $p \in \Pi(x, u)$ is referred to as an *efficiency price* for u at x.[13]

A *mechanism* is a pair $\gamma \equiv (M, g)$, where $M \equiv M_1 \times ... \times M_n$, with each M_i being a (non-empty) set and $g : M \rightarrow \mathbb{R}^{n\ell}$. It consists of a message space M, where M_i is the message space for i, and an outcome function g such that $g(m) = (g_i(m))_{i \in N} \in \mathbb{R}^{n\ell}$ for each $m \in M$. $m_i \in M_i$ denotes a generic message (or strategy) for i. A message profile is denoted by $m \equiv (m_1, ..., m_n) \in M$. For $m \in M$ and j, let $m_{-j} \equiv (m_1, ..., m_{j-1}, m_{j+1}, ..., m_n) \in \times_{i \in N \setminus \{j\}} M_i \equiv M_{-j}$. Given $m_{-j} \in M_{-j}$ and $m_j \in M_j$, (m_j, m_{-j}) is the message profile consisting of m_j and m_{-j}. $g(M_i, m_{-i})$ is the attainable set of i at m_{-i}: That is, the set of bundles that i can induce when the other agents select m_{-i}.

Throughout the paper, we focus our attention on *completely feasible mechanisms* (Saijo et. al, 1996, 1999). A mechanism $\gamma = (M, g)$ is:

Individually feasible if $g(m) \in \mathbb{R}^{n\ell}_+$ for each $m \in M$;

Balanced if $\sum_{i \in N} g_i(m) = \Omega$ for each $m \in M$.

A mechanism $\gamma = (M, g)$ is *completely feasible* if it is individually feasible and balanced. Such a mechanism γ has the property that $g(m) \in A$ for each $m \in M$.

2.2 *Partially honest implementation*

For any mechanism γ and i, a *truth-telling correspondence* is a correspondence $T^\gamma_i : U_N \times \mathcal{F} \twoheadrightarrow M_i$ such that, for each $(u, F) \in U_N \times \mathcal{F}$, $\varnothing \neq T^\gamma_i(u, F) \subseteq M_i$. An interpretation of the set $T^\gamma_i(u, F)$ is that given the mechanism γ and a pair $(u, F) \in U_N \times \mathcal{F}$, i behaves truthfully at the message profile $m \in M$ if and only if $m_i \in T^\gamma_i(u, F)$. We refer the reader to Definition 5 and Definition 6 of section 3 for two different definitions of truthful messages for price-quantity mechanisms.

For any $(i, u) \in N \times U_N$, let $\succcurlyeq^u_i$ be i's weak order over M under the economy u. The asymmetric part of $\succcurlyeq^u_i$ is denoted by $\succ^u_i$, while the symmetric part is denoted by $\sim^u_i$. For any $u \in U_N$, $\succcurlyeq^u$ is the profile of weak orders over M under the economy u; in other words, $\succcurlyeq^u \equiv (\succcurlyeq^u_i)_{i \in N}$. As in Dutta and Sen (2012), partially honest behavior is defined as follows.

[13] If $u_i \in U_i$ is differentiable for all $i \in N$ and $F \in \mathcal{F}$ selects only interior allocations, $\varnothing \neq F(u) \subseteq A \cap \mathbb{R}^{n\ell}_{++}$ for any $u \in U_N$, then the set $\Pi(x, u)$ is a singleton whenever $x \in F(u)$; in particular, the set $\Pi(x, u)$ has the form of $\{p\} \subseteq \Delta$ such that $\nabla u_i(x_i) = p$ for all $i \in N$, where $\nabla u_i(x_i)$ denotes the gradient vector at x_i which is normalized to belong to the unit simplex Δ.

Definition 1. An agent $h \in N$ is a *partially honest agent* if for any mechanism γ, any $u \in U_N$, any $F \in \mathcal{F}$, any $m \equiv (m_h, m_{-h})$ and any $m' \equiv (m'_h, m_{-h}) \in M$, the following properties hold:
(i) If $m_h \in T^{\gamma}_h(u, F)$, $m'_h \notin T^{\gamma}_h(u, F)$ and $u_h(g_h(m)) \geq u_h(g_h(m'))$, then $(m, m') \in \succ^u_h$;
(ii) otherwise, $(m, m') \in \succcurlyeq^u_h$ if and only if $u_h(g_h(m)) \geq u_h(g_h(m'))$.

Assume also that, for any i who is *not partially honest*, it follows that for any mechanism γ, any $u \in U_N$, any $F \in \mathcal{F}$, any $m \equiv (m_i, m_{-i})$, and any $m' \equiv (m'_i, m_{-i}) \in M$, $(m, m') \in \succcurlyeq^u_i$ if and only if $u_i(g_i(m)) \geq u_i(g_i(m'))$.

The following informational assumption holds throughout the paper.

Assumption 1. *There are* partially honest agents *in N. The mechanism designer knows that there are partially honest agents in N, though she does not know their identities or their exact number.*

The mechanism designer cannot exclude any agent from being partially honest on the basis of information given by Assumption 1. To formalize this fact, let $\varnothing \neq \mathcal{H} \subseteq 2^N \setminus \{\varnothing\}$ be a class of non-empty subsets of N. Given the truly limited information injected by Assumption 1, in what follows we shall view $\mathcal{H}$ as the *class of conceivable sets of partially honest agents*. Although Assumption 1 only implies that $\#\mathcal{H} \geq 2$, we might generally view $\mathcal{H}$ as being identical to $2^N \setminus \{\varnothing\}$.

A mechanism γ induces a class of (*non-cooperative*) *games with partially honest agents* $\{(\gamma, \succcurlyeq^u) \mid (u, H) \in U_N \times \mathcal{H}\}$. Given a game $(\gamma, \succcurlyeq^u)$, we say that $m^* \in M$ is a (pure strategy) *Nash equilibrium with partially honest agents* at u if and only if for all i, $\left(m^*, \left(m_i, m^*_{-i}\right)\right) \in \succcurlyeq^u_i$ for all $m_i \in M_i$. Given a game $(\gamma, \succcurlyeq^u)$, $NE(\gamma, \succcurlyeq^u)$ denotes the set of (Nash) equilibrium message profiles of $(\gamma, \succcurlyeq^u)$, whereas $NA(\gamma, \succcurlyeq^u)$ represents the corresponding set of (Nash) equilibrium allocations.

Definition 2. A mechanism γ *partially honest implements* $F \in \mathcal{F}$ in *Nash equilibria*, or simply *partially honest implements* F, if and only if $F(u) = NA(\gamma, \succcurlyeq^u)$ for all $u \in U_N$ and all $H \in \mathcal{H}$.

If such a mechanism exists, then F is *partially honest* (*Nash*) *implementable*.

Definition 2 is similar, but not identical to, the standard definition of implementation.[14] First, the equilibrium allocations are given by the game $(\gamma, \succcurlyeq^u)$ rather than by the game (γ, u). Second, the equivalence of the set of SCC-optimal allocations with the set of Nash equilibrium allocations is required not only for any economy $u \in U_N$ but also for any conceivable set $H \in \mathcal{H}$.

3 Natural price-quantity mechanisms

A *price-quantity* mechanism is a mechanism in which each agent's message consists of reporting a price vector as well as her own consumption bundle. Moreover, it is said to be *natural* if some elementary but important properties hold (Saijo et al.,

[14] The two definitions are however identical if $\mathcal{H} = \varnothing$.

1996, 1999). In addition to the property of complete feasibility (defined in subsection 2.1), a natural price-quantity mechanism satisfies the *best response property* and the condition of *forthrightness*.

The property of non-emptiness of the agent's best response function, introduced by Jackson et al. (1994), can be formulated as follows:

DEFINITION 3. (Jackson, Palfrey and Srivastava, 1994) A mechanism γ satisfies the *best response property* if, for all i, all $u_i \in U_i$, and all $m_{-i} \in M_{-i}$, there exists $m_i \in M_i$ such that $u_i\left(g_i\left(m_i, m_{-i}\right)\right) \geq u_i\left(g_i\left(m_i', m_{-i}\right)\right)$ for all $m_i' \in M_i$.

Before introducing forthrightness a little more notation is needed: For each $u \in U_N$, each $x \in F(u)$, and each $p \in \Pi(x, u)$, define the set $F^{-1}(x, p)$ as

$$F^{-1}(x, p) \equiv \{u' \in U_N \mid x \in F(u') \text{ and } p \in \Pi(x, u')\},$$

whereas the sets $F_!^{-1}(x, p)$ and $\Pi^F(x, u)$ are defined respectively as follows:

$$F_!^{-1}(x, p) \equiv \{u' \in U_N \mid x \in F(u') \text{ and } \{p\} = \Pi(x, u')\}, \tag{1}$$

$$\Pi^F(x, u) \equiv \left\{p \in \Pi(x, u) \mid F_!^{-1}(x, p) \neq \varnothing\right\}. \tag{2}$$

Then, for each $u \in U_N$ and each $x \in F(u)$, define the set $\pi^F(x, u)$ as follows:

$$\pi^F(x, u) \equiv \begin{cases} \Pi^F(x, u) & \text{if it is non-empty,} \\ \Pi(x, u) & \text{otherwise.} \end{cases} \tag{3}$$

Any $p \in \pi^F(x, u)$ is referred to as an *efficiency price* for u and F at x.[15]

It is important to observe that the difference between $\pi^F(x, u)$ and $\Pi(x, u)$ centers on whether it is possible to select efficiency prices from the set $\Pi(x, u)$ which are relevant for the given u and F at the F-optimal allocation x. The reason for this selection is that not all efficiency prices are equally important from the standpoint of an F-optimal allocation. This is particularly true when F is the (constrained) Walrasian.

DEFINITION 4. An *SCC* $F \in \mathcal{F}$ is *partially honest implementable by a natural price-quantity mechanism* if there exists a mechanism $\gamma = (M, g)$ such that:
(i) γ partially honestly implements F.
(ii) For each i, $M_i = \Delta \times Q$, where $Q \equiv \left\{x_i \in \mathbb{R}_+^\ell \mid x_i \leq \Omega\right\}$.
(iii) For each $u \in U_N$, each $x \in F(u)$, and each $p \in \pi^F(x, u)$, if $m_i = (p, x_i)$ for each i, then $m \in NE(\gamma, \succcurlyeq^u)$ and $g(m) = x$.
(iv) γ is completely feasible.
(v) γ satisfies the best response property.

[15] Note that, for any $p \in \Pi(x, u)$ with $x \in F(u)$, there always exists $u' \in U_N$ such that $\{p\} = \Pi(x, u')$ whenever U_N contains the set of all *linear* utility functions. Indeed, if $u_i'(x_i) = p \cdot x_i$ for all $i \in N$, then $\{p\} = \Pi(x, u')$. However, note that $F_!^{-1}(x, p) \neq \varnothing$ does not necessarily follow from the case $\{p\} = \Pi(x, u')$. For instance, if $F = W_c$, it is possible that $x \in W_c(u)$, $p \in \Pi(x, u)$, $\{p\} = \Pi(x, u')$, but $x \notin W_c(u')$, whenever p is not a (constrained) Walrasian equilibrium price vector corresponding to $x \in W_c(u)$.

Call γ a *natural price-quantity mechanism* if γ satisfies requirements (ii)-(v) of Definition 4.

Requirement (iii) of the definition just stated is the forthrightness condition for a price-quantity mechanism: For every economy u and F-optimal allocation x at u, if each i reports an efficiency price p for u and F at x and her assigned consumption bundle x_i, then the profile m constitutes an equilibrium in which i receives x_i. The condition requires therefore, that in equilibrium each agent receives what she has announced as a bundle.[16] Observe that our forthrightness condition differs from that of Saijo et al. (1999) because it applies only to efficiency prices in $\pi^F(x,u)$ rather than to those in $\Pi(x,u)$. We shall make use of this difference in sub-section 3.3, which will be relevant for the case that F is the (constrained) Walrasian.[17]

We shall now introduce two different notions of honesty for a price-quantity mechanism, called *weak* honesty and *strong* honesty. The weak honesty requires that *only* the consumption bundle x_i reported by i be consistent with $F(u)$: That is, $(x_i, x_{-i}) \in F(u)$ for some x_{-i}. The strong honesty requires not only that x_i be consistent with $F(u)$, but also that the price vector reported by i be an efficiency price for u and F at an F-optimal allocation $\left(x_i, x'_{-i}\right)$. The difference between the weak and strong honesty centers therefore, on whether or not the price announcement component p of the message must also be consistent with the true economy u and the social goal F. The notions can be stated as follows:

Definition 5 (Weak honesty). Given a natural price-quantity mechanism γ, an economy $u \in U_N$ and an *SCC* $F \in \mathcal{F}$, a partially honest agent h has a *weak* intrinsic preference towards honesty if *the range of her truth-telling correspondence* is $T_h^\gamma(u,F) \equiv \left\{(p, x_h) \in \Delta \times Q \mid \exists x_{-h} \in \mathbb{R}^{(n-1)\ell} : (x_h, x_{-h}) \in F(u)\right\}$.

Definition 6 (Strong honesty). Given a natural price-quantity mechanism γ, an economy $u \in U_N$ and an *SCC* $F \in \mathcal{F}$, a partially honest agent h has a *strong* intrinsic preference towards honesty if *the range of her truth-telling correspondence* is

$$\begin{aligned} T_h^\gamma(u,F) \equiv \; & \left\{(p, x_h) \in \Delta \times Q \mid \exists x_{-h} \in \mathbb{R}^{(n-1)\ell} : (x_h, x_{-h}) \in F(u), \;\&\right. \\ & \left. \exists x'_{-h} \in \mathbb{R}^{(n-1)\ell} : \left(x_h, x'_{-h}\right) \in F(u) \;\; \& \; p \in \pi^F\left(\left(x_h, x'_{-h}\right), u\right)\right\}. \end{aligned}$$

3.1 *Axiomatic characterization: Weak honesty*

In this sub-section, we shall consider the case of weak honesty. Thus, in addition to Assumption 1, the mechanism designer knows that partially honest agents

[16] The reader should refer to Dutta et al. (1995) and Saijo et al. (1996) for a much fuller discussion on forthrightness than is presented here.

[17] Indeed, in Saijo et al. (1999), it is possible that, if x is an interior Walrasian allocation for some u, and every i announces the consumption bundle x_i and the same non-Walrasian equilibrium price vector $p \in \Pi(x,u) \setminus \pi^{W_c}(x,u)$, then x must be a Nash equilibrium outcome. However, such an unanimously false announcement of the Walrasian equilibrium price makes it difficult to approximate every agent's true budget set which should be induced by the truthful Walrasian equilibrium price. Such a difficulty can be avoided in our formulation of the forthrightness.

have weak preferences towards honesty. We shall propose two conditions which are together necessary and sufficient for an efficient *SCC* to be partially honest implementable by a natural price-quantity mechanism.

The first condition is called *Monotonicity with Weak Honesty* (*M-WH*). Before introducing it, a little more notation is needed: For each $u \in U_N$, $x \in F(u)$, and $p \in \pi^F(x,u)$, let $\Lambda_i^F(x,p) \equiv \cap_{u' \in F^{-1}(x,p)} L(x_i, u'_i)$ for each i.

Condition *M-WH*. *For any given* $H \in \mathcal{H}$, *for all* $u, u^* \in U_N$, *all* $x \in F(u)$, *and all* $p \in \pi^F(x,u)$, *if* $\Lambda_i^F(x,p) \subseteq L(x_i, u_i^*)$ *for all* i *and if* $x \notin F(u^*)$, *then for some* $h \in H$, $x_h \neq x_h^*$ *for all* $x^* \in F(u^*)$.

For any $H \in \mathcal{H}$, any $u \in U_N$, any $x \in F(u)$ and any $p \in \pi^F(x,u)$, *M-WH* requires that if the economy u moves to a new economy $u^* \in U_N$ in such a way that for each i the weak lower contour set for u_i^* at x_i contains the set $\Lambda_i^F(x,p)$ and if $x \notin F(u^*)$, then for some $h \in H$ the bundle x_h can *never* be a *weakly truthful* F-optimal one for u^*, in the sense that $x_h \neq x_h^*$ holds for all $x^* \in F(u^*)$.

M-WH is weaker than the *Generalized Monotonicity* (*GM*) introduced by Saijo et al. (1999). In fact, *M-WH* applies to price vectors belonging to $\pi^F(x,u)$ rather than to $\Pi(x,u)$. Moreover, while *GM* requires that an F-optimal allocation x for u should be F-optimal for u^* if $\Lambda_i^F(x,p) \subseteq L(x_i, u_i^*)$ holds for each i, *M-WH* requires such an invariant property if, in addition, it holds that for each $h \in H$ the bundle x_h is a *weakly truthful* F*-optimal one for* u^*, in the sense that $\left(x_h, x_{-h}^*\right) \in F(u^*)$ holds for some $x_{-h}^* \in Q^{n-1}$.

The second condition, called *Punishment with Weak Honesty* (*P-WH*), is a weaker variant of condition *GPQ* introduced by Saijo et al. (1999). Before stating *P-WH*, a more notation is needed. Let $p \in \Delta$ and $x = (x_i)_{i \in N} \in Q \times ... \times Q$ be given. Let $x^i \equiv \left(\Omega - \sum_{j \neq i} x_j, x_{-i}\right)$. Define the set $\bar{F}^{-1}(x^i, p)$ as follows:

$$\bar{F}^{-1}\left(x^i, p\right) \equiv \begin{cases} F_!^{-1}(x^i, p) & \text{if } F_!^{-1}(x^i, p) \neq \varnothing, \\ F^{-1}(x^i, p) & \text{if } F_!^{-1}(x^i, p) = \varnothing, \text{ and } \nexists p' \in \Delta : F_!^{-1}(x^i, p') \neq \varnothing. \end{cases}$$

Let $I^F(p,x) \equiv \left\{i \in N | \bar{F}^{-1}(x^i, p) \neq \varnothing\right\}$. For each $x \notin A$, a member of the set $I^F(p,x)$ is a *potential deviator*. For each i, $u_i \in U_i$, and $x_i \in \mathbb{R}_+^{\ell}$, let $\partial L(x_i, u_i)$ denote the upper boundary of $L(x_i, u_i)$, that is, $\partial L(x_i, u_i) \equiv \{x'_i \in L(x_i, u_i) \mid u_i(x_i) = u_i(x'_i)\}$. Finally, for each i, $x_i \in Q$ is *weakly truthful for* $u' \in U_N$ if and only if $(x_i, \hat{x}_{-i}) \in F(u')$ for some $\hat{x}_{-i} \in Q^{n-1}$.

Condition *P-WH*. *For any given* $H \in \mathcal{H}$, *and any* $(p,x) \in \Delta \times Q^n$ *such that* $I^F(p,x) = N$ *and* $x \notin A$, *there exists* $z(p,x) \in A$ *such that:*
(i) $z_i(p,x) \in \Lambda_i^F(x^i, p)$ *for all* i.
(ii) *For each* i, *there exists a function* $S_i(\cdot; (p, x_{-i})) : \Delta \times Q \to Q$ *such that for all* $(p', x'_i) \in \Delta \times Q$,
(a) *if there exist* $u' \in U_N$ *and* $x_i^* \in Q$ *such that* $I^F(p, x^*) = N$, *where* $x^* \equiv (x_i^*, x_{-i})$, *and* $\Lambda_j^F(x^{*j}, p) \subseteq L\left(z_j(p, x^*), u'_j\right)$ *for all* $j \in N$ *and if* x'_i *is weakly truthful for* u', *then* $S_i((p', x'_i); (p, x_{-i})) \in \Lambda_i^F(x^i, p)$;

(b) *otherwise,* $S_i((p', x'_i); (p, x_{-i})) = \mathbf{0}$.
(iii) *For all* $u^* \in U_N$*, if* $\Lambda_i^F(x^i, p) \subseteq L(z_i(p, x), u_i^*)$ *for all* i *and if* $z(p, x) \notin F(u^*)$*, then there exists* $h \in H$ *such that:*
(a) x_h *is not weakly truthful for* u^**, and*
(b) *there exists* $(p', x'_h) \in \Delta \times Q$*, with* x'_h *weakly truthful for* u^**, such that* $S_h((p', x'_h); (p, x_{-h})) \in \partial L(z_h(p, x), u_h^*)$.

Let us briefly discuss *P-WH*. This condition applies to the case in which each agent announces the same price vector p, the announced quantities profile is not a feasible allocation and all agents are potential deviators: That is, for each i, $m_i = (p, x_i)$ where $(x_1, ..., x_n) \notin A$, and $I^F(p, x) = N$. As Saijo et al. (1996, 1999) state, an important problem for implementation by natural price-quantity mechanisms is to identify who should be punished in such a case. A solution to this problem is to punish simultaneously all agents by assigning to each i a feasible bundle which is not better than the bundle $\Omega - \sum_{j \neq i} x_j$, regardless of i's true utility function. Parallelly to part (i) of *GPQ*, part (i) of *P-WH* requires that a feasible punishment allocation $z(p, x)$ exists in such a case.[18]

To discuss part (ii) and part (iii) of *P-WH*, let us consider a situation in which a punishment allocation $z(p, x)$ has already been assigned to agents. In some cases it is necessary to identify the property of bundles that i can achieve when she changes her initial announcement (p, x_i) into weakly truthful ones. In this regards, part (ii) of *P-WH* requires that for each i there should exist a function $S_i(\cdot; (p, x_{-i}))$ that assigns feasible bundles to i when she changes her announcement (p, x_i), while keeping the other agents' announcements fixed. This function has the property of punishment, in that i can never get from the assigned bundle a utility higher than that derived from the bundle $\Omega - \sum_{l \neq i} x_l$, regardless of i's true utility function. Finally, the contrapositive of part (iii) of *P-WH* reads as follows: If for an economy u^* it is the case that for each i the weak lower contour set for u_i^* at $z_i(x, p)$ contains the intersection set $\Lambda_i^F(x, p)$, and if for each $h \in H$ the bundle x_h is weakly truthful for u^* or $u_h^*(S_h((p', x'_h); (p, x_{-h}))) < u_h^*(z_h(p, x))$ for all pairs (p', x'_h) such that x'_h is weakly truthful for u^*, then $z(p, x)$ must be F-optimal for u^*. In other words, if the punishment allocation $z(p, x)$ corresponds to a truthful equilibrium for u^* and the given set H, then it must be F-optimal for u^*.

We shall now establish our first characterization result.

[18] Note that, although *P-WH*-(i) is very similar to *GPQ*-(i) of Saijo et al. (1999), there is a small difference: In Saijo et al. (1999), the set $I^F(p, x)$ and the intersection set $\Lambda_i^F(x^i, p)$ are respectively defined by taking $F^{-1}(x^i, p)$ as the domain of potential economies, whereas we take $\bar{F}^{-1}(x^i, p)$ as the domain. The essence of this difference is that $u \in \bar{F}^{-1}(x^i, p)$ implies not only that x^i is F-optimal for u, but also that p is an *important and relevant* efficiency price in that it can *support* x^i *as an* F*-optimal allocation for* u. In contrast, $u \in F^{-1}(x^i, p)$ implies that x^i is F-optimal for u, and p is *simply* an efficiency price for u at x^i.

For instance, if $F = W_c$, then $\bar{W}_c^{-1}(x^i, p) \neq \varnothing$ implies that x^i becomes a constrained Walrasian equilibrium allocation for an economy, and in such an economy, p becomes a constrained Walrasian equilibrium price vector. In contrast, although $W_c^{-1}(x^i, p) \neq \varnothing$ implies that x^i becomes a constrained Walrasian equilibrium allocation for an economy, it may correspond to the case where p can never be a constrained Walrasian equilibrium price vector.

Theorem 1. *Let $n \geq 3$ and suppose that* Assumption 1 with Definition 5 *holds. An SCC $F \in \mathcal{F}$ is* partially honest implementable *by a* natural price-quantity mechanism *if and only if it satisfies* M-WH *and* P-WH.

The reader will have observed that the rationales behind *M-WH* and *P-WH* differ. In fact, *M-WH* can be viewed as a condition to suggest that the behavior of an efficient *SCC* must be consistent with truthful announcements of bundles, whereas *P-WH* can be viewed as a condition to suggest that an *SCC* must allow the existence of feasible allocations in order to construct a punishment scheme.

3.2 *Axiomatic characterization: Strong honesty*

In this subsection, we shall be concerned with the case of strong honesty. Thus, in addition to Assumption 1, the mechanism designer knows that partially honest agents have strong preferences towards honesty. We propose below a condition called *Monotonicity and Punishment with Strong Honesty* (*MP-SH*), which is necessary and sufficient for an efficient *SCC* to be partially honest implementable by a natural price-quantity mechanism.

To facilitate our discussion, we introduce some terminology and notation that will be convenient: For each i, a pair $(p, x_i) \in \Delta \times Q$ is *strongly truthful for $u' \in U_N$* if and only if $(x_i, \hat{x}_{-i}) \in F(u')$ for some $\hat{x}_{-i} \in \mathbb{R}_+^{(n-1)\ell}$, and $p \in \pi^F((x_i, \bar{x}_{-i}), u')$ for some $\bar{x}_{-i} \in \mathbb{R}_+^{(n-1)\ell}$ with $(x_i, \bar{x}_{-i}) \in F(u')$. For each i and $(p^j, x_j)_{j \in N} \in (\Delta \times Q)^n$, with $p = p^j$ for all j, let $x \equiv (x_j)_{j \in N}$, $(p, x) \equiv (p^j, x_j)_{j \in N}$, and $(p, x_{-i}) \equiv (p^j, x_j)_{j \in N \setminus \{i\}}$. For each $x = (x_i)_{i \in N} \in Q \times ... \times Q$, we shall recall that $x^i \equiv \left(\Omega - \sum_{j \neq i} x_j, x_{-i}\right)$.

Condition *MP-SH*. *There exists a map $z : (\Delta \times Q)^n \to Q^n$ such that, for any given $H \in \mathcal{H}$ and any $(p^j, x_j)_{j \in N} \in (\Delta \times Q)^n$:*
(i) *The following requirements are satisfied:*
(i.a) *If $p^j = p$ for each j, then:*
(i.a.1) *If $I^F(p, x) = N$, then $z(p, x) \in A$, with $z(p, x) = x$ for $x \in A$, such that $z_i(p, x) \in \Lambda_i^F(x^i, p)$ for each i.*
(i.a.2) *If $1 \leq |I^F(p, x)| \leq n - 1$, then $z(p, x) \in A$ such that for each $i \in I^F(p, x)$,*

$$\begin{cases} z_i(p, x) \in \Lambda_i^F(x^i, p) & \text{if } (p, x_i) \text{ is strongly truthful for some } u' \in U_N, \\ z_i(p, x) = \mathbf{0} & \text{otherwise.} \end{cases}$$

(i.a.3) *If $|I^F(p, x)| = 0$, then $z(p, x) = \mathbf{0}$.*
(i.b) *If there exists $i \in N$ such that $p^j = p$ for each $j \in N \setminus \{i\}$ and $p^i \neq p$, then:*
(i.b.1) *If there exists $(u', x_i^*) \in U_N \times Q$ with $I^F(p, x^*) = N$, where $x^* \equiv (x_{-i}, x_i^*)$, and $\Lambda_j^F(x^{*j}, p) \subseteq L(z_j(p, x^*), u_j')$ for each $j \in N$ and if (p^i, x_i) is strongly truthful for u', then $z((p^i, x_i), (p, x_{-i})) \in A$ with $z_i((p^i, x_i), (p, x_{-i})) \in \Lambda_i^F(x^i, p)$.*
(i.b.2) *Otherwise, $z((p^i, x_i), (p, x_{-i})) = \mathbf{0}$.*
(i.c) *For any other case, $z((p^i, x_i), (p, x_{-i})) = \mathbf{0}$.*

(ii) *Moreover, for all* $u^* \in U_N$*, if* $p^i = p$ *for all* i*,* $I^F(p, x) = N$ *and* $\Lambda_i^F(x^i, p) \subseteq L(z_i(p, x), u_i^*)$ *for all* i *and if* $z(p, x) \notin F(u^*)$*, then there exists* $h \in H$ *such that:*
(a) (p, x_h) *is not strongly truthful for* u^**, and*
(b) *there exists* $(p', x'_h) \in \Delta \times Q$*, which is strongly truthful for* u^**, such that* $z_h((p', x'_h), (p, x_{-h})) \in \partial L(z_h(p, x), u_h^*)$.

The rationale behind *MP-SH* for the case that $x \in A$ (resp., $x \notin A$) is that of *M-WH* (resp., *P-WH*). Furthermore, the condition just stated is almost parallel to *M-WH* and *P-WH* with differences directly linked to the different notions of honesty that they refer to. The main differences can be summarized as follows: First, part (i.b.1) of *MP-SH* identifies the property of bundles that i can achieve when she changes her initial announcement (p, x_i) into strongly truthful ones, while keeping the other agents' announcements fixed. Second, this identification is also made for cases in which the initial situation is that each agent announced the same price vector p and the announced quantities profile is F-optimal. Finally, and most significantly, the contrapositive of part (ii) of *MP-SH* reads as follows: The punishment allocation $z(p, x)$ is F-optimal for u^* if for each i the weak lower contour set for u_i^* at $z_i(x, p)$ contains the intersection set $\Lambda_i^F(x, p)$ and if for each $h \in H$ the announcement (p, x_h) is strongly truthful for u^* or $u_h^*(z_h((p', x'_h), (p, x_{-h}))) < u_h^*(z_h(p, x))$ for all strongly truthful pairs (p', x'_h) for u^*, that is, if $z(p, x)$ corresponds a truthful equilibrium for u^* and the given set H. Note also that part (i.a) of *MP-SH* extends the existence of feasible punishment allocations to cases in which the number of agents in $I^F(p, x)$ is less than n.

We shall now establish our second characterization result.

Theorem 2. *Let* $n \geq 3$*, and suppose* Assumption 1 *with* Definition 6 *holds. Then,* $F \in \mathcal{F}$ *is* partially honest implementable *by a* natural price-quantity mechanism *if and only if* F *satisfies* MP-SH*.*

3.3 *Implications*

For the remainder of the present section, we shall blend our characterization results and derive several important theorems. Unless there is specific mention to the contrary, the domain of each *SCC* is the set $\bar{U} \equiv \underbrace{U \times \ldots \times U}_{n\text{-times}}$ of all profiles of utility functions that are continuous, quasi-concave and strictly monotonic. Furthermore, unless otherwise specified, we shall let $\mathcal{H} = 2^N \setminus \{\varnothing\}$. Although some other specifications of $\mathcal{H}$ are possible, our specification is naturally connected with Assumption 1 and is of the utmost importance from the viewpoint of partially honest implementation. None of the positive results presented here can be derived from the conventional natural implementation setting.

It is well-known that the *Walrasian correspondence*, W, is not natural implementable. Moreover, Saijo et al. (1999; Lemma 3) show that the *constrained Walrasian correspondence*, W_c, is not implementable by any natural mechanism in economies endowed with more than two commodities. A natural question, then, is whether or not W and W_c are natural implementable when agents have intrinsic preferences towards honesty. While a positive answer is provided for W_c, the answer for W depends on the type of honesty of partially honest agents and on the number of commodities in the economy. The answers are based on the following lemmata:

Lemma 1. *Let $n \geq 3$; suppose that* Assumption 1 *with* Definition 5 *holds; and let the domain of W_c be $\bar{U}$. Then, W_c satisfies* M-WH*.*

Lemma 2. *Let $n \geq 3$; suppose that Assumption 1 with* Definition 5 *holds; and let the domain of W and W_c be $\bar{U}$. Then:*
(i) *W satisfies* P-WH*.*
(ii) *W_c satisfies* P-WH*.*

Lemma 3. *Let $n \geq 3$; suppose that* Assumption 1 *with* Definition 5 *holds; let $\mathcal{H} = 2^N \backslash \{\varnothing\}$ and let the domain of W be $\bar{U}$. Then:*
(i) *W satisfies* M-WH *when $\ell = 2$.*
(ii) *W does not satisfy* M-WH *when $\ell \geq 3$ and $\omega_i = \frac{\Omega}{n}$ for all i.*

Lemma 4. *Let $n \geq 3$; suppose that* Assumption 1 *with* Definition 6 *holds; let $\mathcal{H} = 2^N \backslash \{\varnothing\}$ and let the domain of W be $\bar{U}$. Then, W satisfies* MP-SH*.*

The following results are a direct consequence of the combination of the above lemmata with either Theorem 1 or Theorem 2.

Theorem 3. *Let $n \geq 3$; suppose that* Assumption 1 *with* Definition 5 *holds; and let the domain of W and W_c be $\bar{U}$. Then:*
(i) *W is* partially honest implementable *by a* natural price-quantity mechanism *when $\ell = 2$.*
(ii) *W is* not partially honest implementable *by any* natural price-quantity mechanism *when $\ell \geq 3$.*
(iii) *W_c is* partially honest implementable *by a* natural price-quantity mechanism*.*

Theorem 4. *Let $n \geq 3$; suppose that* Assumption 1 *with* Definition 6 *holds; and let the domain of W be $\bar{U}$. Then, W is* partially honest implementable *by a* natural price-quantity mechanism.

We shall now discuss the partially honest implementability by natural price-quantity mechanisms of the Pareto correspondence, P, the no-envy and efficient correspondence, NP, and the efficient egalitarian-equivalent correspondence, EE. Saijo et al. (1999) show that these SCCs are not implementable by any natural price-quantity mechanism. A natural question then, is whether or not these SCCs are partially honest implementable. A negative answer is provided by means of the following lemmata.

Lemma 5. *Let $n \geq 3$; suppose that* Assumption 1 *holds; and let the domain of P be $\bar{U}$. Then:*

(i) *P does not satisfy* part (i) of P-WH *if* Definition 5 *holds.*
(ii) *P does not satisfy* part (i.a.1) of MP-SH *if* Definition 6 *holds.*

Proof. This is due to the fact that the Pareto correspondence does not satisfy condition PQ(i), as shown in Saijo et al. (1996). ■

Lemma 6. *Let $n \geq 3$; suppose that* Assumption 1 *holds; let $\mathcal{H} = 2^N \setminus \{\varnothing\}$ and let the domain of NP be $\bar{U}$. Then:*
(i) *NP does not satisfy* part (iii) of P-WH *if* Definition 5 *holds.*
(ii) *NP does not satisfy* part (ii) of MP-SH *if* Definition 6 *holds.*

Lemma 7. *Let $n \geq 3$; suppose that* Assumption 1 *holds; let $\mathcal{H} = 2^N \setminus \{\varnothing\}$ and let the domain of EE be $\bar{U}$. Then:*
(i) *EE does not satisfy* M-WH *if* Definition 5 *holds.*
(ii) *EE does not satisfy* MP-SH *if* Definition 6 *holds.*

Our last theorem is a direct consequence of the above lemmata, Theorem 1 and Theorem 2.

Theorem 5. *Let $n \geq 3$; suppose that* Assumption 1 *holds with either* Definition 5 *or* Definition 6*; let $\mathcal{H} = 2^N \setminus \{\varnothing\}$ and let the domain of P, NP, and EE be $\bar{U}$. Then, none of P, NP, and EE are* partially honest implementable *by any* natural price-quantity mechanism*.*

4 Concluding remarks

In this paper, which introduces the weak and the strong types of partially honest agent, we examined partially honest implementation by natural price-quantity mechanisms in pure exchange economies. For each of the two types, the class of efficient *SCC*s that are implementable is fully identified. The implications of our characterization results, coupled with the findings for the two-agent economies of the companion paper (Lombardi and Yoshihara, 2012b) are summarized in Table 1 below and compared with those of Saijo et al. (1996, 1999)'s characterization result in the standard set-up. Two main conclusions can be drawn from this table.

Firstly, the main difficulty in implementing the Walrasian allocations in the standard set-up can be found in the lack of the virtue of honesty in the society, rather than in the failure to design reasonable mechanisms. This view is consistent with the standard general equilibrium theory and the fundamental theorems of welfare economics, in which all agents are assumed to be *sincere*. In other words, honesty seems to be indispensable for the implementability of the Walrasian correspondence.

Secondly, honesty does not seem to be a major determinant of human behavior in solving implemenation problems of fair allocations. The failure to partially honestly implement the no-envy and efficient correspondence is to be attributed to the impossibility of successful design of punishment schemes consistent with natural price-quantity mechanisms (see the proof of Lemma 6). On the other hand, the failure to partially honestly implement the efficient egalitarian-equivalent correspondence is to be attributed to the impossibility of nullifying the appeal to any variant of Maskin monotonicity. No degree of honesty can help to resolve these impossibilities.

We shall close this section with some remarks for two-agent economies: It is well-known that the efficient egalitarian-equivalent correspondence violates Maskin monotonicity. Thus, it is not implementable. In our companion paper, we found that this *SCC* is partially honest implementable by a natural price-quantity mechanism in economies with more than two commodities. Although this is a positive and important result, this *SCC* is not implementable by any natural price-quantity mechanism in economies with two commodities, for the same reason that it is not implementable in economies with more than two agents. The existence, therefore, of the virtue of honesty in the society is not helpful, in general, for the natural implementation of this *SCC*.

TABLE 1: HONEST SET-UP/STANDARD SET-UP COMPARISON TABLE

A summary of results

Number of agents:		$n=2$	$n\geq 3$	
Types of honesty:		Weak honesty	Weak honesty	Strong honesty
$SCCs:$	W_c	yes/no	yes/no	yes/no
	$W:\ell=2$	yes/no	yes/no	yes/no
	$W:\ell\geq 3$	yes/no	no/no	yes/no
	NP	yes/yes	no/no	no/no
	$EE:\ell=2$	no/no	no/no	no/no
	$EE:\ell\geq 3$	yes/no	no/no	no/no
	P	no/no	no/no	no/no

W_c, the constrained Walrasian correspondence; W, the Walrasian correspondence;
NP, the non-envy and efficient correspondence; EE, the egalitarian-equivalent and efficient correspondence;
P, the Pareto correspondence; $\ell\,(\geq 2)$, number of commodities in the economy.

Appendix

Before proceeding, we shall recall that $x^i \equiv \left(\Omega - \sum_{j\neq i} x_j, x_{-i}\right)$ and that $\partial L\left(x_i, u_i\right) \equiv \{x_i' \in L\left(x_i, u_i\right) \mid u_i\left(x_i\right) = u_i\left(x_i'\right)\}$.

Proof of Theorem 1

First, we suppose that $F \in \mathcal{F}$ is partially honest implementable by a natural price-quantity mechanism $\gamma = (M, g)$, and shall show that F satisfies *M-WH* and *P-WH*.

Let $(H, u, x, p) \in \mathcal{H} \times U_N \times F(u) \times \pi^F(x, u)$ and let $m_i \equiv (p, x_i) \in \Delta \times Q$ for each i. By forthrightness, $g(m) = x$ and $m \in NE\left(\gamma, \succcurlyeq^{u'}\right)$ for all $u' \in F^{-1}(x, p)$, whence it follows that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x, p)$ for all i. If $x \notin F(u^*)$ for some $u^* \in U_N$ and if $\Lambda_i^F(x, p) \subseteq L(x_i, u_i^*)$ for all i, it follows that $m_i \notin T_i^\gamma(u^*, F)$ and $((m_i', m_{-i}), m) \in \succ_i^{u^*}$ for some $i \in H$ and some $m_i' \in T_i^\gamma(u^*, F)$, then $x_i \neq x_i^*$ for all $x^* \in F(u^*)$, as sought. This shows that F satisfies *M-WH* for all $H \in \mathcal{H}$.

Let $H \in \mathcal{H}$, let $\bar{m}_i = (p, x_i) \in \Delta \times Q$ for each i, and let p and $x = (x_i)_{i \in N}$ be such that $I^F(p, x) = N$ and $x \notin A$. For any i, let $m = (\bar{m}_{-i}, m_i)$ where $m_i = \left(p, \Omega - \sum_{l \neq i} x_l\right)$. By forthrightness, $g(m) = x^i$ and $m \in NE\left(\gamma, \succcurlyeq^{u'}\right)$ for all $u' \in \bar{F}^{-1}(x^i, p)$, whence it follows that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x^i, p)$. But since i is arbitrary, it follows that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x^i, p)$ for all i. Now let $z(p, x) = g(\bar{m})$ and use the fact that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x^i, p)$ for all i to obtain that $z_i(p, x) \in \Lambda_i^F(x^i, p)$ for all i. This shows that F satisfies part (i) of *P-WH*.

For any i and $(p, x_{-i}) \in \Delta \times Q^{n-1}$, define the real-valued function $S_i(\cdot; (p, x_{-i})) : \Delta \times Q \to Q$ as follows: For any $(p', x_i') \in \Delta \times Q$,
(i) if there exist $u' \in U_N$ and $x_i^* \in Q$ such that $I^F(p, x^*) = N$, $\Lambda_j^F(x^{*j}, p) \subseteq L\left(z_j(p, x^*), u_j'\right)$ for all $j \in N$, where $x^* \equiv (x_i^*, x_{-i})$, and if x_i' is weakly truthful for u', then $S_i((p', x_i'); (p, x_{-i})) \equiv g_i((p', x_i'), \bar{m}_{-i})$;
(ii) otherwise, $S_i((p', x_i'); (p, x_{-i})) \equiv \mathbf{0}$.
Noting that $x^{*i} = x^i$, so that $\Lambda_i^F(x^{*i}, p) = \Lambda_i^F(x^i, p)$, it is readily verifiable that part (ii) of *P-WH* is satisfied by the just stated definition of $S_i(\cdot; \cdot)$.

If for some $u^* \in U_N$ it holds that $\Lambda_i^F(x^i, p) \subseteq L(z_i(p, x), u_i^*)$ for all i and if $z(p, x) = g(\bar{m}) \notin F(u^*)$, it follows that $\bar{m}_i \notin T_i^\gamma(u^*, F)$ and $((m_i', \bar{m}_{-i}), \bar{m}) \in \succ_i^{u^*}$ for some $i \in H$ and some $m_i' = (p', x_i') \in T_i^\gamma(u^*, F)$, then $u_i^*(g_i(m_i', \bar{m}_{-i})) = u_i^*(g_i(\bar{m}))$, so that x_i is not weakly truthful for u^*, x_i' is weakly truthful for u^*, and $g_i(m_i', \bar{m}_{-i}) \in \Lambda_i^F(x^i, p) \cap \partial L(z_i(p, x), u_i^*)$. By part (i) of the definition of function $S_i(\cdot; (p, x_{-i}))$, it follows that $S_i((p', x_i'); (p, x_{-i})) = g_i(m_i', \bar{m}_{-i})$, in which case $S_i((p', x_i'); (p, x_{-i})) \in \partial L(z_i(p, x), u_i^*)$. This shows that F satisfies part (iii) of *P-WH*.

Conversely, suppose that F satisfies *M-WH* and *P-WH*. Before proceeding, let us introduce some preliminaries. Denote the boundary set of Δ by $\partial\Delta$. For any vertex $\bar{p} \in \partial\Delta$ and any other $p \in \Delta$, let $\mathcal{B}_\epsilon(\bar{p}; p)$ be a closed ball with center $\bar{p}$ and radius $\epsilon \equiv \frac{1}{2}\|\bar{p}, p\| > 0$, where $\|\bar{p}, p\|$ is the Euclidean distance between $\bar{p}$ and p. Let $\overline{\mathcal{B}}_\epsilon(\bar{p}; p) \equiv \mathcal{B}_\epsilon(\bar{p}; p) \cap \Delta$. Since $\overline{\mathcal{B}}_\epsilon(\bar{p}; p)$ and Δ are cardinally equivalent, there is a bijection $\phi : \overline{\mathcal{B}}_\epsilon(\bar{p}; p) \to \Delta$. For any i, any $(p, x_{-i}) \in \Delta \times Q^{n-1}$ and any vertex $\bar{p} \in \Delta$, define the function $\tilde{S}_i(\cdot; (p, x_{-i})) : \overline{\mathcal{B}}_\epsilon(\bar{p}; p) \times Q \to Q$ by $\tilde{S}_i((\hat{p}, \hat{x}_i); (p, x_{-i})) \equiv S_i((\phi(\hat{p}), \hat{x}_i); (p, x_{-i}))$ for all $(\hat{p}, \hat{x}_i) \in \overline{\mathcal{B}}_\epsilon(\bar{p}; p) \times Q$. For any $(x, p) \in Q^n \times \Delta$, $\partial\Lambda_i^F(x, p)$ is the upper boundary of $\Lambda_i^F(x, p)$, that is, $\partial\Lambda_i^F(x, p) \equiv \{y_i \in Q | y_i \in \Lambda_i^F(x, p)$ and $\nexists z_i \in \Lambda_i^F(x, p)$ such that $z_i \gg y_i\}$.

With these preliminaries and given any two fixed and distinct vertices $\bar{p}$ and $\bar{p}'$ of $\partial\Delta$ we now define the outcome function g of a price-quantity mechanism $\gamma = (M, g)$ as follows:

Rule 1: If $m_i = (p, x_i)$ for all i such that $x \in F(u')$ and $p \in \pi^F(x, u')$ for some

$u' \in U_N$, then $g(m) = x$.

Rule 2: If $m_i = (p, x_i)$ for all i such that $x \notin A$ and $I^F(p, x) = N$, then $g(m) = z(p, x)$, where $z(p, x)$ is the allocation specified in part (i) of *P-WH.*

Rule 3: If $m_i = (p, x_i)$ for all i, $1 \leq \left|I^F(p, x)\right| \leq n - 1$, then:

$$g_i(m) = \begin{cases} \frac{\Omega}{n - |I^F(p,x)|} & \text{if } \ i \notin I^F(p, x), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Rule 4: If, for some i, (p, x_l) for all $l \in N \backslash \{i\}$, and (p^i, x_i), with $p \neq p^i$, and $i \in I^F(p, x)$, then:

Rule 4.1: if $p^i = \bar{p}'$, then $g(m) = x^i$;

Rule 4.2: if $p^i \in \overline{\mathcal{B}}_\epsilon(\bar{p}; p)$, then:

$$g(m) = \begin{cases} \left(\tilde{S}_i((p^i, x_i); (p, x_{-i})), \left(\frac{\Omega - \tilde{S}_i\left((p^i, x_i); (p, x_{-i})\right)}{n-1}\right)_{l \neq i}\right) & \text{if } \ \tilde{S}_i((p^i, x_i); (p, x_{-i})) \neq \mathbf{0}, \\ \left(x_i, \left(\frac{\Omega - x_i}{n-1}\right)_{l \neq i}\right) & \text{if } \ \tilde{S}_i((p^i, x_i); (p, x_{-i})) = \mathbf{0} \text{ and } x_i \in \Lambda_i^F(x^i, p), \\ \left(\psi_i(x_i), \left(\frac{\Omega - \psi_i(x_i)}{n-1}\right)_{l \neq i}\right) & \text{otherwise,} \end{cases}$$

where $\psi_i : Q \backslash \Lambda_i^F(x^i, p) \to \partial \Lambda_i^F(x^i, p)$ is a surjective function;

Rule 4.3: otherwise, $g(m) = \left(\hat{x}_i, \left(\frac{\Omega - \hat{x}_i}{n-1}\right)_{l \neq i}\right)$, where $\{\hat{x}_i\} \equiv \partial \Lambda_i^F(x^i, p) \cap \{y_i \in \mathbb{R}_+^\ell \mid \exists \alpha \in \mathbb{R}_+ \text{ s.t. } y_i = \alpha x_i\}$.

Rule 5: Otherwise, $g_{i^*}(m) = \Omega$ and $g_j(m) = 0$ for all $j \neq i^*$, where i^* is defined as follows. Without loss of generality, let us suppose that $\Omega_1 \geq 1$. Let $\sum_{i \in N} y_{i1} = t$. Furthermore, it follows that, for all i, $y_{i1} \in [0, \Omega_1]$. Let v be an integer such that $v \leq t < v + 1$. Therefore, $t = v + s$ where $s \in [0, 1)$. It follows that there is a unique agent $i^* \in N$ such that $s \in \left[\frac{i^* - 1}{n}, \frac{i^*}{n}\right)$.

According to the proposed construction, γ is a natural price-quantity mechanism.

We shall show that $F(u) = NA(\gamma, \succsim^u)$ for all $u \in U_N$ and all $H \in \mathcal{H}$. Since it is a routine exercise to prove that $F(u) \subseteq NA(\gamma, \succsim^u)$ for any u and any H, we shall omit the proof here. Conversely, for any $(u, H) \in U_N \times \mathcal{H}$, let $m \in NE(\gamma, \succsim^u)$ for the given H. Since m cannot correspond to *Rule 3, Rule 4, or Rule 5*, m falls either into *Rule 1* or *Rule 2.*

Suppose that m falls into *Rule 1.* By *Rule 4.3*, we obtain that $\partial \Lambda_i^F(x, p) \subseteq g_i(M_i, m_{-i})$ for all i from which it follows that $\Lambda_i^F(x, p) \subseteq L(x_i, u_i)$ for all i since $m \in NE(\gamma, \succsim^u)$ and since agents' utility functions are strictly monotonic. If $i \in H$ and if $m_i \notin T_i^\gamma(u, F)$, *Rule 4.1* implies that $g_h(m_i', m_{-i}) = x_i$ for any $m_i' = (\bar{p}', x_i') \in T_i^\gamma(u, F)$, so that $((m_i', m_{-i}), m) \in \succ_i^u$. Since this contradicts the assumption that $m \in NE(\gamma, \succcurlyeq^u)$ for the given H, $m_i \in T_i^\gamma(u, F)$ if $i \in H$. Therefore, *M-WH* implies that $g(m) \in F(u)$.

Suppose that m falls into *Rule 2*. By the same reasoning used for *Rule 1*, it is clear that $\Lambda_i^F(x^i, p) \subseteq L(z_i(p, x), u_i)$ for all i. If $i \in H$, if $m_i \notin T_i^\gamma(u, F)$ and if there exists $m_i' \in T_i^\gamma(u, F)$ such that $g_i(m_i', m_{-i}) \in \partial L(z_i(p, x), u_i)$, then $((m_i', m_{-i}), m) \in \succ_i^u$. Since this contradicts the assumption that $m \in NE(\gamma, \succcurlyeq^u)$ for the given H, $g_i(m_i', m_{-i}) \notin \partial L(z_i(p, x), u_i)$ for all $m_i' \in T_i^\gamma(u, F)$ if $i \in H$ and $m_i \notin T_i^\gamma(u, F)$. By the definition of function $\tilde{S}_i(\cdot; \cdot)$ and the fact that ϕ is a bijection from $\overline{\mathcal{B}}_\epsilon(\bar{p}; p)$ to Δ, it follows that $S_i((p', x_i'); (p, x_{-i})) \notin \partial L(z_i(p, x), u_i)$ for all $p' \in \Delta$ and all weakly truthfully bundle $x_i' \in Q$ for u if $i \in H$ and $m_i \notin T_i^\gamma(u, F)$. Therefore, part (iii) of *P-WH* implies that $z(p, x) \in F(u)$.

Since $H \in \mathcal{H}$ and $u \in U_N$ were chosen arbitrarily, the statement follows. ■

Proof of Theorem 2

First, we suppose that $F \in \mathcal{F}$ is partially honest implementable by a natural price-quantity mechanism $\gamma = (M, g)$ and shall show that F satisfies *MP-SH*. Define the real-valued function $z : (\Delta \times Q)^n \to Q^n$ as follows: For any $H \in \mathcal{H}$ and for any $(p^j, x_j)_{j \in N} \in (\Delta \times Q)^n$, where $x \equiv (x_j)_{j \in N}$,

(a) if $p^j = p$ for each j, $I^F(p, x) = N$, then $z(p, x) = g\left((p^j, x_j)_{j \in N}\right)$;

(b) if $p^j = p$ for each j, and $1 \leq \left|I^F(p, x)\right| \leq n - 1$, then for each i:

$$z_i(p, x) = \begin{cases} g_i\left((p^j, x_j)_{j \in N}\right) & \text{if } i \in S, \\ \mathbf{0} & \text{if } i \in I^F(p, x) \backslash S, \\ \frac{\Omega - \sum_{k \in S} g_k\left((p^j, x_j)_{j \in N}\right)}{n - |I^F(p, x)|} & \text{otherwise,} \end{cases}$$

where the set S is defined as follows:

$$S = \left\{i \in I^F(p, x) \mid (p, x_i) \text{ is strongly truthful for some } u' \in U_N\right\}; \qquad (4)$$

(c) if there exists $i \in N$ such that $p^j = p$ for each $j \in N \backslash \{i\}$ and $p^i \neq p$, if there exists $(u', x_i^*) \in U_N \times Q$ with $I^F(p, x^*) = N$, where $x^* \equiv (x_{-i}, x_i^*)$, such that $\Lambda_j^F(x^{*j}, p) \subseteq L\left(z_j(p, x^*), u_j'\right)$ for each $j \in N$, and if (p^i, x_i) is strongly truthful for u', then $z((p^i, x_i), (p, x_{-i})) = g\left((p^i, x_i), (p^j, x_j)_{j \in N \backslash \{i\}}\right)$;

(d) otherwise, $z\left((p^j, x_j)_{j \in N}\right) = \mathbf{0}$.

Let $\bar{m}_i = (p, x_i) \in \Delta \times Q$ for each i. Suppose that p and $x = (x_i)_{i \in N}$ are such that $I^F(p, x) = N$. For any i, let $m = (\bar{m}_{-i}, m_i)$ where $m_i = \left(p, \Omega - \sum_{l \neq i} x_l\right)$. By forthrightness, $g(m) = x^i$ and $m \in NE\left(\gamma, \succcurlyeq^{u'}\right)$ for all $u' \in \bar{F}^{-1}(x^i, p)$, whence it follows that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x^i, p)$. But since i is arbitrary, it follows that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x^i, p)$ for all i. By definition of function z and the fact that $g_i(M_i, m_{-i}) \subseteq \Lambda_i^F(x^i, p)$ for all i, it follows that $z(p, x) = g(\bar{m})$ and $z_i(p, x) \in \Lambda_i^F(x^i, p)$ for all i. Moreover, $z(p, x) = x$ if $x \in A$ by forthrightness. This shows that F satisfies part (i.a.1) of *MP-SH.* Suppose that p and $x = (x_i)_{i \in N}$ are such that $1 \leq \left|I^F(p,x)\right| \leq n-1$. For any $i \in I^F(p,x)$, it follows that $x^i \in NA(\gamma, \succsim^u)$ for all $H' \in \mathcal{H}$ and all $u \in \bar{F}^{-1}(x^i, p)$, so that $g_i(\bar{m}) \in \Lambda_i^F(x^i, p)$. If (4) is non-empty, part (b) of the definition of function z implies that $z_i(p,x) = g_i(\bar{m}) \in \Lambda_i^F(x^i,p)$ for all $i \in S$, $z_i(p,x) = \mathbf{0}$ for all $i \in I^F(p,x) \backslash S$ and $z(p,x) \in A$. If (4) is empty, then part (b) of the definition of function z implies that $z_i(p,x) = \mathbf{0}$ for each $i \in I^F(p,x)$ and $z(p,x) \in A$. This shows that the above definition of function z satisfies part (i.a.2) of *MP-SH.* Noting that $x^{*i} = x^i$ and $\Lambda_i^F(x^{*i}, p) = \Lambda_i^F(x^i, p)$, it is readily verifiable that part (i.a.3), part (i.b) and part (i.c) of *MP-SH* are satisfied by the definition of real-valued function z. Thus, we conclude that F satisfies part (i) of *MP-SH.*

If for some $u^* \in U_N$ it holds that $\Lambda_i^F(x^i,p) \subseteq L(z_i(p,x), u_i^*)$ for all i and if $z(p,x) = g(\bar{m}) \notin F(u^*)$, it follows that $\bar{m}_i \notin T_i^{\gamma}(u^*, F)$ and $((m_i', \bar{m}_{-i}), \bar{m}) \in \succ_i^{u^*}$ for some $i \in H$ and some $m_i' = (p', x_i') \in T_i^{\gamma}(u^*, F)$, then $u_i^*(g_i(m_i', \bar{m}_{-i})) = u_i^*(g_i(\bar{m}))$, so that (p, x_i) is not strongly truthful for u^*, (p', x_i') is strongly truthful for u^*, and $g_i(m_i', \bar{m}_{-i}) \in \Lambda_i^F(x^i, p) \cap \partial L(z_i(p,x), u_i^*)$. By part (c) of definition of function z, it follows that $z_i((p', x_i'), (p, x_{-i})) = g_i((p', x_i'), \bar{m}_{-i})$, so that $z_i((p', x_i'), (p, x_{-i})) \in \partial L(z_i(p,x), u_i^*)$. This shows that F satisfies part (ii) of *MP-SH.*

Conversely, suppose that F satisfies *MP-SH.* Before proceeding, let us introduce some preliminaries. Denote the boundary set of Δ by $\partial\Delta$. For any $(x,p) \in Q^n \times \Delta$, $\partial\Lambda_i^F(x,p)$ is the upper boundary of $\Lambda_i^F(x,p)$, that is, $\partial\Lambda_i^F(x,p) \equiv \{y_i \in Q | y_i \in \Lambda_i^F(x,p) \text{ and } \nexists z_i \in \Lambda_i^F(x,p) \text{ such that } z_i \gg y_i\}$. With these preliminaries we now define the outcome function g of a price-quantity mechanism $\gamma = (M, g)$ as follows:

Rule 1: If $m_j = (p, x_j)$ for all j such that $x \equiv (x_j)_{j \in N} \in F(u')$ and $p \in \pi^F(x, u')$ for some $u' \in U_N$, then $g(m) = z(p,x) = x$ is specified in part (i.a.1) of *MP-SH.*

Rule 2: If $m_j = (p, x_j)$ for all j such that $x \equiv (x_j)_{j \in N} \notin A$ and $I^F(p,x) = N$, then $g(m) = z(p,x)$, where $z(p,x)$ is specified in part (i.a.1) of *MP-SH.*

Rule 3: If $m_j = (p, x_j)$ for all j, $1 \leq \left|I^F(p,x)\right| \leq n-1$, where $x \equiv (x_j)_{j \in N}$, then:

$$g_i(m) = \begin{cases} \frac{\Omega - \sum_{k \in I^F(p,x)} z_k(p,x)}{n - |I^F(p,x)|} & \text{if } \quad i \notin I^F(p,x), \\ z_i(p,x) & \text{otherwise,} \end{cases}$$

where $z(p,x)$ is specified in part (i.a.2) of *MP-SH.*

Rule 4: If, for some i, (p, x_j) for all $j \in N \backslash \{i\}$, and (p^i, x_i), with $p \neq p^i$, and $i \in I^F(p,x)$, where $x \equiv (x_j)_{j \in N}$, then:

$$g(m) = \begin{cases} \left(z_i((p^i, x_i), (p, x_{-i})), \left(\frac{\Omega - z_i((p^i, x_i), (p, x_{-i}))}{n-1} \right)_{j \neq i} \right) & \text{if } z_i((p^i, x_i), (p, x_{-i})) \neq \mathbf{0}, \\ \left(x_i, \left(\frac{\Omega - x_i}{n-1} \right)_{j \neq i} \right) & \text{if } z_i((p^i, x_i), (p, x_{-i})) = \mathbf{0} \ \& \ x_i \in \Lambda_i^F(x^i, p), \\ \left(\hat{x}_i, \left(\frac{\Omega - \hat{x}_i}{n-1} \right)_{j \neq i} \right) & \text{otherwise,} \end{cases}$$

where $\{\hat{x}_i\} \equiv \partial\Lambda_i^F(x^i,p)\cap\{y_i \in \mathbb{R}_+^\ell \mid \exists\alpha \in \mathbb{R}_+ \text{ s.t. } y_i = \alpha x_i\}$, and $z((p^i,x_i),(p,x_{-i}))$ is specified in part (i.b) and in part (i.c) of *MP-SH*.

Rule 5: Otherwise, $g_{i^*}(m) = \Omega$ and $g_j(m) = 0$ for all $j \neq i^*$, where i^* is defined as follows. Without loss of generality, let us suppose that $\Omega_1 \geq 1$. Let $\sum_{i\in N} y_{i1} = t$. Furthermore, it follows that, for all i, $y_{i1} \in [0,\Omega_1]$. Let v be an integer such that $v \leq t < v+1$. Therefore, $t = v + s$ where $s \in [0,1)$. It follows that there is a unique agent $i^* \in N$ such that $s \in \left[\frac{i^*-1}{n}, \frac{i^*}{n}\right)$.

According to the proposed construction, γ is a natural price-quantity mechanism. Note that in *Rule 4*, $g_i(m) = \hat{x}_i$ holds whenever $p^i \in \partial\Delta$ and $x_i \in Q\backslash\Lambda_i^F(x^i,p)$, since (p^i,x_i) can never be a truthful message for any $u' \in U_N$ if $p^i \in \partial\Delta$. Thus, agent i can realize any element of $\partial\Lambda_i^F(x^i,p)$ by a suitable choice of $x_i \in Q\backslash\Lambda_i^F(x^i,p)$.

We shall show that $F(u) = NA(\gamma,\succcurlyeq^u)$ for all $u \in U_N$ and all $H \in \mathcal{H}$. Since it is a routine exercise to prove that $F(u) \subseteq NA(\gamma,\succcurlyeq^u)$ for any u and any H, we shall omit the proof here. Conversely, for any $(u,H) \in U_N \times \mathcal{H}$ let $m \in NE(\gamma,\succcurlyeq^u)$ for the given H. Since m cannot correspond to *Rule 3, Rule 4, or Rule 5*, m falls either into *Rule 1* or *Rule 2.*

Suppose that m falls into *Rule 1.* By *Rule 4*, we obtain that $\partial\Lambda_i^F(x,p) \subseteq g_i(M_i,m_{-i})$ for all i from which it follows that $\Lambda_i^F(x,p) \subseteq L(x_i,u_i)$ for all i since $m \in NE(\gamma,\succcurlyeq^u)$ and since agents' utility functions are strictly monotonic. If $i \in H$, if $m_i \notin T_i^\gamma(u,F)$ and if there exists $m_i' = (p',x_i') \in T_i^\gamma(u,F)$ such that $g_i(m_i',m_{-i}) \in \partial L(x_i,u_i)$, then $((m_i',m_{-i}),m) \in \succ_i^u$. Since this contradicts the assumption that $m \in NE(\gamma,\succcurlyeq^u)$ for the given H, $g_i(m_i',m_{-i}) \notin \partial L(x_i,u_i)$ for all $m_i' \in T_i^\gamma(u,F)$ if $i \in H$ and $m_i \notin T_i^\gamma(u,F)$. By the definition of function g, it follows that $z_i\left((p',x_i'),(p,x_j)_{j\in N\backslash\{i\}}\right) \notin \partial L(x_i,u_i)$ for all pairs (p',x_i') which are strongly truthful for u if $i \in H$ and $m_i \notin T_i^\gamma(u,F)$. Therefore, part (ii) of *MP-SH* implies that $x \in F(u)$. Since the case of m falling into *Rule 2* can be dealt similarly, we shall omit the proof here.

Since $H \in \mathcal{H}$ and $u \in U_N$ were chosen arbitrarily, the statement follows. ■

Proof of Lemma 1

We shall first show that $\Pi^{W_c}(x,u)$ is non-empty and consists solely of constrained Walrasian prices p^{W_c} for any $u \in \bar{U}$ and any $x \in W_c(u)$. If $u \in \bar{U}$ and if $x \in W_c(u)$, there exists $p^{W_c} \in \Pi(x,u)$ such that $p^{W_c}\cdot x_i = p^{W_c}\cdot\omega_i$ for all i. Since the domain of W_c is $\bar{U}$, there exists $u' \in \bar{U}$ such that $x \in W_c(u')$ and $\{p^{W_c}\} = \Pi(x,u')$, in which case $u' \in W_{\underline{c!}}^{-1}(x,p^{W_c})$ and $p^{W_c} \in \Pi^{W_c}(x,u)$ by (1) and (2), respectively. If $(x,u) \in \mathbb{R}_+^{n\ell}\times\bar{U}$, if $p \in \Pi(x,u)$ is such that $p\cdot x_i \neq p\cdot\omega_i$ for some i and if $\{p\} = \Pi(x,u')$ for some $u' \in \bar{U}$, then $x \in P(u')\backslash W_c(u')$, whence it follows from (1) that $u' \notin W_{\underline{c!}}^{-1}(x,p)$. Therefore, if $(x,u) \in \mathbb{R}_+^{n\ell}\times\bar{U}$ and if $p \in \Pi(x,u)$ is such that $p\cdot x_i \neq p\cdot\omega_i$ for some i, $W_{\underline{c!}}^{-1}(x,p)$ is an empty set, in which case (2) implies that $p \notin \Pi^{W_c}(x,u)$. By definition (3), $\pi^{W_c}(x,u) = \Pi^{W_c}(x,u)$ for any $u \in \bar{U}$ and any $x \in W_c(u)$. For any tuple $(u,x,p) \in \bar{U}\times W_c(u)\times\pi^{W_c}(x,u)$, the intersection set $\Lambda_i^{W_c}(x,p)$ is $\cap_{u' \in W_c^{-1}(x,p)} L(x_i, u_i') = \{y_i \in Q \mid p \cdot y_i \leq p \cdot \omega_i\}$ for each i. If $u^* \in \bar{U}$ and if $\Lambda_i^{W_c}(x,p) \subseteq L(x_i, u_i^*)$ for all i, it is plain that $x \in W_c(u^*)$ with the equilibrium price p. ■

Proof of Lemma 2

We shall show only part (i) of the statement since part (ii) can be proved similarly. Let $p \in \Delta$, let $\omega \in \mathbb{R}_+^{n\ell}$ and let $\omega^p = (\omega_i^p)_{i \in N} \equiv \left(\left(\frac{p \cdot \omega_i}{p \cdot \Omega}\right)\Omega\right)_{i \in N}$. If $(p,x) \in \Delta \times Q^n$ is such that $I^W(p,x) = N$ and $x \notin A$, it follows from the definition of $I^W(p,x)$ and the fact that $\bar{U}$ is the domain of W that $W_!^{-1}(x^i, p) \neq \varnothing$ for each i. Thus for any i, $u_i(x_i^i) \geq u_i(\omega_i^p)$ for all $u \in W_!^{-1}(x^i, p)$. Moreover, given that $I^W(p,x) = N$ and that agents' utility functions are strictly monotonic, it follows that $p \gg \mathbf{0}$, so that $\omega_i^p \in \Lambda_i^W(x^i, p) \cap \mathbb{R}_{++}^{\ell}$ for all i. Let $z(p,x) \equiv \omega^p$. Therefore, by construction, part (i) of *P-WH* is satisfied. If $u^* \in \bar{U}$ and if $\Lambda_i^W(x^i, p) \subseteq L(z_i(p,x), u_i^*)$ for all i, then $z_i(p,x) \in \arg\max_{p \cdot y_i \leq p \cdot \omega_i} u_j^*(y_i)$ for all i, so that $z(p,x) \in W(u^*)$. Since a suitable definition of function $S_i(\cdot; (p, x_{-i}))$ satisfying part (ii) of *P-WH* exists and part (iii) of *P-WH* is vacuously satisfied, we conclude that W satisfies *P-WH*. ■

Proof of Lemma 3

We shall first prove part (i) and then part (ii).

(i) Similarly to the proof of Lemma 1, it can be shown that $\pi^W(x,u) = \Pi^W(x,u)$ holds for any $u \in \bar{U}$ and any $x \in W(u)$. Furthermore, fix any tuple $(u,x,p) \in \bar{U} \times W(u) \times \pi^W(x,u)$. Then the intersection set $\Lambda_i^W(x,p)$ is $\cap_{u' \in W^{-1}(x,p)} L(x_i, u_i') = \{y_i \in Q \mid p \cdot y_i \leq p \cdot \omega_i\}$ for each i. Note that $\pi^W(x,u) = \{p\}$ since $\ell = 2$. Fix an arbitrary $u^* \in \bar{U}$. Let $\Lambda_i^W(x,p) \subseteq L(x_i, u_i^*)$ for all i and let $x \notin W(u^*)$. Then, x is a boundary point; moreover, $x_j' \not\leq \Omega$ if $x_j' \in \arg\max_{p \cdot y_j \leq p \cdot \omega_j} u_j^*(y_j)$ for some j, in which case p cannot be paired with any Walrasian allocation for u^*. Take any $x^* \in W(u^*)$ paired with the unique equilibrium price p^W. If $x_i^* = x_i$ for some i, then $p = p^W$, which contradicts the hypothesis that $x^* \in W(u^*)$. Thus $x_i^* \neq x_i$ for all i. Since $x^* \in W(u^*)$ is arbitrary, it follows that W satisfies *M-WH*.

(ii) For the sake of simplicity, let $n = 3$. Without loss of generality, let $\omega_i = (1,1,1)$ for each i. Choose $u \in \bar{U}$ such that a Walrasian equilibrium allocation at u is $x_1 = \left(3, \frac{1}{2}, \frac{1}{2}\right)$, $x_2 = \left(0, \frac{5}{2}, 0\right)$ and $x_3 = \left(0, 0, \frac{5}{2}\right)$, with the equilibrium price $p = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$. Let $x^* \in A$ be such that $x_1^* = \left(\frac{5}{2}, 0, 0\right)$, $x_2^* = \left(\frac{1}{2}, 3, \frac{1}{2}\right)$ and $x_3^* = x_3$. Furthermore, given our domain supposition, it is possible to choose a differentiable $u^* \in \bar{U}$ such that the gradient vector of u_i^* at x_i for agent i is $\nabla u_1^*(x_1) = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$ for agent 1, $\nabla u_2^*(x_2) = \left(\frac{1}{10}, \frac{3}{5}, \frac{3}{10}\right)$ for agent 2 and $\nabla u_3^*(x_3) = \left(\frac{1}{10}, \frac{3}{10}, \frac{3}{5}\right)$ for agent 3, while the gradient vector of u_i^* at x_i^* is $\nabla u_1^*(x_1^*) = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$ for agent 1 and $\nabla u_2^*(x_2) = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$ for agent 2. Note that $\nabla u_3^*(x_3^*) = \nabla u_3^*(x_3)$ for agent 3 since $x_3 = x_3^*$. By construction, $\Lambda_i^W(x,p) \subseteq L(x_i, u_i^*)$ for all i and $x \in W_c(u^*) \backslash W(u^*)$. On the other hand, $x^* \in W(u^*)$ with the equilibrium price $p^* = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$. We established that $x^* \in W(u^*)$ and $x_3^* = x_3$, in violation of *M-WH* for $H = \{3\}$. ■

Proof of Lemma 4

For any $p \in \Delta$ and $\omega \in \mathbb{R}_{+}^{n\ell}$, define ω^p as in the proof of Lemma 2. Define the real-valued function z of *MP-SH* as follows: For all $(p^j, x_j)_{j\in N} \in (\Delta \times Q)^n$, with $x \equiv (x_j)_{j\in N}$,
(a) if $p^j = p$ for each j, $I^W(p,x) = N$, then $z(p,x) = x$ if $x \in A$, otherwise, $z(p,x) = \omega^p$;
(b) if $p^j = p$ for each j, and $1 \leq \left|I^W(p,x)\right| \leq n-1$, then for each i:

$$z_i(p,x) = \begin{cases} \omega_i^p & \text{if} \quad i \in S, \\ \mathbf{0} & \text{if} \quad i \in I^W(p,x) \backslash S, \\ \frac{\Omega - \sum_{j\in S}\omega_j^p}{n - |I^W(p,x)|} & \quad \text{otherwise;} \end{cases}$$

where the set S is defined as follows:

$$S = \left\{ i \in I^W(p,x) \mid (p,x_i) \text{ is strongly truthful for some } u' \in \bar{U} \right\}$$

(c) if for some i, $p^j = p$ for each $j \in N\backslash\{i\}$ and $p^i \neq p$, and if there exists $(u', x_i^*) \in \bar{U} \times Q$ such that $I^W(p,x^*) = N$, where $x^* = (x_{-i}, x_i^*)$, and $\Lambda_j^W(x^{*j}, p) \subseteq L\left(z_j(p,x^*), u_j'\right)$ for each $j \in N$ and if (p^i, x_i) is strongly truthful for some u', then:
(c.1) if $x^* \in A$, then $z((p^i,x_i),(p,x_{-i})) = x^*$;
(c.2) if $x^* \notin A$, then $z((p^i,x_i),(p,x_{-i})) = \omega^p$;
(d) otherwise, $z\left((p^j,x_j)_{j\in N}\right) = \mathbf{0}$.

By the above definition of function z, it is plain that part (i) of *MP-SH* is satisfied. To show part (ii), take any $(p,x) \in \Delta \times Q^n$ such that $I^W(p,x) = N$. Let $u^* \in \bar{U}$ and let $\Lambda_i^W(x^i,p) \subseteq L(z_i(p,x), u_i^*)$ for all i. If $x \notin A$, it follows from the same reasoning used in the proof of Lemma 2 that $z(p,x) = \omega^p \in W(u^*)$, whence part (ii) of *MP-SH* is vacuously satisfied. Suppose that $x \in A$, so that for some $u \in \bar{U}$, $x \in W(u)$ with the equilibrium price p. Suppose that $x \notin W(u^*)$. First observe that $x \in W_c(u^*)$. We note also that x is a boundary point and that $x_j' \nleq \Omega$ if $x_j' \in \arg\max_{p\cdot y_j \leq p\cdot\omega_j} u_j^*(y_j)$ and $x_j \notin \arg\max_{p\cdot y_j\leq p\cdot\omega_j} u_j^*(y_j)$ for some j. Therefore, p cannot be paired with any Walrasian allocation for u^*, as in the proof of part (i) of Lemma 3. Fix an arbitrary i. The pair (p,x_i) is not strongly truthful for u^*, which shows part (ii.a) of *MP-SH*. Let (p', x_i') be a strongly truthful pair for u^*. By part (c.1) of the definition of function z, it follows that $z_i((p',x_i'),(p,x_{-i})) = x_i$. Since i is arbitrary, this shows that W satisfies *MP-SH*. ■

Proof of Lemma 6

We shall show only part (i) of the statement since part (ii) can be proved in the same way.

Let $n = 3$, let $\ell = 2$ and let $\Omega = (\Omega_1, \Omega_2) = (1,1)$. Following Saijo et al. (1999; proof of Lemma 1), suppose that each i announces (p, x_i), with $x_i = \left(0, \frac{\Omega_2}{2}\right)$. Thus, $\left|I^{NP}(p,x)\right| = n$ and for each i, $\Lambda_i^{NP}(x^i,p)$ is represented by the area of the Figure $\mathbf{0}AB$ plus the line BC (see Figure 1 below). Then, $z(p,x)$ which satisfies part (i) of *P-WH* is such that for any i, $z_i(p,x) = (\Omega_1, 0)$ and $z_j(p,x) = \left(0, \frac{\Omega_2}{2}\right)$ for all $j \neq i$. Without loss of generality, let us focus on the case that $z(p,x) = \left((\Omega_1, 0), \left(0, \frac{\Omega_2}{2}\right), \left(0, \frac{\Omega_2}{2}\right)\right)$. Assume, to the contrary, that NP satisfies *P-WH*. By Saijo et al. (1999; proof of Lemma 1), there exists $u^* \in \bar{U}$ such that $z(p,x) \notin NP(u^*)$ and for any i, $\Lambda_i^{NP}(x^i, p) \subseteq L(z_i(p,x), u_i^*)$. Figure 1 illustrates the economy u^*, where u_1^* induces the indifference curve I_1^* through $A \equiv (\Omega_1, 0)$, represented by the lines AD and DE; u_2^* induces the indifference curve I_2^* through $C \equiv \left(0, \frac{\Omega_2}{2}\right)$, denoted by the line CK, which is orthogonal to p and parallel to AB; u_3^* induces the indifference curve I_3^* through $C = \left(0, \frac{\Omega_2}{2}\right)$ and $F \equiv \left(\frac{\Omega_1}{2}, \frac{\Omega_2}{4}\right)$, denoted by the lines CF and FG. Let u^* represent an economy with homothetic preferences. It follows that I_1^* of u_1^* at $F = \left(\frac{\Omega_1}{2}, \frac{\Omega_2}{4}\right)$ is represented by HF and FG. If $x' \in P(u^*)$, then x'_1 is always on the ray from the origin which passes through D. Then, $z(p,x) \notin NP(u^*)$. Consider $\hat{x} \equiv (\hat{x}_1, x_2, \hat{x}_3) \in A$ such that $\hat{x}_1 = \hat{x}_3 = \left(\frac{\Omega_1}{2}, \frac{\Omega_2}{4}\right)$. Observe that $\hat{x}$ is Pareto superior to $z(p,x)$ at u^* and that $\hat{x} \in NP(u^*)$ with $p \in \pi^{NP}(\hat{x}, u^*)$. We established that x_2 is weakly truthful for u^*, in violation of part (iii.a) of *P-WH* for $H = \{2\}$. ■

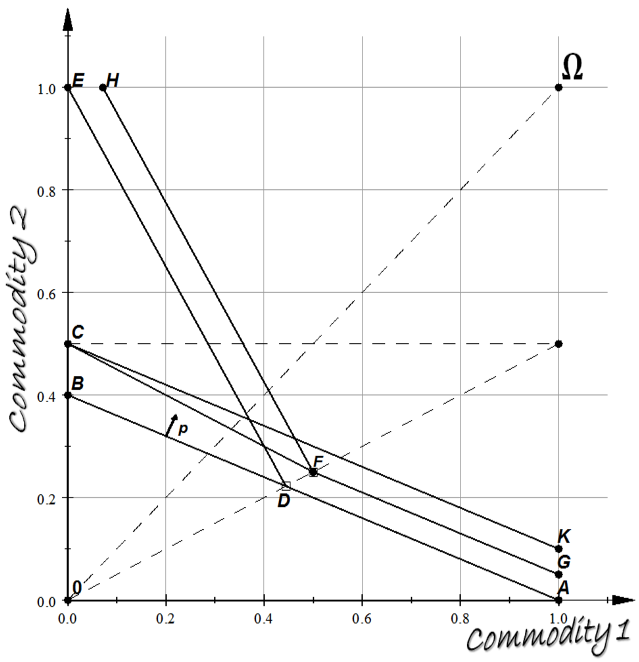

Figure 1: NP violates *P-WH* and *MP-SH*

Proof of Lemma 7

Fix any $u \in \bar{U}$ such that u_i is strictly concave and differentiable for each i. Let $x \in EE(u) \cap \mathbb{R}^{n\ell}_{++}$. Then $\{p\} = \pi^{EE}(x, u)$, from which $\Lambda^{EE}_i(x, p)$ can be constructed for each i. If $u' \in EE^{-1}(x, p)$, then for some $\lambda^{u'} \in (0, 1)$, $u'_i(x_i) = u'_i\left(\lambda^{u'}\Omega\right)$ for each i. Therefore, $\lambda^{u'}\Omega \in \partial L(x_i, u'_i)$ for each i if $u' \in EE^{-1}(x, p)$. Define $\lambda^{(x,p)}$ as $\lambda^{(x,p)} \equiv \min_{u' \in EE^{-1}(x,p)} \lambda^{u'}$. It follows that $\lambda^{(x,p)}\Omega \in \partial\Lambda^{EE}_i(x, p)$ for each i and that i's intersection set $\Lambda^{EE}_i(x, p)$ has the following property: For each i, there is a neighborhood $B(x_i) \subset \partial\Lambda^{EE}_i(x, p)$ of x_i such that for any $y_i \in B(x_i)$, $p \cdot y_i = p \cdot x_i$.

Let $u^* \in \bar{U}$ be such that u^*_i is strictly concave and differentiable for each i and let $\Lambda^{EE}_i(x, p) \subseteq L(x_i, u^*_i)$ for all i. Fix any two distinct agents $j, k \in N$. Suppose that for j and k and for some $\epsilon > 0$ it holds that $u^*_j(x_j + (\epsilon, \mathbf{0})) = u^*_j(\lambda^*\Omega)$ and $u^*_k(x_k - (\epsilon, \mathbf{0})) = u^*_k(\lambda^*\Omega)$ for some $\lambda^* > \lambda^{(x,p)}$, and it holds that the gradient vector of u^*_j at $(x_j + (\epsilon, \mathbf{0}))$ and the gradient vector of u^*_k at $(x_j - (\epsilon, \mathbf{0}))$ are equal to p: That is, $\nabla u^*_j(x_j + (\epsilon, \mathbf{0})) = \nabla u^*_k(x_k - (\epsilon, \mathbf{0})) = p$, where $\mathbf{0} \in \mathbb{R}^{\ell-1}_+$ is the $\ell - 1$-th dimensional zero vector. Let $u^*_i(x_i) = u^*_i(\lambda^*\Omega)$ for each $i \neq j, k$. By construction and the supposition that $u^* \in \bar{U}$, it follows that there exist λ^j and λ^k such that $\lambda^{(x,p)} < \lambda^j < \lambda^* < \lambda^k$, $u^*_j(x_j) = u^*_j\left(\lambda^j\Omega\right)$ and $u^*_k(x_k) = u^*_k\left(\lambda^k\Omega\right)$. Then, $x \in P(u^*) \setminus EE(u^*)$ and $x^* \in EE(u^*)$, where $x^*_j = (x_j + (\epsilon, \mathbf{0}))$, $x^*_k = (x_k - (\epsilon, \mathbf{0}))$, and $x^*_i = x_i$ for each $i \neq j, k$. Since EE is *essentially single-valued* and since each u^*_i is strictly concave,[19] $\{x^*\} = EE(u^*)$. Moreover, $p \in \pi^{EE}(x^*, u^*)$. It follows that for each $i \neq j, k$, x_i is weakly truthful for u^* and (p, x_i) is strongly truthful for u^*, in violation of *M-WH* and *MP-SH* for $i \neq j, k$. ■

References

[1] Bowles, S., Gintis, H., 2000. Walrasian Economics in Retrospect. *Quarterly Journal of Economics*, 115, 1411-1439.

[2] Camerer, C.F., 2003. *Behavioral Game Theory: Experiments in Strategic Interaction.* Princeton University Press, Princeton, New Jersey, USA.

[3] Corchón, L., Herrero, C., 2004. A decent proposal. *Spanish Economic Review,* 6, 107-125.

[4] Dutta, B., Sen, A., 1991. A necessary and sufficient condition for two-person Nash implementation. *Review of Economic Studies,* 58, 121-128.

[5] Dutta, B., Sen, A., 2012. Nash implementation with partially honest individuals. *Games and Economic Behavior*, 74, 154-169.

[6] Dutta, B., Sen, A., Vohra, R., 1995. Nash implementation through elementary mechanisms in economic environments. *Economic Design*, 1, 173-204.

[19] An SCC F is *essentially single-valued* if, for any $u \in U_N$, $x, x' \in F(u)$ implies that $u_i(x_i) = u_i(x'_i)$ for all $i \in N$.

[7] Eliaz, K., 2002. Fault-Tolerant Implementation. *Review of Economic Studies*, 69, 589-610.

[8] Elster, J., 1989. Social Norms and Economic Theory. *Journal of Economic Perspectives*, 3, 99-117.

[9] Elster, J., 1998. Emotions and Economic Theory. *Journal of Economic Literature*, 36, 47-74.

[10] Glazer, J., Rubinstein, A., 1998. Motives and Implementation: On the Design of Mechanisms to Elicit Opinions. *Journal of Economic Theory*, 79, 157-173.

[11] Greene, J. D., Paxton, J.M., 2009. Patterns of Neural Activity Associated with Honest and Dishonest Moral Decisions. *Proceedings of National Academy of Science*, 106, 12506-12511.

[12] Hurwicz, L., 1978. On the Interaction between Information and Incentives in Organizations, in Krippendorff, K. *Communication and Control in Society* (Scientific Publishers, Inc., New York), 123-147.

[13] Hurwicz, L., 1979. Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points. *Review of Economic Studies*, 46, 217-225.

[14] Hurwicz, L., Maskin, E., and Postlewaite A., 1995. Feasible Nash implementation of social choice rules when the designer does not know endowments or production sets, in Ledyard , J. (ed.), *The economics of informational decentralization: Complexity, efficiency, and stability*, Essays in Honor of Stanley Reiter, pp. 367-433. Amsterdam: Kluwer Academic Publishers.

[15] Jackson, M., 1992. Implementation in Undominated Strategies: A Look at Bounded Mechanisms. *Review of Economic Studies*, 59, 757-775.

[16] Kartik, N., Tercieux, O., 2012. Implementation with evidence. *Theoretical Economics*, 7, 323-355.

[17] Kreps, D.M., 1997. Intrinsic Motivation and Extrinsic Incentives. *American Economic Review*, 87, 359-364.

[18] Lombardi, M., Yoshihara, N., 2012a. A Full Characterization of Nash Implementation with Strategy Space Reduction. Forthcoming in *Economic Theory.*

[19] Lombardi, M., Yoshihara, N., 2012b. Natural implementation with partially honest agents. *IER Discussion Paper Series A*, No.561, The Institute of Economic Research, Hitotsubashi University.

[20] Lombardi, M., Yoshihara, N., 2013. Partially-honest Nash implementation: A Full Characterization. Available at *SSRN*: http://ssrn.com/abstract=2232274.

[21] Maskin, E., 1999. Nash equilibrium and welfare optimality. *Review of Economic Studies*, 66, 23-38.

[22] Matsushima, H., 2008a. Role of honesty in full implementation. *Journal of Economic Theory*, 139, 353-359.

[23] Matsushima, H., 2008b. Behavioral Aspects of Implementation Theory. *Economics Letters*, 100, 161-164.

[24] Moore, J., Repullo, R., 1990. Nash implementation: A full characterization. *Econometrica*, 58, 1083-1100.

[25] Otani, Y., Sicilian, J., 1982. Equilibrium of Walras Preference Games. *Journal of Economic Theory*, 27, 47-68.

[26] Postlewaite, A., Wetterstein, D., 1989. Feasible and Continuous Implementation. *Review of Economic Studies*, 56, 603-611.

[27] Saijo, T., 1988. Strategy space reduction in Maskin's theorem: sufficient conditions for Nash implementation. *Econometrica*, 56, 693-700.

[28] Saijo, T., Tatamitani, Y., Yamato, T., 1996. Toward natural implementation. *International Economic Review*, 37, 949-980.

[29] Saijo, T., Tatamitani, Y., Yamato, T., 1999. Characterizing natural implementability: The fair and Walrasian correspondences. *Games and Economic Behavior*, 28, 271-293.

[30] Schmeidler, D., 1980. Walrasian Analysis via Strategic Outcome Functions. *Econometrica*, 48, 1585-1593.

[31] Sen, A.K., 1997. Maximization and the act of choice. *Econometrica*, 65, 745-779.

[32] Sjöström, T., 1996. Implementation by demand mechanisms. *Economic Design*, 1, 343-354.

[33] Suzumura, K., Xu, Y., 2001. Characterizations of Consequentialism and Nonconsequentialism. *Journal of Economic Theory*, 101, 423-436.

[34] Thomson, W., 1984. The Manipulability of Resource Allocation Mechanisms. *Review of Economic Studies*, 51, 447-460.

[35] Tian, G., 1992. Implementation of the Walrasian Correspondence without Continuous, Convex, and Ordered Preferences, *Social Choice and Welfare,* 9, 117-130.

[36] Tian, G., 2000. Incentive mechanism design for production economies with both private and public ownerships, *Games and Economic Behavior,* 33, 294-320.