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The Measurement of Labour Content: A General Approach

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Abstract

This paper analyses the theoretical issues related to the measurement of labour content in the context of general technologies with heterogeneous labour. A novel axiomatic framework is used in order to formulate the key properties of the notion of labour content and analyse its theoretical foundations. Then, a simple measure of labour content is uniquely characterised, which is consistent with common practice in input-output analysis and with a number of recent approaches in value theory.

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1 Introduction

The measurement of the labour content of produced goods plays a central role in many different fields in economics, and it is rather controversial. It is important in input-output theory, in classical political economy, and also in a number of empirical analyses. The literature is too vast for a comprehensive list of references, but examples include productivity analysis;[1] structural macrodynamic models;[2] and studies of the relation between technical change and profitability.[3]

In normative economics, the notion of labour content is fundamental in the theory of exploitation as the unequal exchange of labour,[4] but it also plays a pivotal - albeit often implicit - role in Kantian approaches to distributive justice.[5]

Last but not least, labour content is a critical concept in classical and Marxian price and value theory, and many debates have revolved around the notion of labour embodied.[6]

Outside of simple models adopting a Leontief technology with a single type of homogeneous labour, however, the concept of labour content is elusive and controversial, and there exists no widely accepted approach. In productivity analysis, for example, various alternative indices of quality-adjusted labour inputs have been proposed in studies of total factor productivity.[7] Further, it is well known that in the various strands of the literature mentioned above, many of the insights that hold in simple linear models are not necessarily valid in the context of more general technologies, especially if heterogeneous labour inputs are allowed for. It is for this reason that the

[1] See Gollop and Jorgenson ([18]), Jorgenson, Gollop, and Fraumeni ([23]), Bureau of Labor Statistics ([1]), and Ho and Jorgenson ([20]). See also the contributions in footnote 7 below. Gupta and Steedman ([19]) and Flaschel et al. ([12]) provide an analysis of labour content and labour productivity within an input-output theoretic framework.

[2] The classic reference is Pasinetti ([35], [36], [37]). More recent contributions include Lavoie ([27]) and Trigg and Hartwig ([45]).

[3] See Roemer's ([39], [40]) analysis of technical change in classical linear models.

[4] See Roemer's classic contributions (Roemer [41]) and, more recently, Fleurbaey ([13], [14]), Yoshihara ([50], [51]), Veneziani ([46], [47]), and Veneziani and Yoshihara ([48])

[5] See the analysis of Kantian allocations and the so-called *proportional solution* in Roemer ([42], [43]).

[6] For a thorough discussion, see Desai ([6]) and Flaschel ([11]).

[7] The literature here is vast too: an illustrative but far from comprehensive selection of contributions includes: Denison ([5]), Jorgenson and Griliches ([24]), Chinloy ([2]), Wolff and Howell ([49]), Jorgenson ([22]) and Ho and Jorgenson ([21]).

analysis is often restricted to models with one type of labour.

This paper tackles the issue of the appropriate *measure of labour content* (henceforth, MLC) for general production technologies and heterogeneous labour inputs (described in section 2), by rigorously stating and explicitly discussing some foundational properties that a MLC should satisfy.

One key, novel contribution of the paper is methodological: rather than proposing a MLC and comparing it with other measures in the literature, an axiomatic approach is adopted and the appropriate way of measuring labour content is discussed starting from first principles. Although this approach is standard in theories of inequality and poverty measurement (Foster [16]), this paper provides the first application of axiomatic analysis to the measurement of labour content and to quality-adjusted indices of labour inputs, and one of the first applications to classical political economy.[8]

To be specific, in section 3, a MLC is conceptualised as a *binary relation* defined over pairs of bundles of goods, associated production activities and price vectors such that it is possible and meaningful to say that a certain bundle produced with a certain activity at some prices contains more or less labour than another one.

In sections 4 and 5, we study MLCs that are transitive and complete when comparing the labour content of produced goods *at given prices* - called, (p, w)-labour orderings. Four axioms are analysed which capture key properties of (p, w)-labour orderings. *Dominance* says that if the production of a bundle of goods requires a strictly higher amount of each type of labour, then labour content is strictly higher. *Labour trade-offs* rules out the possibility that there exists one type of labour that always determines the labour content of any bundle of produced goods. *Mixture invariance* restricts the way in which the measurement of labour content varies when different production techniques are combined. Finally, *Consistency with Progressive Technical Change* incorporates a classical intuition that capital-using labour-saving technical change should increase labour productivity and decrease labour content.

The first substantive contribution of the paper is the proof that, perhaps strikingly, there is only one (p, w)-labour ordering that satisfies the four properties (Theorem 4). According to this MLC (formally defined in section

[8] Relevant exceptions include recent analyses of labour productivity (Flaschel et al. [12]) and of exploitation as the unequal exchange of labour (Yoshihara [51]; Yoshihara and Veneziani [52], [53]; Veneziani and Yoshihara [48]).

4), in the measurement of labour content, different types of labour should be converted into a single unit by using relative wages. Thus, the labour contained in a vector of net output is higher than the labour contained in another bundle if and only if its total wage costs are greater.

Section 6 generalises this result to comparisons of the labour content of produced goods *when prices may change*: Theorem 7 proves that the wage additive MLC is the only ordering that satisfies the above axioms and an additional property - called *Consistency with Labour-Saving Technical Change* - according to which if technical change does not affect capital input requirements but reduces labour inputs - e.g., due to improvement in the organisation of labour, - then labour content goes down.

This paper therefore provides sound theoretical foundations to the standard practice of measuring labour inputs based on wage costs in input-output theory, and to the main quality-adjusted indices of labour input developed in productivity analysis.[9]

The MLC characterised in Theorems 4 and 7 is also consistent with the received conception of a "quantity of labour" in classical political economy, whereby "the different kinds of labour are to be aggregated via the (gold) money wage rates" (Kurz and Salvadori [26], p.324).[10] Further, unlike in some classic approaches to value theory, such as Morishima ([32]) and Roemer ([41]), which are based on counterfactuals, the wage additive measure is entirely based on observable data concerning production processes and prices and wages.[11]

Certainly, the axioms discussed below are not the only conceivable properties that might be imposed on a MLC. Other axioms can be identified, which would in principle lead to different results. Yet the axioms formalised in this paper have robust theoretical foundations and impose rather weak

[9] See Denison ([5]), Jorgenson and Griliches ([24]), Chinloy ([2]), Jorgenson, Gollop, and Fraumeni ([23]), Jorgenson ([22]), and Ho and Jorgenson ([21]). For an alternative approach focusing on job-based measures of labour skill requirements, see Wolff and Howell ([49]).

[10] Despite some debates on the concept of "abstract labour", the wage-additive measure is consistent also with Marx's ([29], pp.51-2) views on the conversion of complex labour into simple labour, although he refers to a social process, fixed by custom. See Morishima ([32]), Kurz and Salvadori ([26], p.324), Dumenil et al. ([9]).

[11] In this respect, the definition characterised in Theorem 4 is closely related to monetary approaches to value theory that emphasise the importance of actual economic data, such as the 'New Interpretation' (Duménil [8]; Foley [15]; Mohun [31]; Duménil et al. [9]) and the definition of 'actual labour values' by Flaschel ([10], [11]).

restrictions on MLCs. Indeed, they incorporate properties often explicitly or implicitly advocated in the literature.

Perhaps more importantly, from a methodological viewpoint, the explicit statement of the properties that a MLC should satisfy clarifies the intuitions behind alternative measures and the normative and positive differences between them. As a result, the axiomatic approach adopted in this paper can be seen as laying the ground for a fruitful discussion of the foundations of the notion of labour content.

2 The basic framework

Consider general economies in which the production of commodities requires the use of produced goods and of different types of labour. There are n produced goods in the economy, which may be consumed and/or used as inputs in different production activities. The set of types of labour inputs (potentially) used in production is denoted as $\mathcal{T} = \{1, ..., T\}$, with generic elements $\nu, \mu \in \mathcal{T}$.

For any integer $m > 0$, let $\mathbb{R}^m$ (resp., $\mathbb{R}^m_+$, $\mathbb{R}^m_{++}$) denote the (resp., non-negative, strictly positive) m-dimensional Euclidean space. Production technology is described by a production set P, which has elements - activities - of the form $a = (-a_l, -\underline{a}, \overline{a})$, where $a_l \equiv (a_{l\nu})_{\nu \in \mathcal{T}} \in \mathbb{R}^T_+$ is a profile of labour inputs used in the production process and measured in hours; $\underline{a} \in \mathbb{R}^n_+$ are the inputs of the produced goods used; and $\overline{a} \in \mathbb{R}^n_+$ are the outputs of the n goods. Thus, elements of P are vectors in $\mathbb{R}^{2n+T}$. The net output vector arising from activity a is denoted as $\widehat{a} \equiv \overline{a} - \underline{a}$.

This modelling of production allows for any type of heterogeneity in labour inputs and the standard production technologies with homogeneous labour are contained as special cases.[12] Different technologies requiring different types of heterogeneous labour can be represented by different production sets P. For instance, the difference in *labour intensity* of each type of labour due to the difference in skill or human capital within the same type is reflected in the difference of production sets, since labour input vectors of

[12] For example, economies with homogenous labour inputs but agents with heterogenous labour skills are a special case. In such a case, there exists a production set $P^{\mathbf{s}} \subseteq \mathbb{R}_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+$ with a profile of labour skills $\mathbf{s} = (s_1, \ldots, s_T)$. Then, $a = \left(-\left(a_{l\nu}\right)_{\nu \in \mathcal{T}}, -\underline{a}, \overline{a}\right) \in P$ if and only if $a = (-a_l, -\underline{a}, \overline{a}) \in P^{\mathbf{s}}$ with $a_l = \sum_{\nu \in \mathcal{T}} s_\nu a_{l\nu}$.

production activities are measured in hours.[13]

In what follows, some economically meaningful and weak restrictions are imposed on the admissible class of production technologies.[14] Let $\mathbf{0} = (0, ..., 0)$ denote the null vector.

Assumption 0 (A0). P is a closed convex cone in $\mathbb{R}^{2n+T}$ and $\mathbf{0} \in P$.

Assumption 1 (A1). For all $a \in P$, if $\overline{a} \geq \mathbf{0}$ then $a_l \geq \mathbf{0}$.

Assumption 2 (A2). For all $c \in \mathbb{R}^n_+$, there is a $a \in P$ such that $\widehat{a} \geqq c$.

Assumption 3 (A3). For all $a \in P$, and for all $(-\underline{a}', \overline{a}') \in \mathbb{R}^n_- \times \mathbb{R}^n_+$, if $(-\underline{a}', \overline{a}') \leqq (-\underline{a}, \overline{a})$ then $(-a_l, -\underline{a}', \overline{a}') \in P$.

A0 allows for general constant returns to scale technologies with joint production. **A1** implies that labour is indispensable to produce any non-negative output vector. **A2** states that any non-negative commodity vector is producible as net output. **A3** is a standard *free disposal* condition. The set of all production sets that satisfy **A0**-**A3** is denoted as $\mathcal{P}$.

These assumptions are standard in production theory; they are rather general and include standard input-output models (such as the Leontief and von Neumann models) as well as the standard neoclassical growth model as special cases. It is worth emphasising, however, that the key insights of the paper do not crucially depend on the specific shape of the production set implied by them.

For any non-negative bundle $c \in \mathbb{R}^n_+$, the set of activities available in $P \in \mathcal{P}$ that produce at least c as net output is:

$$\phi^P(c) \equiv \{a \in P \mid \widehat{a} \geqq c\}.$$

Let $p = (p_1, ..., p_n) \in \mathbb{R}^n_+$ be the vector of prices of the n produced commodities and let $w = (w_1, ..., w_T) \in \mathbb{R}^T_+$ be the vector of the wages of the T types of labour. The price vector $(p, w) \in \mathbb{R}^{n+T}_+$ may either be part of some (classical or neoclassical) equilibrium concept, or it may be the vector

[13] Alternatively, one may define activity vectors by measuring each type of labour input in efficiency units, so that the amount of type-ν labour $a_{l\nu}$ would be the product of labour hours times the *intensity* of this type of labour. The two formulations are formally and - if market wages reflect differences in productivity and intensities - even observationally equivalent. However, we prefer the formulation in the main text since it is based on observable magnitudes and easily available and reliable data.

[14] The notation for vector inequalities is: for all $x, y \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$.

of observed market prices and wages. In the following discussion, we will focus on $w \in \mathbb{R}^T_{++}$, as typical and usual cases of economic environments.

3 Comparing labour content

The main purpose of our analysis is to identify some theoretically robust and widely shared intuitions about the measurement of labour content, and then analyse what they imply in terms of the appropriate MLC. Consequently, we aim to define axioms that impose restrictions on MLCs that are a priori as weak as possible, from both a formal and a theoretical viewpoint.

Thus, although one may think of many properties that a MLC should possess (including, for example, identifying a meaningful, *cardinal amount* of labour contained in a bundle), as a starting point, and consistently with the literature, we simply require that a MLC be able to compare the labour content of produced goods. This choice has two important implications.

First, the existence of an appropriate definition of labour content for non-produced goods is set aside. This is an interesting theoretical question with relevant implications, for example, in the analysis of environmental issues or in the economics of the household, but it is not the main focus of our analysis. As noted earlier, from an axiomatic perspective, in the first stage of the investigation it is appropriate to restrict the domain of the analysis in order to identify a set of theoretically robust properties and formally weak restrictions that are widely (albeit possibly implicitly) endorsed in the literature.

Second, if a key property of a MLC is to allow one to make meaningful statements of the form: "the bundle of produced goods c contains more labour than the bundle c'", then it can be conceptualised as *a binary relation* that allows us to compare the labour contained in different produced goods.

It is a priori unclear what type of information is necessary in order to make such comparisons. For example, it is not obvious whether only observable variables should matter, or rather one should focus on (possibly counterfactual) equilibrium allocations; whether or not price information should enter the definition of labour content; and so on. At this stage, we shall adopt the most general approach and allow the MLC to depend on all the potentially relevant information.

Formally, we consider *profiles* $(c, a; p, w)$, where $c \in \mathbb{R}^n_+$ is a non-negative bundle of goods producible as net output by using activity $a \in \phi^P(c)$ for some $P \in \mathcal{P}$ at the price vector $(p, w) \in \mathbb{R}^{n+T}_+$. Observe that this notation comprises all the information that might be *potentially* relevant to the measurement of labour content, but it does *not* imply, for example, that price information must enter the definition of the MLC.

Observe further that very few restrictions are imposed on the variables in the admissible profiles. For example, they might be based purely on actual data, or they might be determined (possibly counterfactually) from optimal, equilibrium behaviour. Indeed, the only restriction imposed on the profiles $(c, a; p, w), (c', a'; p', w')$ is that the vectors c and c' be productively feasible according to *some* technologies - a and a', respectively, - but a and a' are not even restricted to be *in the same production set*. In fact, it may be desirable in principle to compare the labour content of one (or more) vectors of net outputs, say, in nations with different technologies, or - in a dynamic perspective - as technology evolves over time.

Let the set of such profiles $(c, a; p, w)$ be denoted by $\mathcal{CP}$. Then:

Definition 1 *A measure of labour content is a binary relation* $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$ *such that for any* $(c, a; p, w), (c', a'; p', w') \in \mathcal{CP}$*, vector* c *produced with* a *at* (p, w) contains at least as much labour *as vector* c' *produced with* a' *at* (p', w') *if and only if* $(c, a; p, w) \succcurlyeq (c', a'; p', w')$.

Definition 1 provides a rigorous, general framework to study MLCs. For the specification of the desirable properties of a MLC can be seen as the identification of a set of axioms capturing different properties of the binary relation $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$. Note, for example, that it imposes no restrictions on the transitivity and completeness of the relation $\succcurlyeq$.[15] This is important because different views can be expressed concerning the comparability of labour content when prices vary, especially if the analysis is not restricted to equilibrium allocations.

Similarly, Definition 1 imposes no restriction on the role of prices in the measurement of labour content. A central question concerns whether prices should enter the definition of labour content and, if so, whether only equilibrium prices should matter. This is a rather controversial issue and various views have been proposed in the literature, depending also on the focus of the analysis. Definition 1 is compatible with different views: at this stage, we simply allow for the *possibility* that the measurement of labour content depends on (equilibrium or disequilibrium) prices. Further, as noted above,

[15] Let $x \equiv (c, a; p, w)$. For any $x, x', x'' \in \mathcal{CP}$, $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$ is *transitive* if and only if $x \succcurlyeq x'$ and $x' \succcurlyeq x''$ implies $x \succcurlyeq x''$; and it is *complete* if and only if $x \succcurlyeq x'$ or $x' \succcurlyeq x$.

by allowing the binary relation $\succcurlyeq$ to be potentially incomplete, Definition 1 allows for the possibility that the measurement of labour content be restricted to comparing bundle/technology pairs $(c, a), (c', a')$ only at given prices (p, w).

The analysis of these issues, and in general of the desirable properties that $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$ should possess, is the topic of the next sections. In what follows, for any $(c, a; p, w), (c', a'; p', w') \in \mathcal{CP}$, the asymmetric and the symmetric factors of $\succcurlyeq$ are denoted, respectively, as $\succ$ and $\sim$. They stand, respectively, for "contains strictly more labour than" and "contains the same amount of labour as".[16]

4 The foundations of labour measurement

The main aim of this paper is to identify some basic, minimal properties that a MLC should satisfy. The axioms presented have robust theoretical foundations and incorporate properties often explicitly or implicitly advocated in the literature, and they impose rather weak restrictions on the MLC. Thus, these axioms arguably form the core of the measurement of labour content.

As a first step, this section focuses on a subset of the set of possible MLCs by restricting attention to measures that can rank any bundles for a *given, constant price vector.* Formally:

Definition 2 *For all (p, w), a measure of labour content $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$ is a (p, w)*-labour ordering *if there exists an ordering $\succcurlyeq_{(p,w)} \subseteq \mathbb{R}_+^T \times \mathbb{R}_+^T$ such that for any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, $(c, a; p, w) \succcurlyeq (c', a'; p, w)$ if and only if $a_l \succcurlyeq_{(p,w)} a'_l$.*

Two properties of Definition 2 should be noted. First, since the binary relation $\succcurlyeq_{(p,w)} \subseteq \mathbb{R}_+^T \times \mathbb{R}_+^T$ is an *ordering*, it is reflexive, transitive and complete. Therefore, Definition 2 implies that, *for any given price vector*, the MLC should be able to compare any two bundles and when several bundles of produced goods are considered, it should be possible to say which one contains more labour. It may be argued that *in general* completeness and transitivity are desirable properties for any MLC, and may even be necessary

[16]Let $x \equiv (c, a; p, w)$. For all $x, x' \in \mathcal{CP}$, the asymmetric part $\succ$ of $\succcurlyeq$ is defined by $x \succ x'$ if and only if $x \succcurlyeq x'$ and $x' \not\succcurlyeq x$; and the symmetric part $\sim$ of $\succcurlyeq$ is defined by $x \sim x'$ if and only if $x \succcurlyeq x'$ and $x' \succcurlyeq x$.

for any consistent evaluation. Definition 2 is less demanding, and possibly less controversial, as it requires these properties to hold only in a given economic environment.[17]

Second, although in Definition 2 the measurement of labour content is based on the vector of *direct* labour inputs used in production, this does not imply that *indirect* labour - that is, the labour contained in produced inputs used in the production process - plays no role in the analysis. The emphasis on direct labour is motivated by the focus on the measurement of the labour content of bundles that are (or can be) produced as *net output* of a production process. By A0-A3, the vector a_l represents the amount of each type of labour used directly in the production of net output $c \leqq \overline{a}-\underline{a}$, *and* also in the production of the capital goods $\underline{a}$ used in the production of c. As is well known in input-output analysis, for example, in the standard Leontief model, the amount of (homogeneous) direct labour used in production corresponds to the total amount of *direct* and *indirect* labour invested to produce a vector of net outputs.

In the rest of this section, we identify some theoretically relevant and formally weak restrictions on (p, w)-*labour orderings*. The first property seems uncontroversial: it states that, given a price vector (p, w), if a bundle of produced goods c requires a strictly higher amount of *every* type of labour than a bundle c', then it contains more labour. Formally:

Dominance (D): For any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, if $a_l > a'_l$, then $a_l \succ_{(p,w)} a'_l$.

It might be argued that, for a given price vector (p, w), it should be sufficient for the amount of *one* type of labour to be strictly greater in a_l than in a'_l to conclude that c contains more labour than c'. This seems reasonable, for example, in an input-output analysis aimed at capturing labour multipliers. Yet classical authors have long argued that one should distinguish productive and unproductive labour and not all types of labour are relevant to capture the labour content of a bundle of produced goods. This is an important issue, but we need not adjudicate it here. Given that we aim to lay out some *minimal* desirable properties that any MLC should satisfy, it is theoretically appropriate to focus on the weaker, and less controversial, condition **D**.

[17] It is worth emphasising, again, that Definition 2 does *not* imply that a MLC must incorporate price information, but only that it *can* do so.

The next property states that the MLC should allow for trade-offs between different types of labour used in production. To be precise, for a given price vector (p, w), for any pair of labour types ν and μ, there exist two production activities which only differ in the amount of labour of types ν and μ used and yield the same labour content, but one of them uses more of type-ν labour while the other uses more of type-μ labour.

Labour Trade-offs (LT): For all $\nu, \mu \in \mathcal{T}$, $\nu \neq \mu$, there are $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$ for each $\zeta \neq \nu, \mu$, and $a_l \sim_{(p,w)} a'_l$.

Theoretically, axiom **LT** rules out the possibility that there exists one type of labour that always determines the labour content of produced goods. This does not preclude the possibility that some types of labour have a (possibly much) bigger weight in the determination of labour content than others. Yet, intuitively, if the amount of type-ν labour decreases, there exists a sufficient increase in the amount of type-μ labour used in production that can *conceivably* compensate for it in the measurement of labour content. Formally, the axiom imposes a rather weak restriction in that it only requires that, for any pair of labour types $\nu, \mu \in \mathcal{T}$, there exists *one* pair of production activities in the set of all *conceivable* production techniques which yield the same amount of labour in producing *some* (possibly different) net output vectors.

The last axiom of this section imposes a minimal requirement of consistency in labour measurement. It states that, for a given price vector (p, w), if two vectors of labour inputs dominate (in terms of corresponding labour content) another pair of vectors, then convex combinations of the former should dominate convex combinations of the latter.

Mixture Invariance (MI): Let $(c, a; p, w), (c', a'; p, w), (\widetilde{c}, \widetilde{a}; p, w), (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$. Given $\tau \in (0, 1)$, let $a^{\tau}_l = \tau a_l + (1 - \tau) \widetilde{a}_l$ and $a'^{\tau}_l = \tau a'_l + (1 - \tau) \widetilde{a}'_l$. Then, $a^{\tau}_l \succ_{(p,w)} a'^{\tau}_l$ holds, whenever $a_l \succ_{(p,w)} a'_l$ and $\widetilde{a}_l \succsim_{(p,w)} \widetilde{a}'_l$.

Note that $a^{\tau}_l \succ_{(p,w)} a'^{\tau}_l$ implies $(c^{\tau}, a^{\tau}; p, w), (c'^{\tau}, a'^{\tau}; p, w) \in \mathcal{CP}$. The latter property is guaranteed by the universal class of production sets $\mathcal{P}$ and the convexity of the production sets, without loss of generality.

To see why **MI** is a desirable property, suppose that both a and $\widetilde{a}$ produce bundle c as net output, while a' and $\widetilde{a}'$ produce c'. If **MI** were violated, then it would be possible to conclude that, overall, c' contains more labour than

c when, say, a proportion $\tau \in (0,1)$ of the firms use a and a' to produce, respectively, c and c' (and a proportion $(1-\tau)$ use $\widetilde{a}$ and $\widetilde{a}'$ to produce, respectively, c and c'), even though for each individual activity (and firm) using a and a', c contains more labour than c', and the same holds for $\widetilde{a}$ and $\widetilde{a}'$. Or, consider firms 1 and 2 producing, respectively, c and c', and suppose that firm 1 (respectively, 2) uses technique a for a part $\tau \in (0,1)$ of the year and $\widetilde{a}$ for the rest of the year (respectively, a' and $\widetilde{a}'$). Then it would be possible to conclude that, overall, the labour contained in 1's net output is lower than that contained in 2's, despite the fact that in each part of the production period the opposite holds.

Observe that **MI** restricts the way in which a MLC ranks mixtures, starting from original bundles. However, it does *not* require that the amount of labour in a bundle should remain the same, nor does it impose significant restrictions on the way in which such amount should vary.

In order to illustrate the implications of the axioms, consider the standard definition of labour content. Let the Leontief technology with a $n \times n$ non-negative and productive matrix, A, and a $1 \times n$ positive vector, L, of homogeneous labour requirements be represented by

$$P_{(A,L)} \equiv \left\{ a \in \mathbb{R}_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+ \mid \exists x \in \mathbb{R}^n_+ : a \leqq (-Lx, -Ax, x) \right\},$$

and let $\mathcal{P}_{(\mathcal{A},\mathcal{L})} \subset \mathcal{P}$ denote the set of all Leontief technologies.

In input-output theory and classical approaches, the vector of labour multipliers is defined as $v = L(I-A)^{-1}$. Therefore, for any $(c, a; p, w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$ such that $a = (-Lx, -Ax, x)$, the labour content of c is defined as $vc = Lx$. It is then immediate to show that the standard MLC satisfies the axioms in this section. To see that **D** is satisfied, note that for any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$, $Lx > L'x'$ immediately implies $a_l \succ_{(p,w)} a'_l$. To see that **MI** is satisfied, consider $(c, a; p, w), (c', a'; p, w), (\widetilde{c}, a; p, w), (\widetilde{c}', a'; p, w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$ such that $Lx > L'x'$ and $\widetilde{L}\widetilde{x} \geqq \widetilde{L}'\widetilde{x}'$. Then, for any $\tau \in (0,1)$, $a^\tau_l = \tau Lx + (1-\tau)\widetilde{L}\widetilde{x} > a'^\tau_l = \tau L'x' + (1-\tau)\widetilde{L}'\widetilde{x}'$, and so $a^\tau_l \succ_{(p,w)} a'^\tau_l$. Finally, because there is only one type of labour, **LT** is vacuously satisfied.

We conclude this section by noting that axioms **D**, **LT** and **MI** are analogous to well-known Paretian, anonymity and independence properties in social choice theory. However, the similarity is purely at the formal level: the interpretation and justification are completely different, and indeed some of the axioms are more defensible in the context of the measurement of labour content than in the context of welfare economics. Diamond's [7] classic critique of utilitarianism, for example, is based on the rejection of independence (or 'sure thing') principles analogous to **MI**. For 'mixing' welfare or opportunities across different individuals may produce ethically relevant effects.[18] Clearly, this normative argument does not apply in the context of the measurement of labour content.

5 Labour content: a characterisation

The previous section discusses some key properties that any MLC should satisfy. The axioms presented share one important feature: they are independent of price information. For any pairs of profiles with the same prices and wages (p, w), the axioms focus exclusively on information on productive conditions. As noted in section 5.1 below, these seemingly weak conditions are sufficient to impose significant restrictions on the measurement of labour content. Yet, although they identify one class of MLCs which share an important and intuitive property - namely, additivity in labour amounts, - they do not characterise a unique measure within that class.

In order to derive the main characterisation result, an additional condition is imposed which aims to capture the relation between technical changes and labour content in market economies. The axiom generalises an insight first proved rigorously by Roemer ([39]; see also Roemer [40] and Flaschel et al. [12]): any profitable (cost-reducing at current prices) technical change that is capital-using and labour-saving is *progressive*, that is, it leads to a decrease in labour content (and an increase in labour productivity). In the context of the standard linear models in which these results are derived, the definition of labour content is uncontroversial and so this insight is obtained as a result. However, given the theoretical relevance of the link between technical change, productivity and labour content in the literature, it may be argued that its epistemological status is as a postulate.[19]

The next axiom captures the labour-content-reducing effect of profitable capital-using technical change for profit maximising firms.

[18] For a discussion, see Mariotti and Veneziani ([28]).

[19] The link between labour content and labour productivity, for example, is central in Marx's theory: "In general, the greater the productiveness of labour, the less is the labour-time required for the production of an article, the less is the amount of labour crystallised in that article, and the less is its value; and vice versa. The value of a commodity, therefore, varies directly as the quantity, and inversely as the productiveness, of the labour incorporated in it" (Marx [29], p.48).

Consistency with Progressive Technical Change (CPTC): For any $(c, a; p, w), (c, a'; p, w) \in \mathcal{CP}$, if $p\underline{a} + wa_l > p\underline{a}' + wa'_l$ and $\underline{a} \leq \underline{a}'$, then $a_l \succ_{(p,w)} a'_l$.

Various features of **CPTC** are worth noting. First, the axiom focuses exclusively on innovations that (weakly) increase the amount of *all* physical inputs used in a given process. As a general definition of profitable capital-using technical progress, this may be considered too restrictive. However, our aim is not to provide a general theory of technological change and in the context of an axiomatic analysis of MLCs, focusing on a smaller set of technical changes imposes weaker restrictions on the MLC.

Second, although no condition is explicitly imposed on labour inputs, the changes considered are, in a relevant sense, labour-saving. To see this, consider the special case of economies with only one type of homogeneous labour. In this case, $p\underline{a}+wa_l > p\underline{a}'+wa'_l$ and $\underline{a} \leq \underline{a}'$ imply that $a_l > a'_l$, and so technical change is labour-saving. In economies with heterogeneous labour, cost-reducing and capital-using technical changes are not necessarily labour-saving *for all types of labour*. In other words, $p\underline{a}+wa_l > p\underline{a}'+wa'_l$ and $\underline{a} \leq \underline{a}'$ do not imply $a_l > a'_l$. However, the changes considered in **CPTC** do imply that the amount of at least one type of labour decreases, that is $a_{l\nu} > a'_{l\nu}$ for some $\nu \in \mathcal{T}$, and even if the amount of some labour input increases, this is more than outweighed by decreases in other types of labour. Thus, the axiom nicely captures, for example, some classic Marxian insights about the nature of technical change in market economies (Marx [29], chapter 23): capitalist dynamics always encourages capitalists to implement cost-reducing and capital-using technical change in order to reduce labour demand, which results in a reduction of labour costs.

Third, the axiom focuses on innovations that change the technological conditions of the production of a *given* net output vector c. This is theoretically intuitive, and it makes the axiom weaker, but our key result remains valid even if **CPTC** is strengthened to hold for any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, and allowing for the possibility that $c \neq c'$.

Finally, **CPTC** focuses on innovations that are cost-reducing *at current prices*: the effect of technical change on the price of commodities and on the wage rate is ignored. This is standard in the literature on progressive technical change (e.g., Morishima [32]; Roemer [39], [40]; Flaschel et al. [12]). We shall consider a strengthening of **CPTC** which allows for changes in the price vector in section 6 below.

Again, the standard definition of labour content in Leontief models with homogeneous labour satisfies **CPTC** in $\mathcal{CP}_{(\mathcal{A},\mathcal{L})}$. To see this, given a price vector $(p,w) \in \mathbb{R}_+^{n+1}$, consider any $(c,a;p,w),(c,a';p,w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$, such that $a = (-a_l, -Ax, x)$ and $a' = (-a_l', -A'x', x')$, where $a \in P_{(A,L)}$ and $a' \in P_{(A',L')}$. Suppose that the labour intensity is identical between a and a'. Then, without loss of generality, we can set $Lx = a_l$ and $L'x' = a_l'$. In this setting, if $pAx + wLx > pA'x' + wL'x'$ and $Ax \leq A'x'$, then $Lx > L'x'$ and so $a_l \succ_{(p,w)} a_l'$.

Perhaps strikingly, if one endorses **CPTC** together with the three axioms in section 4, then one must conclude that the labour content of a bundle of produced goods should be measured as the weighted average of the different types of labour used in its production, with the weights given by the relevant wages. Formally:

Definition 3 *For any given $(p,w) \in \mathbb{R}_+^{n+T}$, a (p,w)-labour ordering $\succsim$ is* (p,w)-wage additive *if, for all $(c,a;p,w),(c',a';p,w) \in \mathcal{CP}$, $a_l \succsim_{(p,w)} a_l'$ if and only if $\sum_{\nu \in \mathcal{T}} w_\nu a_{l\nu} \geqq \sum_{\nu \in \mathcal{T}} w_\nu a_{l\nu}'$.*

The main characterisation result demonstrates that the only MLC that satisfies all axioms is indeed wage additive.[20]

Theorem 4 *A (p,w)-labour ordering $\succsim$ satisfies **Dominance**, **Labour Trade-offs**, **Mixture Invariance**, and **Consistency with Progressive Technical Change** if and only if it is (p,w)-wage additive.*

By Theorem 4, the labour content of a bundle of goods produced as net output should be measured as its total wage costs. If a small number of widely (albeit often implicitly) accepted principles of labour measurement with sound theoretical foundations are adopted, which impose rather weak formal restrictions on MLCs, then the vexed issue of how to convert different types of labour into a single measure has a unique, simple and intuitive answer: relative wages should be used to homogenise different types of labour.

[20] The proof of Theorem 4 is in Appendix A.

5.1 Discussion

Theorem 4 provides rigorous axiomatic foundations to the standard practice of measuring labour inputs based on wage costs in the input-output literature as well as in empirical studies on total factor productivity. It is also consistent with the views of classical political economy on the conversion of complex labour into simple labour.[21] Indeed, Theorem 4 suggests that the wage additive measure is the appropriate generalisation of the standard MLC universally used in linear economies with homogeneous labour. For the wage additive measure reduces to the standard MLC in those economies and, as shown above, the standard MLC satisfies all of the axioms on the set $\mathcal{CP}_{(\mathcal{A},\mathcal{L})} \subset \mathcal{CP}$.

Certainly, the characterisation result depends on the specific set of axioms chosen, and alternative axioms would yield different MLCs. As discussed in the concluding section below, we see this as a virtue, rather than a shortcoming of the axiomatic approach, for it helps to clarify the theoretical foundations and properties of different measures.

It is noteworthy, however, that the key conclusions of Theorem 4 are quite robust, and can be obtained with a number of different axioms. For example, given the emphasis on the effect of capitalist behaviour and technological progress on labour productivity in the literature, it is arguably desirable to have an axiom capturing the relation between (cost reducing) technical change and labour content. Axiom **CPTC** is one - particularly clear and

[21] "It is often difficult to ascertain the proportion between two different quantities of labour. The time spent in two different sorts of work will not always alone determine this proportion. The different degrees of hardship endured, and of ingenuity exercised, must likewise be taken into account. There may be more labour in an hour's hard work, than in two hours easy business; or in an hour's application to a trade which it cost ten years labour to learn, than in a month's industry, at an ordinary and obvious employment. But it is not easy to find any accurate measure either of hardship or ingenuity. In exchanging, indeed, the different productions of different sorts of labour for one another, some allowance is commonly made for both. It is adjusted, however, not by any accurate measure, but by the higgling and bargaining of the market, according to that sort of rough equality which, though not exact, is sufficient for carrying on the business of common life." (Smith [44], chapter V, pp.34-35.) "The estimation in which different quantities of labour are held, comes soon to be adjusted in the market with sufficient precision for all practical purposes, and depend much on the comparative skill of the labourer, and intensity of the labour performed." (Ricardo [38], chapter I, section II, p. 11.) See also Marx ([29], pp.51-2). For a comprehensive survey about the treatment of heterogenous labour in the classical theory, see Kurz and Salvadori ([26], chapter 11).

intuitive - way of formalising such relation, but Theorem 4 can be derived under a number of alternative specifications.

Perhaps more interestingly, even if **CPTC** is dropped altogether (possibly on the ground that price information should not directly enter the measurement of labour content), the other three technology-based axioms are sufficient to conclude that the MLC should be additive. Formally:

Definition 5 *For any given* $(p, w) \in \mathbb{R}_{+}^{n+T}$*, a* (p, w)*-labour ordering* $\succcurlyeq$ *is* additive *if, for all* $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$*, there is some* $\sigma_{(p,w)} \in \mathbb{R}_{++}^{T}$ *such that for all* $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$*,* $a_l \succcurlyeq_{(p,w)} a'_l$ *if and only if* $\sum_{\nu \in \mathcal{T}} \sigma_{(p,w)}^{\nu} a_{l\nu} \geqq \sum_{\nu \in \mathcal{T}} \sigma_{(p,w)}^{\nu} a'_{l\nu}$.

The demonstration of Theorem 4 in Appendix A can be used to prove that a (p, w)-labour ordering $\succcurlyeq$ satisfies Dominance, Labour Trade-offs, and Mixture Invariance if and only if it is additive. Although this does not uniquely characterise a MLC, it does identify a class of measures which share an important property: the labour content of a vector of net outputs is a weighted average of the amounts of different types of labour used to produce them. This additive structure is often considered as a fundamental property of a MLC and thus implicitly postulated *as an axiom* (for example, in input-output theory and in classical-Marxian approaches; see Krause [25], Duménil et al. [9], and the thorough discussion in Flaschel [11]).[22] Instead, additivity is here derived as a *result* starting from more foundational principles.

Finally, although the main contribution of this paper is conceptual, it is worth noting in passing that, from a purely formal viewpoint, the arguments in Appendix A provide an independent characterisation of the so-called *weak weighted utilitarian* ordering which is analysed in social choice theory in the context of evaluating welfare profiles.[23]

[22] For instance, both Krause ([25]) and Duménil et al. ([9]) *define* labour content as the weighted sum of the labour hours of all types. In Krause ([25]) the weights are given by the *reduction vector*, which is defined as the Frobenius eigenvector of the matrix $\mathbf{H} =< h_{ij} >$, where h_{ij} is the amount of type-i labour required directly or indirectly to reproduce one unit of type-j labour. See also Okishio ([33], [34]) and Fujimori ([17]), where the former is the first work which proposes a mathematical definition of reduction vector independently of price information, though its definition is different from Krause ([25]). In Dumenil et al. ([9]), in contrast, the weights are not explicitly determined but they are given by the ratio of the wage of each type of labour to the average wage whenever wages are proportional to the capacity of each category of labor to create value.

[23] Actually, standard results in social choice theory highlight the robustness of the main

6 A generalisation

Theorem 4 characterises a measure that allows to compare any pairs of produced bundles, *at a given price vector*. Formally, the MLC is transitive and complete over profiles $(c, a; p, w), (c', a'; p', w') \in \mathcal{CP}$ such that $(p, w) = (p', w')$. However, it is silent whenever profiles with $(p, w) \neq (p', w')$ are considered. This section analyses whether our result can be extended to hold for *any* profiles $(c, a; p, w), (c', a'; p', w') \in \mathcal{CP}$.

As a first step, we reformulate without further discussion the four axioms presented above as restrictions on the MLC $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$, without assuming the latter to be a (p, w)-labour ordering.

Dominance (D): For any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, if $a_l > a'_l$ then $(c, a; p, w) \succ (c', a'; p, w)$.

Labour Trade-offs (LT): For all $\nu, \mu \in \mathcal{T}$, $\nu \neq \mu$, and all $(p, w) \in \mathbb{R}^{n+T}_{+}$, there are $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$ for each $\zeta \neq \nu, \mu$, and $(c, a; p, w) \sim (c', a'; p, w)$.

Mixture Invariance (MI): Let $(c, a; p, w), (c', a'; p, w), (\widetilde{c}, \widetilde{a}; p, w), (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$. Given $\tau \in (0, 1)$, let $a^{\tau}_l = \tau a_l + (1 - \tau) \widetilde{a}_l$ and $a'^{\tau}_l = \tau a'_l + (1 - \tau) \widetilde{a}'_l$. Then, $(c^{\tau}, a^{\tau}; p, w) \succ (c'^{\tau}, a'^{\tau}; p, w)$ holds, whenever $(c, a; p, w) \succ (c', a'; p, w)$ and $(\widetilde{c}, \widetilde{a}; p, w) \succcurlyeq (\widetilde{c}', \widetilde{a}'; p, w)$.

Consistency with Progressive Technical Change (CPTC): For any $(c, a; p, w), (c, a'; p', w) \in \mathcal{CP}$, if $p\underline{a} + wa_l > p\underline{a}' + wa'_l$ and $\underline{a} \leq \underline{a}'$, then $(c, a; p, w) \succ (c, a'; p', w)$.

The next axiom states that if two bundles of produced goods require exactly the *same* vector of direct labour to be produced at the *same* prices, then they have the same labour content.

Equal Labour (EL): For any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, if $a_l = a'_l$ then $(c, a; p, w) \sim (c', a'; p, w)$.

conclusions of this paper. For it is well-known that weak weighted utilitarianism can be characterised based on various different sets of axioms, focusing for example on *invariance conditions*. See d'Aspremont ([3], Theorem 3.3.5, p.51), d'Aspremont and Gevers ([4], Theorem 4.2, p.509), Mitra and Ozbek ([30], Theorem 2, p.14). The axioms used in Theorem 4, however, are more intuitive and economically meaningful in the context of the measurement of labour content.

Axiom **EL** is theoretically related to **D** and it generalises an intuitive property of (p, w)-labour orderings to the larger domain of profiles $\mathcal{CP} \times \mathcal{CP}$.

Finally, we introduce another axiom that captures the relation between technical change and labour content. The theoretical justification of the axiom is similar to **CPTC** but it captures a different type of technological innovations - which alter the amount of labour inputs in production without changing capital requirements - and it allows the vector of wage rates to change. Formally:

Consistency with Labour-Saving Technical Change (CLSTC): For any $(c, a; p, w)$, $(c, a'; p, w') \in \mathcal{CP}$, if $p\underline{a} + wa_l > p\underline{a}' + w'a_l'$, $\underline{a} = \underline{a}'$, and $a_l \neq a_l'$, then $(c, a; p, w) \succ (c, a'; p, w')$.

The technical changes considered in **CLSTC** do not involve any modification in input requirements and so they can be interpreted as innovations in human resource management, or in the organisation of labour in production, that decrease the amount of direct labour necessary in production and therefore - given $\underline{a} = \underline{a}'$ - the overall labour content of a given bundle.

To see why **CLSTC** is an appealing property, suppose first that $w = w'$ (a possibility that is not ruled out in the axiom). In this case, technical change is cost-reducing at current prices and **CLSTC** represents a very mild strenghtening of **CPTC**, and the intuition is exactly the same.

Suppose next that $w \neq w'$ and that relative wages reflect the different productivities of different types of labour. The innovations considered in **CLSTC** imply either that the amount of productivity-adjusted labour of *all* types necessary in production decreases (weakly for all types and strictly for some of them); or that any increase in the amount of labour of some type (for example, managerial or supervisory labour) is more than compensated by the decrease in the labour input of other types. Given that the vector of capital inputs - and therefore, in principle, the amount of labour indirectly required to produce net output - is unchanged, a decrease in direct effective labour should unambiguously decrease labour content.

As in the case of **CPTC**, the standard definition of labour content used in Leontief models satisfies **CLSTC** in $\mathcal{CP}_{(\mathcal{A},\mathcal{L})}$. Let $(c, a; p, w), (c, a'; p, w') \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$ such that $a = (-a_l, -Ax, x) \in P_{(A,L)}$ and $a' = (-a_l', -Ax, x) \in P'_{(A,L')}$. Let $pAx + wa_l > pAx + w'a_l'$. Then, $wa_l > w'a_l'$ holds even if $w < w'$. This implies that there are underlying labour intensities or skills $s > 0$ and $s' > 0$ such that $\frac{w}{w'} = \frac{s}{s'}$ and $sa_l = Lx > L'x = s'a_l'$. Since Lx and $L'x$

are the labour contents in the standard Leontief model with homogeneous labour, $(c, a; p, w) \succ (c, a'; p, w')$ holds.

Together with **D**, **LT**, **MI** and **CPTC**, if one endorses **EL** and **CLSTC**, then one must conclude that the labour content of a bundle of produced goods should be measured as the weighted average of the different types of labour used in its production, with the weights given by the relevant wages, *even when the price vector changes.* Formally:

Definition 6 *A MLC* $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$ *is* wage additive *if for all* $(c, a; p, w), (c', a'; p', w') \in \mathcal{CP}$, $(c, a; p, w) \succcurlyeq (c', a'; p', w')$ *if and only if* $wa_l = \sum_{\nu \in \mathcal{T}} w_\nu a_{l\nu} \geqq \sum_{\nu \in \mathcal{T}} w'_\nu a'_{l\nu} = w'a'_l$.

The next result proves that the only reflexive, transitive and complete MLC that satisfies all axioms is indeed wage additive.[24]

Theorem 7 *A reflexive, transitive and complete MLC* $\succcurlyeq$ *satisfies* ***Dominance****,* ***Labour Trade-offs****,* ***Mixture Invariance****,* ***Equal Labour****,* ***Consistency with Progressive Technical Change*** *and* ***Consistency with Labour-Saving Technical Change*** *if and only if it is wage additive.*

7 Conclusion

This paper analyses the issue of the appropriate measurement of the labour content of produced goods. Measures of labour content are formally conceptualised as binary relations comparing bundles of goods produced with certain activities at a certain price vector. An axiomatic approach is adopted in order to identify some foundational properties that every MLC should satisfy. Strikingly, it is shown that a small number of axioms incorporating either technology-related properties or some widely held intuitions on the relation between technical progress and changes in labour productivity, and labour content, uniquely determine a simple MLC: the labour content of a bundle of goods produced as net output corresponds to the total wage costs of production. As in standard input-output theory, in classical political economy, and

[24] The proof of Theorem 7 is in Appendix B.

in productivity analysis, relative wages are used to convert different types of labour into a single measure.

The axiomatic analysis developed in this paper is motivated by the idea that the theoretical strength of a MLC depends - to a large extent - on the foundational principles that underlie it. There are two important caveats to make about this, which also suggest directions for further research.

First, the axiomatic characterisation of the wage additive MLC does not imply that it provides the only possible definition of labour content. Although the wage additive measure possesses a number of desirable features from both the theoretical and the empirical viewpoint, alternative measures can certainly be proposed that capture different intuitions, and have different properties. From this perspective, the adoption of an axiomatic analysis aims precisely at making the relevant assumptions and intuitions explicit and open to discussion and criticism. The point is not to tinker with alternative specifications of assumptions in order to demonstrate some variations on a theme. Rather, a rigorous statement of the main axioms is helpful in clearing the ground for discussions and in fostering dialogue, and further research, over foundational principles.

Second, it is certainly desirable for a MLC to have sound theoretical fondations. Yet one may argue that its cogency and usefulness ultimately rest on the insights that can be gained from using it. In this case, the fruitfulness of the wage additive measure can only be judged when it is applied to economically relevant problems. From this perspective, too, this paper can be seen only as a first, and preliminary step in a wider research programme.

A Proof of Theorem 4

First of all, we prove two technical Lemmas which are of some interest in their own right. Lemma 8 shows some convexity properties of the (p, w)-*labour ordering* $\succcurlyeq$.

Lemma 8 *Let* $\succcurlyeq_{(p,w)}$ *satisfy* ***Mixture Invariance****. Consider any set* $\left\{a_l^1, ..., a_l^K\right\}$, *such that* $\left(c^k, a^k; p, w\right) \in \mathcal{CP}$*, for all* $k = 1, ..., K$ *and* $a_l^i \sim_{(p,w)} a_l^j$*, for all* $i, j \in \{1, ..., K\}$*. Then, for all* $\{\tau_1, ..., \tau_K\}$ *such that* $\tau_i \in (0, 1)$ *all* $i \in \{1, ..., K\}$ *and* $\sum_{i=1}^K \tau_i = 1$, $\sum_{i=1}^K \tau_i a_l^i \sim_{(p,w)} a_l^j$*, for all* $j \in \{1, ..., K\}$.

Proof. We prove that for any pair a_l^i, a_l^j, $i, j \in \{1, ..., K\}$, $a_l^i \sim_{(p,w)} \tau a_l^i + (1-\tau)a_l^j \sim_{(p,w)} a_l^j$ for all $\tau \in (0,1)$. The desired conclusion then follows from repeated application of this result, given the transitivity of $\succcurlyeq_{(p,w)}$.

Step 1. Note that for any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, if $a_l \succ_{(p,w)} a_l'$, then by **MI**, and noting that by reflexivity of $\succcurlyeq_{(p,w)}$, $a_l \sim_{(p,w)} a_l$ and $a_l' \sim_{(p,w)} a_l'$, it follows that for all $\tau \in (0,1)$, $a_l \succ_{(p,w)} \tau a_l + (1-\tau)a_l' \succ_{(p,w)} a_l'$.

Step 2. Consider any pair $(c^i, a^i; p, w), (c^j, a^j; p, w)$, where $i, j \in \{1, ..., K\}$. Suppose, by way of contradiction, there exists some $\tau \in (0,1)$, such that $\tau a_l^i + (1-\tau)a_l^j \nsim_{(p,w)} a_l^j$. By completeness, suppose $\tau a_l^i + (1-\tau)a_l^j \succ_{(p,w)} a_l^j$, without loss of generality. Let $a^\tau \equiv \tau a^i + (1-\tau)a^j$.

Then, by *Step 1*, for all $t \in (0,1)$, $a_l^\tau \succ_{(p,w)} t a_l^\tau + (1-t)a_l^i \succ_{(p,w)} a_l^i \sim_{(p,w)} a_l^j$. However, by *Step 1*, for any given $t \in (0,1)$, we have that $ta^\tau + (1-t)a_l^i \succ_{(p,w)} \chi a_l^i + (1-\chi)a_l^j$ for all $\chi \in (0,1)$. [This is because by *Step 1* $ta^\tau + (1-t)a_l^i \succ_{(p,w)} h[ta^\tau + (1-t)a_l^i] + (1-h)\, a_l^i$ for all $h \in (0,1)$. However, $h[ta^\tau + (1-t)a_l^i] + (1-h)\, a_l^i = ga^\tau + (1-g)a_l^i = g[\tau a_l^i + (1-\tau)a_l^j] + (1-g)a_l^i$, where $g = ht$ and by setting $\chi = 1 - g(1-\tau)$ the latter expression follows.] Setting $\chi = \tau$ yields the desired contradiction. ■

The next Lemma proves that any two vectors with the same amount of labour content actually identify a direction in the T-dimensional space along which all vectors have the same labour content. This is a very useful property because, once a linear $T-1$ dimensional subset of the state space is identified which contains vectors with the same amount of labour, Lemma 9 allows to extend it in all directions.

Lemma 9 *Let $\succcurlyeq_{(p,w)}$ satisfy **Mixture Invariance**. Suppose $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ and $a_l \sim_{(p,w)} a_l'$. If $(c'', a''; p, w) \in \mathcal{CP}$ and there exists $t \in (0,1)$ such that $a_l = ta_l'' + (1-t)a_l'$, then $a_l'' \sim_{(p,w)} a_l \sim_{(p,w)} a_l'$.*

Proof. Suppose that $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ and $a_l \sim_{(p,w)} a_l'$. Suppose that $(c'', a''; p, w) \in \mathcal{CP}$ and there exists $t \in (0,1)$ such that $a_l = ta_l'' + (1-t)a_l'$, but $a_l'' \nsim_{(p,w)} a_l'$. By completeness, suppose $a_l'' \succ_{(p,w)} a_l'$, without loss of generality. By *Step 1* of the proof of Lemma 8, $a_l'' \succ_{(p,w)} \tau a_l'' + (1-\tau)a_l' \succ_{(p,w)} a_l'$ holds for all $\tau \in (0,1)$. The desired contradiction follows setting $\tau = t$. ■

We can now prove Theorem 4.

Proof of Theorem 4. (*Necessity*) We show that the (p, w)-wage additive MLC $\succcurlyeq$ satisfies the axioms.

To see that **D** is satisfied, note that for any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, if $a_l > a'_l$ then $wa_l > wa'_l$ and so $a_l \succ_{(p,w)} a'_l$.

To see that **LT** is satisfied, consider any $\nu, \mu \in \mathcal{T}$, $\nu \neq \mu$, and any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, with $w_\nu (a_{l\nu} - a'_{l\nu}) = w_\mu \left(a'_{l\mu} - a_{l\mu}\right)$ and $a_{l\zeta} = a'_{l\zeta}$, $\zeta \neq \nu, \mu$. Since $wa_l = wa'_l$, then $a_l \sim_{(p,w)} a'_l$.

To see that **MI** is satisfied, consider any $(c, a; p, w), (c', a'; p, w), (\widetilde{c}, \widetilde{a}; p, w), (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$, and suppose that $wa_l > wa'_l$ and $w\widetilde{a}_l \geqq w\widetilde{a}'_l$. Then for any $\tau \in (0, 1)$, $w (\tau a_l + (1 - \tau) \widetilde{a}_l) > w (\tau a'_l + (1 - \tau) \widetilde{a}'_l)$, and so $\tau a_l + (1 - \tau) \widetilde{a}_l \succ_{(p,w)} \tau a'_l + (1 - \tau) \widetilde{a}'_l$, as sought.

To see that **CPTC** is satisfied, take any $(c, a; p, w), (c, a'; p', w') \in \mathcal{CP}$ such that $p\underline{a} + wa_l > p\underline{a}' + w'a'_l$ and $\underline{a} \leq \underline{a}'$. For any $(p, w), (p', w') \in \mathbb{R}^{n+T}_+$ with $w, w' > \mathbf{0}$, $\underline{a} \leq \underline{a}'$ implies $p\underline{a} \leqq p\underline{a}'$. Therefore, given $p\underline{a}+wa_l > p\underline{a}'+w'a'_l$ it follows that $wa_l > w'a'_l$, and so $a_l \succ_{(p,w)} a'_l$, as sought.

(*Sufficiency*) Consider any (p, w)-labour ordering $\succcurlyeq$ that satisfies **D**, **LT**, **MI** and **CPTC**. In order to show that $\succcurlyeq$ is wage additive, we first show that any (p, w)-labour ordering $\succcurlyeq$ that satisfies **D**, **LT**, and **MI** is additive, according to Definition 5. That is to say, we prove that, given a price vector $(p, w) \in \mathbb{R}^{n+T}_+$ with $w > \mathbf{0}$, if a (p, w)-labour ordering $\succcurlyeq$ satisfies **D**, **LT**, and **MI** then there is some $\sigma_{(p,w)} \in \mathbb{R}^T_{++}$ such that for all $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, $a_l \succcurlyeq_{(p,w)} a'_l$ if and only if $\sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} \geqq \sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu}$. Then, we use **CPTC** to prove that $\sigma_{(p,w)} = w$.

Step 1. First of all, we prove that for any $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, $a_l \succcurlyeq_{(p,w)} a'_l$ implies $a_l + y \succcurlyeq_{(p,w)} a'_l + y$, for all $y \in \mathbb{R}^T$ such that $a_l + y, a'_l + y \in \mathbb{R}^T_+$. To see this, suppose, by way of contradiction, that there exist $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$, and $y \in \mathbb{R}^T$ such that $a_l \succcurlyeq_{(p,w)} a'_l$, but $a_l + y \not\succcurlyeq_{(p,w)} a'_l + y$. By completeness, this implies $a'_l + y \succ_{(p,w)} a_l + y$. Then, by **MI**, for all $\tau \in (0, 1)$, $\tau a_l + (1 - \tau) (a'_l + y) \succ_{(p,w)} \tau a'_l + (1 - \tau) (a_l + y)$. Let $\tau = \frac{1}{2}$, then the latter expression becomes

$$\frac{1}{2}a_l + \frac{1}{2}(a'_l + y) \succ_{(p,w)} \frac{1}{2}a'_l + \frac{1}{2}(a_l + y)$$

which violates reflexivity.

Step 2. By **LT,** for all $\nu, \mu \in \mathcal{T}$, there are $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$, $\zeta \neq \nu, \mu$, and $a_l \sim_{(p,w)} a'_l$. Take $\nu = 1$: by **LT** there are $T - 1$ pairs $(c^\mu, a^\mu; p, w), (c'^\mu, a'^\mu; p, w) \in \mathcal{CP}$ such that $a^\mu_{l1} > a'^\mu_{l1}$, $a^\mu_{l\mu} < a'^\mu_{l\mu}$, and $a^\mu_{l\zeta} = a'^\mu_{l\zeta}$, $\zeta \neq 1, \mu$, and $a^\mu_l \sim_{(p,w)} a'^\mu_l$. Let the set of all such $2(T - 1)$ vectors of direct labour be denoted as I^1. Define $\sigma_{(p,w)} = \left(\sigma^1_{(p,w)}, ..., \sigma^T_{(p,w)}\right)$ as follows: for all $\mu \in \mathcal{T}$, $\mu \neq 1$, $\frac{\sigma^1_{(p,w)}}{\sigma^\mu_{(p,w)}} \equiv \frac{a'^\mu_{l\mu} - a^\mu_{l\mu}}{a^\mu_{l1} - a'^\mu_{l1}}$ and $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} = 1$: by construction $\sigma_{(p,w)} > \mathbf{0}$ and, for all $\mu \in \mathcal{T}$, $\mu \neq 1$, $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a^\mu_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'^\mu_{l\nu}$. We show that, starting from I^1, it is possible to construct one iso-labour surface such that for all $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k$, we have $a_l \sim_{(p,w)} a'_l$.

Step 3. Consider $a^2_l, a'^2_l \in I^1$: by construction $(c^2, a^2; p, w), (c'^2, a'^2; p, w) \in \mathcal{CP}$ are such that $a^2_{l1} > a'^2_{l1}$, $a^2_{l2} < a'^2_{l2}$, and $a^2_{l\zeta} = a'^2_{l\zeta}$, $\zeta \neq 1, 2$, and $a^2_l \sim_{(p,w)} a'^2_l$. Choose $y^2 \in \mathbb{R}^T_+$ such that for all pairs $a^\mu_l, a'^\mu_l \in I^1$, $a^{\max}_l \equiv a^2_l + y^2 \geqq a^\mu_l$, $\mu \in \mathcal{T}$, $\mu \neq 1$. Note that we are allowing for the possibility that $a^2_l \geqq a^\mu_l$ for all $\mu \in \mathcal{T}$, $\mu \neq 1$, and $y^2 = \mathbf{0}$. By Step 1, $a^2_l \sim_{(p,w)} a'^2_l$ implies $a^{\max}_l \equiv a^2_l + y^2 \sim_{(p,w)} a'^2_l + y^2$.

Step 4. Consider any $a^\mu_l, a'^\mu_l \in I^1$, $\mu \in \mathcal{T}$, $\mu \neq 1, 2$: by construction $(c^\mu, a^\mu; p, w), (c'^\mu, a'^\mu; p, w) \in \mathcal{CP}$ are such that $a^\mu_{l1} > a'^\mu_{l1}$, $a^\mu_{l\mu} < a'^\mu_{l\mu}$, and $a^\mu_{l\zeta} = a'^\mu_{l\zeta}$, $\zeta \neq 1, \mu$, and $a^\mu_l \sim_{(p,w)} a'^\mu_l$. For all $\mu \in \mathcal{T}$, $\mu \neq 1, 2$, define $y^\mu \in \mathbb{R}^T_+$ such that for any $a^\mu_l, a'^\mu_l \in I^1$: $a^\mu_l + y^\mu = a^{\max}_l$. By Step 1, $a^\mu_l \sim_{(p,w)} a'^\mu_l$ implies $a^{\max}_l = a^\mu_l + y^\mu \sim_{(p,w)} a'^\mu_l + y^\mu$, for all $\mu \in \mathcal{T}$, $\mu \neq 1, 2$.

Step 5. Noting that the addition of y^μ to each pair of vectors preserves the original inequalities, the procedure in Step 4 yields T linearly independent vectors, $a^{\max}_l, a'^\mu_l + y^\mu, \mu \in \mathcal{T}$, $\mu \neq 1$, such that $a^{\max}_l \sim_{(p,w)} a'^\mu_l + y^\mu$, $\mu \in \mathcal{T}$, $\mu \neq 1, 2$, and, by transitivity, $a'^\eta_l + y^\eta \sim_{(p,w)} a'^\mu_l + y^\mu$, $\mu, \eta \in \mathcal{T}$, $\mu, \eta \neq 1$. Moreover, by the construction of $\sigma_{(p,w)}$ in Step 2, $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \left(a'^\mu_{l\nu} + y^\mu\right) = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a^{\max}_{l\nu} = k > 0$, for all $\mu \in \mathcal{T}$, $\mu \neq 1$. Then, by Lemmas 8 and 9, it follows that for all $(c, a; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a^{\max}_{l\nu} = k$, we have $a_l \sim_{(p,w)} a^{\max}_l$. Therefore, by transitivity, for all $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k$, we have $a_l \sim_{(p,w)} a'_l$.

Step 6. Next, we show that for all $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k' \neq k$, we have $a_l \sim_{(p,w)} a'_l$. Suppose first that $k' > k$. Consider any $(\widetilde{c}, \widetilde{a}; p, w), (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \widetilde{a}_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \widetilde{a}'_{l\nu} = k$. By Step 5, we have $\widetilde{a}_l \sim_{(p,w)} \widetilde{a}'_l$. Let $y = (k' - k, k' - k, ..., k' - k) > 0$. Then, by Step 1, $\widetilde{a}_l + y \sim_{(p,w)} \widetilde{a}'_l + y$. Let the set of all vectors thus constructed be denoted I^2: for all $(c', a'; p, w), (c'', a''; p, w) \in \mathcal{CP}$ such that $a'_l, a''_l \in I^2$, by construction $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a''_{l\nu} = k'$, $a'_l \sim_{(p,w)} a''_l$, and I^2 identifies a $T-1$ dimensional linear space. Then, by Lemmas 8 and 9, it follows that for all $(c, a; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a_{l\nu} = k'$, we have $a_l \sim_{(p,w)} a'_l$ for all $a'_l \in I^2$, and the desired result follows by transitivity.

A similar argument holds for the case $k' < k$, restricting attention to the vectors $(\widetilde{c}, \widetilde{a}; p, w), (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} \widetilde{a}_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} \widetilde{a}'_{l\nu} = k$ and such that if $y = (k' - k, k' - k, ..., k' - k)$ then $\widetilde{a}_l + y, \widetilde{a}'_l + y \in \mathbb{R}^T_+$.

Step 7. The previous arguments prove that if $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ are such that $\sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a'_{l\nu}$, then $a_l \sim_{(p,w)} a'_l$. Then, by **D** and transitivity, it follows that for all $(c, a; p, w), (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a_{l\nu} > \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a'_{l\nu}$, it must be $a_l \succ_{(p,w)} a'_l$.

Step 8. In order to complete the demonstration, we need to prove that for all $\nu, \mu \in \mathcal{T}$, $\frac{w_\nu}{w_\mu} = \frac{\sigma^{\nu}_{(p,w)}}{\sigma^{\mu}_{(p,w)}}$. However, this immediately follows noting that if $\frac{w_\nu}{w_\mu} \neq \frac{\sigma^{\nu}_{(p,w)}}{\sigma^{\mu}_{(p,w)}}$ for some $\nu, \mu \in \mathcal{T}$, then it is straightforward to find $(c, a; p, w), (c, a'; p, w) \in \mathcal{CP}$, such that $p\underline{a} + wa_l > p\underline{a}' + wa'_l$ and $\underline{a} \leq \underline{a}'$, and $wa_l > wa'_l$, but $\sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a_{l\nu} \leqq \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a'_{l\nu}$ and so $a_l \not\succ_{(p,w)} a'_l$, thus violating **CPTC**. ■

The properties in Theorem 4 are independent.

B Proof of Theorem 7

Proof of Theorem 7.

(*Necessity*) A similar argument as in the proof of Theorem 4 shows that the wage additive MLC $\succcurlyeq$ satisfies the axioms.

(*Sufficiency*) Consider any reflexive, transitive and complete MLC $\succcurlyeq \subseteq \mathcal{CP} \times \mathcal{CP}$ that satisfies **D**, **LT**, **MI**, **EL**, **CPTC**, and **CLSTC**. In order to show that $\succcurlyeq$ is wage additive, we first show that any reflexive, transitive and complete MLC $\succcurlyeq$ that satisfies **D**, **LT**, **MI**, and **EL** is additive, according to Definition 5. That is to say, we prove that, given a price vector $(p, w) \in \mathbb{R}^{n+T}_+$ with $w > \mathbf{0}$, if this MLC $\succcurlyeq$ satisfies **D**, **LT**, **MI**, and **EL** then there is some $\sigma_{(p,w)} \in \mathbb{R}^T_{++}$ such that for all $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$, $x \succcurlyeq x'$ if and only if $\sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a_{l\nu} \geqq \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a'_{l\nu}$. Then, we use **CPTC** and **CLSTC** to prove that $\sigma_{(p,w)} = w$. Note that Lemmas 8 and 9 for (p, w)-labour orderings are easily generalised into the lemmas for any reflexive, transitive and complete MLC by replacing $\succcurlyeq_{(p,w)}$, a_l, and a'_l respectively with $\succcurlyeq$, x, and x'.

Step 1. First of all, for any $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$, let us define $x \oplus y \equiv (c, (-(a_l + y), -\underline{a}, \overline{a}); p, w)$ and $x' \oplus y \equiv (c', (-(a'_l + y), -\underline{a}', \overline{a}'); p, w)$ for all $y \in \mathbb{R}^T$ such that $a_l + y, a'_l + y \in \mathbb{R}^T_+$. By the universality of $\mathcal{P}$, $x \oplus y \in \mathcal{CP}$ and $x' \oplus y \in \mathcal{CP}$. Then, we will show that $x \succcurlyeq x'$ implies $x \oplus y \succcurlyeq x' \oplus y$. To see this, suppose, by way of contradiction, that there exist $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$, and $y \in \mathbb{R}^T$ such that $x \succcurlyeq x'$, but $x \oplus y \not\succcurlyeq x' \oplus y$. By completeness, this implies $x' \oplus y \succ x \oplus y$. Then, by **MI**, for all $\tau \in (0, 1)$, $\tau x + (1 - \tau)(x' \oplus y) \succ \tau x' + (1 - \tau)(x \oplus y)$. Let $\tau = \frac{1}{2}$, then the latter expression becomes

$$\frac{1}{2}x + \frac{1}{2}(x' \oplus y) \succ \frac{1}{2}x' + \frac{1}{2}(x \oplus y).$$

However, $\frac{1}{2}x + \frac{1}{2}(x' \oplus y) = \left(\frac{1}{2}c + \frac{1}{2}c', \left(-\frac{1}{2}(a_l + a'_l + y), -\frac{1}{2}(\underline{a} + \underline{a}'), \frac{1}{2}(\overline{a} + \overline{a}')\right); p, w\right) = \frac{1}{2}x' + \frac{1}{2}(x \oplus y)$, which violates reflexivity.

Step 2. By **LT,** for all $\nu, \mu \in \mathcal{T}$, there are $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$ such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$, $\zeta \neq \nu, \mu$, and $x \sim x'$. Take $\nu = 1$: by **LT** there are $T - 1$ pairs $x^\mu = (c^\mu, a^\mu; p, w), x'^\mu = (c'^\mu, a'^\mu; p, w) \in \mathcal{CP}$ such that $a^\mu_{l1} > a'^\mu_{l1}$, $a^\mu_{l\mu} < a'^\mu_{l\mu}$, and $a^\mu_{l\zeta} = a'^\mu_{l\zeta}$, $\zeta \neq 1, \mu$, and $x^\mu \sim x'^\mu$. Let the set of all such $2(T - 1)$ vectors of labour inputs be denoted as I^1. Define $\sigma_{(p,w)} = \left(\sigma^1_{(p,w)}, ..., \sigma^T_{(p,w)}\right)$ as follows: for all $\mu \in \mathcal{T}$, $\mu \neq 1$, $\frac{\sigma^1_{(p,w)}}{\sigma^\mu_{(p,w)}} \equiv \frac{a'^\mu_{l\mu} - a^\mu_{l\mu}}{a^\mu_{l1} - a'^\mu_{l1}}$ and $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} = 1$: by construction $\sigma_{(p,w)} > \mathbf{0}$ and, for all $\mu \in \mathcal{T}$, $\mu \neq 1$, $\sigma^\nu_{(p,w)} a^\mu_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'^\mu_{l\nu}$. We show that, starting from I^1, it is possible to construct one iso-labour surface such that for all $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k$, we have $x \sim x'$.

Step 3. Consider $a^2_l, a'^2_l \in I^1$: by construction $x^2 = (c^2, a^2; p, w), x'^2 = (c'^2, a'^2; p, w) \in \mathcal{CP}$ are such that $a^2_{l1} > a'^2_{l1}$, $a^2_{l2} < a'^2_{l2}$, and $a^2_{l\zeta} = a'^2_{l\zeta}$, $\zeta \neq 1, 2$, and $x^2 \sim x'^2$. Choose $y^2 \in \mathbb{R}^T_+$ such that for all pairs $a^\mu_l, a'^\mu_l \in I^1$, $a^{\max}_l \equiv a^2_l + y^2 \geqq a^\mu_l$, $\mu \in \mathcal{T}$, $\mu \neq 1$. Note that we are allowing for the possibility that $a^2_l \geqq a^\mu_l$ for all $\mu \in \mathcal{T}$, $\mu \neq 1$, and $y^2 = \mathbf{0}$. By Step 1, $x^2 \sim x'^2$ implies $x^{2\max} \equiv x^2 \oplus y^2 \sim x'^2 \oplus y^2$.

Step 4. Consider any $a^\mu_l, a'^\mu_l \in I^1$, $\mu \in \mathcal{T}$, $\mu \neq 1, 2$: by construction $x^\mu = (c^\mu, a^\mu; p, w), x'^\mu = (c'^\mu, a'^\mu; p, w) \in \mathcal{CP}$ are such that $a^\mu_{l1} > a'^\mu_{l1}$, $a^\mu_{l\mu} < a'^\mu_{l\mu}$, and $a^\mu_{l\zeta} = a'^\mu_{l\zeta}$, $\zeta \neq 1, \mu$, and $x^\mu \sim x'^\mu$. For all $\mu \in \mathcal{T}$, $\mu \neq 1, 2$, define $y^\mu \in \mathbb{R}^T_+$ such that for any $a^\mu_l, a'^\mu_l \in I^1$: $a^\mu_l + y^\mu = a^{\max}_l$. By Step 1, $x^\mu \sim x'^\mu$ implies $x^{\mu\max} = x^\mu \oplus y^\mu \sim x'^\mu \oplus y^\mu$, for all $\mu \in \mathcal{T}$, $\mu \neq 1, 2$.

Step 5. Noting that the addition of y^μ to each pair of vectors preserves the original inequalities, the procedure in Step 4 yields T linearly independent vectors, $a_l^{\max}, a_l'^{\mu} + y^\mu, \mu \in \mathcal{T}, \mu \neq 1$, such that $x^{\mu \max} \sim x'^{\mu} \oplus y^\mu$, $\mu \in \mathcal{T}$, $\mu \neq 1, 2$, and, by **EL** and transitivity, $x'^{\eta} \oplus y^\eta \sim x^{\eta \max} \sim x^{\mu \max} \sim x'^{\mu} \oplus y^\mu$ implies $x'^{\eta} \oplus y^\eta \sim x'^{\mu} \oplus y^\mu$ for any $\mu, \eta \in \mathcal{T}$ with $\mu, \eta \neq 1$. Moreover, by the construction of σ in Step 2, $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \left(a'^{\mu}_{l\nu} + y^\mu \right) = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a^{\max}_{l\nu} = k > 0$, for all $\mu \in \mathcal{T}$, $\mu \neq 1$. Then, by Lemmas 8 and 9, it follows that for all $x = (c, a; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a^{\max}_{l\nu} = k$, we have $x \sim x^{2 \max}$. Therefore, by transitivity, for all $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k$, we have $x \sim x'$.

Step 6. Next, we show that for all $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k' \neq k$, we have $x \sim x'$. Suppose first that $k' > k$. Consider any $\widetilde{x} = (\widetilde{c}, \widetilde{a}; p, w), \widetilde{x}' = (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \widetilde{a}_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \widetilde{a}'_{l\nu} = k$. By Step 5, we have $\widetilde{x} \sim \widetilde{x}'$. Let $y = (k' - k, k' - k, ..., k' - k) > 0$. Then, by Step 1, $\widetilde{x} \oplus y \sim \widetilde{x}' \oplus y$. Let the set of all vectors thus constructed be denoted by I^2: for all $x'' = (c'', a''; p, w), x''' = (c''', a'''; p, w) \in \mathcal{CP}$ such that $a''_l, a'''_l \in I^2$, by construction $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a''_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'''_{l\nu} = k'$, $x'' \sim x'''$, and I^2 identifies a $T - 1$ dimensional linear space. Then, by Lemmas 8 and 9, it follows that for all $x = (c, a; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma_\nu a_{l\nu} = k'$, we have $x \sim x'$ for all $a'_l \in I^2$, and the desired result follows by transitivity.

A similar argument holds for the case $k' < k$, restricting attention to the vectors $(\widetilde{c}, \widetilde{a}; p, w), (\widetilde{c}', \widetilde{a}'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \widetilde{a}_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} \widetilde{a}'_{l\nu} = k$ and such that if $y = (k' - k, k' - k, ..., k' - k)$ then $\widetilde{a}_l + y, \widetilde{a}'_l + y \in \mathbb{R}^T_+$.

Step 7. The previous arguments prove that if $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$ are such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu}$, then $x \sim x'$. Then, by **D** and transitivity, it follows that for all $x = (c, a; p, w), x' = (c', a'; p, w) \in \mathcal{CP}$ such that $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} > \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu}$, it must be $x \succ x'$.

Step 8. In order to complete the proof, we need to show that for all $\nu, \mu \in \mathcal{T}$, $\frac{w_\nu}{w_\mu} = \frac{\sigma^\nu_{(p,w)}}{\sigma^\mu_{(p,w)}}$. However, this immediately follows noting that if $\frac{w_\nu}{w_\mu} \neq \frac{\sigma^\nu_{(p,w)}}{\sigma^\mu_{(p,w)}}$ for some $\nu, \mu \in \mathcal{T}$, then it is straightforward to find $x = (c, a; p, w), x' = (c, a'; p, w) \in \mathcal{CP}$, such that $p\underline{a} + wa_l > p\underline{a}' + wa'_l$ and $\underline{a} \leq \underline{a}'$, and $wa_l > wa'_l$, but $\sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} \leqq \sum_{\nu \in \mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu}$ and so $x \not\succ x'$, thus violating **CPTC**. Thus, by **CPTC** and the transitivity of $\succcurlyeq$, it follows that for any $x = (c, a; p, w), x' = (c', a'; p', w) \in \mathcal{CP}$, $wa_l > wa'_l$ if and only if $x \succ x'$; and

$wa_l = wa'_l$ if and only if $x \sim x'$. Moreover, for any $x = (c, a; p, w), x' = (c, a'; p, w') \in \mathcal{CP}$ with $w, w' > \mathbf{0}$, such that $p\underline{a} + wa_l > p\underline{a}' + w'a'_l$, $\underline{a} = \underline{a}'$, and $a_l \neq a'_l$, **CLSTC** implies that $wa_l > w'a'_l$ and $x \succ x'$ hold. Thus, the transitivity and the completeness of $\succsim$, **CPTC** and **CLSTC** together imply that for any $x = (c, a; p, w), x' = (c', a'; p', w') \in \mathcal{CP}$ with $w, w' > \mathbf{0}$, $wa_l > w'a'_l$ if and only if $x \succ x'$; and $wa_l = w'a'_l$ if and only if $x \sim x'$. ■

The properties in Theorem 7 are independent.

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