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**Exploitation and Profits: A General
Axiomatic Approach in Convex Economies
with Heterogeneous Agents**

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Abstract

This paper provides an innovative axiomatic analysis of the notion of exploitation as the unequal exchange of labour. General convex economies with heterogeneous agents endowed with unequal amounts of physical and human capital are considered. An axiomatic characterisation of the class of definitions that satisfy a weak domain condition and the profit-exploitation correspondence principle (PECP) is derived. It is shown that none of the main received definitions preserves the PECP. Instead, a novel definition is presented which satisfies the PECP and allows one to generalise a number of key insights of exploitation theory to complex advanced economies.

JEL Classifications: D63; D70; B51; D51.

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1 Introduction

What is exploitation? In political philosophy, the most general definition affirms that agent A exploits agent B if and only if A takes unfair advantage of B . Despite its intuitive appeal, this definition leaves two major issues in need of a precise specification, namely the kind of unfairness involved and the structure of the relationship between A and B that allows A to take advantage of B . There is considerable debate in the economic and philosophical literature concerning both issues. Although both aspects of exploitative relations are arguably crucial (Yoshihara and Veneziani, 2009), the analytical focus of this paper is on the unfairness, or more precisely, on the economic inequalities involved in the concept of exploitation.

To be specific, this paper analyses the theory of *exploitation as an unequal exchange* (hereafter, UE) *of labour*, according to which exploitative relations are characterised by systematic differences between the amount of labour that individuals contribute to the economy and the amount of labour they receive, in the form of labour contained in some relevant bundle that they do (or can) purchase with their income. There are at least two reasons to focus on *labour* as the measure of the injustice of exploitative relations. First, in a number of crucial economic interactions, the notion of exploitation is inextricably linked with some form of labour exchange (Veneziani, 2008). Second, the UE definition of exploitation captures some inequalities in the distribution of material well-being and free hours that are - at least *prima facie* - of normative relevance. For instance, they are relevant for *inequalities of well-being freedom*, as discussed by Rawls (1971) and Sen (1985, 1985a),¹ because material well-being and free hours are two crucial determinants of individual well-being freedom. Further, it can be proved that in a private-ownership economy with positive profits, class and UE exploitation status are strictly related, and they accurately reflect an unequal distribution of assets

¹The notion of well-being freedom emphasises an individual's ability to pursue the life she values. In the Rawls-Sen theory, inequalities in the distribution of well-being freedom are formulated as *inequalities of capabilities*, whereas they are formulated as *inequalities of (comprehensive) resources* in Dworkin's theory (Dworkin, 2000). The resource allocation problem in terms of equality of capability is analysed in Gotoh and Yoshihara (2003), whereas Roemer (1986) and Yoshihara (2003) analyse it in terms of equality of resources.

(Roemer, 1982; Yoshihara and Veneziani, 2009). That is, in equilibrium the wealthy emerge as exploiters and members of the capitalist class, whereas the poor are exploited and members of the working class. From this perspective, exploitative relations are relevant because they reflect unequal opportunities of life options, due to differential ownership of productive assets.

Although the definition of UE exploitation is seemingly intuitive, it has proved surprisingly difficult to provide a fully satisfactory general theory of exploitation. In fact, outside of standard Leontief economies, the appropriate definition of the amount of labour ‘received’ by an agent is not obvious, and indeed a number of approaches have been proposed (see Yoshihara, 2010). Further, outside of stylised, linear two-class economies, the core insights of exploitation theory do not necessarily hold (Yoshihara and Veneziani, 2011a).

In this paper, exploitation is analysed in general economies with a convex production technology and with maximising agents endowed with heterogeneous preferences and with different amounts of both physical and human capital, as outlined in section 2. These economies are significantly more general than those usually considered in exploitation theory.² One substantive contribution of the paper is to provide a novel definition of exploitation, which extends the core insights of exploitation theory and allows one to characterise the exploitation status of all agents in such general economies. This definition focuses on aggregate social labour performed and on its distribution to agents via market mechanisms, and it is conceptually related to the ‘New Interpretation’ (Duménil, 1980; Foley, 1982; Duménil and Foley, 2008; Duménil, Foley, and Lévy, 2009). According to this definition, an agent is exploited if and only if the amount of labour she contributes is greater than the share of social labour that she receives via her income.

This definition has a number of desirable features. It defines exploitation as a feature of the (competitive) allocation of social labour rather than as the result of productive inefficiencies, or imperfections in the labour market. Unlike the main received approaches, it has a clear empirical content, for it is firmly anchored to the actual data of the economy. Perhaps more importantly, it clearly captures the inequalities arising from exploitative relations, as it identifies exploitation as a social relation between individuals: in equilibrium there are some exploited agents if and only if there are some exploiters. As shown in Yoshihara and Veneziani (2009), none of the main

²An interesting analysis of nonconvexities in Marxian economic theory can be found in Negishi (1998). The latter paper does not focus on exploitation, though.

definitions in the literature satisfy this fundamental *relational* property in general.

Methodologically, this paper provides a new axiomatic analysis of UE exploitation and a general characterisation of the class of definitions satisfying two key axioms.³ An axiomatic approach was long overdue in exploitation theory, where the proposal of alternative definitions has sometimes appeared as a painful process of adjustment of the theory to anomalies and counterexamples. The definitions of exploitation thus constructed have progressively lost the intuitive appeal, normative relevance, and even connection with the actual, observed variables emerging from a competitive mechanism. By adopting an axiomatic approach, this paper suggests to start from first principles, thus explicitly discussing the intuitions behind UE exploitation.

To be precise, in section 3, two axioms are analysed. The first is called *Labour Exploitation for the Working Class* (hereafter, **LEW**), and it restricts the way in which the set of exploited agents is identified. This axiom is interpreted as a minimal necessary condition to capture the core intuitions of exploitation theory, and it is shown that indeed all of the main definitions of exploitation in the literature satisfy it (see Morishima, 1974; Foley, 1982; Roemer, 1982. See also Yoshihara and Veneziani, 2009; and Yoshihara, 2010). The second axiom is the *Profit-Exploitation Correspondence Principle* (hereafter, **PECP**), and it incorporates the intuition that profits are one of the key determinants of the existence of exploitation, and of inequalities in well-being freedom: profits represent the way in which capitalists appropriate social surplus and social labour. Formally, **PECP** states that, in equilibrium, propertyless agents are exploited if and only if profits are positive. Theorem 1 provides the first rigorous characterisation of the class of definitions satisfying **LEW** which meet **PECP**. Based on this characterisation, it is shown that, among all the main definitions, the ‘New Interpretation’ is the only one that preserves **PECP**.

Theorem 1 provides some interesting and innovative insights, as compared to existing contributions investigating the relation between exploitation and profits, such as the literature on the so-called *Fundamental Marxian Theorem* (hereafter, **FMT**; see Morishima, 1974; Roemer, 1980; Krause, 1982; Morishima, 1989; Veneziani, 2004; Mori, 2008; Flaschel, 1983, 2010; Fujimoto and Opocher, 2010). Methodologically, the epistemological status of **PECP** as a postulate is explicitly acknowledged and Theorem 1 provides the first

³For a related approach see Yoshihara and Veneziani (2009) and Yoshihara (2010).

general axiomatic analysis of the relation between exploitation and profits, and a starting point for further research in general convex economies with heterogeneous agents.

Substantively, in the literature on the FMT, the existence of exploitation is proved to be synonymous with positive profits in an arguably small set of linear, two-class economies. Yet a number of counterexamples have been found in more general models and no fully satisfactory definition of exploitation that preserves the FMT has been provided so far. Theorem 1 proves that if the ‘New Interpretation’ is adopted, then a robust correspondence between exploitation and profits can be established in general convex economies with heterogeneous agents. In this sense, Theorem 1 is significantly more general than analogous results on the FMT. However, Theorem 1 is theoretically different from, and arguably more interesting than, standard FMT results: first, as argued below, axiom **PECP** is logically different from the standard FMT. Second, in the standard Okishio-Morishima approach, the existence of (aggregate) labour exploitation is just a numerical representation of the existence of surplus products in a productive economy. Thus, the FMT establishes the equivalence between positive profits and the productiveness of the economy measured in terms of the labour numéraire. However, analogous results can be proved when productiveness is measured in terms of any other good (this is the so-called *Generalised Commodity Exploitation Theorem*; Roemer, 1982), which raises doubts on the significance of the FMT for exploitation theory. If the ‘New Interpretation’ is adopted, instead, positive profits are necessary and sufficient for the existence of exploitative relations, but this holds only if labour exploitation is considered: no equivalent result holds if any other commodity is used to define exploitation.

Given the theoretical relevance of **PECP** in exploitation theory, however, the main implication of Theorem 1 is to provide strong support for the ‘New Interpretation’ as the appropriate formulation of UE exploitation. Thus, it confirms and extends the analysis of Yoshihara and Veneziani (2009), who have shown that in the class of convex subsistence economies - which may be taken as a subset of the economies analysed in this paper - the ‘New Interpretation’ is uniquely characterised by a small number of weak axioms capturing the key insights of UE exploitation.

Two extensions of the analysis are also presented, which provide further support for the ‘New Interpretation.’ First, a focus on the poorest segment of the working class, namely agents without any physical assets, is appropriate from the axiomatic viewpoint: focusing on a strict subset of the set

of agents makes the axiomatic framework rather weak. Yet one may argue that this is reductive and some key characteristics of advanced capitalist economies should be explicitly considered, which make the issue of Marxist exploitation a contentious one today - such as the fact that many workers own some non-labour assets, and even stock in firms, through their pension funds. Second, although exploitation is traditionally analysed by focusing on equilibrium allocations (see Morishima, 1974; Roemer, 1980), one may question general equilibrium-type constructions as representations of allocation and distribution in market economies because they depend on the often tacit assumption of equal-treatment.⁴ In a general theory of exploitation, it would be important to take account of transactions at disequilibrium prices and the resulting inequity in distribution endogenous to market allocation. In section 4, the generality of the model is exploited to show that the ‘New Interpretation’ can be extended, first, to analyse the exploitation status of all agents, in economies with heterogeneous preferences, physical assets, and skills (Theorem 2), and then to establish a relation between exploitation and profits outside of equilibrium allocations (Theorem 3).

These results are encouraging, as they show that the ‘New Interpretation’ has a number of desirable properties in rather general economies and, among other things, it captures the relation between profits and exploitative relations. Indeed, the ‘New Interpretation’ seems to provide the foundations for a *general* theoretical framework, which can deal with many unresolved issues in exploitation theory, including the analysis of unequal exchange and international relations. (See Veneziani and Yoshihara, 2009. For a critique of the standard Marxist analysis, see Negishi, 1999.) Some extensions of the analysis are briefly discussed in the concluding section 5 below.

Finally, the existence of a general equilibrium is proved in Appendix 1. This proof completes the analysis by showing the consistency of the economic framework, but it is also interesting per se because both the structure of Marxian economies and the equilibrium concept adopted are different from the standard Walrasian framework. Indeed, Appendix 1 generalises the existence results derived by Roemer (1980).

⁴This issue has been brought to our attention by Duncan Foley in a private exchange. For an analysis of the implications of trading at disequilibrium prices, see Foley (2010).

2 The Model

An economy consists of N agents. Let \mathbb{R}_+ be the set of nonnegative real numbers. Production technology is freely available to all agents, who can operate any activity in the production set P , which has elements of the form $\alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha})$ where $\alpha_l \in \mathbb{R}_+$ is the *effective* labour input of the process; $\underline{\alpha} \in \mathbb{R}_+^n$ are the inputs of the produced goods used in the process; and $\bar{\alpha} \in \mathbb{R}_+^n$ are the outputs of the n goods. Thus, elements of P are vectors in \mathbb{R}^{2n+1} . The net output vector arising from α is denoted as $\hat{\alpha} \equiv \bar{\alpha} - \underline{\alpha}$. P is assumed to be a closed convex cone containing the origin in \mathbb{R}^{2n+1} . Let the vector with all components equal to zero be denoted as $\mathbf{0}$. The following assumptions on P hold throughout the paper.⁵

Assumption 1 (A1). For all $\alpha \in P$, if $\bar{\alpha} \geq \mathbf{0}$ then $\alpha_l > 0$.

Assumption 2 (A2). For all $c \in \mathbb{R}_+^n$, there is a $\alpha \in P$ such that $\hat{\alpha} \geq c$.

Assumption 3 (A3). For all $\alpha \in P$, and for all $(-\underline{\alpha}', \bar{\alpha}') \in \mathbb{R}_-^n \times \mathbb{R}_+^n$, if $(-\underline{\alpha}', \bar{\alpha}') \leq (-\underline{\alpha}, \bar{\alpha})$ then $(-\alpha_l, -\underline{\alpha}', \bar{\alpha}') \in P$.

A1 implies that labour is indispensable to produce any non-negative output vector. **A2** states that any non-negative commodity vector is producible as a net output. **A3** is a standard *free disposal* condition.

Given P , the set of production activities feasible with $\alpha_l = k$ units of effective labour can be defined as follows:

$$P(\alpha_l = k) \equiv \{(-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in P \mid \alpha_l = k\},$$

and the set of net output vectors feasible with k units of effective labour is:

$$\hat{P}(\alpha_l = k) \equiv \{\hat{\alpha} \in \mathbb{R}^n \mid \text{there is } \alpha \in P(\alpha_l = k) \text{ such that } \bar{\alpha} - \underline{\alpha} \geq \hat{\alpha}\}.$$

For any set $X \subseteq \mathbb{R}^n$, $\partial X \equiv \{x \in X \mid \nexists x' \in X \text{ s.t. } x' > x\}$ is the frontier of X , and $SX \equiv \{x \in X \mid \nexists x' \in X \text{ s.t. } x' \geq x\}$ is the efficient frontier of X .

This paper investigates exploitation when heterogeneous agents are endowed with unequal amounts of physical and human capital. In the economy, agents produce, consume, and trade labour. On the production side, they

⁵For all vectors $x, y \in \mathbb{R}^n$, $x \geq y$ if and only if $x_i \geq y_i$ ($i = 1, \dots, n$); $x \geq y$ if and only if $x \geq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ ($i = 1, \dots, n$).

can either sell their labour-power or hire workers to work on their capital, or they can be self-employed and work on their own assets. More precisely, for all $\nu \in N$, let $s^\nu \in \mathbb{R}_{++}$ be agent ν 's skill level and let $\omega^\nu \in \mathbb{R}_+^n$ be the vector of productive assets inherited by ν . Then, $\alpha^\nu = (-\alpha_l^\nu, -\underline{\alpha}^\nu, \bar{\alpha}^\nu) \in P$ is the production process operated by ν as a self-employed producer, with her own capital, where $\alpha_l^\nu = s^\nu a_l^\nu$ and a_l^ν is the labour *time* expended by ν ; $\beta^\nu = (-\beta_l^\nu, -\underline{\beta}^\nu, \bar{\beta}^\nu) \in P$ is the production process that ν operates by hiring (effective) labour β_l^ν ; $\gamma^\nu = s^\nu l^\nu$ is ν 's effective labour supply, where l^ν is the labour *time* supplied by ν on the market. Thus, let $\lambda^\nu = (a_l^\nu + l^\nu)$ be the total amount of labour time expended by ν , and let $\Lambda^\nu = \alpha_l^\nu + \gamma^\nu = s^\nu \lambda^\nu$ be the total amount of effective labour performed by ν , either as a self-employed producer or working for some other agent.

On the consumption side, let $C \subseteq \mathbb{R}_+^n$ be the consumption space of each agent with generic element c^ν as a consumption vector of agent ν , and assume that total labour hours expended by each agent do not exceed the common endowment L^ν , where units are normalised so that $L^\nu = 1$, for all ν . Agent ν 's welfare is representable by a function $u^\nu : C \times [0, 1] \rightarrow \mathbb{R}_+$, which is monotonic on $C \times [0, 1]$ (increasing in consumption and decreasing in labour time). The function u can be interpreted either as a standard subjectivist neoclassical utility function or as an objectivist index of individual well-being, or status. The latter view is more in line with exploitation theory, but the two interpretations are formally equivalent.⁶ For the sake of simplicity, and with no loss of generality, in what follows, u is assumed to be strictly monotonic on C in at least one argument. The conclusions of the paper do not depend on this assumption, and some extensions of the analysis and the relation with other models in the literature are discussed in section 5 below.

Let p denote the $1 \times n$ vector of commodity prices and let w denote the wage rate *per unit of effective labour*. Given the assumption of perfect contracting in the labour market, the latter is indeed the relevant wage. Given (p, w) , each ν is assumed to choose a plan $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)$ to maximise her welfare subject to the constraint that net income is sufficient for consumption plans; wealth is sufficient for production plans; production plans are technically feasible; and total labour hours expended do not exceed $L^\nu = 1$.

⁶For a discussion of subjective and objective approaches, see Roemer and Veneziani (2004) and, in the context of exploitation theory, Veneziani and Yoshihara (2009).

Formally, each ν solves the following programme MP^ν .⁷

$$\max_{(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)} u^\nu(c^\nu, \lambda^\nu)$$

subject to

$$\begin{aligned} [p(\bar{\alpha}^\nu - \underline{\alpha}^\nu)] + [p(\bar{\beta}^\nu - \underline{\beta}^\nu) - w\beta_l^\nu] + [w\gamma^\nu] &= pc^\nu, \\ p(\underline{\alpha}^\nu + \underline{\beta}^\nu) &\leq p\omega^\nu, \\ \alpha^\nu &\in P; \beta^\nu \in P, \lambda^\nu \leq 1. \end{aligned}$$

MP^ν is a rather standard way of modelling agent ν 's decision problem in microeconomic theory, and thus no detailed discussion is necessary. It is worth noting, however, that MP^ν explicitly incorporates the simultaneous role of economic actors as consumers *and* producers, so that no separate consideration of firms is necessary. In this respect, MP^ν can be interpreted as a generalisation of standard Marxian accumulation economies with identical agents (e.g., Roemer 1980, 1982; Yoshihara, 2010). In Yoshihara (2010), for example, $\mathbf{s} = (1, \dots, 1)$, $C \equiv \mathbb{R}_+^n$, and there is a continuous, quasi-concave, and strictly monotonic real-valued function $f : C \rightarrow \mathbb{R}_+$ such that $u^\nu(c, \lambda) = f(c)$, for all ν and for any $(c, \lambda) \in C \times [0, 1]$. Further, as shown below, although agents are not assumed to maximise profits, profit maximisation is a corollary of MP^ν . Yet in this model individuals are not assumed to be simply ‘agents of capital’ and unlike in traditional Marxian economies (e.g., Roemer, 1982, ch.4), capitalists are not assumed to maximise accumulation per se, or to produce for production’s own sake.

Let $\mathcal{O}^\nu(p, w)$ be the set of plans $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)$ that solve MP^ν at prices (p, w) . Let $\Omega = (\omega^1, \omega^2, \dots, \omega^N)$, $\mathbf{u} = (u^1, u^2, \dots, u^N)$, and $\mathbf{s} = (s^1, s^2, \dots, s^N)$. Let $E(P, N, \mathbf{u}, \mathbf{s}, \Omega)$, or as a shorthand notation E , denote the economy with technology P , agents N , utility functions \mathbf{u} , labour skills \mathbf{s} , and productive endowments Ω . Let the set of all such economies be denoted by \mathcal{E} . Let $c = \sum_{\nu=1}^N c^\nu$ be aggregate consumption; and let a similar notation hold for all other variables. The equilibrium concept can now be defined.

Definition 1: A *reproducible solution* (RS) for $E(P, N, \mathbf{u}, \mathbf{s}, \Omega) \in \mathcal{E}$ is a price vector (p, w) and an associated set of actions such that:

⁷The first constraint is written as equality without loss of generality, given the assumptions on the monotonicity of u .

- (i) $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu) \in \mathcal{O}^\nu(p, w)$ for all ν (optimality);
- (ii) $\widehat{\alpha} + \widehat{\beta} \geq c$ (reproducibility);
- (iii) $\underline{\alpha} + \underline{\beta} \leq \omega$ (feasibility);
- (iv) $\beta_l = \gamma$ (labour market equilibrium).

In other words, at a RS (i) every agent optimises; (iii) there are enough resources for production plans; and (iv) the labour market clears. Condition (ii) states that net outputs should at least suffice for aggregate consumption. This is equivalent to requiring that the vector of social endowments does not decrease component-wise, because (ii) is equivalent to $\omega + (\widehat{\alpha} + \widehat{\beta} - c) \geq \omega$, which states that stocks at the beginning of next period should not be smaller than stocks at the beginning of the current period. Indeed, although the RS is defined as a temporary equilibrium in a static general equilibrium framework, it can be seen as a one-shot slice of a *stationary equilibrium* in a dynamic general equilibrium framework.⁸

Some properties of RSs should be noted. First, by the assumptions on u , it immediately follows that $p \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and $w \geq 0$ at a non-trivial RS. Next, let $\pi^{\max} = \max_{\alpha \in P} \frac{p\widehat{\alpha} - w\alpha_l}{p\alpha}$: by the assumptions on P , π^{\max} is well-defined. Hence, let $P^\pi(p, w) = \left\{ \alpha \in P \mid \pi^{\max} = \frac{p\widehat{\alpha} - w\alpha_l}{p\alpha} \right\}$. It is proven in a straightforward way that, at any non-trivial RS, the maximum profit rate is nonnegative; and only processes yielding the maximum rate of profit are activated.

Lemma 1: *Let (p, w) be a non-trivial RS for $E \in \mathcal{E}$ such that $c \geq \mathbf{0}$. Then, $p\widehat{\alpha} - w\alpha_l \geq 0$ for some $\alpha \in P \setminus \{\mathbf{0}\}$, and $\alpha^\nu, \beta^\nu \in P^\pi(p, w)$ for all ν .*

3 Labour exploitation: an axiomatic approach

In the UE approach, exploitation is conceived of as the unequal exchange of labour between agents: considering an agent $\nu \in N$, exploitative relations are characterised by systematic differences between the labour contributed by ν to the economy and the labour ‘received’ by ν , which is given by the amount of labour contained, or embodied, in some relevant consumption bundle(s). Therefore, for any bundle $c \in \mathbb{R}_+^n$, it is necessary to define the labour value (or labour content) of c . Unlike in standard Leontief economies, the definition

⁸See Veneziani (2007) and Veneziani and Yoshihara (2009) for a thorough analysis.

of the labour content of c is not obvious, and various definitions have, in fact, been proposed. In this section, a general condition - called the axiom of Labour Exploitation for the Working Class, or **LEW** - is proposed which every definition of labour exploitation should satisfy in order to capture the core insights of the theory of exploitation as the UE of labour.

Let $W \equiv \{\nu \in N \mid \omega^\nu = \mathbf{0}\}$: W is the set of agents with no initial endowments. The economies analysed in this paper are more general than the polarised, two-class societies usually considered in the literature, and in the next section the exploitation status of all agents is derived, including those in intermediate class positions. Yet the set W - which can be interpreted as the core of the working class - is of clear focal interest in exploitation theory: theoretically, if *any* agents are exploited, then those in W should be definitely among them, if they work at all. It is therefore opportune, from an axiomatic viewpoint, to focus on the set W in order to provide a domain condition defining a minimum requirement that all definitions of exploitation as the UE of labour should satisfy.⁹

Let $B(p, w\Lambda) \equiv \{c \in \mathbb{R}_+^n \mid pc = w\Lambda\}$ denote the set of consumption bundles that can be (just) afforded, at prices p , by an agent in W , who supplies Λ units of labour at a wage rate w . Let $\phi(c) \equiv \{\alpha \in P \mid \hat{\alpha} \geq c\}$ denote the set of activities that produce at least c as net output. A basic axiom can now be introduced that every formulation of labour exploitation should satisfy.

Labour Exploitation for the Working Class (LEW): *Consider any economy $E \in \mathcal{E}$. Let (p, w) be a RS for E . Given any definition of exploitation, the set of exploited agents $N^{ted} \subseteq N$ is identified at (p, w) . The set N^{ted} should have the following property: there exists a profile $(\bar{c}^1, \dots, \bar{c}^{|W|})$ such that for any $\nu \in W$, $\bar{c}^\nu \in B(p, w\Lambda^\nu)$ and for some $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$ with $\hat{\alpha}^{\bar{c}^\nu} \not\geq \bar{c}^\nu$:*

$$\nu \in N^{ted} \Leftrightarrow \alpha_i^{\bar{c}^\nu} < \Lambda^\nu.$$

Axiom **LEW** requires that, at any RS, the exploitation status of every propertyless worker $\nu \in W$ be characterised by identifying a nonnegative

⁹It might be argued that the appropriate definition of proletarians relates to their financial wealth, rather than their vector of endowments. If this view is adopted, then $W' = \{\nu \in N \mid p\omega^\nu = 0\}$. This distinction is relevant only if $p \not\geq \mathbf{0}$ and it does not make any significant difference for the results of this paper. In fact, since axiom **LEW** aims to provide a weak domain condition to define the set of exploited agents, it is theoretically appropriate to focus on the set of agents $W \subseteq W'$.

reference commodity vector \bar{c}^ν . This reference vector is technically feasible and can be purchased by ν , and it identifies the amount of labour that ν receives. Thus, if $\nu \in W$ supplies Λ^ν , and Λ^ν is more than the labour socially necessary to produce \bar{c}^ν , then ν is regarded as contributing more labour than ν receives. According to **LEW**, all such agents belong to N^{ted} .¹⁰

As a domain condition for the admissible class of exploitation-forms, **LEW** captures some key insights of the UE theory of exploitation that are shared by *all* of the main approaches in the literature.¹¹ In the UE theory, the exploitation status of an agent ν is determined by the difference between the amount of labour that ν ‘contributes’ to the economy, in some relevant sense, and the amount she ‘receives’, in some relevant sense. In the convex economies considered in this paper, the former quantity is given by the amount of labour supplied, Λ^ν , whereas there are many possible UE views concerning the amount of labour that each agent receives, which incorporate different normative and positive concerns. As a domain condition, **LEW** provides some minimal, key restrictions on the definition of the amount of labour that a theoretically relevant subset of agents receives.

First, according to **LEW**, the amount of labour that $\nu \in W$ receives depends on her income, or more precisely, it is determined in equilibrium by some reference consumption vectors that ν can purchase. In the standard approaches, the reference vector corresponds to the bundle actually chosen by the agent. **LEW** is weaker in that it only requires that the reference vector be *potentially* affordable.

Second, **LEW** captures another key tenet of the UE theory of exploitation by stipulating that the amount of labour associated with the reference bundle - and thus ‘received’ by an agent - is related to the production conditions of the economy. More precisely, **LEW** states that the reference bundle be technologically feasible as net output, and it defines its labour content as the amount of labour socially necessary to produce it. It is worth noting that **LEW** requires that the amount of labour associated with each reference bundle be uniquely determined with reference to production conditions, but

¹⁰In axiom **LEW** the case $N^{ted} = N$ is not ruled out: this is theoretically appropriate, given the nature of **LEW** as a minimum domain condition, for even some of the classic definitions of exploitation - such as Morishima’s (1974) - do not exclude this case.

¹¹It should be stressed that **LEW** only applies to labour-based definitions of exploitation. It is not relevant, for example, for Roemer’s (1982) property-relations definition of exploitation. Similar versions of **LEW** are analysed by Yoshihara and Veneziani (2009) and Yoshihara (2010), in the context of different economies.

it does not specify how such amount should be chosen, and there may be in principle many (efficient) ways of producing \bar{c}^ν , and thus of determining $\alpha_l^{\bar{c}^\nu}$.

Third, **LEW** is weak also because it does not provide comprehensive conditions for the determination of exploitation status. As already noted, it only focuses on a subset of agents, namely those who own no physical assets, and it is silent on the exploitation status of all other agents. Further, given any definition of exploitation, and any RS, the *set of exploiters* $N^{ter} \subseteq N$ is also defined, where $N^{ter} \cap N^{ted} = \emptyset$, but axiom **LEW** imposes no restrictions on the determination of N^{ter} .

Finally, it is worth noting that the vector \bar{c}^ν in **LEW** need not be uniquely fixed, and may be a function of (p, w) . Further, once \bar{c}^ν is identified, the existence of $\alpha^{\bar{c}^\nu}$ is guaranteed by **A2** and **A3**.

In sum, **LEW** incorporates several key features of exploitation as the UE of labour, and it sets a weak restriction on the class of admissible definitions. Indeed, all of the main definitions in the literature, suitably extended to economies with heterogeneous labour, satisfy **LEW**. Consider first Morishima's (1974) classic definition. According to Morishima, the labour embodied in a commodity vector c , denoted as $l.v.(c)$, is the minimum amount of (effective) labour necessary to produce c as net output. Formally:

$$l.v.(c) \equiv \min \{ \alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in \phi(c) \}.$$

It is easy to see that $\phi(c)$ is non-empty by **A2** and that the set

$$\{ \alpha_l \mid \alpha = (-\alpha_l; -\underline{\alpha}; \bar{\alpha}) \in \phi(c) \}$$

is bounded from below by 0, by the assumption $\mathbf{0} \in P$ and by **A1**. Hence, $l.v.(c)$ is well-defined and, by **A1**, it is positive whenever $c \neq \mathbf{0}$. Then:

Definition 2 (Morishima, 1974): A worker $\nu \in W$, who supplies Λ^ν and consumes $c^\nu \in \mathbb{R}_+^n$, is *exploited*, i.e. $\nu \in N^{ted}$, if and only if $\Lambda^\nu > l.v.(c^\nu)$.

Definition 2 satisfies **LEW**: at any RS, let $\bar{c}^\nu \equiv c^\nu \in B(p, w\Lambda^\nu)$ and

$$\alpha^{\bar{c}^\nu} \in \arg \min \{ \alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in \phi(\bar{c}^\nu) \}.$$

Unlike Morishima's (1974) definition, Roemer's (1982) definition of labour value depends on prices. Given a price vector (p, w) , let $\phi(c; p, w) \equiv \{ \alpha \in P^\pi(p, w) \mid \hat{\alpha} \geq c \}$ be the set of profit-rate-maximising activities that produce at least c as net output. According to Roemer (1982), the labour value of vector c , denoted

as $l.v.(c; p, w)$, is the minimum amount of (effective) labour necessary to produce c as net output among profit-rate-maximising activities. Formally:

$$l.v.(c; p, w) \equiv \min \{ \alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in \phi(c; p, w) \}.$$

Again, $l.v.(c; p, w)$ is well defined and it is positive for all $c \neq \mathbf{0}$. Then:

Definition 3 (Roemer, 1982): Consider an economy $E \in \mathcal{E}$. Let (p, w) be a RS for E . A worker $\nu \in W$, who supplies Λ^ν and consumes c^ν , is *exploited*, i.e. $\nu \in N^{ted}$, if and only if $\Lambda^\nu > l.v.(c^\nu; p, w)$.

Definition 3 also satisfies **LEW**: at any RS, let $\bar{c}^\nu \equiv c^\nu \in B(p, w\Lambda^\nu)$ and

$$\alpha^{\bar{c}^\nu} \in \arg \min \{ \alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in \phi(\bar{c}^\nu; p, w) \}.$$

In addition to the above two classic definitions, in this paper, a new definition is analysed, which has been recently proposed by Yoshihara and Veneziani (2009, 2011) and Yoshihara (2010). For any $p \in \mathbb{R}_+^n$ and $c \in \mathbb{R}_+^n$, let $\mathcal{B}(p, c) \equiv \{x \in \mathbb{R}_+^n \mid px = pc\}$: $\mathcal{B}(p, c)$ is the set of bundles that cost exactly as much as c at prices p .

Definition 4: Consider an economy $E \in \mathcal{E}$. Let (p, w) be a RS for E such that $\hat{\alpha}^{p,w}$ is aggregate net output and $\alpha_l^{p,w}$ is aggregate (effective) labour expended. For each $c \in \mathbb{R}_+^n$ with $pc \leq p\hat{\alpha}^{p,w}$, let $\tau^c \in [0, 1]$ be such that $\tau^c \hat{\alpha}^{p,w} \in \mathcal{B}(p, c)$. The *labour embodied in c* at the social reproduction point $\hat{\alpha}^{p,w}$ is $\tau^c \alpha_l^{p,w}$.

As in Roemer's (1982) approach, in Definition 4 the labour content of a bundle can be identified only if the price vector is known. Yet social relations play a more central role than in Roemer's theory, because the definition of labour content requires a prior knowledge of the social reproduction point and labour content is explicitly linked to the redistribution of total social labour (total labour employed), which corresponds to the total labour content of national income. Then, the following definition identifies the set of propertyless workers who are exploited.

Definition 5: Consider an economy $E \in \mathcal{E}$. Let (p, w) be a RS for E such that $\hat{\alpha}^{p,w}$ is the social reproduction point. For any $\nu \in W$, who supplies Λ^ν and consumes c^ν , let τ^{c^ν} be defined as in Definition 4. Then, $\nu \in W$ is *exploited*, i.e. $\nu \in N^{ted}$, if and only if $\Lambda^\nu > \tau^{c^\nu} \alpha_l^{p,w}$.

Definition 5 is conceptually related to the ‘New Interpretation’ developed by Duménil (1980) and Foley (1982). In fact, for any $\nu \in W$, τ^{c^ν} represents ν ’s share of national income, and so $\tau^{c^\nu} \alpha_t^{p,w}$ represents the share of social labour which ν receives by earning income barely sufficient to buy pc^ν . Then, as in the New Interpretation, the notion of exploitation is related to the production and distribution of national income and social labour.

In order to show that Definition 5 satisfies **LEW**, given any (p, w) such that $\alpha^{p,w}$ is the social production point, let $\tau^{c^\nu} = \frac{pc^\nu}{p\hat{\alpha}^{p,w}}$, $\bar{c}^\nu \equiv \tau^{c^\nu} \cdot \hat{\alpha}^{p,w} \in B(p, w\Lambda^\nu)$ and $\alpha^{\bar{c}^\nu} \equiv \tau^{c^\nu} \alpha^{p,w}$.

The previous arguments provide strong support to the idea that **LEW** does represent an appropriate domain condition in exploitation theory. **LEW** is formally weak and it incorporates some arguably compelling and widely shared views on exploitation as the UE of labour. Thus, although it can be proved that the axiom is not trivial and not all definitions in the literature satisfy it,¹² all of the major approaches do.¹³ The next question, then, is how to discriminate among the various definitions satisfying **LEW**.

A key tenet of UE exploitation theory is the idea that profits are one of the key determinants of the existence of exploitation, and of inequalities in well-being freedom: profits represent the way in which capitalists appropriate social surplus and social labour. Therefore there should exist in general a correspondence between the exploitation of at least the poorest segments of the working class and positive profits. The next axiom formalises this intuition.

Profit-Exploitation Correspondence Principle (PECP): *Given an economy $E \in \mathcal{E}$ and a RS for E , (p, w) , with aggregate production activity $\alpha^{p,w}$:*

$$[p\hat{\alpha}^{p,w} - w\alpha_t^{p,w} > 0 \Leftrightarrow N^{ted} \supseteq W_+],$$

whenever $W_+ \equiv \{\nu \in W \mid \Lambda^\nu > 0\} \neq \emptyset$.

A number of points are worth noting about **PECP**. First, the axiom is formulated without specifying *any* definition of exploitation: *whatever* the

¹²For example, it can be proved that the subjectivist notion of labour exploitation based on workers’ preferences recently proposed by Matsuo (2008) does not satisfy **LEW**. For a thorough discussion, see Yoshihara and Veneziani (2011).

¹³It is worth noting that based on Flaschel’s (1983) definition of *additive labor values*, it is possible to derive another formulation of labor exploitation that satisfies **LEW**. Similarly, Definition 6 in Yoshihara (2010) also satisfies **LEW**.

definition adopted, propertyless agents should be exploited if and only if profits are positive in equilibrium. Second, **PECP** is more general than in standard two-class models. This is because it both applies to advanced capitalist economies with a complex class structure, and allows for the possibility that propertyless workers in W_+ are a *strict* subset of the set of exploited agents, that is $W_+ \subset N^{ted}$. Note that the axiom focuses only on propertyless workers who perform at least some labour: this is a theoretically appropriate restriction, since the exploitation status of agents who do not engage in any economic activities is unclear. Third, unlike in standard models, **PECP** is general also in the sense that it allows for very general assumptions on agents and technology, including heterogeneous preferences and skills, a convex cone technology, and so on. Finally, unlike in the standard literature, **PECP** explicitly focuses on the exploitation status of a specific set of agents, rather than on the aggregate rate of exploitation in the economy. Indeed, the axiom imposes no constraints on the definition of exploitation at RS's with $W_+ = \emptyset$.

Let $B_{++}(p, w\Lambda) \equiv \{c \in \mathbb{R}_+^n \mid pc > w\Lambda\}$: $B_{++}(p, w\Lambda)$ is the set of consumption bundles that an agent in W supplying Λ units of effective labour cannot afford. Let $\Gamma(p, w; k) \equiv \{\hat{\alpha} \in \partial\hat{P}(\alpha_l = k) \cap \mathbb{R}_+^n \mid \hat{\alpha} \in B_{++}(p, wk)\}$: $\Gamma(p, w; k)$ is the set of net outputs that can be produced efficiently using k units of (effective) labour, which cannot be afforded by propertyless agents supplying k units of effective labour. The next theorem characterises the class of definitions of exploitation that satisfy **LEW** and such that **PECP** holds. Recall that if **LEW** holds, then for any $\nu \in W$, there is a $\bar{c}^\nu \in B(p, w\Lambda^\nu)$ and $\hat{\alpha}^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$ with $\hat{\alpha}^{\bar{c}^\nu} \not\prec \bar{c}^\nu$ such that $[\nu \in N^{ted} \Leftrightarrow \hat{\alpha}_l^{\bar{c}^\nu} < \Lambda^\nu]$.

Theorem 1: *For any definition of labour exploitation satisfying **LEW**, the following two statements are equivalent for any $E \in \mathcal{E}$ and for any RS (p, w) with associated aggregate production activity $\alpha^{p,w}$:*

- (1) **PECP** holds under this definition;
- (2) for each $\nu \in W_+$, $\left[\text{there exists } \underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu) \cup \left\{ \frac{\Lambda^\nu}{\alpha_l^{p,w}} \hat{\alpha}^{p,w} \right\} \text{ such that } \underline{c}^\nu > \hat{\alpha}^{\underline{c}^\nu} \right] \Leftrightarrow \pi^{\max} > 0$.

Proof: First of all, note that at any $E \in \mathcal{E}$ and any RS (p, w) with $\alpha^{p,w}$ such that either $W = \emptyset$ or $\Lambda^\nu = 0$ for all $\nu \in W$, the equivalence is immediately established, for both **PECP** and condition (2) are vacuously satisfied. Therefore in the rest of the proof suppose that $\Lambda^\nu > 0$ for at least some $\nu \in W$, and $W \neq \emptyset$.

(2) \Rightarrow (1): Consider any $E \in \mathcal{E}$ and any RS (p, w) with $\alpha^{p,w}$. Suppose that, for each $\nu \in W_+$, $\left[\text{there exists } \underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu) \cup \left\{ \frac{\Lambda^\nu}{\alpha_i^{p,w}} \widehat{\alpha}^{p,w} \right\} \text{ such that } \underline{c}^\nu > \widehat{\alpha}^{\bar{c}^\nu} \right] \Leftrightarrow \pi^{\max} > 0$.

Let $\pi^{\max} > 0$, so that by Lemma 1, $p\widehat{\alpha}^{p,w} - w\alpha_i^{p,w} > 0$. Note that, for any $\nu \in W_+$, if $\underline{c}^\nu = \frac{\Lambda^\nu}{\alpha_i^{p,w}} \widehat{\alpha}^{p,w}$ then $\underline{c}^\nu \in \partial \widehat{P}(\alpha_l = \Lambda^\nu)$. Then, for all $\nu \in W_+$, since $\underline{c}^\nu \in \partial \widehat{P}(\alpha_l = \Lambda^\nu)$ and $\underline{c}^\nu > \widehat{\alpha}^{\bar{c}^\nu}$, and noting that $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$, we have $\alpha_i^{\bar{c}^\nu} < \Lambda^\nu$. Thus, by **LEW**, $\nu \in N^{ted}$ holds for any $\nu \in W_+$.

Let $\pi^{\max} = 0$, so that by Lemma 1, $p\widehat{\alpha}^{p,w} - w\alpha_i^{p,w} = 0$. First, note that by A2, $\pi^{\max} = 0$ implies $w > 0$. Next, for each $\nu \in W_+$, if $\pi^{\max} = 0$, then $\partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_+^n \subseteq B_-(p, w\Lambda^\nu) \equiv \{c \in \mathbb{R}_+^n \mid pc \leq w\Lambda^\nu\}$, which implies that $\Gamma(p, w; \Lambda^\nu) = \emptyset$. Thus, (2) implies that for each $\nu \in W_+$, for $\underline{c}^\nu = \frac{\Lambda^\nu}{\alpha_i^{p,w}} \widehat{\alpha}^{p,w}$, $\underline{c}^\nu > \widehat{\alpha}^{\bar{c}^\nu}$ does not hold. Then, $\underline{c}^\nu = \frac{\Lambda^\nu}{\alpha_i^{p,w}} \widehat{\alpha}^{p,w}$ and $\bar{c}^\nu \in B(p, w\Lambda^\nu)$ imply that for any $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu)$ with $\widehat{\alpha}^{\bar{c}^\nu} \not\geq \bar{c}^\nu$, $\alpha_i^{\bar{c}^\nu} \geq \Lambda^\nu$. Thus, by **LEW**, $\nu \notin N^{ted}$ holds for any $\nu \in W_+$.

In sum, (2) implies that **PECP** holds under any definition of exploitation satisfying **LEW**.

(1) \Rightarrow (2): Consider any $E \in \mathcal{E}$ and any RS (p, w) with $\alpha^{p,w}$. Suppose that $p\widehat{\alpha}^{p,w} - w\alpha_i^{p,w} > 0 \Leftrightarrow N^{ted} \supseteq W_+$.

Let $p\widehat{\alpha}^{p,w} - w\alpha_i^{p,w} > 0$, so that $\pi^{\max} > 0$. By **LEW** and **PECP**, for each $\nu \in W_+$, there exist $\bar{c}^\nu \in \mathbb{R}_+^n$ and $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$ with $\alpha^{\bar{c}^\nu} \not\geq \bar{c}^\nu$ such that $p\bar{c}^\nu = w\Lambda^\nu$ and $\alpha_i^{\bar{c}^\nu} < \Lambda^\nu$.

Suppose first that $\alpha_i^{\bar{c}^\nu} = 0$ for some $\nu \in W_+$. Then, by **A1** and **LEW**, this implies that $\widehat{\alpha}^{\bar{c}^\nu} = \mathbf{0}$, and therefore $\bar{c}^\nu = \mathbf{0}$, which implies $p\bar{c}^\nu = 0$ and $w = 0$. Then, noting $\Lambda^\nu > 0$ for all $\nu \in W_+$, the set $\partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$ is non-empty by **A2**. Moreover, for any $\underline{c}^\nu \in \partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$, $\underline{c}^\nu \in B_{++}(p, w\Lambda^\nu)$ follows from $p \geq \mathbf{0}$. Therefore, there exists $\underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu)$ such that $\underline{c}^\nu > \widehat{\alpha}^{\bar{c}^\nu}$.

Suppose that $\alpha_i^{\bar{c}^\nu} > 0$ for all $\nu \in W_+$. Then, for all $\nu \in W_+$, let $\delta^\nu > 1$ be such that $\delta^\nu \alpha_i^{\bar{c}^\nu} = \Lambda^\nu$. For each $\nu \in W_+$, let us first consider the case that $\widehat{\alpha}^{\bar{c}^\nu} > \mathbf{0}$. Then, let $\underline{c}^\nu \equiv \delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$. Clearly $\underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu)$ and $\underline{c}^\nu > \widehat{\alpha}^{\bar{c}^\nu}$. Secondly, let us consider the case that $\widehat{\alpha}^{\bar{c}^\nu} \not\geq \mathbf{0}$. Note that, because of A2, $\partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n \neq \emptyset$ holds. Since $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu} \in \partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_+^n$, the convexity of $\widehat{P}(\alpha_l = \Lambda^\nu)$ guarantees that any convex combination of $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$ and any points in $\partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$ is feasible, and any point derived from this convex combination belongs to $\widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$, even if it is very close to $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$. Thus, for any open neighbourhood \mathcal{V} of $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$, $\mathcal{V} \cap \widehat{P}(\alpha_l = \Lambda^\nu) \cap$

$\mathbb{R}_{++}^n \neq \emptyset$, and for some sufficiently small neighbourhood \mathcal{V}^* of $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$, there is $x^\nu \in \mathcal{V}^* \cap \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$ which is sufficiently close to $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$ and $x^\nu > \widehat{\alpha}^{\bar{c}^\nu}$ holds. Then, there is $\epsilon^\nu \geq 1$ such that $\epsilon^\nu x^\nu \in \partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$. Take $\underline{c}^\nu \equiv \epsilon^\nu x^\nu$. Suppose $w > 0$. Then $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu} \in B_{++}(p, w\Lambda^\nu)$, and $\underline{c}^\nu \in B_{++}(p, w\Lambda^\nu)$ follows from the fact that x^ν is sufficiently close to $\delta^\nu \widehat{\alpha}^{\bar{c}^\nu}$, $B_{++}(p, w\Lambda^\nu)$ is open, and $\epsilon^\nu \geq 1$. Thus, $\underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu)$ and $\underline{c}^\nu > \widehat{\alpha}^{\bar{c}^\nu}$. Suppose $w = 0$. If $p\widehat{\alpha}^{\bar{c}^\nu} > 0$, the result follows in a similar manner. If $p\widehat{\alpha}^{\bar{c}^\nu} = 0$, the result follows from the fact that $\underline{c}^\nu \in \mathbb{R}_{++}^n$, noting that $\pi^{\max} > 0$ implies $p \geq \mathbf{0}$.

Let $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} = 0$, so that by Lemma 1, $\pi^{\max} = 0$. By **LEW** and **PECP**, for some $\nu \in W_+$, there exist $\bar{c}^\nu \in \mathbb{R}_+^n$ and $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$ with $\alpha^{\bar{c}^\nu} \not\geq \bar{c}^\nu$ such that $p\bar{c}^\nu = w\Lambda^\nu$ and $\alpha_l^{\bar{c}^\nu} \geq \Lambda^\nu$. Actually, the latter property must hold for all $\nu \in W_+$. For suppose, to the contrary, that for some $\nu \in W_+$, there exist $\bar{c}^\nu \in \mathbb{R}_+^n$ and $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$ with $\alpha^{\bar{c}^\nu} \not\geq \bar{c}^\nu$ such that $p\bar{c}^\nu = w\Lambda^\nu$ and $\alpha_l^{\bar{c}^\nu} < \Lambda^\nu$. Then this implies $p\widehat{\alpha}^{\bar{c}^\nu} \geq p\bar{c}^\nu = w\Lambda^\nu > w\alpha_l^{\bar{c}^\nu}$, which violates the assumption that $\pi^{\max} = 0$. Thus, for any $\nu \in W_+$, there exist $\bar{c}^\nu \in \mathbb{R}_+^n$ and $\alpha^{\bar{c}^\nu} \in \phi(\bar{c}^\nu) \cap \partial P$ with $\alpha^{\bar{c}^\nu} \not\geq \bar{c}^\nu$ such that $p\bar{c}^\nu = w\Lambda^\nu$ and $\alpha_l^{\bar{c}^\nu} \geq \Lambda^\nu$. Then, for each $\nu \in W_+$, let $\underline{c}^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w}$, since $\Gamma(p, w; \Lambda^\nu)$ is empty when $\pi^{\max} = 0$. Since $p\underline{c}^\nu = w\Lambda^\nu = p\bar{c}^\nu$, it follows that $\underline{c}^\nu \not\geq \widehat{\alpha}_i^{\bar{c}^\nu}$ for at least some i with $\widehat{\alpha}_i^{\bar{c}^\nu} > 0$. Note that by **A2**, $\pi^{\max} = 0$ implies that $w > 0$.

In sum, if **PECP** holds, then (2) holds under any definition of exploitation satisfying **LEW**. ■

Theorem 1 can be interpreted as follows. **PECP** states that propertyless workers are exploited if and only if equilibrium profits are positive. According to **LEW**, the exploitation status of propertyless workers is determined by identifying a set of reference bundles (call them the *exploitation-reference bundles*). By Theorem 1, in every convex economy, **PECP** holds if and only if the existence of positive profits in equilibrium is also determined by identifying a set of reference bundles (call them the *profit-reference bundles*). According to **LEW**, the exploitation-reference bundles must be affordable by the workers and must be producible with less than Λ^ν units of labour for all exploited workers. According to condition (2) of Theorem 1, instead, for all workers $\nu \in W_+$, the profit-reference bundles must be producible with a technically efficient process using Λ^ν units of labour, and must be such that they are not affordable by ν and dominate the exploitation-reference vectors if and only if the maximum profit rate is positive. The relevance of Theorem 1, then, is not only in the identification of a general condition for the validity

of the relation between exploitation and profits. Methodologically, Theorem 1 suggests that different views about exploitation, and the analysis of the key features of exploitation theory, should focus on the identification of the relevant vectors of (exploitation and profit) reference bundles.

Theorem 1 does not identify a unique definition of exploitation that meets axiom **PECP**, but rather a class of definitions satisfying condition (2). Yet Theorem 1 has surprising implications concerning the main received approaches in exploitation theory. For there are economies in which for all $\nu \in W_+$, no point in $\Gamma(p, w; \Lambda^\nu) \cup \left\{ \frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w} \right\}$ satisfies condition (2), if $\widehat{\alpha}^{c^\nu}$ is given either by Definition 2 or by Definition 3. In contrast, it can be proved that Definition 5 satisfies condition (2), and thus **PECP** holds in general convex economies with heterogeneous agents.¹⁴

Corollary 1: *There exists an economy $E \in \mathcal{E}$ and a RS (p, w) with associated aggregate production point $\alpha^{p,w}$ such that neither Definition 2 nor Definition 3 satisfies **PECP**. Instead, Definition 5 satisfies **PECP** for all $E \in \mathcal{E}$ and all RS (p, w) .*

Proof. For the proof that neither Definitions 2 nor 3 satisfies **PECP**, see Appendix 2. We need to prove that Definition 5 satisfies condition (2) of Theorem 1. We consider two cases for any $E \in \mathcal{E}$ and any RS (p, w) with $W_+ \neq \emptyset$.

Case 1: $\widehat{\alpha}^{p,w} > \mathbf{0}$. By setting $\underline{c}^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w}$ for all $\nu \in W_+$, it is immediately seen that Definition 5 satisfies condition (2).

Case 2: $\widehat{\alpha}^{p,w} \geq \mathbf{0}$ and $\widehat{\alpha}^{p,w} \not\geq \mathbf{0}$. (Note that the case $\widehat{\alpha}^{p,w} = \mathbf{0}$ can be ruled out at any RS with $W_+ \neq \emptyset$.) First, let this RS (p, w) be associated to $\pi^{\max} = 0$. Then, only $\underline{c}^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w}$ is available for all $\nu \in W_+$, since $\Gamma(p, w; \Lambda^\nu) = \emptyset$ as shown in the proof of Theorem 1. Then, it is immediately seen that Definition 5 does not meet $\underline{c}^\nu > \widehat{\alpha}^{c^\nu}$. Second, let this RS (p, w) be associated to $\pi^{\max} > 0$. Then, for each $\nu \in W_+$, $\frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w} \in \partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \partial \mathbb{R}_+^n$. Further, noting that $\tau^{c^\nu} = \frac{pc^\nu}{p\widehat{\alpha}^{p,w}} = \frac{w\Lambda^\nu}{w\alpha_l^{p,w}} \frac{w\alpha_l^{p,w}}{p\widehat{\alpha}^{p,w}} = \frac{\Lambda^\nu}{\alpha_l^{p,w}} \frac{w\alpha_l^{p,w}}{p\widehat{\alpha}^{p,w}}$ and $\frac{w\alpha_l^{p,w}}{p\widehat{\alpha}^{p,w}} < 1$ by $\pi^{\max} > 0$, it follows that $\tau^{c^\nu} < \frac{\Lambda^\nu}{\alpha_l^{p,w}}$ and $\tau^{c^\nu} \widehat{\alpha}^{p,w} \leq \frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w}$, where $\tau^{c^\nu} \widehat{\alpha}^{p,w} \in \partial \widehat{P}(\alpha_l = k^\nu) \cap \partial \mathbb{R}_+^n$ for some $k^\nu < \Lambda^\nu$. Then, using the same argument as in Theorem 1, it can be shown that for each $\nu \in W_+$, there is

¹⁴An example similar to those analysed in Appendix 2 below is used in Yoshihara (2010; Corollary 2) to prove that the *Class-Exploitation Correspondence Principle* (Roemer, 1982) does not hold under Definition 3.

$x^\nu \in \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$, which is sufficiently close to $\frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w}$, and such that $x^\nu > \tau^{c^\nu} \widehat{\alpha}^{p,w}$. Since $\frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w} \in B_{++}(p, w\Lambda^\nu)$ by $p \frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w} - w\Lambda^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}} (p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w}) > 0$, $x^\nu \in B_{++}(p, w\Lambda^\nu)$ follows from the fact that x^ν is sufficiently close to $\frac{\Lambda^\nu}{\alpha_l^{p,w}} \widehat{\alpha}^{p,w}$ and $B_{++}(p, w\Lambda^\nu)$ is open. Then, let $\epsilon^\nu \geq 1$ be such that $\epsilon^\nu x^\nu \in \partial \widehat{P}(\alpha_l = \Lambda^\nu) \cap \mathbb{R}_{++}^n$. Let $\underline{c}^\nu \equiv \epsilon^\nu x^\nu$ for each $\nu \in W_+$. By construction, $\underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu)$. Furthermore, since $x^\nu > \tau^{c^\nu} \widehat{\alpha}^{p,w}$, then $\tau^{c^\nu} \widehat{\alpha}^{p,w} < \underline{c}^\nu$ and since $\widehat{\alpha}^{\underline{c}^\nu} = \tau^{c^\nu} \widehat{\alpha}^{p,w}$ under Definition 5, $\widehat{\alpha}^{\underline{c}^\nu} < \underline{c}^\nu$ holds for each $\nu \in W_+$.

In summary, condition (2) of Theorem 1 holds for any RS (p, w) . ■

4 Exploitation and Profits: Two extensions

Given the theoretical relevance of **PECP** in exploitation theory, Theorem 1 and Corollary 1 provide strong support for Definition 5 as the appropriate notion of UE exploitation. In this section, two extensions of the analysis are presented, which provide further support to the ‘New Interpretation’. The generality of the model is exploited to show that Definition 5 can be extended to analyse, first, the exploitation status of all agents and the existence of exploitative *relations*; and then the correspondence between exploitation and profits outside of equilibrium allocations, in economies with heterogeneous preferences and unequal endowments of physical and human capital. This suggests that, if the ‘New Interpretation’ is adopted, then exploitation theory can be extended to yield interesting insights on unequal relations between agents in advanced capitalist economies. As a first step, Definition 5 is generalised to identify the exploitation status of all agents.

Definition 6: Consider any economy $E \in \mathcal{E}$. Let (p, w) be a RS for E with aggregate production activity $\alpha^{p,w}$. For any $\nu \in N$, who supplies Λ^ν and consumes $c^\nu \in \mathbb{R}_+^n$, let τ^{c^ν} be defined as in Definition 4. Agent ν is: exploited if and only if $\Lambda^\nu > \tau^{c^\nu} \alpha_l^{p,w}$; an *exploiter* if and only if $\Lambda^\nu < \tau^{c^\nu} \alpha_l^{p,w}$; and *neither exploited nor an exploiter* if and only if $\Lambda^\nu = \tau^{c^\nu} \alpha_l^{p,w}$.

Theorem 2 proves that, based on Definition 6, it is possible to characterise the exploitation status of all agents - and not only of the poorest segments of the working class - and to derive a more general relation between profits and exploitation beyond the subset of propertyless agents. Recall that N^{ted} is the set of exploited agents and N^{ter} is the set of exploiters.

Theorem 2: Consider an economy $E \in \mathcal{E}$. Let (p, w) be a RS for E with $w > 0$ and associated aggregate production activity $\alpha^{p,w}$. Under Definition 6:

(1) if $\pi^{\max} > 0$, agent ν is: exploited if and only if $\frac{p\omega^\nu}{p\omega} < \frac{\Lambda^\nu}{\alpha_l^{p,w}}$; neither exploited nor an exploiter if and only if $\frac{p\omega^\nu}{p\omega} = \frac{\Lambda^\nu}{\alpha_l^{p,w}}$; and an exploiter if and only if $\frac{p\omega^\nu}{p\omega} > \frac{\Lambda^\nu}{\alpha_l^{p,w}}$.

(2) if $\pi^{\max} > 0$, then $\left\{ \nu \in N \mid \frac{p\omega^\nu}{\alpha_l^{p,w}} < \frac{p\omega^\nu}{s^\nu} \right\} \subseteq N^{ter}$. Furthermore, if there is a subsistence bundle $b \in \mathbb{R}_+^n$ such that $c^\nu \geq b$, for all $\nu \in N$, then $\left\{ \nu \in N \mid \frac{p\omega^\nu}{p\omega} < \frac{pb}{p\hat{\alpha}^{p,w}} \right\} \subseteq N^{ted}$.

(3) if $\pi^{\max} = 0$, $N^{ted} = N^{ter} = \emptyset$.

Proof. 1. Consider the case $p\hat{\alpha}^{p,w} > 0$.

Part (1). Let (p, w) be a RS for $E \in \mathcal{E}$. Then by Definition 1-(i), it follows that $p\hat{\alpha}^\nu + [p\hat{\beta}^\nu - w\beta_l^\nu] + w\gamma^\nu = pc^\nu$ for all $\nu \in N$. Since $p(\underline{\alpha}^\nu + \underline{\beta}^\nu) = p\omega^\nu$ for all $\nu \in N$, and noting that only processes yielding the maximum rate of profit are going to be activated, the latter expression can be written as $\pi^{\max}p\omega^\nu + w\Lambda^\nu = pc^\nu$. Then, by Definition 1-(ii) and Definition 1-(iv), it follows that $\pi^{\max}p\omega + w\alpha_l^{p,w} = p\hat{\alpha}^{p,w}$. Therefore $\Lambda^\nu = \tau^{c^\nu} \alpha_l^{p,w}$ if and only if $\Lambda^\nu = \frac{\pi^{\max}p\omega^\nu + w\Lambda^\nu}{\pi^{\max}p\omega + w\alpha_l^{p,w}} \alpha_l^{p,w}$, which yields the desired result. The other two inequalities are proved similarly.

Part (2). Let (p, w) be a RS for $E \in \mathcal{E}$. The first part of the statement follows immediately from part 1, noting that $\lambda^\nu \leq 1$. In order to prove the second part of the statement, note that by Definition 1-(i), it follows that $p\hat{\alpha}^\nu + [p\hat{\beta}^\nu - w\beta_l^\nu] + w\gamma^\nu = pc^\nu$ for all $\nu \in N$. Since $p(\underline{\alpha}^\nu + \underline{\beta}^\nu) = p\omega^\nu$, for all $\nu \in N$, and noting that only processes yielding the maximum rate of profit are going to be activated, the latter expression can be written as $\pi^{\max}p\omega^\nu + w\Lambda^\nu = pc^\nu$. Therefore it follows that $\Lambda^\nu > \tau^{c^\nu} \alpha_l^{p,w}$ if and only if $\left[\frac{pc^\nu - \pi^{\max}p\omega^\nu}{w} \right] > \frac{pc^\nu}{p\hat{\alpha}^{p,w}} \alpha_l^{p,w}$, which is in turn equivalent to $pc^\nu \left[1 - \frac{w\alpha_l^{p,w}}{p\hat{\alpha}^{p,w}} \right] > \pi^{\max}p\omega^\nu$. Then, setting $c^\nu \geq b$, for all $\nu \in N$, gives the desired result.

Part (3). If $\pi^{\max} = 0$, then it follows that $w\Lambda^\nu = pc^\nu$, for all $\nu \in N$, and $w\alpha_l^{p,w} = p\hat{\alpha}^{p,w}$, which yields the desired result.

2. Consider the case $p\hat{\alpha}^{p,w} = 0$. Then, since $p\hat{\alpha}^{p,w} = w\alpha_l^{p,w} + \pi^{\max}p\hat{\alpha}^{p,w}$ holds in the RS, $\alpha_l^{p,w} = 0$ follows from $w > 0$, which together with **A1** imply that $\alpha^{p,w} = \mathbf{0}$. Note that the RS (p, w) with $\alpha^{p,w} = \mathbf{0}$ implies that $\pi^{\max} = 0$,

thus we only examine Part (3). Given $\alpha_l^{p,w} = 0$, $\Lambda^\nu = 0$ holds for any $\nu \in N$. Thus, $\Lambda^\nu = \tau^{c^\nu} \alpha_l^{p,w}$ holds for any $\nu \in N$, which implies $N^{ted} = N^{ter} = \emptyset$. ■

Theorem 2-(1) completely characterises the exploitation structure of an economy in equilibrium: an agent is exploited (respectively, an exploiter) if and only if her share of social wealth is lower (respectively, higher) than her share of social labour. Theorem 2-(2) shows that at the two extremes of the wealth distribution, exploitation status can be determined independently of individual choices, an intuition of standard Marxist theory that is proved to be robust. Indeed, if a subsistence bundle exists, the set of agents that are exploited regardless of their individual choices will be larger than the set of propertyless agents (those who have ‘nothing to lose but their chains’). This set can be sizable if b is not interpreted as a *physical* subsistence bundle, but rather as reflecting moral and social elements. Jointly with Theorem 2-(3), this result proves the correspondence between positive profits and the exploitation of a larger set of agents than the propertyless segment of the working class. Actually, given Definition 6, a very interesting property of the ‘New Interpretation’ immediately derives from Theorems 1 and 2, which can be stated formally as follows.

Corollary 2: *Consider an economy $E \in \mathcal{E}$. Let (p, w) be a RS for E with associated aggregate production activity $\alpha^{p,w}$ such that $W_+ \neq \emptyset$. Under Definition 6, the following statements are equivalent:*

- (1) $\pi^{\max} > 0$;
- (2) $W_+ \subseteq N^{ted} \neq \emptyset$;
- (3) $N^{ter} \neq \emptyset$.

Corollary 2 implies that in equilibrium positive profits are a necessary and sufficient condition for the existence of *exploitative relations*, where the latter notion can be formalised as requiring that $N^{ted} \neq \emptyset$ if and only if $N^{ter} \neq \emptyset$. This seems a weak and obviously reasonable property in exploitation theory: some agents are exploited if and only if there is someone exploiting them. Yet Yoshihara and Veneziani (2009) show that none of the main definitions in the literature satisfy it in general. As shown by Corollary 2, instead, according to the ‘New Interpretation’, exploitation has an inherently relational nature. Further, it captures inequalities between classes of individuals concerning the allocation of *labour*. In fact, it is not difficult to prove that, unlike in other approaches, if some other good is used as the exploitation numéraire in Definition 6, neither **PECP** nor Corollary 2 holds.

Theorem 2 and Corollary 2 complete the analysis of the relation between exploitation and profits in equilibrium. They extend the main insights of UE exploitation theory to all agents in the general economies considered in this paper, under Definition 6. This is crucial given the focal theoretical interest in equilibrium allocations, but one may argue that a robust theory of exploitation should provide insights also on disequilibrium allocations. In the rest of this section, an extension of Definition 5 is proposed, and a general relation between exploitation and profits is derived, at *any feasible allocation*.

The key point to note is that there are various possible ways of conceptualising exploitation at general disequilibrium allocations and, consequently, there is no trivial way of extending Definition 5. For example, outside of a RS, it is unclear whether exploitation status should be determined relative to the *actual* features of the allocation. On the one hand, if individual plans are not realised, coordination failures arise, and perhaps even sheer mistakes are made, then by focusing on *actual* data one may be capturing only purely transient and ephemeral phenomena that do not tell much about the structural features of the economy. On the other hand, one may insist that, even outside of an RS, only the information contained in the *actual* allocation point is relevant to analyse exploitation. For, ultimately, the actual features of the allocation are what matters to the agents.

In the extension of Definition 5 to disequilibrium allocations proposed here, the actual features of the allocation, including the actual price vector, the aggregate production activity, and the individual work and consumption choices of all agents remain central in the definition of the labour content of a bundle of commodities and the exploitation status of propertyless agents. However, the effects of sheer individual mistakes in technical choices, or of purely temporary market imbalances leading to productive inefficiency are discounted. To be precise, given a price vector (p, w) and an associated aggregate production activity $\alpha^{p,w} \in P$, define

$$\begin{aligned} & \phi(c; \alpha^{p,w}) \\ \equiv & \left\{ \alpha' \in P \mid \exists t, \mu \in \mathbb{R}_+ : (\alpha'_l, \hat{\alpha}') = (t\alpha_l^{p,w}, t\mu\hat{\alpha}^{p,w}), \mu\hat{\alpha}^{p,w} \in \hat{P}(\alpha_l = \alpha_l^{p,w}) \ \& \ \hat{\alpha}' \geq c \right\}. \end{aligned}$$

$\phi(c; \alpha^{p,w})$ denotes the set of production activities which are along the ray defined by $(\alpha_l^{p,w}, \hat{\alpha}^{p,w})$ and produce at least c as net output. Then:

$$l.v.(c; \alpha^{p,w}) \equiv \min \{ \alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in \phi(c; \alpha^{p,w}) \}.$$

Clearly, $l.v. (c; \alpha^{p,w})$ is well-defined and bounded below by 0. The labour content of a bundle c at any given allocation can be defined as follows.

Definition 7: Consider an economy $E \in \mathcal{E}$. Let (p, w) be a price vector for E with associated aggregate production activity $\alpha^{p,w}$. For each $c \in \mathbb{R}_+^n$ with $pc \leq p\hat{\alpha}^{p,w}$, let $\tau^c \in [0, 1]$ be such that $\tau^c \hat{\alpha}^{p,w} \in \mathcal{B}(p, c)$. The *labour embodied in c* at the social reproduction point $\alpha^{p,w}$ is $l.v. (\tau^c \hat{\alpha}^{p,w}; \alpha^{p,w})$.

The following definition identifies the set of propertyless workers who are exploited at any given allocation.

Definition 8: Consider an economy $E \in \mathcal{E}$. Let (p, w) be a price vector for E such that $\alpha^{p,w}$ is the associated production point. For any $\nu \in W$, who supplies Λ^ν and consumes c^ν , let τ^{c^ν} be defined as in Definition 7. Then, $\nu \in W$ is *exploited*, i.e. $\nu \in N^{ted}$, if and only if $\Lambda^\nu > l.v. (\tau^{c^\nu} \hat{\alpha}^{p,w}; \alpha^{p,w})$.

From a formal viewpoint, Definitions 7 and 8 generalise Definitions 4 and 5 and they reduce to the latter at a RS. In fact, if (p, w) is a RS for E , then $\alpha^{p,w} \in \partial P$ and $l.v. (\tau^c \hat{\alpha}^{p,w}; \alpha^{p,w}) = \tau^c \alpha_l^{p,w}$ holds. From a theoretical viewpoint, in Definitions 7 and 8, the actual allocation of the economy plays a pivotal role. In order to define labour content and the exploitation status of propertyless agents, the *actual* price vector and the *actual* individual choices on work and consumption are central. The only possible deviation from actual data concerns the focus on technically efficient production activities in the definition of labour content, but the set of admissible efficient activities used in Definitions 7 and 8 is significantly constrained by the *actual* social production point $\alpha^{p,w}$ (unlike in Roemer's or Morishima's definitions).

The focus on efficient aggregate production vectors is theoretically reasonable. For technically inefficient activities in the interior of the production possibility set are the product of transient contingencies and do not reveal much about the structural features of the economy.¹⁵ Moreover, note that, given the nature of **LEW** as a domain condition, in section 3 a weak formulation of the axiom is adopted by restricting its application to RS's. It is straightforward, however, to extend **LEW** to all price vectors (p, w) with associated social production point $\alpha^{p,w}$ and, from a theoretical viewpoint, none of the arguments used to defend **LEW** in section 3 depends on the

¹⁵Indeed, Marx's own notion of Socially Necessary Labour Time may be interpreted as ruling out inefficient technologies and involving a counterfactual analysis. See Sen (1978).

assumption that the allocation is an equilibrium. Therefore one may argue that **LEW** remains an appropriate domain condition to define UE exploitation even at disequilibrium allocations. From this perspective, it is worth noting that Definition 8 satisfies **LEW**, at any (p, w) with associated social production point $\alpha^{p,w}$. To see this, let $\bar{c}^\nu \equiv \tau^{c^\nu} \cdot \hat{\alpha}^{p,w} \in B(p, w\Lambda^\nu)$ and $\alpha^{\bar{c}^\nu} \equiv \arg \min \{\alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha}) \in \phi(\bar{c}^\nu; \alpha^{p,w})\}$.

Let $C^W = \sum_{\nu \in W} c^\nu$ and $\Lambda^W = \sum_{\nu \in W} \Lambda^\nu$. Based on Definition 8, Theorem 3 establishes a general relation between exploitation and profits for any general convex cone economies and at any feasible allocations.

Theorem 3: *For any economy $E \in \mathcal{E}$, any $(p, w) \in \mathbb{R}_+^{n+1}$ with $w > 0$ and any allocation $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)_{\nu \in N}$ with $pc^\nu = p(\hat{\alpha}^\nu + \hat{\beta}^\nu) - w(\beta_l^\nu - \gamma^\nu)$ ($\forall \nu \in N$), and $W_+ \neq \emptyset$, the following statements are equivalent for any $\alpha^* \in \partial P(\alpha_l = \Lambda^W)$ with $\hat{\alpha}^* \in \partial \hat{P}(\alpha_l = \Lambda^W) \cap \mathbb{R}_+^n$:*

- (1) $p\hat{\alpha}^* - w\alpha_l^* > 0$ holds;
- (2) for any $\nu \in W_+$, $\Lambda^\nu > l.v.(\tau^{c^\nu} \hat{\alpha}^*; \alpha^*)$, where $l.v.(\tau^{c^\nu} \hat{\alpha}^*; \alpha^*) = \tau^{c^\nu} \alpha_l^*$ for $\tau^{c^\nu} \in [0, 1)$ with $\tau^{c^\nu} \hat{\alpha}^* \in \mathcal{B}(p, c^\nu)$.

Proof. Taking a point $\hat{\alpha}^*$ from $\partial \hat{P}(\alpha_l = \Lambda^W) \cap \mathbb{R}_+^n$. Let $\alpha^* \in \partial P(\alpha_l = \Lambda^W)$ be a production point corresponding to $\hat{\alpha}^*$.

Suppose (1) holds. Then, $p\hat{\alpha}^* - w\Lambda^W = p(\hat{\alpha}^* - C^W) > 0$, since the budget constraint holds for all agents. Note that, for any $\nu \in W_+$, $pc^\nu = w\Lambda^\nu = w\Lambda^W \frac{\Lambda^\nu}{\Lambda^W} = pC^W \frac{\Lambda^\nu}{\Lambda^W}$. Then, let $\tau^{c^\nu} = \frac{pc^\nu}{p\hat{\alpha}^*}$ for any $\nu \in W_+$. Clearly $\tau^{c^\nu} \in [0, 1)$ with $\tau^{c^\nu} \hat{\alpha}^* \in \mathcal{B}(p, c^\nu)$. Moreover, for any $\nu \in W_+$, $\tau^{c^\nu} \alpha_l^* = \frac{pc^\nu}{p\hat{\alpha}^*} \Lambda^W = \Lambda^\nu \frac{pc^\nu}{p\hat{\alpha}^*} < \Lambda^\nu$, where the latter inequality follows from $p(\hat{\alpha}^* - C^W) > 0$. Finally, since $\alpha^* \in \partial P(\alpha_l = \Lambda^W)$, $l.v.(\tau^{c^\nu} \hat{\alpha}^*; \alpha^*) = \tau^{c^\nu} \alpha_l^*$ holds. Thus, (2) is obtained.

Suppose (2) holds. Then, for any $\nu \in W_+$, $\Lambda^\nu > l.v.(\tau^{c^\nu} \hat{\alpha}^*; \alpha^*)$, where $l.v.(\tau^{c^\nu} \hat{\alpha}^*; \alpha^*) = \tau^{c^\nu} \alpha_l^*$ holds for $\tau^{c^\nu} \in [0, 1)$ with $\tau^{c^\nu} \hat{\alpha}^* \in \mathcal{B}(p, c^\nu)$. Thus, $\Lambda^W > \sum_{\nu \in W_+} \tau^{c^\nu} \alpha_l^*$ holds. Note that for any $\nu \in W_+$, $w\Lambda^\nu = pc^\nu > 0$ by $w > 0$. Then, $\tau^{c^\nu} \hat{\alpha}^* \in \mathcal{B}(p, c^\nu)$ implies $\tau^{c^\nu} > 0$ and $p\hat{\alpha}^* > 0$. Since $\tau^{c^\nu} = \frac{pc^\nu}{p\hat{\alpha}^*}$ for any $\nu \in W_+$, $\Lambda^W > \sum_{\nu \in W_+} \tau^{c^\nu} \alpha_l^*$ implies that $\Lambda^W > \frac{pC^W}{p\hat{\alpha}^*} \Lambda^W$, thus $p(\hat{\alpha}^* - C^W) > 0$ holds. Since $pC^W = w\Lambda^W = w\alpha_l^*$ by the budget constraint, $p\hat{\alpha}^* - w\alpha_l^* > 0$ holds. ■

Theorem 3 states that a general relation between exploitation and profits holds, at any price vector and corresponding allocation, provided that pro-

ductive inefficiencies and temporary disequilibrium phenomena are ruled out: at every technically efficient production vector α^* (which is feasible using actual, effective labour $\Lambda^W = \sum_{\nu \in W} \Lambda^\nu$) society realises positive profits if and only if every propertyless worker is exploited. In order to appreciate the full generality of Theorem 3, it is important to stress that no significant restriction is imposed on individual behaviour (except that the budget constraint holds for all agents) and on the actual allocation. As a result, Theorem 3 does not establish necessary and sufficient conditions for the existence of positive profits and the exploitation of propertyless workers at the *actual* allocation, and the social production point $\alpha^{p,w}$ may, or may not, coincide with one of the vectors α^* . For given the extremely weak restrictions on the set of admissible allocations, the link between profits and exploitation may be somewhat weakened. For instance, if $\frac{\Lambda^W}{\alpha_l + \beta_l} (\hat{\alpha} + \hat{\beta}) \in \hat{P}(\alpha_l = \Lambda^W) \setminus \partial \hat{P}(\alpha_l = \Lambda^W)$ and $C^W \in \hat{P}(\alpha_l = \Lambda^W) \setminus \partial \hat{P}(\alpha_l = \Lambda^W)$ hold at the actual allocation, then the corresponding profit rate may be non-positive while propertyless agents are exploited. However, Theorem 3 derives the general conditions under which the economy can generate positive profits and propertyless workers are exploited, starting from the actual individual consumption/leisure choices, price system, and aggregate production activity. In other words, if one abstracts from temporary disequilibrium phenomena, Theorem 3 does derive a fully general relation between the appropriation of surplus by capitalists and the exploitation of (propertyless) workers, which holds even if exchanges do not take place at equilibrium prices.

5 Conclusions

This paper provides a novel axiomatic analysis of the notion of exploitation as the unequal exchange of labour. General convex economies with agents endowed with heterogeneous preferences and with different amounts of physical and human capital are considered. A definition of exploitation related to the ‘New Interpretation’ is analysed, which emphasises the relational nature of exploitation and the inequalities in the allocation of labour. An axiomatic characterisation of the class of definitions that preserve two weak axioms - a domain condition called Labour Exploitation of the Working Class and the Profit-Exploitation Correspondence Principle - is derived (Theorem 1). Based on this characterisation, it is shown that none of the main received

definitions preserves the link between the appropriation of surplus and the exploitation of (at least some) workers, except for the ‘New Interpretation’. The latter definition also allows one to generalise some key insights of exploitation theory in complex convex economies with heterogeneous agents: it is possible to characterise the exploitation status of all agents in equilibrium (Theorem 2) and to derive a general relation between exploitation and profits even outside of equilibrium allocations (Theorem 3).

Given the relevance of the PECP in exploitation theory, the results presented in this paper provide strong support to the ‘New Interpretation’ as the appropriate notion of exploitation in advanced capitalist economies. Thus, they complement and strengthen the analysis developed by Yoshihara and Veneziani (2009) in the context of convex *subsistence* economies. In fact, as mentioned in section 2 above, the main results of the paper could be derived by assuming the function u^ν to be weakly monotone on $C \times [0, 1]$ and strictly monotone in at least one argument, provided some additional technical conditions to ensure local nonsatiation are added.¹⁶ This assumption encompasses the special case where there is a subsistence bundle $b \in \mathbb{R}_+^n$ such that $C \equiv \{c \in \mathbb{R}_+^n \mid c \geq b\}$, and $u^\nu(c, \lambda) = 1 - \lambda$, for all ν and for any $(c, \lambda) \in C \times [0, 1]$. If \mathbf{u} is given by a profile of functions of the latter type and $\mathbf{s} = (1, \dots, 1)$, then $E(P, N, \mathbf{u}, \mathbf{s}, \Omega)$ is a *subsistence economy* of the type analysed by Roemer (1982) and Yoshihara and Veneziani (2009). But then, it is possible to conclude that the ‘New Interpretation’ provides the unique appropriate definition of exploitation because, as shown above, it preserves PECP in general, and, as shown by Yoshihara and Veneziani (2009), it is fully characterised by a small set of weak and intuitive axioms in the set of subsistence economies which is a subset of the general class of economies considered in this paper. Moreover, the set of axioms is satisfied by the ‘New Interpretation’ definition of exploitation even in the general class of convex economies discussed in this paper.

The results presented above, however, raise some interesting questions. First of all, the paper focuses on economies with heterogenous human capital, or skills, in which only one type of homogenous labour is required in production, but one may argue that a general model of heterogeneous labour should also allow for the possibility different types of labour inputs in the

¹⁶For example, if agents minimise labour over $[0, 1]$, subject to a subsistence constraint, then something like Roemer’s (1982) ‘Non Benevolent Capitalists’ assumption should be made. For a thorough discussion, see Yoshihara and Veneziani (2009).

production set P . This represents an interesting direction for further research and it raises interesting issues concerning the existence of a RS. However, all of the key results on exploitation theory proved in this paper should continue to hold provided each agent's effective labour contribution per unit of time is measured by her marginal productivity. In other words, given a production point $\alpha^{p,w} = ((\alpha_\nu^{p,w})_{\nu \in N}, \underline{\alpha}^{p,w}, \bar{\alpha}^{p,w})$ which supports a RS, there is a profile of individual wage rates $\mathbf{w} = (w^\nu)_{\nu \in N}$ at this RS, and a common wage rate $w > 0$ and $s^\nu(\alpha^{p,w}) > 0$ for each $\nu \in N$ such that $ws^\nu(\alpha^{p,w}) = w^\nu$, that is the marginal rate of productivity of his individual labour. Given this formulation, all the key analytical results of this paper on exploitation should hold.

Second, Theorem 2-(2) confirms the standard Marxist analysis of exploitation at the two ends of the wealth distribution: propertyless agents are exploited and the very wealthy are exploiters. Yet, outside of the two extremes, the exploitation status of an agent is in general determined not only by her endowment of physical capital, but also by her choice of consumption and leisure, as well as her endowment of human capital - namely, her skills. This raises some interesting issues for exploitation theory, in particular from a normative viewpoint: except for the agents at the two extremes of the distribution of productive assets, it may well be the case that agents with nonnegligible amounts of physical assets, who do not work much appear as exploited because they have a large endowment of human capital, which increases their overall labour contribution to the economy.

Third, this paper focuses on exploitation, and on the key relation between profits and exploitation. Another interesting issue concerns the relation between class and exploitation: Roemer (1982), for example, maintains that the correspondence between class and exploitation status is a core tenet of Marxian exploitation theory. Definition 6 above provides interesting results on this issue, too. For example, Yoshihara and Veneziani (2009) and Yoshihara (2010) prove that, unlike in the standard approaches, if the New Interpretation is adopted, it is possible to derive the full class and exploitation structure, and a robust correspondence between class and exploitation status in convex economies with agents endowed with identical preferences and skills. To extend the latter results to general economies with heterogeneous agents is an interesting direction for further research.

6 References

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7 Appendix 1: The existence of a RS

This appendix proves the existence of an equilibrium for a theoretically relevant subset of the set of economies \mathcal{E} . It focuses on the polar case where $C = \mathbb{R}_+^n$ and it generalises the proofs of existence in Roemer (1980, 1982). Yoshihara and Veneziani (2009) prove the existence of a RS for another polar case where $C = \{c \in \mathbb{R}_+^n \mid c \geq b\}$ for some subsistence vector $b \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, u^ν is not strictly increasing on C , and agents minimise labour.

It is assumed that u^ν is continuous, quasi-concave, and strictly increasing on C for all $\nu \in N$. Further, the following standard *boundary condition of utility functions* is assumed: $u^\nu(c, \lambda) > u^\nu(\mathbf{0}, \lambda')$ for any $c \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, and any $\lambda, \lambda' \in [0, 1]$. This assumption implies that any propertyless agent $\nu \in W$ would rather participate in labour markets to earn some revenue and purchase *some* consumption goods, than drop out of labour markets consuming nothing. Finally, **A1** is slightly strengthened to require that *some* produced inputs be used in the production of commodities:

Assumption 1' (A1'). For all $\alpha \in P$, $\bar{\alpha} \geq \mathbf{0} \Rightarrow [\alpha_l > 0 \text{ and } \underline{\alpha} \geq \mathbf{0}]$.

A1' is an essential property of a capitalist economy in the sense that if it is not satisfied, anyone - including propertyless agents - can in principle hire

workers. Given the twin role of agents as consumers and producers, **A1'** also allows us to prove the boundedness of the aggregate demand correspondences.

Let a profile $(c^\nu, \gamma^\nu, \beta^\nu)_{\nu \in N}$ be a *feasible allocation* for $E \in \mathcal{E}$ if and only if $(c^\nu, \gamma^\nu, \beta^\nu)_{\nu \in N}$ satisfies Definition 1-(ii), 1-(iii), and 1-(iv), and $(c^\nu, \gamma^\nu, \beta^\nu) \in C \times [0, s^\nu] \times P$ holds for all $\nu \in N$. If the social endowment of capital ω of an economy $E \in \mathcal{E}$ only allows for feasible allocations with $\sum_{\nu \in N} c^\nu = \mathbf{0}$, then if a RS exists for this economy, it can only be a trivial RS. However, by **A2**, it is always possible to have a non-trivial feasible allocation with $\sum_{\nu \in N} c^\nu \neq \mathbf{0}$ if ω is placed appropriately. Thus, in order to guarantee the existence of non-trivial feasible allocations, the following assumption is made:

Assumption 4 (A4). $E(P, N, \mathbf{u}, \mathbf{s}, \Omega)$ has the following property:

$$\omega \in \left\{ \underline{\alpha} \in \mathbb{R}_+^n \mid \exists \alpha \in P \text{ s.t. } \alpha_l \leq \sum_{\nu \in N} s^\nu \text{ and } \hat{\alpha} \geq \mathbf{0} \right\}.$$

By **A4**, there exists $\alpha' \in P$ with $\alpha'_l \leq \sum_{\nu \in N} s^\nu$ and $\underline{\alpha}' = \omega$ such that for any $p > \mathbf{0}$, $p(\bar{\alpha}' - \omega) > 0$. Thus, for a sufficiently small $w' > 0$, $p(\bar{\alpha}' - \omega) - w\alpha'_l \geq 0$ holds for any $w \leq w'$. This implies that for any $p > \mathbf{0}$, there is $w' > 0$ such that for any $w \leq w'$, $\max_{\alpha \in P: p\alpha = p\omega} p\hat{\alpha} - w\alpha_l$ is non-negative.

For any vector (p, w) , let $\Pi^\nu(p, w) \equiv p\hat{\alpha}^\nu + [p\hat{\beta}^\nu - w\beta_l^\nu] + w\gamma^\nu$ denote agent ν 's net revenue. Note that, for any (p, w) , if the set of optimal solutions $\mathcal{O}^\nu(p, w)$, then it always contains vectors of the form $(\mathbf{0}, \beta^\nu, \gamma^\nu, c^\nu)$ such that $\Pi^\nu(p, w) = \pi^{\max} p\beta^\nu + w\gamma^\nu = pc^\nu$ with $p\beta^\nu = p\omega^\nu$ for all ν . Let $\Delta \equiv \{(p, w) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^n p_i + w = 1\}$ and $\Delta_+ \equiv \{(p, w) \in \Delta \mid p > \mathbf{0}\}$.

In order to analyse the existence of a RS, for all $(p, w) \in \Delta_+$, and for all $\nu \in N$, define the feasibility correspondence

$$B^\nu(p, w) \equiv \{(c^\nu, \beta^\nu, \gamma^\nu) \in C \times P \times [0, s^\nu] \mid pc^\nu \leq \Pi^\nu(p, w); p\beta^\nu \leq p\omega^\nu\}.$$

The next result establishes some basic properties of $B^\nu(p, w)$.

Lemma A1.1: *For each $\nu \in N$, the correspondence B^ν is non-empty, closed-valued and convex-valued, and continuous on Δ_+ . Moreover, every (c^ν, γ^ν) in $B^\nu(p, w)$ is bounded for each $(p, w) \in \Delta_+$.*

Proof. It is obvious that B^ν is non-empty, closed-valued, and convex-valued. Since $pc^\nu \leq \Pi^\nu(p, w) \leq \pi^{\max} p\omega^\nu + ws^\nu$, the boundedness of (c^ν, γ^ν) in $B^\nu(p, w)$ follows from **A1'**, for all $(p, w) \in \Delta_+$.

Finally, we prove the continuity of B^ν . First, we show that B^ν is lower hemi-continuous. Let $\{(p^k, w^k)\} \subseteq \Delta_+$ be a sequence such that $(p^k, w^k) \rightarrow (p, w)$ and $(c^\nu, \beta^\nu, \gamma^\nu) \in B^\nu(p, w)$.

Case 1: Suppose $(p\widehat{\beta}^\nu - w\beta_l^\nu) + w\gamma^\nu > 0$. Then, for each (p^k, w^k) , let $\beta^{k\nu} \equiv \mu^{k\nu}\beta^\nu$ where if $\beta^\nu = \mathbf{0}$ then $\mu^{k\nu} = 1$ and if $\beta^\nu \neq \mathbf{0}$, then

$$\mu^{k\nu} \equiv \min \left\{ \frac{\min \left\{ \max \left\{ p^k\widehat{\beta}^\nu - w^k\beta_l^\nu + w^k\gamma^\nu, 0 \right\}, p\widehat{\beta}^\nu - w\beta_l^\nu + w\gamma^\nu \right\}}{p\widehat{\beta}^\nu - w\beta_l^\nu + w\gamma^\nu}, \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu} \right\},$$

and let $\gamma^{k\nu} = \gamma^\nu$. Note that if $\beta^\nu \neq \mathbf{0}$, then by A1' and $(p^k, w^k) \in \Delta_+$, $p^k\underline{\beta}^\nu > 0$. Moreover, if $c^\nu \neq \mathbf{0}$, then let $\sigma^{k\nu} \equiv \min \left\{ \frac{\mu^{k\nu}(p^k\widehat{\beta}^\nu - w^k\beta_l^\nu) + w^k\gamma^\nu}{p^k c^\nu}, 1 \right\}$ and $c^{k\nu} \equiv \sigma^{k\nu}c^\nu$, whereas if $c^\nu = \mathbf{0}$, then let $c^{k\nu} \equiv c^\nu$. Then, since $\mu^{k\nu} \leq \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}$, $p^k\underline{\beta}^{k\nu} \leq p^k\omega^\nu$, and since $\mu^{k\nu} = 0$ for $p^k\widehat{\beta}^\nu - w^k\beta_l^\nu + w^k\gamma^\nu \leq 0$, $\Pi^\nu(p^k, w^k) \geq 0$ holds. Therefore, $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \in B^\nu(p^k, w^k)$ with $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \rightarrow (c^\nu, \beta^\nu, \gamma^\nu)$ as $(p^k, w^k) \rightarrow (p, w)$. The last convergence property follows from $\mu^{k\nu} \rightarrow 1$ as $(p^k, w^k) \rightarrow (p, w)$.

Case 2: Suppose $(p\widehat{\beta}^\nu - w\beta_l^\nu) + w\gamma^\nu = 0$. In this case, $c^\nu = \mathbf{0}$ holds. Then, for each (p^k, w^k) , let $\gamma^{k\nu} = \gamma^\nu$, $c^{k\nu} = \mathbf{0}$, and $\beta^{k\nu} \equiv \mu^{k\nu}\beta^\nu$ where if $\beta^\nu = \mathbf{0}$ then $\mu^{k\nu} = 1$ and if $\beta^\nu \neq \mathbf{0}$, then

$$\mu^{k\nu} \equiv \begin{cases} \min \left\{ 1, \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu} \right\} & \text{if } (p^k\widehat{\beta}^\nu - w^k\beta_l^\nu) + w^k\gamma^\nu \geq 0, \\ \min \left\{ \frac{w^k\gamma^\nu}{|p^k\widehat{\beta}^\nu - w^k\beta_l^\nu|}, \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu} \right\} & \text{if } (p^k\widehat{\beta}^\nu - w^k\beta_l^\nu) + w^k\gamma^\nu < 0. \end{cases}$$

Then, since $\mu^{k\nu} \leq \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}$, $p^k\underline{\beta}^{k\nu} \leq p^k\omega^\nu$. Also, since $\mu^{k\nu} \leq \frac{w^k\gamma^\nu}{|p^k\widehat{\beta}^\nu - w^k\beta_l^\nu|}$ for $(p^k\widehat{\beta}^\nu - w^k\beta_l^\nu) + w^k\gamma^\nu < 0$, $\Pi^\nu(p^k, w^k) \geq 0$ holds. Therefore, $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \in B^\nu(p^k, w^k)$ with $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \rightarrow (c^\nu, \beta^\nu, \gamma^\nu)$ as $(p^k, w^k) \rightarrow (p, w)$. The last convergence property follows from $\mu^{k\nu} \rightarrow 1$ as $(p^k, w^k) \rightarrow (p, w)$.

The previous arguments show that B^ν is lower hemi-continuous.

To prove that B^ν is upper hemi-continuous, suppose that $\{(p^k, w^k)\} \subseteq \Delta_+$ is a sequence such that $(p^k, w^k) \rightarrow (p, w)$ and $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \in B^\nu(p^k, w^k)$ with $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \rightarrow (c^\nu, \beta^\nu, \gamma^\nu)$ as $(p^k, w^k) \rightarrow (p, w)$, and $(c^\nu, \beta^\nu, \gamma^\nu) \notin B^\nu(p, w)$. Then, either $(c^\nu, \beta^\nu, \gamma^\nu) \notin C \times P \times [0, s^\nu]$, or $pc^\nu > \Pi^\nu(p, w)$, or $p\underline{\beta}^\nu > p\omega^\nu$. Since $C \times P \times [0, s^\nu]$ is closed, $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \rightarrow (c^\nu, \beta^\nu, \gamma^\nu)$

implies that $(c^\nu, \beta^\nu, \gamma^\nu) \in C \times P \times [0, s^\nu]$. Thus, either $pc^\nu > \Pi^\nu(p, w)$ or $p\beta^\nu > p\omega^\nu$. Suppose $p\beta^\nu > p\omega^\nu$. Then, for some (p^k, w^k) close enough to (p, w) , its corresponding $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu})$ is also sufficiently close to $(c^\nu, \beta^\nu, \gamma^\nu)$, which implies $p^k\beta^{k\nu} > p^k\omega^\nu$, which yields a contradiction. This implies that $(c^\nu, \beta^\nu, \gamma^\nu) \in B^\nu(p, w)$. A similar argument holds if $pc^\nu > \Pi^\nu(p, w)$ and therefore B^ν is upper hemi-continuous. ■

Lemma A1.2 analyses optimal choice correspondences.

Lemma A1.2: *For each ν , the correspondence \mathcal{O}^ν is non-empty, closed-valued, convex-valued, and upper hemi-continuous on Δ_+ . Moreover, every (c^ν, γ^ν) in $\mathcal{O}^\nu(p, w)$ is bounded for each $(p, w) \in \Delta_+$.*

Proof. Non-emptiness, closed-valuedness, and convexity can be proved in the standard manner. Since every (c^ν, γ^ν) in $B^\nu(p, w)$ is bounded by Lemma A1.1, every (c^ν, γ^ν) in $\mathcal{O}^\nu(p, w)$ is bounded for any $(p, w) \in \Delta_+$.

We only need to show that \mathcal{O}^ν is upper hemi-continuous. Let $\{(p^k, w^k)\} \subseteq \Delta_+$ be a sequence such that $(p^k, w^k) \rightarrow (p, w)$ and $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \in \mathcal{O}^\nu(p^k, w^k)$ with $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \rightarrow (c^\nu, \beta^\nu, \gamma^\nu)$ as $(p^k, w^k) \rightarrow (p, w)$. Suppose $(c^\nu, \beta^\nu, \gamma^\nu) \notin \mathcal{O}^\nu(p, w)$. This implies that (c^ν, γ^ν) is not a maximizer of u^ν over $B^\nu(p, w)$ and $(c^\nu, \beta^\nu, \gamma^\nu) \in B^\nu(p, w)$ by the upper hemi-continuity of B^ν . Then, there exists $(c'^\nu, \beta'^\nu, \gamma'^\nu) \in B^\nu(p, w)$ such that $u^\nu\left(c'^\nu, \frac{\gamma'^\nu}{s^\nu}\right) > u^\nu\left(c^\nu, \frac{\gamma^\nu}{s^\nu}\right)$. Since B^ν is lower hemi-continuous, there exists a sequence $\{(c'^{k\nu}, \beta'^{k\nu}, \gamma'^{k\nu})\}$ such that for each $(p^k, w^k) \in \Delta_+$, $(c'^{k\nu}, \beta'^{k\nu}, \gamma'^{k\nu}) \in B^\nu(p^k, w^k)$ with $(c'^{k\nu}, \beta'^{k\nu}, \gamma'^{k\nu}) \rightarrow (c'^\nu, \beta'^\nu, \gamma'^\nu)$ as $(p^k, w^k) \rightarrow (p, w)$. Then, for (p^k, w^k) which is sufficiently close to (p, w) , $u^\nu\left(c'^{k\nu}, \frac{\gamma'^{k\nu}}{s^\nu}\right) > u^\nu\left(c^{k\nu}, \frac{\gamma^{k\nu}}{s^\nu}\right)$ holds. However, since $(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}) \in \mathcal{O}^\nu(p^k, w^k)$, this is a contradiction. Thus, $(c^\nu, \beta^\nu, \gamma^\nu) \in \mathcal{O}^\nu(p, w)$, and so \mathcal{O}^ν is upper hemi-continuous. ■

For any $\nu \in N$, if $(p, w) \in \Delta_+$ is associated with $p\hat{\alpha} - w\alpha_l < 0$ for all $\alpha \in P \setminus \{\mathbf{0}\}$, then $(c^\nu, \beta^\nu, \gamma^\nu) \in \mathcal{O}^\nu(p, w)$ implies $\beta^\nu = \mathbf{0}$. However, by A4, for any $p > \mathbf{0}$, there is $w' > 0$ such that for any $w \leq w'$, $\max_{\alpha \in P: p\alpha = p\omega} p\hat{\alpha} - w\alpha_l$ is non-negative, so that there is $(c^\nu, \beta^\nu, \gamma^\nu)_{\nu \in N} \in \times_{\nu \in N} \mathcal{O}^\nu(p, w)$ with $\sum_{\nu \in N} \beta^\nu \neq \mathbf{0}$.

For each $(p, w) \in \Delta_+$, let $P(p, w; \omega) \equiv \{\alpha \in \arg \max_{\alpha' \in P: p\alpha' \leq p\omega} p\hat{\alpha}' - w\alpha'_l\}$. Let $\alpha_l^m \equiv \max_{(p', w') \in \Delta} \min \{\alpha_l \mid \alpha \in P(p', w'; \omega)\}$. Then, let $P^*(p, w; \omega) \equiv \{\alpha \in P(p, w; \omega) \mid \alpha_l \leq \max \{\alpha_l^m, \sum_{\nu \in N} s^\nu\}\}$ for each $(p, w) \in \Delta_+$. By this

definition, $P^*(p, w; \omega)$ is non-empty, convex, compact, and upper hemi-continuous at every $(p, w) \in \Delta_+$.

For each $(p, w) \in \Delta_+$, define the aggregate excess demand correspondence:

$$Z(p, w) \equiv \left\{ \left(\sum_{\nu \in N} c^\nu - \sum_{\nu \in N} \widehat{\beta}^\nu, \sum_{\nu \in N} \beta_l^\nu - \sum_{\nu \in N} \gamma^\nu \right) \mid \sum_{\nu \in N} \beta^\nu \in P^*(p, w; \omega) \right. \\ \left. \& (c^\nu, \beta^\nu, \gamma^\nu) \in \mathcal{O}^\nu(p, w) \ (\forall \nu \in N) \right\}.$$

Given the above Lemmas and the definition of $P^*(p, w; \omega)$, it follows that Z is compact-valued, convex-valued, and upper hemi-continuous on Δ_+ . To see that it is non-empty, firstly suppose that $(p, w) \in \Delta_+$ is such that $p\widehat{\alpha} - w\alpha_l < 0$ for all $\alpha \in P \setminus \{\mathbf{0}\}$. Then, $P(p, w; \omega) = \{\mathbf{0}\} = P^*(p, w; \omega)$, so that there exists $(\beta^\nu)_{\nu \in N}$ such that $\beta^\nu = \mathbf{0}$ for all ν . Next, if $p\widehat{\alpha} - w\alpha_l \geq 0$ for some $\alpha \in P \setminus \{\mathbf{0}\}$, $P(p, w; \omega) \supseteq \{\alpha \in \arg \max_{\alpha' \in P: p\alpha' = p\omega} p\widehat{\alpha}' - w\alpha_l'\}$ holds by A1', so that $P^*(p, w; \omega) \setminus \{\mathbf{0}\} \neq \emptyset$, and so if $\alpha \in P^*(p, w; \omega) \setminus \{\mathbf{0}\}$ then there is $(\beta^\nu)_{\nu \in N}$ such that $\sum_{\nu \in N} \beta^\nu = \alpha$, and $p\beta^\nu = p\omega^\nu$ for all ν . In either case, for (c^ν, γ^ν) in $\mathcal{O}^\nu(p, w)$, it follows that $(c^\nu, \beta^\nu, \gamma^\nu) \in \mathcal{O}^\nu(p, w)$ for each $\nu \in N$. By definition, since $\sum_{\nu \in N} \beta^\nu \in P^*(p, w; \omega)$, $Z(p, w)$ is non-empty. Then:

Lemma A1.3: *There exists a price vector $(\bar{p}, \bar{w}) \in \Delta_+$ such that $\mathbf{0} \in Z(\bar{p}, \bar{w})$.*

Proof. 1. First, we prove that Z satisfies the Strong Walras Law (SWL), namely for each $(p, w) \in \Delta_+$, and each $(z_1, z_2) \in Z(p, w)$, $pz_1 + wz_2 = 0$. In fact, for each $(p, w) \in \Delta_+$, and each $(z_1, z_2) \in Z(p, w)$,

$$pz_1 + wz_2 = p \left(\sum_{\nu \in N} c^\nu - \sum_{\nu \in N} \widehat{\beta}^\nu \right) + w \left(\sum_{\nu \in N} \beta_l^\nu - \sum_{\nu \in N} \gamma^\nu \right) \\ = \sum_{\nu \in N} \left[pc^\nu - \left\{ \left(p\widehat{\beta}^\nu - w\beta_l^\nu \right) + w\gamma^\nu \right\} \right] = 0,$$

since $pc^\nu = \left(p\widehat{\beta}^\nu - w\beta_l^\nu \right) + w\gamma^\nu$ for every ν , by the strict monotonicity of u^ν .

2. Next, we prove that Z satisfies the following Boundary condition: there is a $(\tilde{p}, \tilde{w}) \in \Delta_+$ such that for every sequence $\{(p^k, w^k)\} \subseteq \Delta_+$ with $(p^k, w^k) \rightarrow (p, w) \in \Delta \setminus \Delta_+$, there is an M such that for every $k \geq M$, $(\tilde{p}, \tilde{w}) \cdot (z_1^k, z_2^k) > 0$ holds for every $(z_1^k, z_2^k) \in Z(p^k, w^k)$. Take a sufficiently small but positive real number ε , and define $(\tilde{p}, \tilde{w}) \in \Delta_+$ as $\tilde{w} = \varepsilon > 0$, and for all j , $\tilde{p}_j = \frac{1-\varepsilon}{n} > 0$. Then, consider any price vector $(p, w) \in \Delta \setminus \Delta_+$,

such that $p_i = 0$ for one i . Firstly, note that because $\{(p^k, w^k)\} \subseteq \Delta_+$, it is possible that $w^k = 0$ for sufficiently large k . Thus, in this case, $c^{k\nu} = \mathbf{0}$ for any $\nu \in W$. However, in this case, the corresponding $\pi^{\max k}$ is strictly positive by **A4**, and so $\Pi^\nu(p^k, w^k) > 0$ for any $\nu \in N \setminus W$. Hence, by the strict monotonicity of utility functions, $c^{k\nu} \geq \mathbf{0}$ for any $\nu \in N \setminus W$, and in particular, $c_i^{k\nu}$ is sufficiently large at p^k for sufficiently large k . Secondly, $\{(p^k, w^k)\} \subseteq \Delta_+$ may also contain the case that $w^k > 0$ but $\pi^{\max k}$ is zero for sufficiently large k . In this case, because of the boundary condition for utility functions, any $\nu \in N$ optimally supplies a positive amount of labour, so that $\Pi^\nu(p^k, w^k) > 0$. Thus, by the strict monotonicity of utility functions, $c^{k\nu} \geq \mathbf{0}$ for any $\nu \in N$, and in particular, $c_i^{k\nu}$ is sufficiently large at p^k for sufficiently large k . In sum, noting that $\beta^k \in P^*(p^k, w^k; \omega)$ is bounded above, it follows that $z_{1i}^k > 0$ is sufficiently large for p^k sufficiently close to p . Then, even if $\tilde{w} > 0$, $\tilde{w}z_2^k$ will never compensate for $\tilde{p}z_1^k > 0$, since z_2^k is bounded below by $-\sum_{\nu \in N} s^\nu$ whereas $\tilde{p}z_1^k$ grows infinitely large due to a sufficiently large $z_{1i}^k > 0$. Thus, there is a neighbourhood $\mathcal{N}((p, w), \delta)$ of (p, w) such that $(\tilde{p}, \tilde{w}) \cdot (z_1^k, z_2^k) > 0$ for all $(p^k, w^k) \in \mathcal{N}((p, w), \delta) \cap \Delta_+$. A similar argument holds if $(p, w) \in \Delta \setminus \Delta_+$, with $p_i = 0$, for more than one i .

3. Set $K_m \equiv \text{co}\{(q, w) \in \Delta_+ \mid \text{dist}((q, w), \Delta \setminus \Delta_+) \geq \frac{1}{m}\}$. Then, $\{K_m\}$ is an increasing family of compact convex sets and $\Delta_+ = \cup_m K_m$. Then, as in Border (1985, Theorem 18.13, p. 85), it follows that there exist $(\bar{p}, \bar{w}) \in \Delta_+$ and $\bar{z} \in Z(\bar{p}, \bar{w})$ such that $\bar{z} \leq \mathbf{0}$. This fact together with (SWL) imply that $\bar{z} = \mathbf{0}$. In fact, since $\bar{p} > \mathbf{0}$, (SWL) and $\bar{z} \leq \mathbf{0}$ imply that $\bar{z}_1 = \mathbf{0}$. Second, if $\bar{w} > 0$, then $\bar{z}_2 = 0$ holds by (SWL) and $\bar{z} \leq \mathbf{0}$. Thus, suppose $\bar{w} = 0$ and $\bar{z}_2 \equiv \sum_{\nu \in N} \beta_l^{*\nu} - \sum_{\nu \in N} \gamma^{*\nu} < 0$. Given that every agent's utility function u^ν is strictly monotonic on C , the real-valued function $V^\nu(\Pi^\nu(p, w), \gamma^\nu) \equiv \max_{(c^\nu, \beta^\nu, \gamma^\nu) \in B^\nu(p, w)} u^\nu(c^\nu, \gamma^\nu)$ is strictly monotonic on $\Pi^\nu(p, w)$, for all ν .

Since $\Pi^\nu(\bar{p}, \bar{w}) = \pi^{\max} \bar{p} \beta^{*\nu} + \bar{w} \gamma^{*\nu} = \pi^{\max} \bar{p} \beta^{*\nu}$, then $V^\nu(\Pi^\nu(\bar{p}, \bar{w}), \gamma^{*\nu}) = V^\nu(\Pi^\nu(\bar{p}, \bar{w}), 0)$ because u^ν is (weakly) decreasing in γ^ν on $[0, 1]$. Thus, whenever $(c^{*\nu}, \beta^{*\nu}, \gamma^{*\nu}) \in \mathcal{O}^\nu(\bar{p}, \bar{w})$ for all $\nu \in N$, then for any $\gamma^{**\nu} \in [0, \gamma^{*\nu}]$, we have $(c^{*\nu}, \beta^{*\nu}, \gamma^{**\nu}) \in \mathcal{O}^\nu(\bar{p}, \bar{w})$, which implies that, for any $(\gamma^{**\nu})_{\nu \in N} \in \times_{\nu \in N} [0, \gamma^{*\nu}]$ with $\sum_{\nu \in N} \gamma^{**\nu} = \sum_{\nu \in N} \beta_l^{*\nu}$, $(c^{*\nu}, \beta^{*\nu}, \gamma^{**\nu}) \in \mathcal{O}^\nu(\bar{p}, \bar{w})$ holds for any $\nu \in N$. Let $\bar{z}'_2 \equiv \sum_{\nu \in N} \beta_l^{*\nu} - \sum_{\nu \in N} \gamma^{**\nu} = 0$. Then, $(\bar{z}_1, \bar{z}'_2) \in Z(\bar{p}, \bar{w})$, which yields the desired result. ■

Lemma A1.3 proves the existence of a fixed point for the aggregate excess demand correspondences: there exists a price vector $(\bar{p}, \bar{w}) \in \Delta_+$ such that conditions (i), (ii) and (iv) of Definition 1 are satisfied. In order to complete

the proof of existence of a RS, it is necessary to show that condition (iii) also holds. Theorem A1.1 provides a condition on aggregate social endowments under which the capital constraint (iii) is satisfied.

Theorem A1.1: *Let $\mathbf{A1}' \sim \mathbf{A3}$ hold and let u^ν be continuous, quasi-concave, strictly increasing on C , and satisfying the boundary condition for all $\nu \in N$. For any profile $\Omega = (\omega^\nu)_{\nu \in N}$ with $\sum_{\nu \in N} \omega^\nu = \omega \geq \mathbf{0}$ which satisfies $\mathbf{A4}$, there exist a distribution $\Omega' = (\omega'^\nu)_{\nu \in N}$ with $\sum_{\nu \in N} \omega'^\nu = \omega'$ and a RS $(p, w) \in \Delta_+$ for the economy $E(P, N, \mathbf{u}, \mathbf{s}, \Omega')$ with $p\omega'^\nu = p\omega^\nu$ for all $\nu \in N$.*

Proof. Let P, N, \mathbf{s} , and $\Omega = (\omega^\nu)_{\nu \in N}$ satisfy $\mathbf{A1}' \sim \mathbf{A4}$, and let \mathbf{u} be such that for all $\nu \in N$, u^ν is continuous, quasi-concave, strictly increasing on C , and it satisfies the boundary condition. Then, we can apply Lemmas A1.1-A1.3, to prove that there exists $(p^*, w^*) \in \Delta_+$ such that $(\sum_{\nu \in N} c^{*\nu} - \sum_{\nu \in N} \widehat{\beta}^{*\nu}) = \mathbf{0}$ and $(\sum_{\nu \in N} \beta_l^{*\nu} - \sum_{\nu \in N} \gamma^{*\nu}) = 0$.

Thus, (p^*, w^*) is associated with $p^*\widehat{\alpha} - w^*\alpha_l \geq 0$ for some $\alpha \in P \setminus \{\mathbf{0}\}$. In fact, if (p^*, w^*) is such that $p^*\widehat{\alpha} - w^*\alpha_l < 0$ for all $\alpha \in P \setminus \{\mathbf{0}\}$, then $\beta^{*\nu} = \mathbf{0}$ for all $\nu \in N$, but $\gamma^{*\nu} > 0$ and $c^{*\nu} \neq \mathbf{0}$ follow from $w^* > 0$ and the boundary condition for utility functions. (Note that if $p^*\widehat{\alpha} - w^*\alpha_l < 0$ for all $\alpha \in P \setminus \{\mathbf{0}\}$, then $w^* > 0$.) Hence, $(\sum_{\nu \in N} c^{*\nu} - \sum_{\nu \in N} \widehat{\beta}^{*\nu}) \geq \mathbf{0}$ and $(\sum_{\nu \in N} \beta_l^{*\nu} - \sum_{\nu \in N} \gamma^{*\nu}) < 0$ follow if $p^*\widehat{\alpha} - w^*\alpha_l < 0$ for all $\alpha \in P \setminus \{\mathbf{0}\}$, which is a contradiction. Thus, $p^*\widehat{\alpha} - w^*\alpha_l \geq 0$ for some $\alpha \in P \setminus \{\mathbf{0}\}$.

Since $p^*\widehat{\alpha} - w^*\alpha_l \geq 0$ for some $\alpha \in P \setminus \{\mathbf{0}\}$, $(\mathbf{0}, \beta^{*\nu}, \gamma^{*\nu}, c^{*\nu})_{\nu \in N}$ is a profile of optimal solutions of all MP^ν with $p^*\beta^{*\nu} = p^*\omega^\nu$ for all $\nu \in N$, thus $p^*\beta^* = p^*\omega$ at (p^*, w^*) . By $\mathbf{A4}$, the existence of such profile is guaranteed.

Let us define $\Omega' = (\omega'^\nu)_{\nu \in N}$ as $\omega'^\nu = \underline{\beta}^{*\nu}$ for each $\nu \in N$. Then, since $p^*\omega'^\nu = p^*\omega^\nu$ holds for each $\nu \in N$, it follows that $(\mathbf{0}, \beta^{*\nu}, \gamma^{*\nu}, c^{*\nu})_{\nu \in N}$ remains a profile of optimal solutions of all MP^ν such that $(\sum_{\nu \in N} c^{*\nu} - \sum_{\nu \in N} \widehat{\beta}^{*\nu}) = \mathbf{0}$ and $(\sum_{\nu \in N} \beta_l^{*\nu} - \sum_{\nu \in N} \gamma^{*\nu}) = 0$. Moreover $\underline{\beta}^* = \omega'$, and therefore condition (iii) of Definition 1 is also satisfied. Hence, for the economy $E(P, N, \mathbf{u}, \mathbf{s}, \Omega')$, (p^*, w^*) is a RS with associated profile $(\mathbf{0}, \beta^{*\nu}, \gamma^{*\nu}, c^{*\nu})_{\nu \in N}$. ■

Note that the existence of a RS requires an appropriate position of the initial endowment vector, ω , as shown in Roemer (1980; Appendix II) and Yoshihara (2010). This is because a RS is a kind of one-shot slice of a stationary-state dynamic competitive equilibrium, which may be infeasible

under an arbitrary position of the initial endowment vector. Theorem A1.1 does not provide the necessary and sufficient characterisation for a suitable position of the endowment vector, since this paper considers an economy with heterogeneous preferences, in which the balanced growth path cannot be identified by the sole information of the production possibility set, unlike Roemer (1980; Proposition 1.1) and Yoshihara (2010; Proposition 1). However, the theorem shows that even starting from any arbitrary position of the initial endowment vector, there exists an equilibrium price vector such that the economy can ‘purchase’ a suitable endowment vector with that prices, under which the above mentioned stationary-state is feasible.

8 Appendix 2: Definitions 2 and 3

Lemma A2.1: *There exists an economy $E \in \mathcal{E}$ and a RS (p, w) with associated aggregate production point $\alpha^{p,w}$ such that neither Definition 2 nor Definition 3 satisfies condition (2) of Theorem 1.*

Proof. Consider the following von Neumann technology:

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4.5 & 5.25 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3.5 \end{bmatrix}, L = (1 \ 1 \ 1),$$

where A is the input matrix; B is the output matrix; and L is the vector of labour coefficients. Define the production possibility set $P_{(A,B,L)}$ by

$$P_{(A,B,L)} \equiv \{ \alpha \in \mathbb{R}^5 \mid \exists x \in \mathbb{R}_+^3 : \alpha \leq (-Lx, -Ax, Bx) \}.$$

$P_{(A,B,L)}$ is a closed convex cone in \mathbb{R}^5 with $\mathbf{0} \in P_{(A,B,L)}$ and it satisfies A1~A3. Let $\mathbf{e}_j \in \mathbb{R}_+^3$ be a unit column vector with 1 in the j -th component and 0 in any other component. Let $\alpha^1 \equiv (-L\mathbf{e}_1, -A\mathbf{e}_1, B\mathbf{e}_1)$, $\alpha^2 \equiv (-L\mathbf{e}_2, -A\mathbf{e}_2, B\mathbf{e}_2)$, and $\alpha^3 \equiv (-L\mathbf{e}_3, -A\mathbf{e}_3, B\mathbf{e}_3)$. Then,

$$\begin{aligned} \hat{\alpha}^1 &\equiv (B - A)\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{\alpha}^2 \equiv (B - A)\mathbf{e}_2 = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}, \\ \hat{\alpha}^3 &\equiv (B - A)\mathbf{e}_3 = \begin{pmatrix} 0 \\ 1.75 \end{pmatrix}. \end{aligned}$$

Also, we have $\hat{P}(\alpha_l = 1) = co\{(1, 0), (1, 1.5), (0, 1.75), \mathbf{0}\}$.

Let $W \neq \emptyset$ and let N be such that $|N| > |W|$. Let $c^* = (1, 1)$ and let the social endowment of capital be given by $\omega = (2|N|, 3|N|)$. Let $\mathbf{u} \equiv (u, \dots, u)$ with $u(c, \lambda) \equiv c_1 + c_2$, and $\mathbf{s} \equiv (1, \dots, 1)$. Finally, let $\omega^\nu = \left(\frac{2|N|}{|N|-|W|}, \frac{3|N|}{|N|-|W|} \right)$ for all $\nu \in N \setminus W$, so that $\sum_{\nu \in N} \omega^\nu = \omega$. This completely defines the economy $E(N, P_{(A,B,L)}, \mathbf{u}, \mathbf{s}, \Omega)$. Then, a pair $(p, 1)$ with $p = (0.5, 0.5)$ constitutes a RS for $E(N, P_{(A,B,L)}, \mathbf{u}, \mathbf{s}, \Omega)$ associated with a social production point $|N| \alpha^2$. To see this, note first that

$$\begin{aligned} \frac{[p(B-A) - L] \mathbf{e}_1}{pA\mathbf{e}_1} &= -\frac{1}{2}, \quad \frac{[p(B-A) - L] \mathbf{e}_2}{pA\mathbf{e}_2} = \frac{1}{10}; \\ \frac{[p(B-A) - L] \mathbf{e}_3}{pA\mathbf{e}_3} &= -\frac{1}{14}. \end{aligned}$$

Thus, for all $\nu \in N \setminus W$, $\beta^\nu = \frac{|N|}{|N|-|W|} \alpha^2$, $c^\nu = \left(1, \frac{1.5|N|-|W|}{|N|-|W|} \right)$, and $\lambda^\nu = 1$ is an optimal solution to MP^ν . Further, for every $\nu \in W$, $(c^\nu, \lambda^\nu) = (c^*, 1)$ is an optimal solution to MP^ν , so that at this RS, $W_+ = W$. Then, it is immediate to check that conditions (ii)-(iv) of Definition 1 are all satisfied.

Since $c^\nu = c^*$, then in both Definition 2 and Definition 3, $\bar{c}^\nu = c^*$ and $\hat{\alpha}^{\bar{c}^\nu} = c^*$ hold for every $\nu \in W$. Then it is immediate to show that for all $\nu \in W$, there exists no $\underline{c}^\nu \in \Gamma(p, w; \Lambda^\nu) \cup \left\{ \frac{\Lambda^\nu}{\alpha_i^{p,w}} \hat{\alpha}^{p,w} \right\}$ such that $\underline{c}^\nu > (1, 1) = \hat{\alpha}^{\bar{c}^\nu}$ even though $\pi^{\max} = \frac{1}{10} > 0$, which implies that in this economy, neither Definition 2 nor Definition 3 satisfies condition (2) of Theorem 1.¹⁷ ■

¹⁷The constructed economy in this proof does not satisfy ‘independence of production,’ which was introduced by Roemer (1980) as the necessary and sufficient condition for preserving FMT under Definition 2 in convex cone economies with homogeneous agents. Note, however, that in convex cone economies with heterogeneous agents, independence of production is no longer necessary nor sufficient, since it is easy to find an economy and a RS with heterogenous consumption bundles $(c^\nu)_{\nu \in W}$ among propertyless workers, in which the production set satisfies independence of production while condition (2) of Theorem 1 does not hold for some propertyless worker.

In contrast, FMT does not hold in general under Definition 3 even if the production set satisfies independence of production and the economy has homogeneous agents. This can be seen by Yoshihara and Veneziani (2011a; Theorem 1).