The Behavior of Solutions to Bargaining Problems on the Basis of Solidarity*

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Abstract

The main purpose of this paper is to provide a systematic study of the solidarity-type axioms for classical convex bargaining problems. As a consequence, we present alternative characterizations for some well-known solutions in the literature. Instead of using the Monotonicity axiom, the paper provides characterizations of the egalitarian solution and the Kalai-Somorodinsky solution using slightly weaker versions of Nash's original IIA for convex bargaining problems with a fixed population.

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1 Introduction

A persistent question asked in axiomatic bargaining problems is the following: when bargaining problems change from A to B, how should solutions to them respond? Many of the axioms in the axiomatic bargaining literature are of this type. Among them, the most prominent ones are perhaps Nash's IIA (1950) and Kalai's Monotonicity (1977). Though they have been used mainly for understanding and characterizing different solutions to bargaining problems, a moment's reflection may convince us that they all have one thing in common: they are types of a broad class of axioms that may be called *solidarity*. The main idea underlying a solidarity-type axiom is that, when bargaining problems change from A to B, the "utility gains" by any two players should not be in opposite directions: if one player gains from moving A to B, then no player should become worse off from such a move; and if one player loses from moving A to B, then no player should gain from such a move. The solidarity-type axioms have been fruitfully studied in the literature on fair allocations (see, for example, Fleurbaey and Maniquet (1999)). In the standard bargaining model, on the other hand, there exists no systematic study on solutions to bargaining problems based explicitly on the idea of solidarity.[1] The main purpose of this paper is to provide a systematic study of the solidarity-type axioms for classical convex bargaining problems. As a consequence, we present alternative characterizations for some well-known solutions in the literature. Instead of using the Monotonicity axiom, the paper provides a characterization of the egalitarian solution using a slightly weaker version of Nash's original IIA for convex bargaining problems with a fixed population. Though our alternative characterizations are simple and easy to be obtained, we hope that they will provide a further understanding of the behavior of solutions to bargaining problems in terms of solidarity.

2 Basic Model

The set of players is denoted by $N = \{1, 2, \ldots, n\}$ where $n \geq 2$. We use $\mathbb{R}_+$ to denote the set of all non-negative real numbers, while $\mathbb{R}^n_+$ is used to denote the n-fold Cartesian product of $\mathbb{R}_+$. For each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, we write $\mathbf{x} \geq \mathbf{y}$ as $[x_i \geq y_i$ for all $i \in N]$, $\mathbf{x} > \mathbf{y}$ as $[x_i \geq y_i$ for all $i \in N$ and $\mathbf{x} \neq \mathbf{y}]$, and $\mathbf{x} \gg \mathbf{y}$

[1] Yoshihara (2003, 2005) discusses axiomatic characterizations of bargaining solutions in terms of solidarity in a specific model of production economies.

as $[x_i > y_i$ for all $i \in N]$.

Let π be a permutation of N. For each $\mathbf{x} = (x_i)_{i \in N} \in \mathbb{R}^n_+$, let $\pi(\mathbf{x}) = (x_{\pi(i)})_{i \in N}$. Let Π be the set of all permutations of N.

Let $\mathcal{B}$ be the set of all compact, convex, and comprehensive subsets of $\mathbb{R}^n_+$, each of which contains an interior point of $\mathbb{R}^n_+$. Elements in $\mathcal{B}$ are interpreted as *normalized (bargaining) problems.* For each $A \in \mathcal{B}$ and each $\pi \in \Pi$, let $\pi(A) = \{\pi(\mathbf{a}) \mid \mathbf{a} \in A\}$. For each $A \in \mathcal{B}$, A is a *symmetric problem* if $A = \pi(A)$ for all $\pi \in \Pi$.

For each $A \in \mathcal{B}$ and each $i \in N$, let $m_i(A) = \max\{a_i \mid (a_1, \cdots, a_i, \cdots, a_n) \in A\}$. Therefore, $\mathbf{m}(A) \equiv (m_i(A))_{i \in N}$ is the *ideal point* of A.

For each $\mathbf{x} \in \mathbb{R}^n_+$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, let $\boldsymbol{\alpha}(\mathbf{x}) \equiv (\alpha_i x_i)_{i \in N}$. Given $A \in \mathcal{B}$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, let $\boldsymbol{\alpha}(A) \equiv \left\{\boldsymbol{\alpha}(\mathbf{x}) \in \mathbb{R}^n_+ \mid \mathbf{x} \in A\right\}$. For each A in $\mathbb{R}^n_+$, we define the *comprehensive hull of* A by

$$compA \equiv \left\{\mathbf{z} \in \mathbb{R}^n_+ \mid \exists \mathbf{x} \in A : \mathbf{z} \leq \mathbf{x}\right\}.$$

Let the *convex hull of* A be denoted by $conA$.

A *solution* F is a single-valued mapping from $\mathcal{B}$ to $\mathbb{R}^n_+$ such that for each problem $A \in \mathcal{B}$, $F(A) \in A$. For given $F(A) \in A$, let $F_i(A) \in \mathbb{R}_+$ be its i-th component. The following three are well-known solutions.

Nash Solution F^{NA}: *For each $A \in \mathcal{B}$, $F^{NA}(A) = \arg\max_{(a_1,\ldots,a_n) \in A} \prod_{i \in N} a_i$.*

Kalai-Smorodinsky Solution F^{KS}: *For each $A \in \mathcal{B}$, $F^{KS}(A) \in A$ implies that: (1) there is no other $\mathbf{a} \in A$ such that $\mathbf{a} \gg F^{KS}(A)$; and (2) there exists $\gamma \in (0,1]$ such that $F^{KS}(A) = \gamma \cdot \mathbf{m}(A)$.*

Egalitarian Solution F^E: *For each $A \in \mathcal{B}$, $F^E(A) \in A$ implies that: (1) there is no other $\mathbf{a} \in A$ such that $\mathbf{a} \gg F^E(A)$; and (2) $F^E_i(A) = F^E_j(A)$ for each $i, j \in N$.*

3 Axioms and their relations

We group axioms into two categories: non-solidarity type and solidarity type. We consider first the non-solidarity type. They are fairly standard in the literature on convex problems (see, for example, Peters (1992) and Thomson (1994) for discussions).

Efficiency (E): *For each $A \in \mathcal{B}$, there is no $\mathbf{x} \in A$ such that $\mathbf{x} > F(A)$.*

Weak Efficiency (WE): *For each* $A \in \mathcal{B}$*, there is no* $\mathbf{x} \in A$ *such that* $\mathbf{x} \gg F(A)$.

Symmetry (S): *For each* $A \in \mathcal{B}$*, if* A *is symmetric, then* $F_i(A) = F_j(A)$ *for all* $i, j \in N$.

Scale Invariance (SI): *For each* $A, B \in \mathcal{B}$*, and each* $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$*, if* $B = \boldsymbol{\alpha}(A)$*, then* $F(B) = \boldsymbol{\alpha}(F(A))$.

Next, we turn to solidarity type axioms.

Solidarity (SOL): *For each* $A, B \in \mathcal{B}$, *either* $F(A) \gg F(B)$ *or* $F(B) \gg F(A)$ *or* $F(A) = F(B)$.

Restricted Solidarity (RSOL): *For each* $A, B \in \mathcal{B}$ *with* $\mathbf{m}(A) = \mathbf{m}(B)$, *either* $F(A) \gg F(B)$ *or* $F(B) \gg F(A)$ *or* $F(A) = F(B)$.

Weak Solidarity (WSOL): *For each* $A, B \in \mathcal{B}$, *either* $F(A) \geq F(B)$ *or* $F(B) \geq F(A)$.

Contraction Independence (CI): *For each* $A, B \in \mathcal{B}$*, if* $A \subseteq B$ *and* $F(B) \in A$ is efficient on A*, then* $F(B) = F(A)$.

Restricted Contraction Independence (RCI): *For each* $A, B \in \mathcal{B}$ *such that* $\mathbf{m}(A) = \mathbf{m}(B)$*, if* $A \subseteq B$ *and* $F(B) \in A$ is efficient on A*, then* $F(B) = F(A)$.

Expansion Independence (EI): *For each* $A, B \in \mathcal{B}$*, if* $A \subseteq B$ *and* $F(A)$ *is efficient on* B*, then* $F(B) = F(A)$.

Restricted Expansion Independence (REI): *For each* $A, B \in \mathcal{B}$ *such that* $\mathbf{m}(A) = \mathbf{m}(B)$*, if* $A \subseteq B$ *and* $F(A)$ *is efficient on* B*, then* $F(B) = F(A)$.

(SOL) is the most general form of a solidarity-type axiom. It requires that, whenever bargaining problems change from A to B, either $F(A) \gg F(B)$ or $F(B) \gg F(A)$ or $F(A) = F(B)$, meaning that if one player gains (loses) as a result from the move, every player gains (loses) as well. (RSOL) is a restricted version of (SOL). (WSOL) conveys the same idea as (SOL) in a weak form in which, when bargaining problems change from A to B, it should be the case in which no player's gain is at the expense of some other player's loss. It may be noted that these three axioms are new in the literature.

(CI) and (RCI) each deal with situations where a bargaining problem A shrinks to another problem B. (CI) requires that, when a problem B shrinks to another problem A, if the solution to B is efficient on A, then $F(B)$ should continue to be the solution to A. The solidarity idea embedded in the axiom is that, given that $F(B)$ is efficient on A, any movement away from $F(B)$ will make at least one player worse off, and as a consequence, to keep the spirit of solidarity, $F(B)$ should continue to be the solution to A. It may be noted that (CI) is formally slightly weaker than Nash's IIA (1950) in that $F(B)$ is required to be *efficient* on A, and also that (RCI) is formally slightly weaker than Restricted IIA proposed by Yu (1973) by stipulating that $F(B)$ is *efficient* on A.

On the other hand, (EI) and (REI) each deal with situations involving the expansion of bargaining problems from A to B. For example, (EI) requires that, when a problem A is enlarged to another problem B, if the solution $F(A)$ to A is efficient on B, then $F(A)$ should continue to be the solution to the problem B. The idea is that, even though there is an enlargement of "opportunities" from A to B, given that $F(A)$ is both efficient on A and on B, and that $F(A)$ is already the solution to the original problem A, any movement away from $F(A)$ will hurt at least one player, and thus the solution to the enlarged problem B should continue to be $F(A)$. This requirement is consistent with the solidarity idea embedded in the solution. The property can also be seen as stating a certain inertia of the choice process. (REI) is weaker than (EI) in that it restricts its applicability to situations where the ideal point remains unchanged. The spirit of our expansion-type axioms may be traced back to the axiom *Independence of Undominating Alternatives* (IUA) proposed by Thomson and Myerson (1980) in which in the premise of (IUA), they require $F(A)$ of A be *weakly* efficient on B.[2]

The following proposition summarizes the logical relationships between and/or among the solidarity-type axioms discussed above. The proof is simple and we leave it to the reader.

[2] There is another axiom similar to IUA, *Independence of Irrelevant Expansion* (IIE), in Thomson (1981), though Thomson (1981) formulates IIE solely for two-person bargaining problems of compact and strictly convex sets. In such a restricted class of problems, IIE implies EI, but the inverse does not hold.

Proposition 1. (i) (SOL) $\Rightarrow$ (RSOL) and (WSOL); (ii) (SOL) + (WE) $\Rightarrow$ (CI), (RCI), (EI), and (REI); (iii) (RSOL) + (WE) $\Rightarrow$ (RCI) and (REI); (iv) (WSOL) + (WE) $\Rightarrow$ (CI) $\Rightarrow$ (RCI); (v) (WSOL) + (WE) $\Rightarrow$ (EI) $\Rightarrow$ (REI).

4 Results and Their Proofs

This section presents our main results and their proofs follow.

Theorem 1: The following statements are equivalent: (1.i) $F = F^E$; (1.ii) F satisfies (WE), (S), (CI) and (EI); (1.iii) F satisfies (WE), (S), (SOL).

Proof. We first establish the equivalence of (1.i) and (1.ii). It can be checked that F^E satisfies (WE), (S), (CI) and (EI). We now show that if F satisfies (WE), (S), (CI) and (EI), then $F = F^E$.

Let F be a solution satisfying (WE), (S), (CI) and (EI). By non-emptiness of F and (WE), we need only to show the following:

> For each $A \in \mathcal{B}$, all $\mathbf{x}, \mathbf{a} \in A$ that are weakly efficient in A, if $[a_i = a_j$ for all $i, j \in N]$, but $[x_i \neq x_j$ for some $i, j \in N]$, then $\mathbf{x} \neq F(A)$.

Let $\mathbf{x}$ and $\mathbf{a}$ be such that both are weakly efficient on A, $[a_i = a_j$ for all $i, j \in N]$, and $[x_i \neq x_j$ for some $i, j \in N]$. Suppose to the contrary that $\mathbf{x} = F(A)$. Consider $B \equiv comp\{\mathbf{x}\}$. Note that $B \subseteq A$ and that $\mathbf{x} \in A$ is efficient on B. By (CI), $\mathbf{x} = F(B)$.

Consider the set $con\left[\cup_{\pi \in \Pi} \pi(B)\right]$, and denote it by C. By construction, C is a symmetric convex set having $C \supseteq B$. By the construction of B and C, $\mathbf{x}$ is efficient on C. Therefore, noting that $\mathbf{x} = F(B)$, $B \subseteq C$ and $\mathbf{x}$ is efficient on C, $\mathbf{x} = F(C)$ follows from (EI). Since C is symmetric, by (WE) and (S), $F(C)$ must be weakly efficient and be the equal utility point, which is a contradiction. Therefore, $\mathbf{x} \neq F(A)$. This proves (1.ii) implies (1.i), and thus the equivalence of (1.i) and (1.ii).

Insert Figure 1 around here.

To complete the proof, we note that, F^E satisfies (SOL), and that (SOL) and (WE) imply (CI) and (EI). $\diamond$

Theorem 2: The following statements are equivalent: (2.i) $F = F^{KS}$; (2.ii) F satisfies (WE), (S), (SI), (RCI) and (REI); (2.iii) F satisfies (WE), (S), (SI) and (RSOL).

Proof. It can be checked that $F = F^{KS}$ satisfies (WE), (S), (SI), (RCI) and (REI). We next show that, if F satisfies (WE), (S), (SI), (RCI) and (REI), then $F = F^{KS}$.

Let F be a solution satisfying (WE), (S), (SI), (RCI) and (REI). By (WE) and (SI), and from the non-emptiness of F, we need only to show the following:

> For each $A \in \mathcal{B}$ with $\mathbf{m}(A) = (1, \ldots, 1)$, all $\mathbf{x}, \mathbf{a} \in A$ that are weakly efficient on A, if $[a_i = a_j$ for all $i, j \in N]$, but $[x_i \neq x_j$ for some $i, j \in N]$, then $\mathbf{x} \neq F(A)$.

Let $A \in \mathcal{B}$, $\mathbf{m}(A) = (1, \ldots, 1)$, and $\mathbf{x}$ and $\mathbf{a} \in A$ be weakly efficient on A. Suppose to the contrary that $\mathbf{x} = F(A)$. For any $i \in N$, let e^i be the vector in which $e^i_j = 1$ if $i = j$ and $e^i_j = 0$ for $j \neq i$. Consider $B = con\{\mathbf{0}, e^1, \ldots, e^i, \ldots, e^n, \mathbf{x}\}$. Note that $B \subseteq A$, $\mathbf{m}(B) = \mathbf{m}(A)$, and that $\mathbf{x} \in B$ is efficient on B. By (CI), $\mathbf{x} = F(B)$.

Now, consider the set $C = con(\bigcup_{\pi \in \Pi} \pi(B))$. Note that C is convex, $B \subseteq C$, and $\mathbf{x} \in C$ is efficient on C. By (REI) and from $\mathbf{x} = F(B)$, we obtain $\mathbf{x} = F(C)$. Since C is symmetric, by (WE) and (S), it follows that $F(C)$ must be such that $F_i(C) = F_j(C)$ for all $i, j \in N$, a contradiction with $\mathbf{x} = F(C)$. Therefore, $\mathbf{x} \neq F(A)$.

To summarize the above, we have shown that (2.i) and (2.ii) are equivalent. To complete the proof, we need only to note that F^{KS} satisfies (RSOL) and that (RSOL) and (WE) imply (RCI) and (REI). $\diamond$

It is of interest to note that our characterizations of F^E and F^{KS} are new. Since we use variants of the solidarity-type axioms, our results offer a new perspective on the two solutions. Noting that F^{NA} can be characterized by our (CI), together with (E), (S) and (SI), and that (CI) is a solidarity-type axiom, F^{NA} has a certain sense of the spirit of solidarity. From our characterizations of F^E and our above remarks on F^{NA}, the main difference between F^E and F^{NA} may be attributed to axioms (EI) and (SI): F^E satisfies (EI) but violates (SI), whereas F^{NS} satisfies (SI) but violates (EI).[3]

[3] It can be checked that F^E is also characterized by (WE), (S) and (IUA), though in

Among the three solutions F^{NS}, F^E and F^{KS}, clearly, F^E fares the best in terms of solidarity since it can be characterized by using the strongest form of the solidarity-type axioms.

In Theorems 1 and 2, the respective independence of the axioms can be checked. For example, to see the independence of (WE), (S), (CI) and (EI) in Theorem (1.ii), check the indispensability of (CI).[4] For this purpose, consider the solution F^1 to be defined below. Let $F^L : \Sigma \to \mathbb{R}^n_+$ be the lexicographic egalitarian solution defined as usual. Given $\lambda \in [0,1]$, define the bargaining solution $F^{\lambda LE}$ as $F^{\lambda LE}(A) \equiv \lambda \cdot F^E(A) + (1-\lambda) \cdot F^L(A)$ for each $A \in \Sigma$. Note that $F^{\lambda LE}(A) = F^E(A)$ if and only if $F^E(A)$ is efficient on $A \in \Sigma$. Let Σ^{sc} be the set of all bargaining problems in Σ each of which is also strictly comprehensive. Given $\lambda \in (0,1)$, define F^1 as follows: for each $A \in \Sigma$,

(1) if $A \in \Sigma^{sc}$ or $A = comp\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+$, then $F^1(A) = F^E(A)$;

(2) otherwise, $F^1(A) = F^{\lambda LE}(A)$.

Then, F^1 can be shown to satisfy (WE), (S) and (EI), but not (CI).

The independence of the axioms (WE), (S), (SI) and (RSOL) in characterizing F^{KS} can be readily checked. For $\#N > 2$, the axioms (WE), (S), (SI), (RCI) and (REI) are independent. If, however, $\#N = 2$, then F^{KS} is characterized by (WE), (S), (SI) and (REI). Thus, (RCI) is no longer indispensable to the characterization of F^{KS} for two-person problems. A bargaining solution which satisfies (WE), (S), (SI) and (REI), but violates (RCI) for $\#N > 2$ is introduced below. Consider $\#N = 3$. Let $\Sigma^u \equiv \{A \in \Sigma \mid \forall i \in N : m_i(A) = 1\}$. Let

$$\triangle_3 \equiv con\{\mathbf{0}, (1,0,0), (0,1,0), (0,0,1), (1,0,1), (0,1,1)\}.$$

Note $\triangle_3 \in \Sigma^u$ and $F^K(\triangle_3) = F^E(\triangle_3) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Given $\lambda \in (0,1)$, define F^2 as: for each $A \in \Sigma$,

(1) if $A \in \Sigma^u$ and,

(1-1) if $A \subseteq \triangle_3$ with $F^K(\triangle_3) = F^E(\triangle_3) \in A$, then $F^2(A) = F^{\lambda LE}(A)$;

(1-2) if otherwise, then $F^2(A) = F^K(A)$;

this case, the difference between F^E and F^{NS} is not as clear-cut as in our characterization (1.ii).

[4] It is easy to see the indispensability of each of the other three axioms.

(2) if $A \notin \Sigma^u$, then $F^2(A) = \boldsymbol{\alpha}(F^2(B))$ for some $B \in \Sigma^u$ and some $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$ such that $\boldsymbol{\alpha}(B) = A$.

It can be checked that F^2 satisfies (WE), (S), (SI) and (REI), but violates (RCI).

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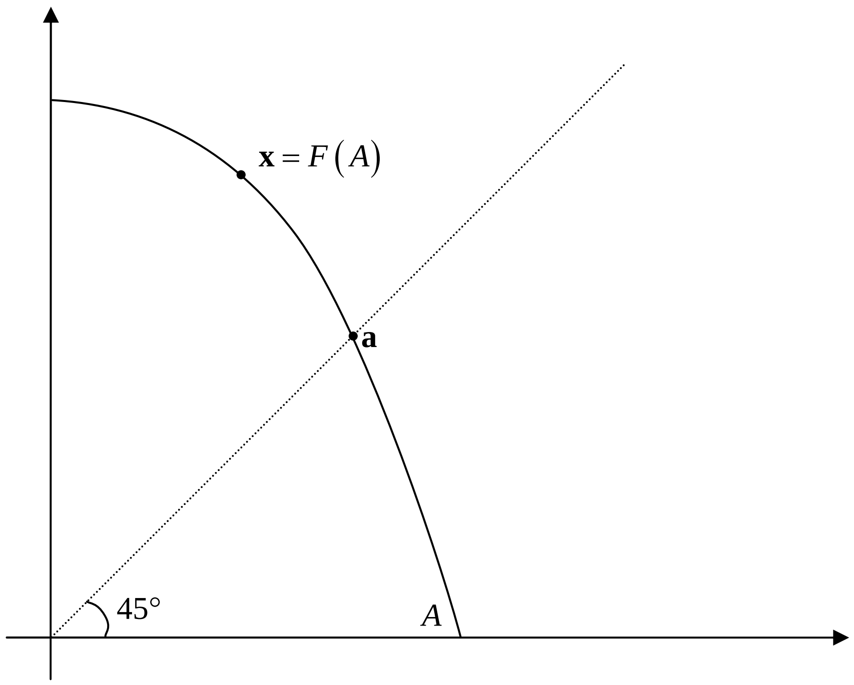
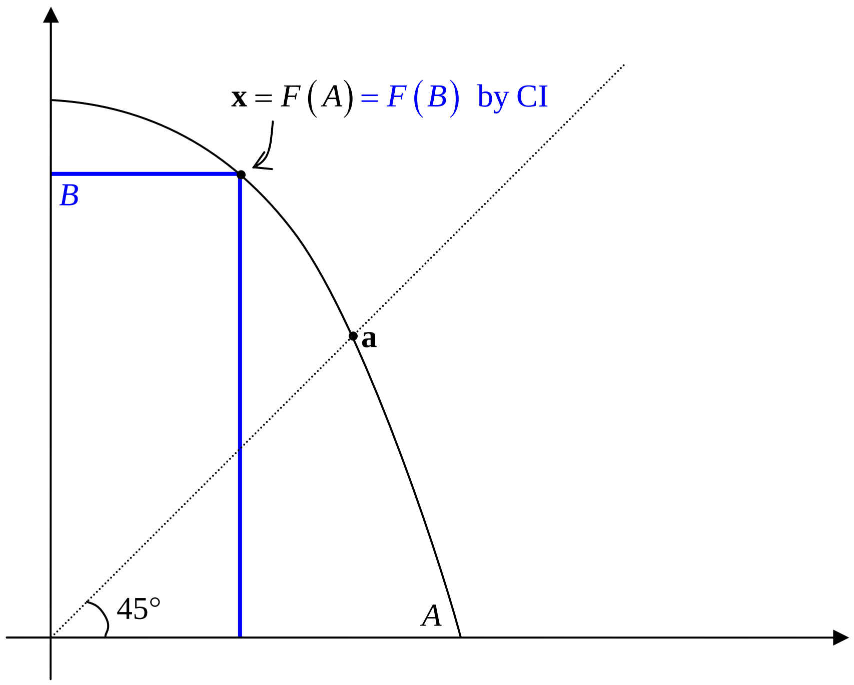
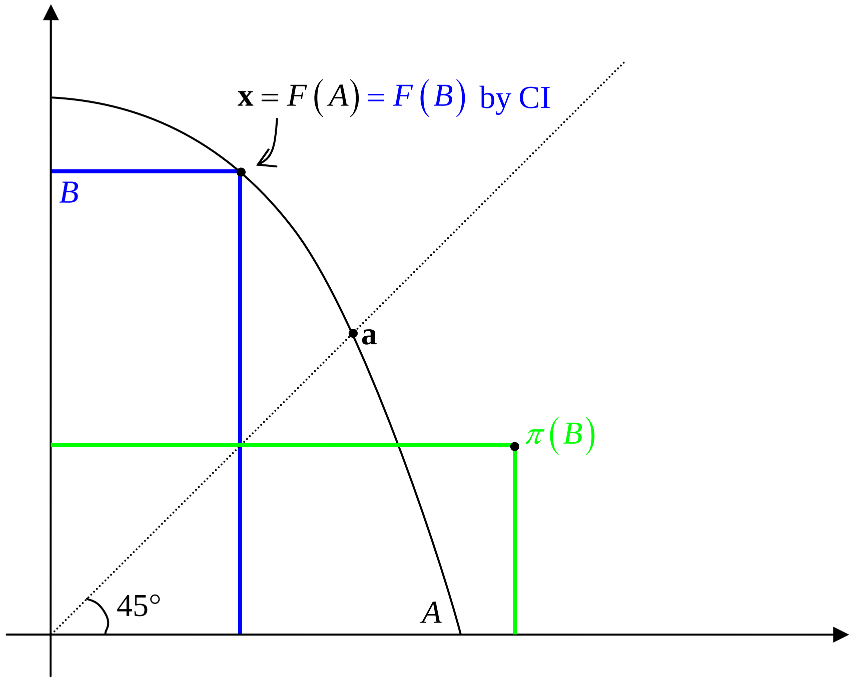


Proof of Theorem 1 – (1)
Figure 1(1)



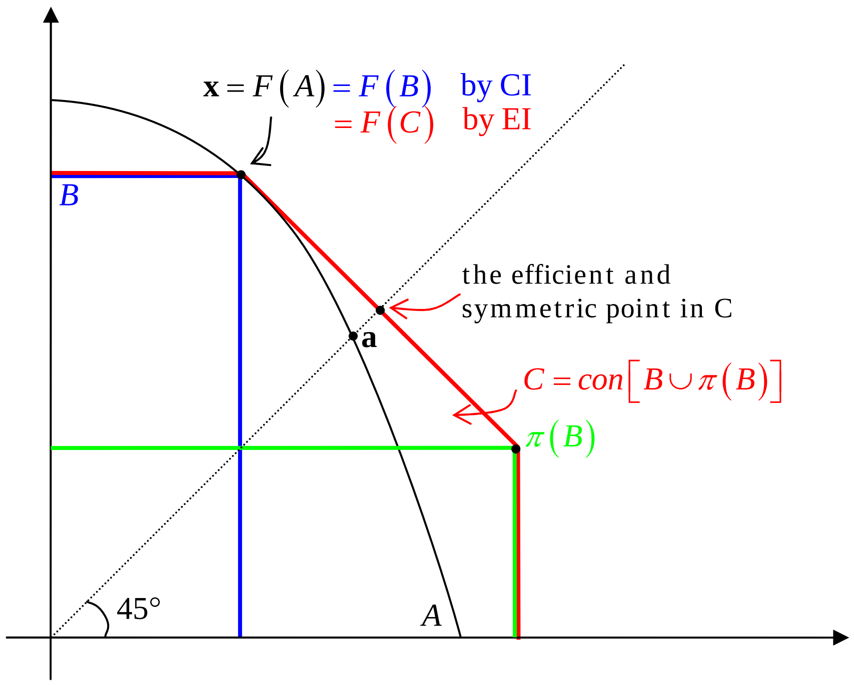
Proof of Theorem 1-(2)

Figure 1-(2)



Proof of Theorem 1-(3)

Figure 1-(3)



Proof of Theorem 1-(4)

Figure 1-(4)