Alternative characterizations of the proportional solution for nonconvex bargaining problems with claims*

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Abstract

We provide three alternative characterizations of the proportional solution defined on *compact* and *comprehensive* bargaining problems with claims that are not necessarily convex. One characterization result is obtained by using, together with other standard axioms, two solidarity axioms. Another characterization theorem shows that the single-valuedness axiom is dispensable even within the class of nonconvex problems if the standard symmetry axiom is imposed.

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1 Introduction

By considering the class of *bargaining problems* (feasible utility sets) *with claims* that are *compact* and *comprehensive* but *not necessarily convex*, we axiomatize the *proportional solution* in terms of solidarity.[1] The aforementioned class enriches the classical Nash (1950) bargaining domain by adding an unfeasible point representing the claims of bargainers.[2] The proportional rule assigns to bargainers payoffs proportional to their claims relative to the disagreement point. This rule was first defined and axiomatically studied by Kalai (1977) in convex bargaining domain (with symmetric claims) and extended by Chun and Thomson (1992) into convex bargaining domain with possibly asymmetric claims.

Nonconvex bargaining problems with claims are not unnatural. If agents involved in some bargaining situation are not all expected utility maximizers, then the feasible utility set is not convexifiable by randomization. Moreover, randomization is not always reasonable or possible in all bargaining situations. For instance, consider a principal-agent relationship with moral hazard where preferences of the transacting parties are represented by von Neumann-Morgenstern utility functions and their expectations (claims) have utility values.[3] The utility possibility set is not convex in general unless random contracts are allowed [see, for example, Ross (1973)].[4]

The solidarity-type axioms are systematically studied by Xu and Yoshihara (2008) for classical convex bargaining problems. In this paper, we propose two new axioms of solidarity for nonconvex problems with claims, by which a new characterization of the proportional solution is provided. This new result strengthens the characterization of Chun and Thomson (1992), which was by means of a version of Kalai's monotonicity axiom [Kalai (1977)].

The paper is organized as follows. First, we provide some basic notations and definitions. Our axioms and results are laid down next. Finally, we provide the independence of axioms.

[1] Noncovex bargaining problems have been considered for the three classical bargaining solutions: Nash solution [Nash (1950)], Kalai-Smorodinsky solution [Kalai and Smorodinsky (1975)], and Egalitarian solution [Kalai (1977)] (see, for instance, Mariotti (1998, 1999), Xu and Yoshihara (2006), along with references cited therein).

[2] For an excellent and easy introduction to the axiomatic bargaining theory, see, for instance, Thomson (1994).

[3] Expectations may come from their experience and/or observation of related contracts.

[4] The utility surface is not convex because agents' incentive constraints are not convex in general.

2 Preliminaries

Let $N = \{1, ..., n\}$ be the set of agents with $n \geqq 2$. For all $x \in \mathbb{R}^n_+$ and $\alpha \in \mathbb{R}_+$, we write $y = (\alpha; x_{-i}) \in \mathbb{R}^n_+$ to mean that $y_i = \alpha$ and $y_j = x_j$ for all $j \in N \backslash \{i\}$.[5] A *positive affine transformation* is a function $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that there exist numbers $a_i \in \mathbb{R}_{++}$ and $b_i \in \mathbb{R}$ for each $i \in N$, with $\lambda_i(x) = a_i x_i + b_i$ for all $x = (x_i)_{i \in N} \in \mathbb{R}^n$. The class of all positive affine transformations is denoted by Λ. For all $S \subseteq \mathbb{R}^n$ and any $\lambda \in \Lambda$, let $\lambda(S) \equiv \{\lambda(x) \mid x \in S\}$. Let π be a *permutation* of N, and Π be the set of all permutations of N. For all $x = (x_i)_{i \in N} \in \mathbb{R}^n$, let $\pi(x) = \left(x_{\pi(i)}\right)_{i \in N}$ be a permutation of x. For all $S \subseteq \mathbb{R}^n$ and all $\pi \in \Pi$, let $\pi(S) \equiv \{\pi(x) \mid x \in S\}$. For all $S \subseteq \mathbb{R}^n$, S is *symmetric* if $S = \pi(S)$ for all $\pi \in \Pi$; S is *comprehensive* if for all $x, y \in \mathbb{R}^n$, $[x \geqq y$ and $x \in S] \Rightarrow y \in S$.[6] For all $x^1, ..., x^k \in \mathbb{R}^n$, let $ch(\{x^1, ..., x^k\}) \equiv \{y \in \mathbb{R}^n | y \leqq x$ for some $x \in \{x^1, ..., x^k\}\}$ denote the *comprehensive hull* of $x^1, ..., x^k \in \mathbb{R}^n$. For all $i \in N$, let $\mathbf{e}_i \in \mathbb{R}^n_+$ be the unit vector with 1 in the i-th component, and 0 in all other components.

A *n-person bargaining problem with claim* (or simply a *problem*) is a triple (S, d, c), where S is a subset of $\mathbb{R}^n_+$, the disagreement outcome $d \in S$, and c is a point in $\mathbb{R}^n_+$ such that (i) S is compact and comprehensive, (ii) there exists $x \in S$ such that $x > d$, (iii) there exists $p \in \mathbb{R}^n_{++}$ and $r \in \mathbb{R}$ such that for all $x \in S : p \cdot x \leq r$, and (iv) $c \notin S$, $c \geq d$, and $c \leqq \overline{x}(S) = (\overline{x}_1(S), ..., \overline{x}_n(S))$, where $\overline{x}_i(S) \equiv \max\{x_i | x \in S\}$ for all $i \in N$ if this maximum exists, otherwise $\overline{x}_i(S) = \infty$.

Let Σ^n be the class of all n-person problems. Given a problem $(S, d, c) \in \Sigma^n$ and $\lambda \in \Lambda$, let $\lambda(S, d, c) \equiv (\lambda(S), \lambda(d), \lambda(c))$. Similarly, given a problem $(S, d, c) \in \Sigma^n$ and $\pi \in \Pi$, let $\pi(S, d, c) \equiv (\pi(S), \pi(d), \pi(c))$. Let $WPO(S) \equiv \{x \in S | \ \forall y \in \mathbb{R}^n,\ y > x \Rightarrow y \notin S\}$ be the set of *weakly Pareto optimal points of* S. Similarly, let $PO(S) \equiv \{x \in S | \ \forall y \in \mathbb{R}^n,$ $y \geq x \Rightarrow y \notin S\}$ be the set of *Pareto optimal points of* S.

A (bargaining) solution with claims is a correspondence $F : \Sigma^n \twoheadrightarrow \mathbb{R}^n_+$ such that, for every $(S, d, c) \in \Sigma^n$, $F(S, d, c) \subseteq S$ and $x \leq c$ for all $x \in F(S, d, c)$.

Definition 1 *A solution F over Σ^n is the proportional (bargaining) solution,*

[5] Note that $\mathbb{R}$ is the set of all real numbers; $\mathbb{R}_+$ (respectively, $\mathbb{R}_{++}$) is the set of all non-negative (respectively, positive) real numbers; $\mathbb{R}^n$ is the n-fold Cartesian product of $\mathbb{R}$; whilst $\mathbb{R}^n_+$ (respectively, $\mathbb{R}^n_{++}$) is the n-fold Cartesian product of $\mathbb{R}_+$ (respectively, $\mathbb{R}_{++}$).

[6] Given $x, y \in \mathbb{R}^n$, we write $x \geqq y$ to mean $[x_i \geqq y_i$ for all $i \in N]$, $x \geqslant y$ to mean $[x \geqq y$ and $x \neq y]$, and $x > y$ to mean $[x_i > y_i$ for all $i \in N]$.

denoted by F^P, if for all $(S,d,c) \in \Sigma^n$, $F(S,d,c)$ consists of all maximal points of S on the segment connecting d and c.

3 Axioms and Results

We are interested in a solution F that satisfies the following axioms, in the statement of which (S,d,c) and (T,d,c) are arbitrary feasible elements of its domain Σ^n:

Single Valuedness (**SV**). $|F(S,d,c)| = 1$.

Weak Pareto Optimality (**WPO**). For all $x \in F(S,d,c)$, $y > x \Rightarrow y \notin S$.

Anonymity (**AN**). For all $\pi \in \Pi$, $F(\pi(S,d,c)) = \pi(F(S,d,c))$.

Symmetry (**S**). $(S,d,c) = \pi(S,d,c)$ for all $\pi \in \Pi \Rightarrow [x \in F(S,d,c) \Rightarrow x_i = x_j$ for all $i,j \in N]$.

Scale Invariance (**SINV**). For all $\lambda \in \Lambda$, $F(\lambda(S,d,c)) = \lambda(F(S,d,c))$.

Strong Monotonicity (**SMON**). $S \subseteq T \Rightarrow [\forall y \in F(S,d,c), \exists x \in F(T,d,c)$ s.t. $x \geqq y$; and $\forall x \in F(T,d,c), \exists y \in F(S,d,c)$ s.t. $x \geqq y]$.

Contraction Independence other than Disagreement and Claims (**CIDC**). $S \subseteq T$, $F(T,d,c) \cap S \neq \varnothing \Rightarrow F(S,d,c) = S \cap F(T,d,c)$.

Weak Contraction Independence other than Disagreement and Claims (**WCIDC**). $S \subseteq T$, $F(T,d,c) \cap S \neq \varnothing$, and $F(T,d,c) \cap S \subseteq PO(S) \Rightarrow F(S,d,c) = S \cap F(T,d,c)$.

Expansion Independence other than Disagreement and Claims (**EIDC**). $S \subseteq T$ and $F(S,d,c) \subseteq PO(T) \Rightarrow F(S,d,c) = F(T,d,c)$.

The first seven axioms are standard. Note that (SMON) is a version applied to possibly multi-valued bargaining solutions. If we restrict our attention to single-valued solutions, then (SMON) is reduced to the standard monotonicity axiom discussed by Chun and Thomson (1992).[7]

[7] For all (S,d,c),$(T,d,c) \in \Sigma^n$ with $S \subseteq T$, $F(S,d,c) \leqq F(T,d,c)$.

Note that (WCIDC) is a solidarity axiom, which requires that whenever a problem (T, d, c) shrinks to another problem (S, d, c), and there are solutions to the problem (T, d, c) which are also Pareto optimal on (S, d, c), then $F(T, d, c) \cap S$ should continue to be the only solution set of (S, d, c). The solidarity idea embedded in this axiom is that, given that $F(T, d, c) \cap S$ is Pareto optimal on (S, d, c), any movement away from $F(T, d, c) \cap S$ will make at least one player worse off, and as a consequence, to keep the spirit of solidarity, $F(T, d, c) \cap S$ should continue to be the solution set of (S, d, c). (WCIDC) is slightly weaker than Nash's original contraction independence in that $F(T, d, c)$ is required to be Pareto optimal on S.

Note that (EIDC) is another type of solidarity axiom, which requires that whenever a problem (S, d, c) expands to another problem (T, d, c), and all solutions to the problem (S, d, c) are Pareto optimal on (T, d, c), then $F(T, d, c)$ should coincide with $F(S, d, c)$. The solidarity idea embedded in this axiom is that, given that any element in $F(S, d, c)$ is still Pareto optimal on (T, d, c), any movement away from it will hurt at least one player, and so the solution set of this enlarged problem (T, d, c) should continue to be $F(S, d, c)$ by the spirit of solidarity. (EIDC) is a weaker formulation of *Independence of Undominating Alternatives* suggested by Thomson and Myerson (1980), which requires that $F(S)$ to be weakly Pareto optimal on T. However, (EIDC), combined with (SV), is stronger than *Independence of Irrelevant Expansions* suggested by Thomson (1981).

Theorem 1. *A solution F over Σ^n is the proportional solution F^P if and only if it satisfies (SV), (WPO), (AN), (WCIDC), (EIDC), and (SINV).*

Proof. It can be easily checked that if $F = F^P$ over Σ^n then it satisfies (SV), (WPO), (AN), (WCIDC), (EIDC), and (SINV). Thus, we need only to show that if a solution F over Σ^n satisfies (SV), (WPO), (AN), (WCIDC), (EIDC), and (SINV), then it must be the proportional solution.

Let F satisfy (SV), (WPO), (AN), (WCIDC), (EIDC), and (SINV). Let $(S, d, c) \in \Sigma^n$. Assume that $\{x\} = F^P(S, d, c)$. We will show that $F(S, d, c) = \{x\}$ holds. By (SINV), let $\{\lambda(x)\} = F^P(\lambda(S), \mathbf{0}, \mathbf{1})$, with $\lambda(d) \equiv \mathbf{0}$ and $\lambda(c) \equiv \mathbf{1}$, for some $\lambda \in \Lambda$. Clearly, $\lambda(x) \in WPO(\lambda(S))$ and $\lambda(x) \equiv (\alpha, ..., \alpha) \leq \mathbf{1}$. Assume, to the contrary, that $\lambda(x) \notin F(\lambda(S), \mathbf{0}, \mathbf{1})$. Let $\{y\} = F(\lambda(S), \mathbf{0}, \mathbf{1})$ by (SV). Let $\pi(\lambda(S), \mathbf{0}, \mathbf{1})$ be a permutation of $(\lambda(S), \mathbf{0}, \mathbf{1})$. It follows from (AN) that $F(\pi(\lambda(S), \mathbf{0}, \mathbf{1})) = \{\pi(y)\}$ holds for all $\pi \in \Pi$. By (WPO), $y \in WPO(\lambda(S))$ and $\pi(y) \in WPO(\pi(\lambda(S)))$ for all $\pi \in \Pi$. Let us consider $T \equiv ch(\{y, \mathbf{e}_1, \ldots, \mathbf{e}_n\})$. Then, $(T, \mathbf{0}, \mathbf{1}) \in \Sigma^n$ and by (WCIDC), $\{y\} = F(T, \mathbf{0}, \mathbf{1})$. Then, by (AN), $\{\pi(y)\} = F(\pi(T, \mathbf{0}, \mathbf{1}))$ for all $\pi \in \Pi$. Now, define $V \equiv \cup_{\pi \in \Pi} \pi(T)$. Then, for all $\pi \in \Pi$, $\pi(y) \in PO(V)$. Thus, by (EIDC), $F(V, \mathbf{0}, \mathbf{1}) = \{\pi(y) | \pi \in \Pi\}$. However, since y is not a symmetric outcome, there exist $\pi, \pi' \in \Pi$ such that $\pi(y) \neq \pi'(y)$, which is a contradiction by (SV). Hence, $\{\lambda(x)\} = F(\lambda(S), \mathbf{0}, \mathbf{1})$, and (SINV) implies $\{x\} = F(S, d, c)$. ■

Defining F as a single-valued solution, Chun and Thomson (1992) provided a characterization of the proportional solution in the domain of convex problems by means of (WPO), (S), (SINV), and (SMON) formulated for single-valued solutions. Note that this characterization still holds even if the domain of problems is extended to nonconvex problems. By replacing the monotonicity axiom discussed by Chun and Thomson (1992) with (CIDC), we obtain an alternative characterization of the proportional solution.

Theorem 2. *A solution F over Σ^n is the proportional solution F^P if and only if it satisfies (WPO), (S), (SINV), and (CIDC).*

Proof. It is clear that if $F = F^P$ over Σ^n, then it satisfies (WPO), (S), (SINV), and (CIDC). Next, we show that if F over Σ^n satisfies (WPO), (S), (SINV), and (CIDC), then it must be the proportional solution.

Let F satisfy (WPO), (S), (SINV), and (CIDC). Let $(S, d, c) \in \Sigma^n$, and assume that $\{x\} = F^P(S, d, c)$. We show that $\{x\} = F(S, d, c)$. By (SINV), let $\{\lambda(x)\} = F^P(\lambda(S), \mathbf{0}, \mathbf{1})$ with $\lambda(d) \equiv \mathbf{0}$ and $\lambda(c) \equiv \mathbf{1}$ for some $\lambda \in \Lambda$. Clearly, $\lambda(x) \in WPO(\lambda(S))$, and it is a symmetric outcome, i.e. $\lambda(x) \equiv (\alpha, ..., \alpha) \leq \mathbf{1}$. Define the real number β as $\beta \equiv \max\{\overline{x}_i(\lambda(S)) | i \in N\}$, and the vectors $y^i = (\beta; \alpha_{-i})$ for all $i \in N$. Let $T \equiv ch(\{y^1, ..., y^n\})$, and observe that $\lambda(S) \subseteq T$. By definition of Σ^n, $(T, \mathbf{0}, \mathbf{1}) \in \Sigma^n$. Since T is symmetric, $(T, \mathbf{0}, \mathbf{1})$ is a symmetric problem. Thus, by (WPO) and (S), $F(T, \mathbf{0}, \mathbf{1}) = \{\lambda(x)\}$. It follows from (CIDC) that $F(\lambda(S), \mathbf{0}, \mathbf{1}) = \{\lambda(x)\}$, so that $F(S, d, c) = \{x\}$ by (SINV). ■

Remark: In the above theorem, the axiom (CIDC) is indispensable, and the weaker axiom (WCIDC) is insufficient to characterize F^P together with (WPO), (S), and (SINV). In fact, as the following figure indicates, the situation that $F(T, \mathbf{0}, \mathbf{1}) = \{x\}$ and $F(S, \mathbf{0}, \mathbf{1}) = \{y\}$, where $S \subseteq T$, and T is symmetric, is consistent with (WCIDC), but inconsistent with (CIDC).

Insert Figure around here.

Thus, there exists a solution $F \neq F^P$ satisfying (WPO), (S), (SINV), and (WCIDC), but not in case of (WPO), (S), (SINV), and (CIDC).

An interesting aspect of Theorem 2 is that it is obtained without imposing (SV) on F. This property does no longer hold if (S) is replaced with (AN). Thus, with respect to the aforementioned characterization offered by Chun and Thomson (1992), another alternative characterization of the proportional solution is obtained by replacing (S) with (AN) and by adding (SV).

Theorem 3. *A solution F over Σ^n is the proportional solution F^P if and only if it satisfies (SV), (WPO), (AN), (SINV), and (SMON).*

Proof. It is clear that if $F = F^P$ over Σ^n, then it satisfies (SV), (WPO), (AN), (SINV), and (SMON). Next we show that if F over Σ^n satisfies (SV), (WPO), (AN), (SINV), and (SMON), then it must be the proportional solution.

Let F satisfy (SV), (WPO), (AN), (SINV), and (SMON). Let $(S, d, c) \in \Sigma^n$. Assume that $\{x\} = F^P(S, d, c)$. We show that $F(S, d, c) = \{x\}$. By (SINV), let $\{\lambda(x)\} = F^P(\lambda(S), \mathbf{0}, \mathbf{1})$ with $\lambda(d) \equiv \mathbf{0}$ and $\lambda(c) \equiv \mathbf{1}$ for some $\lambda \in \Lambda$. Clearly, $\lambda(x) \in WPO(\lambda(S))$ and $\lambda(x) \equiv (\alpha, ..., \alpha) \leq \mathbf{1}$. Assume, to the contrary, that $\lambda(x) \notin F(\lambda(S), \mathbf{0}, \mathbf{1})$. Let $\{y\} = F(\lambda(S), \mathbf{0}, \mathbf{1})$, by (SV). (AN) implies that $F(\pi(\lambda(S), \mathbf{0}, \mathbf{1})) = \{\pi(y)\}$ for all $\pi \in \Pi$. Moreover, it follows from (WPO) that $y \in WPO(\lambda(S))$ and $\pi(y) \in WPO(\pi(\lambda(S)))$ for all $\pi \in \Pi$. Thus, we consider the following cases: (i) $\lambda(x) \in PO(\lambda(S))$ and $y \leq \lambda(x)$, (ii) $\lambda(x) \in PO(\lambda(S))$ and $y \nleq \lambda(x)$, (iii) $\lambda(x) \in WPO(\lambda(S))$ and $y \leq \lambda(x)$, and (iv) $\lambda(x) \in WPO(\lambda(S))$ and $y \nleq \lambda(x)$.

Consider (i) or (iii). Then, for all $\pi \in \Pi$, $\pi(y) \leq \lambda(x)$. Let $T \equiv \cap_{\pi \in \Pi} \pi(\lambda(S))$. Obviously, $(T, \mathbf{0}, \mathbf{1}) \in \Sigma^n$. Let $F(T, \mathbf{0}, \mathbf{1}) = \{z\}$ by (SV). By (SMON) and (SV), $z \leqq (\wedge_{\pi \in \Pi} \pi(y))$.[8] By the property of permutation, $(\wedge_{\pi \in \Pi} \pi(y))$ is a symmetric outcome. By the way, $(\vee_{\pi \in \Pi} \pi(y)) \leqq \lambda(x)$. Since $(\wedge_{\pi \in \Pi} \pi(y))$ and $(\vee_{\pi \in \Pi} \pi(y))$ are symmetric outcomes, but y is not a symmetric outcome, it follows that $(\wedge_{\pi \in \Pi} \pi(y)) < (\vee_{\pi \in \Pi} \pi(y))$. Thus, $z < \lambda(x)$, and (WPO) implies that $\lambda(x) \notin T$, a contradiction.

Consider (ii) or (iv). We proceed according to whether $y \geq \lambda(x)$ or $[y \ngeq \lambda(x)$ and $y \nleq \lambda(x)]$.

Suppose $y \geq \lambda(x)$. Then, for all $\pi \in \Pi$, $\pi(y) \geq \lambda(x)$. Observe that $(\wedge_{\pi \in \Pi} \pi(y)) = \lambda(x) < (\vee_{\pi \in \Pi} \pi(y))$. Let $T \equiv \cup_{\pi \in \Pi} \pi(\lambda(S))$, and observe

[8] For all $a, b \in \mathbb{R}^n_+$, $a \wedge b = (\min\{a_i, b_i\})_{i=\{1,...,n\}}$, $a \vee b = (\max\{a_i, b_i\})_{i=\{1,...,n\}}$.

that $(T, \mathbf{0}, \mathbf{1}) \in \Sigma^n$. Let $F(T, \mathbf{0}, \mathbf{1}) = \{z\}$, by (SV). By (SMON) and (SV), $\pi(y) \leqq z$ for all $\pi \in \Pi$. This implies $(\vee_{\pi \in \Pi} \pi(y)) \leqq z$. Since $(\vee_{\pi \in \Pi} \pi(y))$ is a symmetric outcome and $\lambda(x) \in WPO(\pi(\lambda(S)))$ for all $\pi \in \Pi$, it follows that $z \notin \pi(\lambda(S))$ for all $\pi \in \Pi$, so that $z \notin T$, a contradiction.

Otherwise, consider $y \not\geqq \lambda(x)$ and $y \not\leqq \lambda(x)$. Then, there is at least one player $i \in N$ such that $y_i < \lambda_i(x)$. Thus, $(\wedge_{\pi \in \Pi} \pi(y)) < \lambda(x)$. Let $T \equiv \cap_{\pi \in \Pi} \pi(\lambda(S))$, and observe that $(T, \mathbf{0}, \mathbf{1}) \in \Sigma^n$. Moreover, let $\{z\} = F(T, \mathbf{0}, \mathbf{1})$, by (SV). (SMON) and (SV) imply that $z \leqq (\wedge_{\pi \in \Pi} \pi(y)) < \lambda(x)$, so that $\lambda(x) \notin T$ by (WPO), a contradiction.

Hence, $\{\lambda(x)\} = F(\lambda(S), \mathbf{0}, \mathbf{1})$, and so $\{x\} = F(S, d, c)$ by (SINV). ■

4 Independence of Axioms

The axioms used in Theorems 1-3 are independent. To do this, let $F^{LP} : \Sigma^n \to \mathbb{R}^n_+$ be the lexicographic proportional solution defined as usual. Given $\lambda \in [0, 1]$, define the solution $F^{\lambda LP}$ as $F^{\lambda LP}(S, d, c) \equiv \lambda \cdot F^P(S, d, c) + (1 - \lambda) \cdot F^{LP}(S, d, c)$ for all $(S, d, c) \in \Sigma^n$. Note that $F^{\lambda LP}(S, d, c) = F^P(S, d, c)$ if and only if $F^P(S, d, c)$ is efficient on $S \in \Sigma^n$. Let Σ^n_{sc} be the set of all problems in Σ^n each of which is also *strictly comprehensive*.[9] Given $\lambda \in (0, 1)$, define F as follows: for all $(S, d, c) \in \Sigma^n$,
(1) if $(S, d, c) \in \Sigma^n_{sc}$ or $S \equiv ch(\{x\} \cup \{(c_i; \mathbf{0}_{-i})\, i \in N\})$ for some $x \in \mathbb{R}^n_+$, then $F(S, d, c) = F^P(S, d, c)$;
(2) otherwise, $F(S, d, c) = F^{\lambda LP}(S, d, c)$.

Then, (WCIDC) and (CIDC) are indispensable, since F satisfies (SV), (WPO), (S), (AN), (EIDC), and (SINV), but violates (WCIDC) and (CIDC); (EIDC) is indispensable, since F^{LP} satisfies (SV), (WPO), (AN), (WCIDC), and (SINV), but violates (EIDC); (SMON) is indispensable, since F^{LP} satisfies (SV), (WPO), (AN), and (SINV), but violates (SMON); (SINV) is indispensable, since the egalitarian solution F^E satisfies (SV), (WPO), (AN), (WCIDC), (EIDC), (S), (CIDC), and (SMON), but violates (SINV); (AN) and (S) are indispensable, since the dictatorial solution satisfies (SV), (WPO), (SINV), (CIDC), (WCIDC), (EIDC), and (SMON), but violates (AN) and (S); (WPO) is indispensable, since a solution which always chooses d as the solution outcome satisfies (SV), (AN), (S), (SINV), (WCIDC), (CIDC), (EIDC), and (SMON), but violates (WPO).

[9] $S \subseteq \mathbb{R}^n$ is strictly comprehensive if and only if for all $x \in S$, $y \in \mathbb{R}^n$: $x \geq y \Rightarrow [y \in S$ and $\exists z \in S$ such that $z > y]$.

Finally, for an example violating (SV), let $F : \Sigma^n \twoheadrightarrow \mathbb{R}^n_+$ be defined for all $(S, d, c) \in \Sigma^n$ by:
(1) if $PO(S) \cap \{x \in S | d < x \leq c\}$ is non-empty, then $F(S, d, c) = PO(S) \cap \{x \in S | d < x \leq c\}$;
(2) otherwise, $F(S, d, c) = \max_{s \in S}\{WPO(S) \cap \{x \in S | d < x \leq c\}\}$.
It can be shown that F satisfies (WPO), (SINV), (SMON), (WCIDC), (EIDC), and (AN), but it violates (SV). Thus, (SV) is indispensable.

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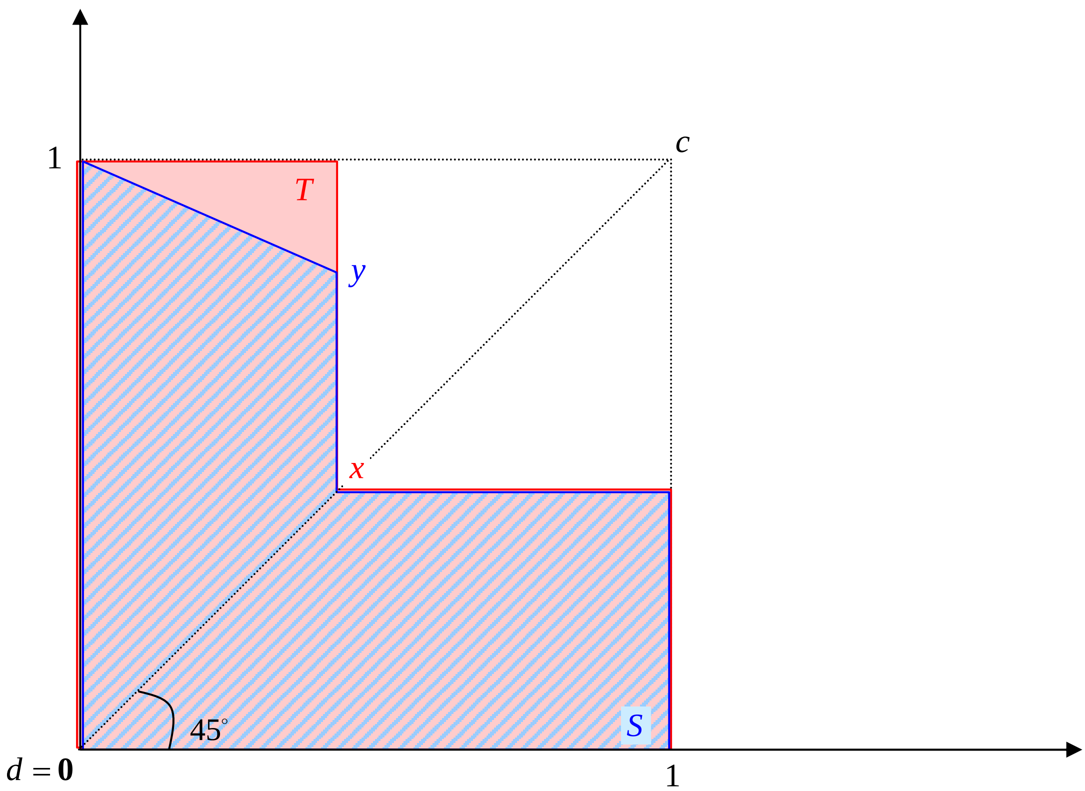

Figure:

$$F(T,\mathbf{0},\mathbf{1})=\{x\} \text{ and } F(S,\mathbf{0},\mathbf{1})=\{y\}.$$