Partially Honest Nash Implementation: A Full Characterization

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Abstract

Given the framework introduced by Dutta and Sen (2012), this paper offers a comprehensive analysis of (Nash) implementation with partially honest agents when there are three or more participants. First, it establishes a condition which is necessary *and* sufficient for implementation. Second, it provides simple *tests* for checking whether or not a social choice correspondence can be implemented. Their usefulness is shown by examining implementation in a wide variety of environments.

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1 Introduction

The main practical aim of adopting an axiomatic approach to (Nash) implementation theory is to draw a demarcation line between which social choice correspondences (SCCs) are or are not implementable. Drawing from the recent literature on implementation with *partially honest agents* (Matsushima, 2008a, 2008b; Dutta and Sen, 2012), this paper identifies necessary and sufficient conditions for implementation when there are three or more agents.[1] Existing results on implementation with partially honest agents identify only sufficient conditions.[2]

The axiomatization result is derived within the classical implementation model (Maskin, 1999), and enriched by the following two suppositions in Matsushima (2008a, 2008b) and Dutta and Sen (2012). Firstly, among agents involved in the mechanism, there are agents who lie to the mechanism designer only when they prefer the outcome obtained from false-telling over the outcome obtained from truth-telling. This paper refers to these agents as being partially honest. Secondly, the mechanism designer knows that there are partially honest participants involved in the devised mechanism, but does not know the identity of these participants or their exact number. These elements modify the implementation problem in a fundamental way: The mechanism must be designed such that, for each state of the world *and* set of agents that are presumed to be partially honest, only the SCC-optimal allocations emerge as the (pure strategy) equilibrium outcomes. If such a design is possible, we shall call an SCC implementable with partially honest agents *partially honest implementable*.

The necessary and sufficient condition for implementation is derived by using the approach developed by Moore and Repullo (1990). It is, then, stated in terms of existence of certain sets. To overcome the difficulties related to the existential clauses, the paper provides simple procedures for how to prove or disprove the existence of these sets. Their usefulness is illustrated by examining the partially honest implementability in marriage problems, rationing problems under single peaked/plateaued preferences, bargaining problems, and in coalitional games. All SCCs considered here are not implementable in the standard setting, and Dutta and Sen (2012)'s result is silent with respect to their implementability.

The paper is organized as follows. Section 2 describes the formal environment. Section 3 reports our characterization result, algorithms and briefly discusses its implications. Section 4 concludes briefly. The appendix includes proofs omitted from the text.

2 Notation and general definitions

2.1 Preliminaries

The set of outcomes is denoted by X and the set of agents is $N = \{1, ..., n\}$. The cardinality of X is $\#X \geq 2$, while the cardinality of N is $n \geq 3$. Let $\mathcal{R}(X)$ be the set of all possible weak orders on X.[3] Let $\mathcal{R}_\ell \subseteq \mathcal{R}(X)$ be the (non-empty) set of all admissible weak orders for agent $\ell \in N$. Let $\mathcal{R}^n \subseteq \mathcal{R}_1 \times ... \times \mathcal{R}_n$ be the set of all admissible profiles of weak orders. A generic element of $\mathcal{R}^n$ is denoted by R, where its ℓth component is $R_\ell \in \mathcal{R}_\ell$ for each $\ell \in N$. The symmetric and asymmetric parts of any $R_\ell \in \mathcal{R}_\ell$ are, in turn, denoted by I_ℓ and P_ℓ, respectively. For any $R_\ell \in \mathcal{R}_\ell$ and any $x \in Y \subseteq X$, let $I_\ell(x, Y)$ denote agent ℓ's set of outcomes in Y which are indifferent to x according to R_ℓ, that is, $I_\ell(x, Y) = \{y \in Y | (y, x) \in I_\ell\}$. For any $R \in \mathcal{R}^n$ and any $\ell \in N$, let $R_{-\ell}$ be the list of elements of R for all agents except ℓ, i.e., $R_{-\ell} \equiv (R_1, ..., R_{\ell-1}, R_{\ell+1}, ..., R_n)$. Given a list $R_{-\ell}$ and $R_\ell \in \mathcal{R}_\ell$, we denote by $(R_{-\ell}, R_\ell)$ the preference profile consisting of these R_ℓ and $R_{-\ell}$. Let $L(R_\ell, x)$ denote agent ℓ's lower contour set at $(R_\ell, x) \in \mathcal{R}_\ell \times X$, that is, $L(R_\ell, x) \equiv$

[1] Two-agent case is studied in Lombardi and Yoshihara (2011).

[2] In a related paper, Kartik and Tercieux (2012) observe that an intrinsic motivation towards honesty of one of participants renders any social choice function evidence-monotonic on the extended space of outcomes-plus-evidences.

[3] A weak order over X is a complete and transitive binary relation over X.

$\{y \in X | (x,y) \in R_\ell\}$, while $L(P_\ell, x)$ denote the strict lower contour set at $(R_\ell, x) \in \mathcal{R}_\ell \times X$, that is, $L(P_\ell, x) \equiv \{y \in X | (x,y) \in P_\ell\}$. For any $R_\ell \in \mathcal{R}_\ell$ and $Y \subseteq X$, let $\max_{R_\ell} Y$ be agent ℓ's set of optimal outcomes in Y according to R_ℓ, that is, $\max_{R_\ell} Y \equiv \{x \in Y | (x,y) \in R_\ell \text{ for all } y \in Y\}$. For any $R_\ell \in \mathcal{R}_\ell$, any $Y \subseteq X$, and any $y \in \max_{R_\ell} Y$, let R_ℓ^p be agent ℓ's weak order which has the property that y is the unique optimal outcome in Y according to R_ℓ^p, that is, $\{y\} = \max_{R_\ell^p} Y$.

A *social choice correspondence* (*SCC*) F on $\mathcal{R}^n$ is a correspondence $F : \mathcal{R}^n \twoheadrightarrow X$ with $F(R) \neq \varnothing$ for all $R \in \mathcal{R}^n$. For each $R \in \mathcal{R}^n$, the subset $F(R) \subseteq X$ is the set of F-optimal outcomes associated with the configuration R. Denote the class of admissible SCCs by $\mathcal{F}$.

A *mechanism* or *game form* is a pair $\gamma \equiv (M, g)$, where $M \equiv M_1 \times ... \times M_n$, with each M_ℓ being a (non-empty) set, and $g : M \to X$ is a function; then, γ consists of a message space M, where M_ℓ is the message space for agent $\ell \in N$, and an outcome function g. Let $m_\ell \in M_\ell$ denote a generic message (or strategy) for agent ℓ. A message profile is denoted by $m \equiv (m_1, ..., m_n) \in M$. For any $m \in M$ and $\ell \in N$, let $m_{-\ell} \equiv (m_1, ..., m_{\ell-1}, m_{\ell+1}, ..., m_n)$. Let $M_{-\ell} \equiv \times_{i \in N \setminus \{\ell\}} M_i$. Given an $m_{-\ell} \in M_{-\ell}$ and an $m_\ell \in M_\ell$, denote by $(m_\ell, m_{-\ell})$ the message profile consisting of these m_ℓ and $m_{-\ell}$.

A mechanism γ induces a class of (*non-cooperative*) *games* $\{(\gamma, R) | R \in \mathcal{R}^n\}$. Given a game (γ, R), we say that $m \in M$ is a (pure strategy) *Nash equilibrium* at R if and only if, for all $\ell \in N$, $(m, (m'_\ell, m_{-\ell})) \in R_\ell$ for all $m'_\ell \in M_\ell$. Given a game (γ, R), let $NE(\gamma, R)$ denote the set of Nash equilibrium message profiles of (γ, R), whereas $NA(\gamma, R)$ represents the corresponding set of Nash equilibrium outcomes.

A mechanism $\gamma \in \Gamma$ *implements* F in *Nash equilibria*, or simply *implements* F, if and only if $F(R) = NA(\gamma, R)$ for all $R \in \mathcal{R}^n$. If such a mechanism exists, then F is (*Nash-*)*implementable*.

2.2 Partially honest participants

For any mechanism γ and any agent $\ell \in N$, a *truth-telling correspondence* T_ℓ^γ on $\mathcal{R}^n \times \mathcal{F}$ is a correspondence $T_\ell^\gamma : \mathcal{R}^n \times \mathcal{F} \twoheadrightarrow M_\ell$ with $T_\ell^\gamma(R, F) \neq \varnothing$ for all $(R, F) \in \mathcal{R}^n \times \mathcal{F}$. An interpretation of the set $T_\ell^\gamma(R, F)$ is that, given a mechanism γ and a pair (R, F), participant ℓ behaves truthfully at the message profile $m \in M$ if and only if $m_\ell \in T_\ell^\gamma(R, F)$. Note that the type of elements of M_ℓ constituting $T_\ell^\gamma(R, F)$ depends on the type of mechanism γ that one may consider.

For any $\ell \in N$ and any $R \in \mathcal{R}^n$, let $\succcurlyeq_\ell^R$ be agent ℓ's weak order over M under the profile R. The asymmetric part of $\succcurlyeq_\ell^R$ is denoted $\succ_\ell^R$, while the symmetric part is denoted $\sim_\ell^R$.

Definition 1. An agent $h \in N$ is a *partially honest agent* if, for any mechanism γ, any $R \in \mathcal{R}^n$, and any $m \equiv (m_h, m_{-h}), m' \equiv (m'_h, m_{-h}) \in M$, the following properties hold:

(i) if $m_h \in T_h^\gamma(R, F)$, $m'_h \notin T_h^\gamma(R, F)$, and $(g(m), g(m')) \in R_h$, then $(m, m') \in \succ_h^R$;

(ii) otherwise, $(m, m') \in \succcurlyeq_h^R$ if and only if $(g(m), g(m')) \in R_h$.

Definition 2. If agent $\ell \in N$ is not partially honest, i.e., $\ell \neq h$, then for any mechanism γ, any $R \in \mathcal{R}^n$, and any $m, m' \in M$, the following property holds: $(m, m') \in \succcurlyeq_\ell^R$ if and only if $(g(m), g(m')) \in R_\ell$.

For any $R \in \mathcal{R}^n$, let $\succcurlyeq^R$ denote the profile of weak orders over M under the profile R, that is, $\succcurlyeq^R \equiv \left(\succcurlyeq_\ell^R\right)_{\ell \in N}$.

2.3 Partially honest implementation

Throughout the paper, the following informational assumption holds.

Assumption 1. *There are* partially honest agents *in N. The mechanism designer knows that there are partially honest agents in N, though she does not know their identities or their exact number.*

Let $\varnothing \neq \mathcal{H} \subseteq 2^N \setminus \{\varnothing\}$ be a class of non-empty subsets of N. The family $\mathcal{H}$ is viewed as the class of potential groups of partially honest agents. Each $H \in \mathcal{H}$ then, represents a *conceivable set of partially honest agents* in N. By Assumption 1, the mechanism designer simply knows that $\mathcal{H}$ is non-empty, but she may not know what subsets of N belong to $\mathcal{H}$ and she never knows which element of $\mathcal{H}$ is the true set of partially honest agents in the society. Given this interpretation, throughout the paper, we assume that $\mathcal{H} = 2^N \setminus \{\varnothing\}$ holds from the point of view of the mechanism designer.

A mechanism γ induces a class of (*non-cooperative*) *games with partially honest agents* $\{(\gamma, \succcurlyeq^{R,H}) \,|R \in \mathcal{R}^n, H \in \mathcal{H}\}$. Given a game $(\gamma, \succcurlyeq^{R,H})$, we say that $m^* \in M$ is a (pure strategy) *Nash equilibrium with partially honest agents* at (R, H) if and only if, for all $\ell \in N$, $(m^*, (m_\ell, m^*_{-\ell})) \in \succcurlyeq^{R,H}_\ell$ for all $m_\ell \in M_\ell$. Given a game $(\gamma, \succcurlyeq^{R,H})$, let $NE(\gamma, \succcurlyeq^{R,H})$ denote the set of Nash equilibrium message profiles of $(\gamma, \succcurlyeq^{R,H})$, whereas $NA(\gamma, \succcurlyeq^{R,H})$ represents the corresponding set of Nash equilibrium outcomes.

Since by Assumption 1 the mechanism designer knows that there are partially honest agents in N but does not know who these agents are, this raises the question of what is an appropriate notion of implementation in such a setting. To enable the mechanism designer to implement *SCC*s with partially honest agents, this paper amends the standard definition of implementation as follows.

Definition 3. An *SCC* $F \in \mathcal{F}$ is *partially honest (Nash) implementable* if there exists a mechanism $\gamma = (M, g)$ such that:

$$\text{for all } R \in \mathcal{R}^n \text{ and all } H \in \mathcal{H},\ F(R) = NA(\gamma, \succcurlyeq^{R,H}).$$

In contrast to the standard definition of implementation, to achieve the partially honest implementability of F, the mechanism designer must design a mechanism in which the equivalence between the set of equilibrium outcomes and the set of F-optimal outcomes holds not only for each admissible state R, but also for each set $H \in \mathcal{H}$. Note that the gap between the two definitions becomes closed when no agent in N is partially honest.

3 Partially Honest Implementation

3.1 Characterization result

When there are three or more agents, Moore and Repullo (1990) established that an *SCC* F is implementable if and only if it satisfies Condition μ defined below.[4]

Condition μ: For each $F \in \mathcal{F}$, there is a non-empty set $Y^F \subseteq X$; furthermore, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R, x))_{\ell \in N}$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y^F$ for each $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following conditions (i)-(iii) are satisfied:

(i) if $C_\ell(R, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, then $x \in F(R^*)$;

(ii) for all $i \in N$, if $y \in C_i(R, x) \subseteq L(R^*_i, y)$ and $Y^F \subseteq L(R^*_\ell, y)$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;

(iii) if $y \in \max_{R^*_\ell} Y^F$ for all $\ell \in N$, then $y \in F(R^*)$.

Condition μ(i) is equivalent to (Maskin) monotonicity,[5] while Condition μ(ii) and Condition μ(iii) are weaker versions of no veto-power.[6] If F is implementable by a mechanism $\gamma \equiv (M, g)$, the set

[4] Moore and Repullo (1990), Dutta and Sen (1991), Lombardi and Yoshihara (2012), and Sjöström (1991) refined Maskin's characterization result by providing necessary and sufficient conditions for an *SCC* to be implementable. For respected introductions to the theory of implementation, see Jackson (2001) and Maskin and Sjöström (2002).

[5] An *SCC* F on $\mathcal{R}^n$ is *monotonic* if, for all $R, R' \in \mathcal{R}^n$, with $x \in F(R)$, $x \in F(R')$ holds whenever $L(R_\ell, x) \subseteq L(R'_\ell, x)$ for all $\ell \in N$.

[6] An *SCC* F on $\mathcal{R}^n$ satisfies *no veto-power* if, for all $R \in \mathcal{R}^n$, $x \in F(R)$ holds whenever $x \in \max_{R_\ell} X$ for at least $n-1$ agents.

Y^F is simply the range of the outcome function g, while $C_i(R,x)$ represents the set of outcomes that agent i can attain by varying her own strategy, keeping the other agents' strategy choices fixed.

The task of finding necessary and sufficient conditions for an *SCC* to be partially honest implementable is particularly complicated for two reasons: First, the presence of partially honest agents breaks down the equivalent relationship between agents' preferences over outcomes and their preferences over message profiles. Second, conditions on F are to be formulated only in terms of agents' outcome-preferences. Taking these difficulties into account, we obtain the following condition, Condition μ^*, which must be applied to any set $H \in \mathcal{H}$.

CONDITION μ^*: For each $F \in \mathcal{F}$, there is a non-empty set $Y^F \subseteq X$; furthermore, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R,x))_{\ell \in N}$ such that $x \in C_\ell(R,x) \subseteq L(R_\ell,x) \cap Y^F$ for each $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$ and all $\ell \in N$, there exists a non-empty set $S_\ell(R^*;x,R) \subseteq C_\ell(R,x)$ such that for all $H \in \mathcal{H}$, the following conditions (i)-(iii) are satisfied:

(i) if $R^* = R$ and $x \notin S_i(R;x,R)$, then $(x,z) \in P_i$ for all $z \in S_i(R;x,R)$;

(ii) for all $i \in N$, if $y \in C_i(R,x) \subseteq L(R_i^*,y)$, $y \in \max_{R_\ell^*} Y^F$ for all $\ell \in N \backslash \{i\}$, and:

(ii.a) if $H = \{i\}$ and *either* $y \in S_i(R^*;x,R)$ *or* $S_i(R^*;x,R) \subseteq L(P_i^*,y)$, then $y \in F(R^*)$;

(ii.b) if $i \notin H$, $H = \{h\}$, $R^* = R$, and $x \in S_h(R;x,R)$, then $y \in F(R)$;

(iii) if $y \in \max_{R_\ell^*} Y^F$ for all $\ell \in N$, and $y \notin F(R^*)$, then for some $h \in H$, $R_h^* \neq R_h^{*p}$, $y \notin S_h\left(R^*;y,\left(R_{-h}^*,R_h^{*p}\right)\right)$, and there exists $y' \in Y^F \backslash \{y\}$ such that $y' \in S_h\left(R^*;y,\left(R_{-h}^*,R_h^{*p}\right)\right) \cap I_h^*\left(y,Y^F\right)$.[7]

Note that in the above Condition μ^*, the set Y^F coincides with the set X if F is *unanimous* (Sjöström, 1991).[8]

The novelty of Condition μ^* is the introduction of the set $S_i(R^*;x,R) \subseteq C_i(R,x)$. Whilst traditionally the set $C_i(R,x)$ represents the set of outcomes that agent i can generate by varying her own strategy, keeping the other agents' strategy choices fixed, the set $S_i(R^*;x,R)$ represents the set of outcomes that this agent can attain by reporting the agents' true preferences when those preferences change from R to R^*.

Our characterization result hinges upon a condition on the class of admissible preferences, which basically requires that the class of available profiles of agents' preferences is sufficiently rich. To introduce such a condition, for each $F \in \mathcal{F}$, let $X^F \subseteq X$ be defined by:

$$X^F = \begin{cases} X & \text{if } F \text{ is unanimous} \\ F(\mathcal{R}^n) & \text{otherwise.} \end{cases}$$

The condition can be stated as follows.

RICH DOMAIN (**RD**). For all $i \in N$, all $F \in \mathcal{F}$, all $R \in \mathcal{R}^n$, and all $y \in X$, if $y \in \max_{R_\ell} X^F$ for all $\ell \in N$, then there exists $R_i^p \in \mathcal{R}_i$ such that $L(R_i^p,y) = L(R_i,y)$, with $\{y\} = \max_{R_i^p} L(R_i^p,y)$, and $(R_i^p,R_{-i}) \in \mathcal{R}^n$ holds.

Examples of preference domains satisfying such a condition would be the set of all profiles of weak orders, linear orders, and single peaked/plateaued non-private preferences on X. Finally, condition

[7] We shall refer to the condition that requires only one of the conditions (i)-(iii) in Condition μ^* as Conditions μ^*(i)-μ^*(iii) each.

[8] An *SCC* satisfies *unanimity* if, for all $R \in \mathcal{R}^n$, $x \in F(R)$ holds whenever $x \in \max_{R_i} X$ for all $i \in N$.

RD is vacuously satisfied in those classical economic environments with strong monotonic preferences.[9] From the perspective of the applications of implementation theory, therefore, condition **RD** basically represents a mild requirement.

The following theorem states that Condition μ^* is necessary and sufficient for partially honest implementation when the domain of preferences is sufficiently rich (its proof is deferred to Appendix).

THEOREM. *Let* $n \geq 3$; *let* $\mathcal{R}^n$ *satisfy* **RD***; and, let* Assumption 1 *hold. An SCC* $F \in \mathcal{F}$ *is* partially honest implementable *if and only if* F *satisfies* Condition μ^*.

We remark that condition **RD** is relevant only for the case that F is not unanimous, since **RD** is applied only to the necessity and sufficiency of Condition μ^*(iii) as shown in Appendix. Hence, the above theorem has following formulation for *unanimous SCC*s.

COROLLARY 1. *Let* $n \geq 3$; *and, let* Assumption 1 *hold. Any unanimous SCC* $F \in \mathcal{F}$ *is* partially honest implementable *if and only if it satisfies* Conditions μ^*(i)-μ^*(ii).

The Theorem established above does not impose any restriction on mechanisms. However, in implementation theory, the following types of mechansims are usually considered and sometimes useful.

DEFINITION 4. A mechanism $\gamma = (M, g)$ for implementing an *SCC* $F \in \mathcal{F}$ is *forthright* if the following property holds: $M_\ell \equiv \mathcal{R}^n \times Y^F \times N$ for all $\ell \in N$, and for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, if $m_\ell \in \{R\} \times \{x\} \times N$ for all $\ell \in N$, then $m \in NE(\gamma, R)$ and $g(m) = x$.[10]

That is, a forthright mechanism has the property that, if x is F-optimal at the state R, and each agent announces truthfully the state R and an F-optimal outcome x, then such a message profile constitutes a Nash equilibrium, and its corresponding equilibrium outcome should be the announced F-optimal outcome. Because of this simple structure, the canonical mechanisms constructed in the sufficiency proofs of Nash implementation are usually forthright. There is no loss of generality in relying on such mechanisms in the standard setting since Nash implementation and Nash implementation by forthright mechanisms are equivalent (Lombardi and Yoshihara, 2012). We shall now remark that this restriction is not innocuous in our set-up.

COROLLARY 2. *Let* $n \geq 3$; *let* $\mathcal{R}^n$ *satisfy* **RD***; and, let* Assumption 1 *hold. An SCC* $F \in \mathcal{F}$ *is* partially honest implementable by a forthright mechanism *if and only if* F *satisfies* Condition μ^* such that for all $R \in \mathcal{R}^n$, all $x \in F(R)$, and all $i \in N$, $x \in S_i(R; x, R)$.

The additional requirement that $x \in S_i(R; x, R)$ for all agents $i \in N$ restricts the class of *SCC*s that are partially honest implementable. We shall discuss this point further at the end of sub-section 3.3.

3.2 Algorithms for testing (*non-*)implementability

In this sub-section we shall derive from the complete algorithm of Lombardi and Yoshihara (2013b) three conditions for checking partially honest implementation. The first two conditions consist of simple tests for non-implementability of *SCC*s. The third condition yields a test for implementability and says that F is implementable if the constructed sets $S_i(R^*; x, R)$ are non-empty. Hence, in

[9] Note that any *SCC* in the classical economic environments with strong monotonic preferences is vacuously unanimous, in that there is no $R \in \mathcal{R}^n$ such that $\cap_{i \in N} \max_{R_i} X \neq \varnothing$.

[10] The set N can be replaced by any arbitrary set.

order to check for implementation, it is sufficient to construct the set $S_i(R^*; x, R)$ and to check it for non-emptiness.

We shall use the following definitions in the formulation of the conditions: For any $F \in \mathcal{F}$, and any $R \in \mathcal{R}^n$,

$$\bar{Y}^F \equiv X \backslash \{x \in X \backslash F(\mathcal{R}^n) \mid \text{for some } R \in \mathcal{R}^n : X \subseteq L(R_\ell, x) \text{ for all } \ell \in N\}; \tag{1}$$

and for any $i \in N$, and any R, $R^* \in \mathcal{R}^n$, with $x \in F(R)$,

$$\bar{O}_i(x, R, R^*) \equiv \left\{ z \in \bar{Y}^F \backslash F(R^*) \mid z \in L(R_i, x) \cap \bar{Y}^F \subseteq L(R_i^*, z),\, z \in \max_{R_\ell^*} \bar{Y}^F \text{ for all } \ell \in N \backslash \{i\} \right\}, \tag{2}$$

$$\bar{Q}_i(x, R, R^*) \equiv \left\{ z \in F(R^*) \mid z \in L(R_i, x) \cap \bar{Y}^F \subseteq L(R_i^*, z),\, z \in \max_{R_\ell^*} \bar{Y}^F \text{ for all } \ell \in N \backslash \{i\} \right\},$$

and

$$\bar{V}_i(x, R, R^*) \equiv \left\{ z \in \bar{Y}^F \mid z \in L(R_i, x) \cap \bar{Y}^F \subseteq L(R_i^*, z),\, z \notin \max_{R_\ell^*} \bar{Y}^F \text{ for some } \ell \in N \backslash \{i\} \right\}.$$

The first condition is useful as a first step in checking for non-implementability and concerns the violation of Condition μ^*(iii). This condition can be formulated as follows.

Lemma 1. *Let $\mathcal{R}^n$ satisfy* **RD**. *If $F \in \mathcal{F}$ satisfies* Condition μ^*, *then*
(i) $Y^F \subseteq \bar{Y}^F$;
(ii) *for all $R \in \mathcal{R}^n$, and all $x \in \bar{Y}^F$, if $x \in \max_{R_\ell} \bar{Y}^F$ for all $\ell \in N$, then $x \in F(R_i^p, R_{-i})$ for all $i \in N$.*

If an *SCC* has the properties of Lemma 1(i) and Lemma 1(ii), then either implementability or non-implementability may take place. In these cases, it is convenient to have a more powerful test for non-implementability. Here is one that is frequently useful.

Lemma 2. *Let $\mathcal{R}^n$ satisfy* **RD**. *For all $F \in \mathcal{F}$, all $R, R^* \in \mathcal{R}^n$, with $x \in F(R)$, and all $i \in N$, if F satisfies* Condition μ^*, *then there exists $y \in L(R_i, x)$, with $(y, x) \in R^*$, such that $y \in F(R^*)$* or *$y \notin \max_{R_\ell^*} \bar{Y}^F$ for some $\ell \in N \backslash \{i\}$.*

Let us now discuss this result: The contrapositive of the above lemma gives us the following easy way for checking non-implementability. If the set union of the sets $\bar{Q}_i(x, R, R^*)$ and $\bar{V}_i(x, R, R^*)$ is empty, then F violates Condition μ^*(ii.a). We shall exploit this result in the next sub-section.

Let us devise an auxiliary result to guide us in the formulation of our last condition. Before stating this auxiliary result a little more notation is needed: For any $i \in N$, and any $R \in \mathcal{R}^n$, with $x \in F(R)$, define the set $O_i(x, R, R)$ as follows.

$$O_i(x, R, R) \equiv \left\{ z \in X \backslash F(R) \mid C_i(R, x) \subseteq L(R_i, z),\, z \in \max_{R_\ell} Y^F \text{ for all } \ell \in N \backslash \{i\} \right\}.$$

Lemma 3. *For all $F \in \mathcal{F}$, all $R \in \mathcal{R}^n$, with $x \in F(R)$, and all $i \in N$, if F satisfies* Condition μ^* *and the set $O_i(x, R, R)$ is* non-empty, *then $x \in S_i(R; x, R)$ and for all $\ell \in N \backslash \{i\}$, $x \notin S_\ell(R; x, R) \subseteq L(P_\ell, x)$ and the set $O_\ell(x, R, R)$ is* empty.

The usefulness of this result is twofold. First, it partially characterizes the structure of the set $S_i(R; x, R)$ by showing under what conditions this set contains the F-optimal outcome x. Second, when read in combination with Corollary 2, it says that F is not partially honest implementable by any forthright mechanisms when the set $O_i(x, R, R)$ is not empty for some agent i.

We shall now present a deeper, more useful, and more interesting method for checking implementability based on information derived from the above lemmata. We construct the sets Y^F, $C_i(R,x)$ and $S_i(R^*;x,R)$ of Condition μ^* explicitly in the following way.

Take any $F \in \mathcal{F}$. For any $R \in \mathcal{R}^n$, with $x \in F(R)$, let us distinguish the following two cases.

Case I There exists at most one agent $j \in N$ such that $\bar{O}_j(x,R,R) \neq \varnothing$.

Case II Any other case.

For any R,$R^* \in \mathcal{R}^n$, with $x \in F(R)$, let $Y^F \equiv \bar{Y}^F$; furthermore, for any $i \in N$, define the set $C_i(R,x)$ as follows:

$$C_i(R,x) \equiv \begin{cases} L(R_i,x) \cap \bar{Y}^F & \text{if } \textbf{Case I}; \\ \left(L(R_i,x) \cap Y^F\right) \backslash \bar{O}_i(x,R,R) & \text{if } \textbf{Case II}; \end{cases}$$

finally, for any $i \in N$, define the set $S_i(R^*;x,R)$ as follows:

1. if $y \in C_i(R,x) \subseteq L(R_i^*,y)$ and $y \in \max_{R_\ell^*} Y^F$ for all $\ell \in N\backslash\{i\}$, then:
   (a) if $R = R^*$, then:
      i. if there exists a unique agent $j \in N\backslash\{i\}$ such that $\bar{O}_j(x,R,R) \neq \varnothing$, then $S_i(R^*;x,R) = L(P_i,x)$;
      ii. otherwise, $S_i(R^*;x,R) = \{x\} \cup Q_i(x,R,R^*)$;
   (b) if $R \neq R^*$, then
$$S_i(R^*;x,R) = \begin{cases} Q_i(x,R,R^*) & \text{if } Q_i(x,R,R^*) \neq \varnothing; \\ V_i(x,R,R^*) & \text{otherwise}; \end{cases}$$
2. otherwise,
   (a) if $R = R^*$ and there exists a unique agent $j \in N\backslash\{i\}$ such that $\bar{O}_j(x,R,R) \neq \varnothing$, then $S_i(R^*;x,R) = L(P_i,x) \cap C_i(R,x)$;
   (b) otherwise, $S_i(R^*;x,R) = C_i(R,x)$,

where

$$Q_i(x,R,R^*) \equiv \left\{ z \in F(R^*) \mid z \in C_i(R,x) \subseteq L(R_i^*,z),\ z \in \max_{R_\ell^*} Y^F \text{ for all } \ell \in N\backslash\{i\} \right\}, \tag{3}$$

and

$$V_i(x,R,R^*) \equiv \left\{ z \in C_i(R,x) \mid C_i(R,x) \subseteq L(R_i^*,z),\ z \notin \max_{R_\ell^*} Y^F \text{ for some } \ell \in N\backslash\{i\} \right\}. \tag{4}$$

Now, we prove that the above construction really performs the task for which it was intended.

LEMMA 4. *Let* $\mathcal{R}^n$ *satisfy* **RD**. $F \in \mathcal{F}$ *satisfies* Condition μ^* *if*
(i) *the above construction of* $S_i(R^*;x,R)$ *is* non-empty *for all* $i \in N$, *all* $R, R^* \in \mathcal{R}^n$, *with* $x \in F(R)$, *and*
(ii) *for all* $R \in \mathcal{R}^n$ *and all* $x \in \bar{Y}^F$ *such that* $x \in \max_{R_\ell} \bar{Y}^F$ *for all* $\ell \in N$, $x \in F(R_i^p, R_{-i})$ *for all* $i \in N$.

We shall remark that in the preceding lemma the requirement that the domain is sufficiently rich and the premises in part (ii) can be dropped for unanimous *SCC*.

It should be realized that if F passes the tests of Lemma 1 and Lemma 2, and if for any two profiles R and R^* such that x is F-optimal at R and the set $S_i(R^*; x, R)$ constructed as above is non-empty for each agent i, then we can be assured of implementability of F. Then, to check for implementability, care needs to be exercised to check non-emptiness of the constructed set $S_i(R^*; x, R)$, in particular for cases (1.a.i), (1.b), and (2.a). For cases (1.a.i) and (2.a), non-emptiness of $S_i(R^*; x, R)$ is not assured only for the case that x is the worst outcome for agent i at R. For case (1.b), non-emptiness is assured if *either* the set $Q_i(x, R, R^*)$ *or* $V_i(x, R, R^*)$ is non-empty. In view of its pratical importance, we shall exploit extensively Lemma 4 in the next sub-section.

3.3 Implications

In this subsection, we shall blend the results of sub-section 3.1 and sub-section 3.2 and derive a number of propositions in marriage problems, rationing problems under single peaked/plateaued preferences, bargaining problems, and in coalitional games. All positive results presented here cannot be reaped from the conventional implementation setting (Maskin, 1999), and from Dutta and Sen (2012)'s result. The reason is that all *SCC*s studied here violate monotonicity and no veto-power.

3.3.1 Applications to marriage problems

A *marriage problem* is an ordered triplet (M, W, R), where M and W are two non-empty and disjoint finite sets such that $M \cup W = N$, with cardinality $n = |W \cup M| \geq 3$, while R is a profile such that R_j is a linear order on $W \cup \{j\}$ if $j \in M$ or on $M \cup \{j\}$ if $j \in W$.[11] A matching φ on $W \cup M$ is a one-to-one correspondence from the set $W \cup M$ onto itself of order two (that is, $\varphi^2(j) = j$) having the following properties: *a*) for any $m \in M$, $\varphi(m) \neq m$ implies $\varphi(m) \in W$, and *b*) for any $w \in W$, $\varphi(w) \neq w$ implies $\varphi(w) \in M$. Let us denote the set of all matchings on $W \cup M$ by $\mathcal{M}$. In the context of marriage problems, the set of outcomes X is the set of feasible matchings $\mathcal{M}$.

Given any $m \in M$ and any R_m on $W \cup \{m\}$, R_m can be extended to the set $\mathcal{M}$ as follows: for all φ,$\varphi' \in \mathcal{M}$, $(\varphi(m), \varphi'(m)) \in R_m$ if and only if $(\varphi, \varphi') \in R_m$. Abusing notation, hereafter we use R_m to represent both. The same can be done for each $w \in W$. $\mathcal{P}^n$ denotes the set of admissible profiles of preferences for women and men. In what follows, we consider a situation in which the mechanism designer does not know agents' preferences. This situation is modeled by the quadruple $(M, W, \mathcal{M}, \mathcal{P}^n)$, which we refer to as a class of *marriage problems*.

A matching φ is blocked by agent i under R if $(i, \varphi(i)) \in P_i$. Furthermore, a matching φ is blocked by a pair (m, w) under R if

$$(w, \varphi(m)) \in P_m \text{ and } (m, \varphi(w)) \in P_w.$$

A matching φ is *stable under* R if it is not blocked by any individual or any pair. For any $R \in \mathcal{P}^n$, define the *stable SCC* $St(R)$ as the set of all stable matchings under R. A stable matching $\varphi_M^R \in St(R)$ under R is the *man-optimal stable matching for* R if

$$\text{for all } m \in M : \left(\varphi_M^R(m), \varphi(m)\right) \in R_m \text{ for all } \varphi \in St(R).$$

The *woman-optimal stable matching for* R, φ_W^R, is defined similarly. St_M (St_W) denotes the *man-optimal* (*woman-optimal*) *stable SCC*, that is, $St_M(R) = \varphi_M^R$ ($St_W(R) = \varphi_W^R$) for all $R \in \mathcal{P}^n$. The optimal-man (optimal-women) stable *SCC* is a *proper sub-correspondence* of the stable *SCC*.[12] A

[11] A linear order over a set Z is a complete, transitive, and anti-symmetric binary relation over Z.

[12] If F and F' are *SCC*s such that $F'(R) \subseteq F(R)$ for all $R \in \mathcal{R}^n$, then we say that F' is a *sub-correspondence* of F. If furthermore $F'(R) \neq F(R)$ for some $R \in \mathcal{R}^n$, then F' is a *proper sub-correspondence* of F.

matching φ is *Pareto optimal under* R if there is no other matching μ' such that $(\varphi'(i), \varphi(i)) \in R_i$ for all $i \in M \cup W$ and $(\varphi'(i), \varphi(i)) \in P_i$ for some $i \in M \cup W$. For any $R \in \mathcal{P}^n$, define the *Pareto Optimal SCC* $PO(R)$ as the set of all Pareto optimal matchings under R.

Consider the class of marriage problems with *singles*, where being single is a feasible choice and is not necessarily always the last choice of every agent. Within the traditional framework of implementation theory, it is known that the man-optimal (*resp.* woman-optimal) stable *SCC* violates monotonicity and no veto-power. Roth (1982) showed that no selection of the stable *SCC* is strategy-proof, whilst Kara and Sönmez (1996) showed that no proper sub-correspondence of the stable correspondence is implementable. In contrast with these results, however, the man-optimal (*resp.* woman-optimal) stable *SCC* on the class of marriage problems with singles is partially honest implementable, which is provided below.

Proposition 1. *Let* $(M, W, \mathcal{M}, \mathcal{P}^n)$ *be any class of* marriage problems with singles. *The* $St_M \in \mathcal{F}$ *satisfies* Condition μ^*.

Proof. Since St_M is a unanimous *SCC*, let $\mathcal{M} = Y^{St_M}$. Since $St_M(R) \in PO(R)$, then $\bar{O}_i(\varphi, R, R) = \varnothing$ for all $i \in N$. Following the algorithm of Lemma 4, let $C_i(R, \varphi) \equiv L(R_i, \varphi)$ for any $(R, R^*, \varphi, i) \in \mathcal{P}^n \times \mathcal{P}^n \times \mathcal{M} \times N$, with $\varphi \in St_M(R)$. Moreover, by case (1.a.ii) of the algorithm, $S_i(R; \varphi, R) = \{\varphi\}$ for all $i \in N$. Finally, to apply Lemma 4, we shall only confirm that $S_i(R^*; x, R)$ is non-empty for case (1.b).

For any $(R, R^*, \varphi, i) \in \mathcal{P}^n \times \mathcal{P}^n \times \mathcal{M} \times N$, with $R \neq R^*$ and $\varphi = St_M(R)$, let us suppose that $\varphi' \in C_i(R, \varphi) \subseteq L(R_i^*, \varphi')$, and $\mathcal{M} \subseteq L\left(R_j^*, \varphi'\right)$ for all $j \in N \backslash \{i\}$. We show that $\varphi' \in Q_i(\varphi, R, R^*)$. Assume, to the contrary, that $\varphi' \notin Q_i(\varphi, R, R^*)$ which implies that $\varphi' \notin St_M(R^*)$. Since φ' cannot be blocked by any individual or any pair, that is, $\varphi' \in St(R^*)$, we have that for some $\varphi'' \in St(R^*)$ and some $k \in M$, $(\varphi''(k), \varphi'(k)) \in P_k^*$. By our suppositions, i is the unique man such that $(\varphi''(i), \varphi'(i)) \in P_i^*$, otherwise we fall into a contradiction. We also have that $\varphi''(i) \in W$, otherwise, $\varphi' \notin St(R^*)$, which is a contradiction. Since $\varphi''(i) = q \in W$ and $\varphi''(i) \neq \varphi'(i)$, $(\varphi'(q), \varphi''(q)) \in P_q^*$, otherwise, either $\varphi'(i) = \varphi''(i) = q$ or $\varphi' \notin \max_{R_q^*} \mathcal{M}$, and so we fall into a contradiction in either case. Given that $(\varphi'(q), \varphi''(q)) \in P_q^*$ and $\varphi'' \in St(R^*)$, it follows that $\varphi'(q) = k' \in M \backslash \{i\}$, and so $(k', \varphi''(q)) \in P_q^*$ and $\varphi'(k') = q$. Given that $\mathcal{M} \subseteq L\left(R_{k'}^*, \varphi'\right)$, we have also that $(q, \varphi''(k')) \in P_{k'}^*$. Therefore, $(q, \varphi''(k')) \in P_{k'}^*$ and $(k', \varphi''(q)) \in P_q^*$, and so the matching φ'' is blocked by the pair (k', q), which gives us a contradiction. Thus, $\varphi' \in Q_i(\varphi, R, R^*)$. The statement follows from lemma 4. ■

Since by symmetry the parallel result holds for the woman-optimal stable *SCC*, St_W, the following corollary is readily obtained from Proposition 1 and Corollary 1.

Corollary 3. *Let* $(M, W, \mathcal{M}, \mathcal{P}^n)$ *be any class of* marriage problems with singles*; suppose that* Assumption 1 *holds. The* man-optimal (*resp.* woman-optimal) stable SCC *is* partially honest implementable.

On the class of *pure* marriage problems, where being single is not a feasible choice or it is always the last choice of every agent, Tadenuma and Toda (1998) established the impossibility theorem that there exists no single-valued sub-correspondence of the stable *SCC* that is implementable whenever there are at least three men and three women. Their impossibility result no longer holds when agents have intrinsic preferences towards honesty, since the man-optimal (*resp.* woman-optimal) stable *SCC* on pure marriage problems satisfies Condition μ^*, which is shown analogously to the proof of Proposition 1. Therefore:

Corollary 4. *Let* $(M, W, \mathcal{M}, \mathcal{P}^n)$ *be any class of* pure marriage problems*, with* $|M| = |W| \geq 2$*; suppose that* Assumption 1 *holds. There exists a* single-valued sub-correspondence of *St that is* partially honest implementable.

3.3.2 Applications to rationing problems under single peaked/plateaued preferences

The following model can be regarded as a model of rationing problems (Sprumont, 1991; Thomson, 1994). A *social endowment* $M \in \mathbb{R}_{++}$ of an infinitely divisible commodity has to be allocated among a set of agents N, which has cardinality $n \geq 3$. Each agent $i \in N$ is equipped with a continuous and *single peaked* preference defined over the interval $[0, M]$: This means that there is a real number in $[0, M]$, denoted $p(R_i)$, and called *peak amount*, such that for each pair $x_i, x_i' \in [0, M]$, if $x_i' < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x_i'$, then $(x_i, x_i') \in P_i$. Given $x_i \in [0, M]$, let $r_i(x_i)$ be the consumption bundle on the other side of agent i's peak amount that she finds indifferent to x_i if such consumption exists; otherwise, it gives the endpoint of $[0, M]$ on the other side of her peak amount. For each agent i, $\mathcal{R}_{sp_i}$ denotes the class of all preference relations on $[0, M]$ that satisfy continuity and single peakedness. Whenever the social endowment is kept fixed, we simply refer to an *economy* as a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}_{sp}^n$. $p(R)$ denotes the profile of peak amounts of R, $(p(R_i))_{i \in N}$. A feasible allocation is a list $x \equiv (x_i)_{i \in N} \in \mathbb{R}_+^n$ such that $\sum x_i = M$.[13] Note that we do not assume that the commodity can be disposed of. Let $X \equiv \left\{x \in \mathbb{R}_+^n | \sum x_i = M\right\}$ be the set of feasible allocations.[14] Note that the set of feasible allocation X is the n-fold product of $[0, M]$. Note that each R_i is *private* in that whose domain is not X but $[0, M]$. Hence, without loss of generality, when we say $L(R_i, x) = X$, it implies that $L(R_i, x_i) = [0, M]$.

In what follows, we consider a situation in which the mechanism designer does not know agents' preferences. This situation is modeled by the triple $\left(N, X, \mathcal{R}_{sp}^n\right)$, which we refer to as a class of *rationing problems with single peaked preferences.*

A number of *SCC*s that have been frequently discussed in the literature violate monotonicity and no veto-power. Some of these *SCC*s are the following ones (Thomson, 1995).

Proportional SCC, *PR*: For all $R \in \mathcal{R}_{sp}^n$, $x = PR(R)$ if $x \in X$, and (i) when $\sum p(x_i) \geq 0$, and there exists $\lambda \in \mathbb{R}_+$ such that for all $i \in N$, $x_i = \lambda p(R_i)$; and (ii) when $\sum p(x_i) = 0$, $x_i = \left(\frac{M}{n}\right)$ for all $i \in N$.

Equal Distance SCC, *ED*: For all $R \in \mathcal{R}_{sp}^n$, $x = ED(R)$ if $x \in X$, and (i) when $\sum p(x_i) \geq M$, and there exists $d \in \mathbb{R}_+$ such that for all $i \in N$, $x_i = \max\{0, p(R_i) - d\}$; and (ii) when $\sum p(x_i) \leq M$, and there exists $d \in \mathbb{R}_+$ such that for all $i \in N$, $x_i = p(R_i) + d$.

Equal Sacrifice SCC, *ES*: For all $R \in \mathcal{R}_{sp}^n$, $x = ES(R)$ if $x \in X$, and (i) when $\sum p(x_i) \geq M$, and there exists $\sigma \in \mathbb{R}_+$ such that for all $i \in N$, $r_i(x_i) - x_i \leq \sigma$, strict inequality holding only if $x_i = 0$; and (ii) when $\sum p(x_i) \leq M$, and there exists $\sigma \in \mathbb{R}_+$ such that for all $i \in N$, $x_i - r_i(x_i) = \sigma$.

Since the above *SCC*s do not satisfy monotonicity, they are not implementable in the standard framework. Moreover, since they violate no veto-power, Dutta and Sen (2012)'s result does not apply. A natural question, then, is whether or not these *SCC*s are partially honest implementable.

PROPOSITION 3. *Let* $\left(N, X, \mathcal{R}_{sp}^n\right)$ *be any class of* rationing problems with single peaked preferences. *The PR on* $\mathcal{R}_{sp}^n$ *does not satisfy* Condition μ^*.

PROOF. Since PR is unanimous, we can set $X \equiv Y^{PR}$. Take a profile $R \in \mathcal{R}_{sp}^n$ such that $p(R_\ell) = 0$ for any agent $\ell \in N \setminus \{i\}$ and $0 < p(R_i) < M$ and M is the uniquely least preferable over $[0, M]$ for agent i at R_i. Then, by definition of PR, $\{x\} = PR(R)$ where $x_i = M$ and $x_\ell = 0$ for any agent $\ell \in N \setminus \{i\}$. Moreover, $L(R_i, x) = \{x\}$ by $\{x\} = \min_{R_i} X$.

Let $R^* \in \mathcal{R}_{sp}^n$ be such that $p(R_i^*) = 0$ and there exists $y \in X$ with $y \in L(R_i, x) \subseteq L(R_i^*, y)$ and $y \in \max_{R_\ell^*} X$ for any agent $\ell \in N \setminus \{i\}$. Since $L(R_i, x) = \{x\}$, $y = x$. Thus, $\bar{V}_i(x, R, R^*) = \varnothing$,

[13] When its bounds are not explicitly indicated, a summation should be understood to cover all agents.

[14] For a study of consistent solutions to the problem of fair division when preferences are single-peaked, see Thomson (1994).

since $y \in \max_{R^*_\ell} X$ for $\ell \in N\backslash\{i\}$. However, since $p(R^*_\ell) = 0 = y_\ell$ holds for any $\ell \in N\backslash\{i\}$ by $y = x$, $\{z\} = PR(R^*)$ holds, where $z_\ell = \frac{M}{n}$ for all $\ell \in N$. Thus, $y \notin PR(R^*)$, and so $y \in \bar{O}_i(x, R, R^*)$ and $y \notin \bar{Q}_i(x, R, R^*)$. Then, $\{x\} = L(R_i, x)$, $y = x$, and $y \notin \bar{Q}_i(x, R, R^*)$ imply that $\bar{Q}_i(x, R, R^*) \cup \bar{V}_i(x, R, R^*) = \varnothing$. The statement follows from Lemma 2. ■

Proposition 4. *Let* $(N, X, \mathcal{R}^n_{sp})$ *be any class of* rationing problems with single peaked preferences. *The ED on* $\mathcal{R}^n_{sp}$ *satisfies* Condition μ^*.

Proof. Since ED is a unanimous SCC, let $Y^{ED} \equiv X$. Since ED is single-valued and Pareto optimal, then $\bar{O}_i(x, R, R) = \varnothing$ for all $i \in N$. Then, following the algorithm of Lemma 4, for any $R \in \mathcal{R}^n_{sp}$ and any $x \in ED(R)$, $C_i(R, x) \equiv L(R_i, x)$ for each $i \in N$. Moreover, by case (1.a.ii) of the algorithm, $S_i(R; x, R) = \{x\}$ for all $i \in N$. Finally, to apply Lemma 4, we shall only confirm that $S_i(R^*; x, R)$ is non-empty for case (1.b). Take any R, $R^* \in \mathcal{R}^n_{sp}$, with $x \in ED(R)$, and suppose that $y \in L(R_i, x) \subseteq L(R^*_i, y)$ and $y \in \max_{R^*_\ell} X$ for any agent $\ell \in N\backslash\{i\}$.

If $x_i = M$, then $p(R_i) = M$. For otherwise, there should be $d > 0$ such that $p(R_i) + d = M$, which also implies $x_\ell = p(R_\ell) + d$ for any other $\ell \neq i$, that is a contradiction. Moreover, if $p(R_i) = M$, then $p(R_\ell) = 0$ for any other $\ell \neq i$. Indeed, if $p(R_j) > 0$, then there should be $d > 0$ such that $x_j \geq p(R_j) - d$. Then, $x_i = p(R_i) - d < M$ holds, which is a contradiction. Thus, if $x_i = M$, then x is the unanimous allocation at R. Thus, since $L(R_i, x) = X$, it follows from $y \in L(R_i, x) \subseteq L(R^*_i, y)$ and $y \in \max_{R^*_\ell} X$ for any agent $\ell \in N\backslash\{i\}$ that y is the unanimous allocation at R^*. Hence, $y \in ED(R^*)$. Thus, if $x_i = M$, then $y \in Q_i(x, R, R^*) = S_i(R^*; x, R)$.

Let $x_i < M$. If $p(R_i) = x_i$, then $L(R_i, x) = X$, which implies that y is the unanimous allocation at R^* as the same reasoning as the case of $x_i = M$. Thus, let $p(R_i) \neq x_i$. Without loss of generality, let $x_i < r_i(x_i)$. Suppose that $p(R^*_i) \in [0, x_i] \cup [r_i(x_i), M]$. Then, $L(R_i, x) \subseteq L(R^*_i, y)$ implies that $y_i = p(R^*_i)$. Thus, by unanimity, $y \in ED(R^*)$, which implies $y \in Q_i(x, R, R^*) = S_i(R^*; x, R)$. Thus, let us consider the case that $p(R^*_i) \in (x_i, r_i(x_i))$. Then, $y_i \in \{x_i, r_i(x_i)\}$ must hold by $y \in L(R_i, x) \subseteq L(R^*_i, y)$. In this case, since $y_\ell = p(R^*_\ell)$ for any agent $\ell \in N\backslash\{i\}$ and $y_i \neq p(R^*_i)$, $y \notin ED(R^*)$ holds. Thus, $y \notin Q_i(x, R, R^*)$. Then, even if $Q_i(x, R, R^*) = \varnothing$, there always exists $z \in X$ with $z_i \in \{x_i, r_i(x_i)\}$ and there is at least one agent $j \neq i$ such that $z \notin \max_{R^*_j} X$. This implies $V_i(x, R, R^*) \neq \varnothing$. The same argument applies if $r_i(x_i)$ does not exist. The statement follows from lemma 4. ■

Proposition 5. *Let* $(N, X, \mathcal{R}^n_{sp})$ *be any class of* rationing problems with single peaked preferences; *suppose that* Assumption 1 *holds. The ES on* $\mathcal{R}^n_{sp}$ *satisfies* Condition μ^*.

Proof. Since the proof can be obtained as in the proof of Proposition 4, we shall omit it here. ■

Combining with Corollary 1, the above propositions can be summarized as follows.

Corollary 5. *Let* $(N, X, \mathcal{R}^n_{sp})$ *be any class of* rationing problems with single peaked preferences; *suppose that* Assumption 1 *holds. Then, ED and ES are* partially honest implementable, *while PR is* not.

Single plateaued preferences generalize single peaked preferences by allowing for *multiple* peak amounts. Formally, for each agent $i \in N$, the preference relation R_i defined on $[0, M]$ is called *single plateaued* when there exist two numbers $\bar{x}_i, \underline{x}_i \in [0, M]$ such that $\underline{x}_i \leq \bar{x}_i$ and for all $x_i, y_i \in [0, M]$: (i) if $x_i < y_i \leq \underline{x}_i$ or $x_i > y_i \geq \bar{x}_i$, then $(y', x') \in P_i$ for any $x', y' \in X$, with $x'_i = x_i$ and $y'_i = y_i$; (ii) if $x_i, y_i \in [\underline{x}_i, \bar{x}_i]$, then $(x', y') \in I_i$ for any $x', y' \in X$, with $x'_i = x_i$ and $y'_i = y_i$. The interval $p(R_i) \equiv [\underline{x}_i, \bar{x}_i]$ is the *plateau* of R_i, where $\underline{x}$ is the left end-point of the plateau of R_i, and $\bar{x}$ is the right end-point. For each agent i, $\mathcal{R}_{\overline{sp}_i}$ denotes the class of all preference relations on $[0, M]$ that satisfy continuity and single plateauedness. Let $\mathcal{R}^n_{\overline{sp}}$ be the class of admissible preference profiles.

With obvious adaptations, the notation spelled out above for single peaked preferences is carried over single plateaued preferences.

Single plateaued preferences have played an important role in areas such as voting, public good economies, and matching problems. Since Maskin's original result, it is known that the *Pareto Optimal SCC* is monotonic and satisfies no veto-power when the class of admissible preference profiles consists only of single peaked preferences. This conclusion does not extend, however, to single plateaued preferences since this *SCC* satisfies neither monotonicity nor Condition μ(ii) of Moore and Repullo (1990). In what follows, we show that the *Pareto Optimal SCC* (on $\mathcal{R}^n_{\overline{sp}}$) is not yet implementable even in the case that there are partially honest individuals.

Pareto SCC, PO: For all $R \in \mathcal{R}^n_{\overline{sp}}$, $PO(R) \equiv \{x \in X |$There is no $y \in X$: $(y,x) \in R_i$ for all $i \in N$ and $(y,x) \in P_i$ for some $i \in N\}$.

PROPOSITION 6. *Let* $\left(N, X, \mathcal{R}^n_{\overline{sp}}\right)$ *be any class of* rationing problems with single plateaued preferences*; suppose that* Assumption 1 *holds. The PO on* $\mathcal{R}^n_{\overline{sp}}$ *does not satisfy* Condition μ^**.*

PROOF. Since PO is unanimous, we can set $X = Y^{PO}$. In what follows, let us suppose that $n = 3$ and $M = 1$. Let $R \equiv (R_1, R_2, R_3) \in \mathcal{R}^n_{\overline{sp}}$ be such that $p(R) = \left(\frac{1}{4}, 1, [0,1]\right)$. Then, $\left(\frac{1}{6}, \frac{5}{6}, 0\right) \in PO(R)$. For the sake of brevity, let $x \equiv \left(\frac{1}{6}, \frac{5}{6}, 0\right)$. Let $R^* \in \mathcal{R}^n_{\overline{sp}}$ be such that $R^*_{-2} = R_{-2}$, and $R^*_2 \neq R_2$ with $p(R^*_2) = \left[0, \frac{5}{6}\right]$. Then, for each $j = 2,3$, $L\left(R^*_j, x\right) = X$; moreover, $x \in L(R_1, x) = L(R^*_1, x)$, where $L(R^*_1, x) = \left\{z \in X \mid 0 \leq z_1 \leq \frac{1}{6} \text{ or } r_1(x_1) \leq z_1 \leq 1\right\}$.[15] Then, $x \notin PO(R^*)$, since $y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ Pareto-dominates x at R^*. Thus, $x \notin Q_i(x, R, R^*)$. Moreover, take any $z \in X$ such that $L(R_1, x) \subseteq L(R^*_1, z)$ and $L\left(R^*_j, z\right) = X$ for each $j = 2,3$. This implies $z_1 \in \left[\frac{1}{6}, r_1(x_1)\right]$ by $R^*_1 = R_1$. However, by $p(R^*_2) = \left[0, \frac{5}{6}\right]$, $z_1 = \frac{1}{6}$ must hold. Hence, y Pareto-dominates z at R^*. Thus, $\bar{Q}_i(x, R, R^*) = \varnothing$.

Suppose $z \in \bar{V}_i(x, R, R^*)$. Then, $z_1 = \frac{1}{6}$ or $z_1 = r_1(x_1)$. Let $z_1 = \frac{1}{6}$. Then, $z_2 + z_3 = \frac{5}{6}$ must hold, which implies that $L\left(R^*_j, z\right) = X$ for each $j = 2,3$ from $p(R^*_2) = \left[0, \frac{5}{6}\right]$ and $p(R^*_3) = [0,1]$, a contradiction. Thus, let $z_1 = r_1(x_1)$. Since $r_1(x_1) > \frac{1}{4}$, we have $z_2 + z_3 < \frac{3}{4}$. This implies that $L\left(R^*_j, z\right) = X$ for each $j = 2,3$, a contradiction. Thus, $\bar{V}_i(x, R, R^*) = \varnothing$. The statement follows from Lemma 2.[16] ■

As a direct corollary of Proposition 6 and Corollary 1, we have the following result.

COROLLARY 6. *Let* $\left(N, X, \mathcal{R}^n_{\overline{sp}}\right)$ *be any class of* rationing problems with single plateaued preferences*; suppose that* Assumption 1 *holds. Then, the PO on* $\mathcal{R}^n_{\overline{sp}}$ *is not* partially honest implementable*.*

Note that neither rationing problems with single peaked nor with single plateaued preferences satisfy condition **RD**, because of the private preference property of those domains. Hence, partially honest implementability of non-unanimous *SCC*s in those problems cannot be examined by means of Condition μ^*.

3.3.3 On the impossibility of the strong core

A coalitional game contains a finite set of agents N with cardinality $n \geq 3$, a non-empty set of outcomes X, a preference profile $R \in \mathcal{R}^n$, and a characteristic function $v : 2^N \backslash \{\varnothing\} \to 2^X$. A

[15] Without loss of generality, we suppose that $r_1(x_1)$ exists.

[16] We also see that the *Pareto SCC* violates monotonicity and Condition μ(ii) of Moore and Repullo (1990).

coalition, denoted S, is a non-empty subset of the set N. Given a coalitional game (N, X, R, v), an outcome $x \in X$ is *weakly blocked* by S if there is $y \in v(S)$ such that $(y, x) \in R_i$ for each $i \in S$, and $(y, x) \in P_i$ for some $i \in S$. If there is an outcome which is not weakly blocked by any coalition S, then (N, X, R, v) is referred to as a *coalitional game with non-empty strong core.*

In what follows, we consider a situation in which the mechanism designer knows what is feasible for each coalition, that is, the characteristic function v, but she does not know agents' preferences. This situation is modeled by the quadruple $(N, X, \mathcal{R}^n, v)$, which we refer to as a *coalitional game environment.* Given a coalitional game environment $(N, X, \mathcal{R}^n, v)$, the *strong core correspondence* C^S is defined as the *SCC* on $\mathcal{R}^n$ with

$$C^S(R) \equiv \left\{x \in v(N) \,|\, x \text{ is not weakly blocked by any coalition } S \in 2^N \backslash \{\varnothing\}\right\}.$$

We say that $(N, X, \mathcal{R}^n, v)$ is a *coalitional game environment with non-empty strong core* if $C^S(R) \neq \varnothing$ for each $R \in \mathcal{R}^n$.

It is well-known that the strong core correspondence C^S violates monotonicity and no veto-power. This *SCC* is not, therefore, implementable. Moreover, Dutta and Sen (2012)'s result is silent with respect to the partially honest implementability of the strong core. A natural question, then, is whether or not the strong core is implementable when agents have intrinsic preferences towards honesty. A negative answer is provided in the following proposition.

Proposition 7. *Let* $(N, X, \mathcal{R}^n, v)$ *be any* coalitional environment with non-empty strong core. *The* $C^S \in \mathcal{F}$ *does not satisfy* Condition μ^*.

Proof. Let $n \geq 3$; suppose that Assumption 1 holds. Let us suppose that $N = \{1, 2, 3\}$ with cardinality $n = 3$, $X \equiv \{w, x, y, z\}$ with cardinality $|X| = 4$, and $\{R, R^*\} = \mathcal{R}^n$, where profiles R and R^* are as follows:

R			R^*		
1	2	3	1	2	3
y, z	x	w	y	w, x, y, z	w, x, y
x	w, y, z	x, z	x		z
w		y	w, z		

where, as usual, $\substack{x \\ y}$ means that the agent in question strictly prefers x to y, while x, y means that the agent at issue is indifferent between x and y. Let us define v as

$$v(\{1,2\}) = \{x, z\},\ v(\{1,3\}) = \{w, y\},\ v(\{2,3\}) = \{w, z\},\ v(N) = X,$$

and $v(S) = \varnothing$ for all other coalitions $S \in 2^N \backslash \{\varnothing\}$.

By definition, $(N, X, \mathcal{R}^n, v)$ is a coalitional game environment with non-empty strong core since $C^S(R) = \{x\}$ and $C^S(R^*) = \{y\}$.[17] Since C^S is unanimous, $X = Y^{C^S}$. Since by definition, $L(R_1, x) \subseteq L(R_1^*, x)$ and $X \subseteq L\left(R_j^*, x\right)$ for $j = 2, 3$, $C^S(R^*) = \{y\}$ implies that $x \notin \bar{Q}_1(x, R, R^*)$. Since $y \notin L(R_1, x)$, $y \notin \bar{Q}_1(x, R, R^*)$. Thus, $\bar{Q}_1(x, R, R^*) = \varnothing$. Moreover, since there is no outcome which is indifferent to x at R_1, $\bar{V}_1(x, R, R^*) = \varnothing$. The statement follows from Lemma 2. ■

The following result is a direct consequence of Proposition 7 and Corollary 1.

Corollary 7. *Let* $(N, X, \mathcal{R}^n, v)$ *be any* coalitional environment with non-empty strong core*; suppose that* Assumption 1 *holds. The* C^S *is not* partially honest implementable.

[17] An easy compuation yields that $C^S(R) = \{x\}$ since the outcome w is weakly blocked by $\{1, 2\}$, and y and z are weakly blocked by $\{2, 3\}$; on the other hand, $C^S(R') = \{y\}$ since the outcomes w, x and z are weakly blocked by $\{1, 3\}$.

3.3.4 On the possibility of the Nash bargaining solution

Let us examine the implementability of bargaining solutions: Suppose that there is a perfectly divisible cake of size 1, to be shared among $n \geq 3$ agents. The set of possible feasible allocations is $A \equiv \left\{(a_i)_{i\in N} \in \mathbb{R}_+^n \mid \sum_{i\in N} a_i \leq 1\right\}$. Moreover, we allow the lottery over A, so that X is the set of all probability measures on the Borel sigma algebra of A with generic element x, y, and z. For each $i \in N$, let R_i be a von Neumann-Morgenstern preference over X such that there is a corresponding von Neumann-Morgenstern (henceforth, vNM) utility function $u_i : [0,1] \to \mathbb{R}_+$ which is continuous and monotonic, and represents R_i in the sense that for any $x, y \in X$, xR_iy if and only if $U_i(x) \geq U_i(y)$, where $U_i(x) \equiv \int_A u_i(a_i)\mathrm{d}x(a)$. Without loss of generality, let $u_i(0) = 0$ for each $i \in N$. Let $\mathcal{U}$ be the set of all continuous and monotonic vNM utility functions having $u_i(0) = 0$ and $u_i(a) > 0$ for some $a > 0$. Given this, let us take the disagreement point as $\mathbf{d} = (d_i)_{i\in N} \equiv \mathbf{0} = \underbrace{(0,\ldots,0)}_{n \text{ times}}$. Then, a bargaining problem is given by a pair $(S, \mathbf{d})$ corresponding to a profile $(N, X, \mathbf{u}, \mathbf{d})$ where $\mathbf{u} = (u_i)_{i\in N} \in \mathcal{U}^n$ and $S = \left\{\mathbf{s} = (s_i)_{i\in N} \in \mathbb{R}_+^n \mid \exists x \in X : s_i = \int_A u_i(a_i)\,\mathrm{d}x(a) \quad (\forall i \in N)\right\}$. Thus, let $(N, X, \mathcal{U}^n, \mathbf{d})$ be the available class of bargaining problems. A *bargaining correspondence* is defined as a *SCC* $F : \mathcal{U}^n \twoheadrightarrow X$ such that for any $\mathbf{u} \in \mathcal{U}^n$, $F(\mathbf{u}) \neq \varnothing$ and the following property holds: (1) *essentially single-valuedness*: for any $x, y \in F(\mathbf{u})$, $U_i(x) = U_i(y)$ holds for all $i \in N$; and (2) *fullness*: for any $x \in F(\mathbf{u})$ and any $y \in X$, if $U_i(x) = U_i(y)$ holds for all $i \in N$, then $y \in F(\mathbf{u})$. The *Nash bargaining correspondence* is a bargaining correspondence F^N such that for any $\mathbf{u} \in \mathcal{U}^n$, $F^N(\mathbf{u}) = \arg\max_{y\in X} \prod_{i\in N} U_i(y)$.

As Vartiainen (2007) shows, in such cake sharing contexts the Nash bargaining correspondence F^N violates monotonicity. Hence, this *SCC* is not Nash implementable. Moreover, as shown below, it also violates no veto-power, so that Dutta and Sen (2012)'s result is silent with respect to partially honest implementability of F^N. However, F^N is implementable when agents have intrinsic preferences towards honesty, as provided in the following.

Proposition 8. *Let* $(N, X, \mathcal{U}^n, \mathbf{d})$ *be any* class of bargaining problems. *Then,* F^N *does not satisfy* no veto-power*, but satisfies* Condition μ^*.

Proof. First, let us show that F^N violates no veto-power. Let $n = 3$, and consider a bargaining problem $(S, \mathbf{d})$ such that $S \equiv con\{(0.5, 0, 0), (0.5, 0.5, 0), (0, 0.5, 0), (0, 0, 1), \mathbf{0}\}$, where $conX$ means the *convex hull* of the set X. Since S is convex, comprehensive, and compact, there is a suitable $\mathbf{u} = (u_i)_{i\in N} \in \mathcal{U}^n$ from which S is generated as the corresponding utility possibility set, as is well-known in the literature. Then, there is $x \in X$ such that $(U_i(x))_{i\in N} = (0.5, 0.5, 0)$. This x is thus the best outcome for agents 1 and 2, thus no veto-power requests that x is optimal at $\mathbf{u}$. However, $x \notin F^N(\mathbf{u})$, thus F^N violates no veto-power.

Second, let us show that F^N satisfies Condition μ^* by applying the algorithm. Let $Y^{F^N} \equiv X$, since F^N is unanimous. By the fullness and the essential single-valuedness of bargaining correspondence, $\bar{O}_i(x, \mathbf{u}, \mathbf{u}) = \varnothing$ for all $i \in N$. Thus, by the algorithm constructed for proving Lemma 4, for each $\mathbf{u} \in \mathcal{U}^n$ and each $x \in F^N(\mathbf{u})$, $C_i(\mathbf{u}, x) \equiv L(u_i, x) \cap X$ for each $i \in N$. Also, by case (1.a.ii) of the algorithm, $S_i(\mathbf{u}; x, \mathbf{u}) = \{x\} \cup Q_i(x, \mathbf{u}, \mathbf{u})$ for all $i \in N$. Finally, to apply Lemma 4, we shall only confirm that $S_i(R^*; x, R)$ is non-empty for case (1.b).

Take any $\mathbf{u}^* \in \mathcal{U}^n \backslash \{\mathbf{u}\}$. Suppose that there is $y \in X$ such that $y \in C_i(\mathbf{u}, x) \subseteq L(u_i^*, y)$ and $y \in \max_{u_\ell^*} Y^{F^N}$ for all $\ell \in N \backslash \{i\}$. If $y \in F^N(\mathbf{u}^*)$, then $y \in Q_i(x, \mathbf{u}, \mathbf{u}^*) = S_i(\mathbf{u}^*; x, \mathbf{u})$ by the algorithm. Let $y \notin F^N(\mathbf{u}^*)$. Then, for any $y' \in X$ with $y' \in C_i(\mathbf{u}, x) \subseteq L(u_i^*, y')$ and $y' \in \max_{u_\ell^*} Y^{F^N}$ for all $\ell \in N \backslash \{i\}$, $y' \notin F^N(\mathbf{u}^*)$ holds by the fullness property of bargaining correspondences. Hence, according to the algorithm, we have to check $V_i(x, \mathbf{u}, \mathbf{u}^*) \neq \varnothing$. Let $w \in X$ be such that the probability distribution over agent i's sharings is identical to those of y, while j receives 0 with probability 1. Note that such a probability distribution is always available over A.

Thus, $\int_A u_i^* (a_i) \mathrm{d}w(a) = \int_A u_i^* (a_i) \mathrm{d}y(a)$ and $w \notin \max_{R_j^*} Y^F$. This implies $V_i(x, \mathbf{u}, \mathbf{u}^*) \neq \varnothing$. The statement follows from lemma 4. ■

As a direct corollary of Proposition 8 and Corollary 1, we have the following important result.

Corollary 8. *Let* $(N, X, \mathcal{U}^n, \mathbf{d})$ *be any* class of bargaining problems*; suppose that* Assumption 1 *holds. The* $F^N \in \mathcal{F}$ *is* partially honest implementable.

Note that the above proof of Proposition 8 suggests that *any strictly individual rational and unanimous bargaining correspondence* defined on $(N, X, \mathcal{U}^n, \mathbf{d})$ is partially honest implementable. Thus, the *Kalai-Smorodinsky correspondence* is also partially honest implementable. In contrast, partially honest implementability of non-unanimous *SCC*s, such as the *egalitarian bargaining correspondence*, in bargaining problems considered here, cannot be examined by means of Condition μ^*, since the class of such problems does not satisfy condition **RD**.

Finally, in this cake sharing model, we can also define an extended class of *SCC*s by admitting *non*-bargaining *SCC*s. For instance, let $\overline{F}^N$ be an essentially single-valued but *non-full* conrrespondence such that for any $\mathbf{u} \in \mathcal{U}^n$, $\varnothing \neq \overline{F}^N(\mathbf{u}) \subseteq F^N(\mathbf{u})$ and there is a unique agent $i \in N$ such that for some $\mathbf{u}^* \in \mathcal{U}^n$, there is $x^i \in F^N(\mathbf{u}^*) \backslash \overline{F}^N(\mathbf{u}^*)$ in which all agents $\ell \in N$ *except* i enjoys $x^i \in \max_{u_\ell^*} X$. We may find a justification of this $\overline{F}^N$ to exclude this x^i from optimal outcomes, based on a *non-welfaristic viewpoint*. It can be shown by means of Lemma 3 that for $x \in \overline{F}^N(\mathbf{u}^*)$, $x \notin S_\ell(\mathbf{u}^*; x, \mathbf{u}^*)$ holds for all $\ell \in N \backslash \{i\}$. Thus, $\overline{F}^N$ is partially honest implementable, but not by forthright mechanisms.

4 Concluding remarks

While this paper sets solid foundations for implementation with partially honest agents, it falls short in many important aspects. For example, while this paper specified the set of properties that an *SCC* should satisfy in order to be partially honest implementable, the devised mechanisms present the disadvantage of involving complex strategy spaces. In particular, strategies include either whole preference profiles or whole indifference sets for several agents. This implies that the message space is of infinite dimension in many economic applications. Furthermore, the components of the strategy space do not have a straightforward economic interpretation such as consumption bundles, allocations, and prices. Therefore, there is a need to specify the scope of the analysis reported herein away from abstract social choice environments. In this regard, the exploration of the rich set of implications that arise from the injection of a minimal honesty to economic agents involved in a mechanism can take many directions. One interesting direction is explored in a recent work of Lombardi and Yoshihara (2013a) in which implementation of efficient *SCC*s by natural mechanisms is analyzed in classical exchange economies.

5 Appendix

5.1 Proof of Theorem

Let Assumption 1 hold; let $n \geq 3$; let $F \in \mathcal{F}$; and, let us suppose that $\mathcal{R}^n$ satisfies **RD**.

Let us suppose that F is partially honestly implemented by a mechanism $\gamma \equiv (M, g)$. We show that F satisfied Condition μ^*.

First, let us define Y^F as $Y^F \equiv \{y \in X | y = g(m) \text{ for some } m \in M\}$. Take any $R \in \mathcal{R}^n$ and any $x \in F(R)$, so that $x \in NA\left(\gamma, \succcurlyeq^{R,H'}\right)$ for each $H' \in \mathcal{H}$. Then, there is a strategy $m^N \in NE\left(\gamma, \succcurlyeq^{R,N}\right)$ such that $g\left(m^N\right) = x$, given that $N \in \mathcal{H}$. Then, $\{x\} \subseteq g\left(M_\ell, m_{-\ell}^N\right) \subseteq$

$L(R_\ell, x) \cap Y^F$ for each $\ell \in N$. For each $\ell \in N$, let us define $C_\ell(R,x)$ as $C_\ell(R,x) \equiv g\left(M_\ell, m^N_{-\ell}\right)$; therefore, $x \in C_\ell(R,x) \subseteq L(R_\ell, x) \cap Y^F$ for each $\ell \in N$, as required.

Fix an arbitrary $R^* \in \mathcal{R}^n$, and an arbitrary $H \in \mathcal{H}$. Pick any $i \in N$. Let us define the set $S_i(R^*; x, R)$ as

$$S_i(R^*; x, R) \equiv \left\{g\left(m_i, m^N_{-i}\right) \in C_i(R,x) \,|\, g\left(m^N\right) = x \in F(R),\, m_i \in T_i^\gamma(R^*, F)\right\}. \tag{5}$$

Obviously, $S_i(R^*; x, R)$ is a non-empty set. Moreover, let us suppose that $R = R^*$ and $x \notin S_i(R; x, R)$. Then, by (5), $m^N_i \notin T_i^\gamma(R, F)$; moreover, since $m^N \in NE\left(\gamma, \succcurlyeq^{R,N}\right)$, $\left(x, g\left(m_i, m^N_{-i}\right)\right) \in P_i$ for each $m_i \in T_i^\gamma(R, F)$.[18] Since i is arbitrary, this verifies Condition μ^*(i).

In what follows, we first show that F verifies Condition μ^*(ii), and then show that it satisfies Condition μ^*(iii) too.

To show that F satisfies Condition μ^*(ii), let us suppose that $y \in C_i(R,x) \subsetneq L(R^*_i, y)$ and $y \in \max_{R^*_\ell} Y^F$ for each $\ell \in N \backslash \{i\}$. Then, there exists $m_i \in M_i$ such that $g\left(m_i, m^N_{-i}\right) = y$.

To verify Condition μ^*(ii.a), let us assume that $H = \{i\}$. Suppose that $y \in S_i(R^*; x, R)$. Then, by (5), there is $m'_i \in T_i^\gamma(R^*, F)$ such that $g\left(m'_i, m^N_{-i}\right) = y$. It follows that $\left(m'_i, m^N_{-i}\right) \in NE\left(\gamma, \succcurlyeq^{R^*,\{i\}}\right)$; hence, $y \in F(R^*)$. Next, let us assume that, $y \notin S_i(R^*; x, R)$ and $S_i(R^*; x, R) \subseteq L(P^*_i, y)$. Then, by (5), for each $m'_i \in T_i^\gamma(R^*, F)$, $g\left(m'_i, m^N_{-i}\right) \neq y = g\left(m_i, m^N_{-i}\right)$. Assume, to the contrary, that $y \notin F(R^*)$. By our suppositions, it follows that there exists $m'_i \in T_i^\gamma(R^*, F)$ such that $\left(g\left(m'_i, m^N_{-i}\right), g\left(m_i, m^N_{-i}\right)\right) \in I^*_i$; otherwise, we fall into a contradiction. Then, by (5), $g\left(m'_i, m^N_{-i}\right) \in S_i(R^*; x, R)$, which is a contradiction; hence, F satisfies Condition μ^*(ii.a.2).

To verify Condition μ^*(ii.b), let us assume that $i \notin H$, $H = \{h\}$, $R^* = R$, and $x \in S_h(R; x, R)$. Then, since $m^N \in NE\left(\gamma, \succcurlyeq^{R,N}\right)$, $m^N_h \in T_h^\gamma(R, F)$ holds. Thus, since $\left(m_i, m^N_{-i}\right) \in NE\left(\gamma, \succcurlyeq^{R,\{h\}}\right)$, the statement follows. Hence, F satisfies Condition μ^*(ii.b).

To show that F satisfies Condition μ^*(iii), let us suppose that $y \in \max_{R^*_\ell} Y^F$ for each $\ell \in N$, and $y \notin F(R^*)$. It follows that the only agents who find a profitable deviation are those in the set H. Select any of such agents h. Moreover, by **RD**, let $R^{*p}_h \in \mathcal{R}_h$ be such that $\{y\} = \max_{R^{*p}_h} Y^F$; and, finally, let $I^*_h\left(y, Y^F\right)$ denotes the set $\{y' \in Y^F | \, (y', y) \in I^*_h\}$. Since $y \notin F(R^*)$, it follows that for any $m \in g^{-1}(y)$, $m \notin NE\left(\gamma, \succcurlyeq^{R^*,\{h\}}\right)$. Furthermore, for any $m \in g^{-1}(y)$, $m_h \notin T_h^\gamma(R^*, F)$, and there exists $m'_h \in T_h^\gamma(R^*, F)$ such that $(m'_h, m_{-h}) \notin g^{-1}(y)$, with $(g(m'_h, m_{-h}), g(m)) \in I^*_h$; otherwise, we fall into a contradiction. It cannot be that $R^*_h = R^{*p}_h$; otherwise, given that $(g(m'_h, m_{-h}), g(m)) \in I^*_h$, $(m'_h, m_{-h}) \in g^{-1}(y)$, which gives us a contradiction. Therefore, $R^*_h \neq R^{*p}_h$, as sought. Moreover, by **RD**, $\left(R^{*p}_h, R^*_{-h}\right) \in \mathcal{R}^n$. For the sake of brevity, let $\left(R^{*p}_h, R^*_{-h}\right) \equiv R^{*p}$. Since F is partially honestly implemented by γ, it can be shown that $y \in NA\left(\gamma, \succcurlyeq^{R^{*p},\{h\}}\right) = F(R^{*p})$. Then, there is a strategy $m^{*N} \in NE\left(\gamma, \succcurlyeq^{R^{*p},N}\right)$ such that $g\left(m^{*N}\right) = y$, given that $N \in \mathcal{H}$. Let us define the set $S_h(R^*; y, R^{*p})$ as

$$S_h(R^*; y, R^{*p}) = \left\{g\left(m_h, m^{*N}_{-h}\right) \in C_h(R^{*p}, y) \,|\, g\left(m^{*N}\right) = y \in F(R^{*p}),\, m_h \in T_h^\gamma(R^*, F)\right\}.$$

Since for any $m \in g^{-1}(y)$, $m \notin NE\left(\gamma, \succcurlyeq^{R^*,\{h\}}\right)$, it follows that $\left(m_h, m^{*N}_{-h}\right) \notin g^{-1}(y)$ for each $m_h \in T_h^\gamma(R^*, F)$, and so $y \notin S_h(R^*; y, R^{*p})$, as sought. Since $m^{*N} \notin NE\left(\gamma, \succcurlyeq^{R^*,\{h\}}\right)$, $m^{*N}_h \notin T_h^\gamma(R^*, F)$; otherwise, a contradiction can be derived. Moreover, there exists $m_h \in T_h^\gamma(R^*, F)$ such that $\left(m_h, m^{*N}\right) \notin g^{-1}(y)$, with $\left(g\left(m_h, m^{*N}_{-h}\right), g\left(m^{*N}\right)\right) \in I^*_h$; otherwise, we fall into a contradiction.

[18] It can be shown that $x \in S_i(R; x, R)$ for each $R \in \mathcal{R}^n$ and each $x \in F(R)$ if it is assumed that F is partially honestly implemented by a forthright mechanism.

Then, there exists an outcome $g\left(m_h, m^{*N}_{-h}\right) \in Y^F \backslash \{y\}$ such that $g\left(m_h, m^{*N}_{-h}\right) \in S_h\left(R^*; y, R^{*p}\right) \cap I^*_h\left(y, Y^F\right)$, as sought. We conclude that F satisfies Condition μ^*(iii).

Next, we prove sufficiency. Suppose that F satisfies Condition μ^*. Let $\gamma \equiv (g, M)$ be a mechanism whereby for each $i \in N$, the message space is $M_i \equiv (\mathcal{R}^n \cup \mathcal{S}) \times Y^F \times N$, under the specification that $Y^F \subseteq X$, $\mathcal{S} \cap \mathcal{R}^n = \varnothing$, and there exists a bijection $\phi : \mathcal{R}^n \rightarrow \mathcal{S}$.[19] Thus, each agent i announces a preference profile, R^i, or an element of $\mathcal{S}$, $\phi\left(R^i\right)$. Moreover, she announces an outcome, x^i, and an agent index, k^i. For each $i \in N$, and each $R \in \mathcal{R}^n$, the set of truth-telling messages is $T^\gamma_i(R, F) = \{R\} \times Y^F \times N$. Before defining the outcome function g of γ, a little more notation is needed.

Take any $R \in \mathcal{R}^n$, and any $x \in F(R)$. Let us define a profile of strategy sets $(\sigma_\ell(R, x))_{\ell \in N}$ corresponding to $x \in F(R)$ as follows:

Property I**:** if $x \in S_k(R; x, R)$ for some $k \in N$, and $x \notin S_j(R; x, R)$ for each $j \in N \backslash \{k\}$, then $\sigma_k(R, x) \equiv \{R\} \times \{x\} \times N$, and $\sigma_j(R, x) \equiv \{\phi(R)\} \times \{x\} \times N$ for each $j \in N \backslash \{k\}$, where $\phi(R) \in \mathcal{S}$;

Property II**:** otherwise, $\sigma_j(R, x) \equiv \{R\} \times \{x\} \times N$ for each $j \in N$.

For the sake of notation, let

$$\sigma_{j1}(R, x) = \begin{cases} \phi(R) & \text{if } (\sigma_j(R, x))_{j \in N} \text{ corresponds to Property I and } x \notin S_j(R; x, R); \\ R & \text{otherwise.} \end{cases}$$

For any message profile $m \in M$:

Rule 1: If for some $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y^F$, with $x \in F\left(\bar{R}\right)$, $m_\ell \in \sigma_\ell\left(\bar{R}, x\right)$ for all $\ell \in N$, then $g(m) = x$;

Rule 2: If for some $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y^F$, with $x \in F\left(\bar{R}\right)$, there exists a unique agent $i \in N$ such that $m_\ell \in \sigma_\ell\left(\bar{R}, x\right)$ for all $\ell \in N \backslash \{i\}$, $m_i \notin \sigma_i\left(\bar{R}, x\right)$, then:

Rule 2.1: if $R^i = \bar{R} = \sigma_{i1}\left(\bar{R}, x\right)$, or $\phi\left(R^i\right) = \phi\left(\bar{R}\right)$, then $g(m) = x$;

Rule 2.2: if $R^i \neq \bar{R}$ or $\phi\left(R^i\right) \neq \phi\left(\bar{R}\right)$, then:

$$g(m) = \begin{cases} x^i & \text{if } x^i \in S_i\left(R^i; x, \bar{R}\right); \\ x^i & \text{if } x^i \in C_i\left(\bar{R}, x\right) \backslash S_i\left(R^i; x, \bar{R}\right),\ S_i\left(R^i; x, \bar{R}\right) \subseteq L\left(P^i_i, x^i\right); \\ y & \text{if } x^i \in C_i\left(\bar{R}, x\right) \backslash S_i\left(R^i; x, \bar{R}\right),\ y \in S_i\left(R^i; x, \bar{R}\right) \cap I^i_i\left(x^i\right); \\ z \in S_i\left(R^i; x, \bar{R}\right) & \text{otherwise;} \end{cases}$$

where $I^i_i\left(x^i\right) = \left\{y \in C_i\left(\bar{R}, x\right) \mid \left(x^i, y\right) \in I^i_i\right\}$;

[19] The reported indices in a mechanism are used to rule out undesired equilibrium outcomes as equilibria of the mechanism. This type of device, common in the constructive proofs of the literature, is, however, subject to criticism on several fronts. For a systematic criticism of the use of "modulo games" and "integer games" in the literature, see Jackson (1992). Note that these critics do not defeat the point of our theorem, which is to draw a demarcation line between which *SCC*s are or are not partially honest implementable. In addition, if we do not have access to "modulo games" nor "integer games", the range of *SCC*s that can be implemented is severely limited (again, see Jackson, 1992).

Rule 2.3: if $R^i = \bar{R} \neq \sigma_{i1}\left(\bar{R}, x\right)$, then:

$$g(m) = \begin{cases} x^i & \text{if } x^i \in S_i\left(R^i; x, \bar{R}\right); \\ z \in S_i\left(R^i; x, \bar{R}\right) & \text{otherwise;} \end{cases}$$

Rule 3: Otherwise, $g(m) = \tilde{x}^{\ell^*(m)}$ where $\ell^*(m) = \sum_{i \in N} k^i \pmod{n}$,[20] and

$$\tilde{x}^{\ell^*(m)} = \begin{cases} & \text{if } x^{\ell^*(m)} \in \max_{\hat{R}_\ell} Y^F \text{ for all } \ell \in N, \text{ where } \hat{R} = R^{\ell^*(m)} \in \mathcal{R}^n,\ x^{\ell^*(m)} \notin F\left(\hat{R}\right), \\ \hat{x} & \text{and } \hat{x} \in S_{\ell^*(m)}\left(\hat{R}; x^{\ell^*(m)}, \left(\hat{R}^p_{\ell^*(m)}, \hat{R}_{-\ell^*(m)}\right)\right) \cap \hat{I}_{\ell^*(m)}\left(x^{\ell^*(m)}, Y^F\right); \\ x^{\ell^*(m)} & \text{otherwise.} \end{cases}$$

where $Y^F \subseteq L\left(\hat{R}^p_{\ell^*(m)}, x^{\ell^*(m)}\right) = L\left(\hat{R}_{\ell^*(m)}, x^{\ell^*(m)}\right)$, and $\max_{\hat{R}^p_{\ell^*(m)}} Y^F = \left\{x^{\ell^*(m)}\right\}$.

Observe that *Rule 3* is well-defined because F satisfies Condition μ^*(iii), $\mathcal{R}^n$ satisfies **RD**, and $\{\ell\} \in \mathcal{H}$ for all $\ell \in N$.

Take any $R \in \mathcal{R}^n$. We show that $F(R) = NA\left(\gamma, \succcurlyeq^{R,H'}\right)$ for all $H' \in \mathcal{H}$.

To show that $F(R) \subseteq NA\left(\gamma, \succcurlyeq^{R,H'}\right)$ for all $H' \in \mathcal{H}$, take any $H \in \mathcal{H}$. Let us suppose that $x \in F(R)$. We proceed according to whether Property I is applied or not.

Case A: Property II is applied.

Then, let $m_\ell = \left(R, x, k^\ell\right) \in T^\gamma_\ell(R, F)$ for each $\ell \in N$. By the definition of g, the message profile m falls into *Rule 1*, where $\sigma_\ell(R, x) \subseteq T^\gamma_\ell(R, F)$ for each $\ell \in N$; therefore, $g(m) = x$. We can show that any deviation of agent $\ell \in N$ will get her to an outcome in $C_\ell(R, x)$ by *Rule 2.1* and *Rule 2.2*,[21] that is, $g(M_\ell, m_{-\ell}) \subseteq C_\ell(R, x)$; then, $g(M_\ell, m_{-\ell}) \subseteq L(R_\ell, x)$, by Condition μ^*. Obviously, such deviations are not profitable for any $\ell \in N$. We conclude that $m \in NE\left(\gamma, \succcurlyeq^{R,H}\right)$, and so $x \in NA\left(\gamma, \succcurlyeq^{R,H}\right)$.

Case B: Property I is applied.

Then, let $m_k = \left(R, x, k^k\right) \in \sigma_k(R, x) \subseteq T^\gamma_k(R, F)$, and $m_j = \left(\phi(R), x, k^j\right) \in \sigma_j(R, x)$, for each $j \in N \backslash \{k\}$. The corresponding message profile m falls into *Rule 1*, and so $g(m) = x$. By definition of g, we can show that any deviation of agent $i \in N$ will get her to an outcome in $C_i(R, x)$, that is, $g(M_i, m_{-i}) \subseteq C_i(R, x)$; then, $g(M_i, m_{-i}) \subseteq L(R_i, x)$, by Condition μ^*. Obviously, such deviations are not profitable for any $i \in N \backslash H$. To see that such deviations are also not profitable for any $h \in H \backslash \{k\}$ whenever h deviates to a truthful message, take any $h \in H \backslash \{k\}$. Take any deviation of agent h from m_h to $m'_h \in T^\gamma_h(R, F)$. This deviation will get her to an outcome in $S_h(R; x, R)$ via *Rule 2.3*. Since $x \notin S_h(R; x, R)$ and Condition μ^*(i) holds, such a deviation is not profitable. Since m'_h is arbitrary, agent h cannot find any profitable deviation. We conclude that $m \in NE\left(\gamma, \succcurlyeq^{R,H}\right)$, and so $x \in NA\left(\gamma, \succcurlyeq^{R,H}\right)$.

[20] If the remainder is *zero*, the winner of the game is agent n. See Saijo (1988).

[21] *Rule 2.3* can never be induced in this case.

Since H is arbitrary, we conclude that $x \in NA\left(\gamma, \succcurlyeq^{R,H'}\right)$ for each $H' \in \mathcal{H}$.

To show that $NA\left(\gamma, \succcurlyeq^{R,H'}\right) \subseteq F(R)$ for all $H' \in \mathcal{H}$, fix an arbitrary $H \in \mathcal{H}$; let $m \in NE\left(\gamma, \succcurlyeq^{R,H}\right)$ and let us consider the following cases.

Case 1: m corresponds to *Rule 1.*

Then, $g(m) = x$. Assume, to the contrary, that $x \notin F(R)$. We proceed according to whether Property I is applied or not.

Sub-case 1.1: Property II is applied.

Since each partially honest participant can obtain a profitable deviation via *Rule 2.2* whenever $R \neq \bar{R}$, it follows that $R = \bar{R}$, and so $x \in F(R)$, which is a contradiction.[22]

Sub-case 1.2: Property I is applied.

Then, $m_k = \left(\bar{R}, x, k^k\right) \in \sigma_k\left(\bar{R}, x\right)$, and $m_j = \left(\phi\left(\bar{R}\right), x, k^j\right) \in \sigma_j\left(\bar{R}, x\right)$ for each $j \in N\backslash\{k\}$. An immediate contradiction is derived if $\bar{R} = R$. Let us suppose therefore, that $\bar{R} \neq R$. We show that this case is not admissible. First, $k \notin H$; otherwise, she can obtain a profitable deviation via *Rule 2.2* by changing m_k into $m'_k = \left(R, x, k^k\right) \in T^{\gamma}_k(R, F)$, which gives us a contradiction. It follows that $H \subseteq N\backslash\{k\}$. Take an arbitrary $j \in H$; by changing $m_j = \left(\phi\left(\bar{R}\right), x, k^j\right) \notin T^{\gamma}_j(R, F)$ into $m'_j = \left(R, x, k^j\right) \in T^{\gamma}_j(R, F)$, agent j obtains a profitable deviation via *Rule 2.2*, which gives us a contradiction. Since $j \in H$ is an arbitrary agent, and $H \subseteq N\backslash\{k\}$, it follows that H is an empty set, which is a contradiction.

Case 2: m corresponds to *Rule 2.*

Then, $g(m) \in C_i\left(\bar{R}, x\right)$. Since m cannot correspond to *Rule 2.3*, m falls either into *Rule 2.1* or *Rule 2.2.*

Case 2.1: m corresponds to *Rule 2.1.*

Then, $g(m) = x$. Assume, to the contrary, that $x \notin F(R)$. We proceed according to whether Property I is applied or not.

Case 2.1.a: Property II is applied.

Then, $\bar{R} = \sigma_{i1}\left(\bar{R}, x\right)$. Let us suppose that $R \neq \bar{R}$. Then, agent $\ell \in H\backslash\{i\}$ (resp., $i \in H$) can induce *Rule 3* (resp., *Rule 2.2*) by changing $m_\ell \notin T^{\gamma}_\ell(R, F)$ (resp., $m_i \notin T^{\gamma}_i(R, F)$) into $m'_\ell = \left(R, x, k^\ell\right) \in T^{\gamma}_\ell(R, F)$ (resp., $m'_i = \left(R, x, k^i\right) \in T^{\gamma}_i(R, F)$) - with the caution that agent ℓ chooses k^ℓ so as to win the modulo game. In this way, agent ℓ (resp., i) obtains an outcome $g\left(m'_\ell, m_{-\ell}\right)$ (resp., $g\left(m'_i, m_{-i}\right)$) such that $\left(g\left(m'_\ell, m_{-\ell}\right), x\right) \in I_\ell$ (resp., $\left(g\left(m'_i, m_{-i}\right), x\right) \in I_i$), which gives us a contradiction. We conclude that $R = \bar{R}$, and so $x \in F(R)$, which is a contradiction.

Case 2.1.b: Property I is applied.

Then, $\sigma_{1k}\left(\bar{R}, x\right) = \bar{R}$, and $\sigma_{1j}\left(\bar{R}, x\right) = \phi\left(\bar{R}\right)$ for each $j \in N\backslash\{k\}$. First, notice that for the unique agent i identified by *Rule 2*, either $\sigma_{1i}\left(\bar{R}, x\right) = \bar{R}$ or $\sigma_{1i}\left(\bar{R}, x\right) = \phi\left(\bar{R}\right)$. Next, let us suppose that $R \neq \bar{R}$. Let us also suppose that $k \in H$. Then, by changing $m_k \notin T^{\gamma}_k(R, F)$ into $m'_k = \left(R, x, k^k\right) \in T^{\gamma}_k(R, F)$, agent $k \neq i$ (resp., $k = i$) can induce *Rule 3* (resp., *Rule 2.2*) - with the caution that agent k chooses k^k so as to win the modulo game if *Rule 3* is triggered. In this way, agent k obtains an outcome $g\left(m'_k, m_{-k}\right)$ such that $\left(g\left(m'_k, m_{-k}\right), x\right) \in I_k$, which gives us a contradiction. Therefore, $H \subseteq N\backslash\{k\}$, otherwise, we fall into a contradiction. Take an arbitrary

[22] *Rule 2.2* always applies to $\left(m_{-h}, m'_h\right)$ since by our contradiction hypothesis $x \notin F(R)$, and so the profile of strategy sets $\left(\sigma_j(R, x)\right)_{j \in N}$ is not defined when a partially honest agent $h \in H$ deviates from m_h to $m'_h = \left(R, x, k^h\right) \in T^{\gamma}_h(R, F)$. We caution the reader that similar reasoning applies below.

$j \in H$; by changing $m_j \notin T_j^{\gamma}(R,F)$ into $m_j' = (R, x, k^j) \in T_j^{\gamma}(R,F)$, agent $j \neq i$ (resp., $j = i$) can induce *Rule 3* (resp., *Rule 2.2*) - with the caution that agent j chooses k^j so as to win the modulo game if the modulo game is triggered. In this way, agent j obtains an outcome $g\left(m_j', m_{-j}\right)$ such that $\left(g\left(m_j', m_{-j}\right), x\right) \in I_j$, which gives us a contradiction. Since $j \in H$ is an arbitrary agent, and $H \subseteq N \backslash \{k\}$, it follows that H is an empty set, which is a contradiction. Hence, $\bar{R} = R$, and so $x \in F(R)$, which is a contradiction.

Case 2.2: m corresponds to *Rule 2.2.*

In what follows, we first show that $Y^F \subseteq L(R_\ell, g(m))$ for each $\ell \in N \backslash \{i\}$, and then we show that $C_i(\bar{R}, x) \subseteq L(R_i, g(m))$.

To show that $Y^F \subseteq L(R_\ell, g(m))$ for each $\ell \in N \backslash \{i\}$, take an arbitrary $\ell \in N \backslash \{i\}$ and an arbitrary $x' \in Y^F$. Since m falls into *Rule 2.2*, $R^i \neq \bar{R}$ or $\phi(R^i) \neq \phi(\bar{R})$. By changing m_ℓ into $m_\ell' = (\phi(R^i), x', k^\ell) \notin \sigma_\ell(\bar{R}, x)$, agent ℓ can induce *Rule 3*. To attain x', agent ℓ has only to adjust k^ℓ so as to win the modulo game. Since x' is arbitrary, $Y^F \subseteq g(M_\ell, m_{-\ell})$. Moreover, since $m \in NE(\gamma, \succcurlyeq^{R,H})$, it follows that $Y^F \subseteq L(R_\ell, g(m))$. Since ℓ is an arbitrary agent in $N \backslash \{i\}$, $Y^F \subseteq L(R_\ell, g(m))$ for each $\ell \in N \backslash \{i\}$.

Next, to show that $C_i(\bar{R}, x) \subseteq L(R_i, g(m))$, we proceed according to whether $R = \bar{R}$ or not. First, let us notice that agent i can induce *Rule 1* and obtain x by announcing either $m_i' = (\bar{R}, x, k^i)$ or $m_i' = (\phi(\bar{R}), x, k^i)$. Let us suppose that $R \neq \bar{R}$. Agent i can change m_i into $m_i' = (\phi(R), x^i, k^i)$ so as to obtain $g(m_i', m_{-i}) = x^i$ if $x^i \in S_i(R; x, \bar{R})$, by *Rule 2.2*. Therefore, $S_i(R; x, \bar{R}) \cup \{x\} \subseteq g(M_i, m_{-i})$, and so, by our suppositions, $S_i(R; x, \bar{R}) \cup \{x\} \subseteq L(R_i, g(m))$. Assume, to the contrary, that there exists $x^i \in C_i(\bar{R}, x) \backslash S_i(R; x, \bar{R})$ such that $x^i \notin L(R_i, g(m))$, so that $(x^i, g(m)) \in P_i$. By transitivity, $S_i(R_i; x, \bar{R}) \cup \{x\} \subseteq L(P_i, x^i)$. Then, agent i can change m_i into $m_i' = (\phi(R), x^i, k^i)$ so as to obtain $g(m_i', m_{-i}) = x^i$, by *Rule 2.2*, which gives us a contradiction. Therefore, $C_i(\bar{R}, x) \subseteq L(R_i, g(m))$ if $R \neq \bar{R}$. Let us suppose that $R = \bar{R}$. Then, $(x, g(m)) \in R_i$. Moreover, since $\{x\} \subseteq g(M_i, m_{-i})$, and, by our suppositions, $(g(m), x) \in R_i$, it follows that $(x, g(m)) \in I_i$. Assume, to the contrary, that there exists $x^i \in C_i(R, x)$ such that $x^i \notin L(R_i, g(m))$, so that $(x^i, g(m)) \in P_i$. By transitivity, $(x^i, x) \in P_i$, which gives us a contradiction. We conclude that $C_i(\bar{R}, x) \subseteq L(R_i, g(m))$.

Next, we show that $g(m) \in F(R)$. Assume, to the contrary, that $g(m) \notin F(R)$. We proceed according to whether Property I is applied or not. First, let us notice that the profile of strategy sets $(\sigma_j(R, g(m)))_{j \in N}$ is not defined, given our contradiction hypothesis.

Case 2.2.a: Property II is applied.

Then, $\sigma_{j1}(\bar{R}, x) = \bar{R}$ for all $j \in N$. Let us suppose that $m_h \notin T_h^{\gamma}(R,F)$ for some $h \in H \backslash \{i\}$ - where agent i is the agent identified by *Rule 2*. By changing m_h into $m_h' = (R, g(m), k^h) \in T_h^{\gamma}(R,F)$, agent h can induce *Rule 3* - regardless of whether agent i is announcing the true preference profile or not, and whether $g(m) = x$ or not. It is true even if $\#N = 3$ and $m_i = (R, g(m), k^i)$, since $g(m) \notin F(R)$ by supposition. To attain $g(m)$ or an outcome which is indifferent to $g(m)$ according to her true preferences, agent h has only to adjust k^h so as to win the modulo game, which gives us a contradiction. Therefore, $m_h \in T_h^{\gamma}(R,F)$ for each $h \in H \backslash \{i\}$. In what follows, we proceed according to whether $\#H > 1$ and $i \in H$ or not.

Sub-case 2.2.a.1: $\#H > 1$ and $i \in H$.

Then, $R = \bar{R}$, $x \in F(R)$, and $(x, g(m)) \in I_i$. Since m falls into *Rule 2.2*, $m_i \notin T_i^{\gamma}(R,F)$. Therefore, agent i can deviate to $m_i' = (R, x, k^i) \in T_i^{\gamma}(R,F)$ so that *Rule 1* applies to (m_i', m_{-i}) - recall that Property II is applied; then, $g(m_i', m_{-i}) = x$, which gives us a contradiction.

Sub-case 2.2.a.2: not$[\#H > 1$ and $i \in H]$.

Let us suppose that $H = \{i\}$. Let us first notice that $R \neq \bar{R}$; otherwise, $x \in F(R)$, $(x, g(m)) \in I_i$, and a contradiction can be derived by applying the reasoning used in *Sub-case 2.2.a.1*. Furthermore, $m_i \in T_i^\gamma(R, F)$; otherwise, agent i can deviate to $m_i' = \left(R, g(m), k^i\right) \in T_i^\gamma(R, F)$ so as to obtain $(g(m), g(m_{-i}, m_i')) \in I_i$, by *Rule 2.2*, which gives us a contradiction. Let us suppose that $g(m) \in S_i\left(R; x, \bar{R}\right)$. Then, Condition μ^*(ii.a) implies that $g(m) \in F(R)$, which is a contradiction. Then, let us assume that $g(m) \notin S_i\left(R; x, \bar{R}\right)$. By definition of g, $g(m) \in C_i\left(\bar{R}, x\right) \backslash S_i\left(R; x, \bar{R}\right)$ and $S_i\left(R; x, \bar{R}\right) \subseteq L(P_i, g(m))$; Condition μ^*(ii.a) implies that $g(m) \in F(R)$, which is a contradiction.

Let us suppose that $\#H \geq 1$ and $i \notin H$. Then, $R = \bar{R}$. Let us suppose that $x \notin S_i(R; x, R)$. Then, $g(m) \notin S_i(R; x, R)$; otherwise, since Condition μ^*(i) holds, agent i can find a profitable deviation by inducing *Rule 1*, which gives us a contradiction. Thus, since $\{i\} \in \mathcal{H}$ holds, $(x, g(m)) \in I_i$, Condition μ^*(i), and Condition μ^*(ii.a) together imply that $g(m) \in F(R)$, which is a contradiction. Therefore, let us suppose that $x \in S_i(R; x, R)$. Since, by our supposition, PROPERTY II is applied, $x \in S_\ell(R; x, R)$ for some $\ell \in N \backslash \{i\}$. Since $\{\ell\} \in \mathcal{H}$, Condition μ^*(ii.b) implies that $g(m) \in F(R)$, which is a contradiction.

Case 2.2.b: PROPERTY I is applied.

Then, $\sigma_{1k}\left(\bar{R}, x\right) = \bar{R}$, and $\sigma_{1j}\left(\bar{R}, x\right) = \phi\left(\bar{R}\right)$ for each $j \in N \backslash \{k\}$. First, let us notice that agent k may or may not coincide with agent i identified by *Rule 2*; second, since, by our contradiction hypothesis, $g(m) \notin F(R)$, it follows that $R \neq \bar{R}$ or $g(m) \neq x$. Finally, by our contradiction hypothesis, the profile of strategy sets $(\sigma_j(R, g(m)))_{j \in N}$ is not defined.

Let us suppose that there exists $j \in N \backslash \{i, k\}$ such that $j \in H$. Then, $m_j \notin T_j^\gamma(R, F)$. By changing m_j into $m_j' = \left(R, g(m), k^j\right) \in T_j^\gamma(R, F)$, agent j can induce *Rule 3* - regardless of whether $i = k$ or not. To attain $g(m)$ or an outcome which is indifferent to $g(m)$ according to her true preferences, agent j has only to adjust k^j so as to win the modulo game, which gives us a contradiction. Since j is arbitrary, we conclude that $H \subseteq \{i, k\}$. We proceed according to whether $i \neq k$ or not.

Sub-case 2.2.b.1: $i \neq k$.

Then, $x \notin S_i\left(R; x, \bar{R}\right)$. We proceed according to whether $\bar{R} = R$ or not.

Let us suppose that $\bar{R} = R$. Then, $g(m) \notin S_i(R; x, R)$; otherwise, since $x \notin S_i(R; x, R)$ and Condition μ^*(i) hold, by changing m_i into $m_i' \in \sigma_i(R, x)$, agent i can obtain x via *Rule 1*, which gives us a contradiction. Thus, since $\{i\} \in \mathcal{H}$ holds, $(x, g(m)) \in I_i$, Condition μ^*(i), and Condition μ^*(ii.a) together imply that $g(m) \in F(R)$, which is a contradiction.

Let us suppose that $\bar{R} \neq R$. Let us suppose that $k \in H$. By changing m_k into $m_k' = \left(R, g(m), k^k\right) \in T_k^\gamma(R, F)$, agent k can induce *Rule 3*. To attain $g(m)$ or an outcome which is indifferent to $g(m)$ according to her true preferences, agent k has only to adjust k^k so as to win the modulo game, which gives us a contradiction. Therefore, $k \notin H$. Since $H \in \mathcal{H}$ and $H \subseteq \{i, k\}$, it follows that $H = \{i\}$. Let us suppose that $m_i \notin T_i^\gamma(R, F)$. By changing m_i into $m_i' = \left(R, g(m), k^i\right) \in T_i^\gamma(R, F)$, agent i obtains an outcome $g(m_{-i}, m_i')$ such that $(g(m_{-i}, m_i'), g(m)) \in I_i$, by *Rule 2.2*, which gives us a contradiction. Hence, $m_i \in T_i^\gamma(R, F)$. Let us suppose that $g(m) \in S_i\left(R; x, \bar{R}\right)$. Then, Condition μ^*(ii.a) implies that $g(m) \in F(R)$, which is a contradiction. Then, let us suppose that $g(m) \notin S_i\left(R; x, \bar{R}\right)$. By definition of g, $g(m) \in C_i\left(\bar{R}, x\right) \backslash S_i\left(R; x, \bar{R}\right)$ and $S_i\left(R; x, \bar{R}\right) \subseteq L(P_i, g(m))$; Condition μ^*(ii.a) implies that $g(m) \in F(R)$, which is a contradiction.

Sub-case 2.2.b.2: $i = k$.

Then, $H = \{i\}$, given that $H \subseteq \{i, k\}$. Let us suppose that $\bar{R} = R$. Then, $(g(m), x) \in I_i$; moreover, $m_i \notin T_i^\gamma(R, F)$. By changing m_i into $m_i' = \left(R, x, k^i\right) \in T_i^\gamma(R, F) \cap \sigma_i(R, x)$, agent i can attain x via *Rule 1*, which gives us a contradiction. Therefore, let us suppose that $\bar{R} \neq R$. By the same arguments used in *Sub-case 2.2.b.1,* when $\bar{R} \neq R$, it can be shown that $m_i \in T_i^{\gamma}(R, F)$. Condition μ^*(ii.a) implies that $g(m) \in F(R)$, which is a contradiction.

Case 3: m corresponds to *Rule 3.*

It can be shown that for each $\ell \in N$, $Y^F \subseteq L(R_\ell, g(m))$ - see addendum. To show that $g(m) \in F(R)$, assume, to the contrary, that $g(m) \notin F(R)$. Given that $\{\ell\} \in \mathcal{H}$ for $\ell \in N$, let us consider the case $\{\ell^*(m)\} = \hat{H}$. Let us suppose that $m_{\ell^*(m)} \notin T_{\ell^*(m)}^{\gamma}(R, F)$. By changing $m_{\ell^*(m)}$ into $m'_{\ell^*(m)} = \left(R, g(m), k^{\ell^*(m)}\right) \in T_{\ell^*(m)}^{\gamma}(R, F)$, agent $\ell^*(m)$ can induce *Rule 3* - the reason being that $m'_{\ell^*(m)} \notin \sigma_{\ell^*(m)}(R', x')$ for all $(R', x') \in \mathcal{R}^n \times Y^F$, with $x' \in F(R')$, combined with our suppositions that $g(m) \notin F(R)$ and *Rule 3* applies to m. To attain $g(m)$ or an outcome which is indifferent to $g(m)$ according to her true preferences, agent $\ell^*(m)$ has only to adjust $k^{\ell^*(m)}$ so as to win the modulo game, which gives us a contradiction. Thus, $m_{\ell^*(m)} \in T_{\ell^*(m)}^{\gamma}(R, F)$.

We proceed according to whether or not the following requirements are met:

$$x^{\ell^*(m)} \in \max_{R_\ell} Y^F \text{ for all } \ell \in N, \tag{6}$$

$$x^{\ell^*(m)} \notin F(R), \text{ and} \tag{7}$$

$$S_{\ell^*(m)}\left(R; x^{\ell^*(m)}, R^p\right) \cap I_{\ell^*(m)}\left(x^{\ell^*(m)}, Y^F\right) \neq \varnothing, \tag{8}$$

where $R = R^{\ell^*(m)}$, and $R^p = \left(R^p_{\ell^*(m)}, R_{-\ell^*(m)}\right)$, with $\max_{R^p_{\ell^*(m)}} Y^F = \left\{x^{\ell^*(m)}\right\}$.

Let us suppose that requirements (6)-(8) are met. Then, let us suppose that $\hat{x} \in S_{\ell^*(m)}(R; x^{\ell^*(m)}, R^p) \cap I_{\ell^*(m)}\left(x^{\ell^*(m)}, Y^F\right)$; therefore, $g(m) = \hat{x}$, by *Rule 3*. Since $\mathcal{R}^n$ satisfies **RD**, $R^p \in \mathcal{R}^n$. Since $\left(y', x^{\ell^*(m)}\right) \notin I^p_{\ell^*(m)}$ for all $y' \in Y^F \setminus \left\{x^{\ell^*(m)}\right\}$, Condition μ^*(iii) implies that $x^{\ell^*(m)} \in F(R^p)$. Condition μ^*(ii.a), in turn, implies that $\hat{x} = g(m) \in F(R)$, which is a contradiction. Therefore, let us suppose that at least one of the requirements listed above is not met. Then, $g(m) = x^{\ell^*(m)}$, by *Rule 3*, and so requirement (6) is met; furthermore, by our contradiction hypothesis, requirement (7) is met as well. Finally, Condition μ^*(iii) implies that (8) is met too, producing a contradiction.

As the above arguments hold for any $H \in \mathcal{H}$ and any $R \in \mathcal{R}^n$, the statement follows. ■

5.2 Proofs of Lemmata

Proof of Lemma 1. Suppose that the premises hold. Take any $x \in Y^F$. Assume, to the contrary, that $x \notin \bar{Y}^F$. Then, by (1), for some $R \in \mathcal{R}^n$, $X \subseteq L(R_\ell, x)$ for all $\ell \in N$, and $x \notin F(\mathcal{R}^n)$. Since for any $i \in N$, $(R_i^p, R_{-i}) \in \mathcal{R}^n$ by **RD**, Condition μ^*(iii) implies $x \in F(R_i^p, R_{-i}) \subseteq F(\mathcal{R}^n)$, producing a contradiction. This completes the proof of part (i). To prove part (ii), take any $R \in \mathcal{R}^n$ and any $x \in \bar{Y}^F$; and suppose that $x \in \max_{R_\ell} \bar{Y}^F$ for all $\ell \in N$. By (1), $x \in F(\mathcal{R}^n)$, and $x \in \max_{R_\ell} Y^F$ for all $\ell \in N$ by part (i). Condition μ^*(iii) implies that $x \in F(R_i^p, R_{-i})$ for all $i \in N$, as sought. ■

Proof of Lemma 2. Suppose that the premises hold. Assume, to the contrary, that for any $y \in L(R_i, x)$, with $(y, x) \in R_i^*$, $y \notin F(R^*)$ and $y \in \max_{R_\ell^*} \bar{Y}^F$ for all $\ell \in N \setminus \{i\}$. Since F satisfied Condition μ^* and R_i^* is an ordering, there exists $z \in C_i(R, x) \subseteq L(R_i^*, z)$. Condition μ^*(ii.a) implies that $z \notin S_i(R^*; x, R)$, and there exists $y \in S_i(R^*; x, R)$ such that $(y, z) \in I_i^*$, so that $y \in C_i(R, x) \subseteq L(R_i^*, y)$. Condition μ^*(ii.a) implies that $y \in F(R)$, which is a contradiction. ■

Proof of Lemma 3. Suppose that the premises hold, with $y \in O_i(x, R, R)$. Then, $(x, y) \in I_i$ and $x \neq y$. Moreover, Condition μ^*(ii.a) implies that $y \notin S_i(R; x, R)$ and there exists $z \in S_i(R; x, R)$

such that $(z,y) \in I_i$. Then, $(x,z) \in I_i$. If $x = z$, then $x \in S_i(R;x,R)$; otherwise, Condition μ^*(i) implies that $x \in S_i(R;x,R)$. Since $y \notin F(R)$, Condition μ^*(ii.b) implies that $x \notin S_\ell(R;x,R)$ for $\ell \in N\backslash\{i\}$. Then, Condition μ^*(i) implies that $S_\ell(R;x,R) \subseteq L(P_\ell,x)$. Finally, since $x \notin S_\ell(R;x,R)$, it is also plain by the above arguments that the set $O_\ell(x,R,R)$ is empty. ■

Proof of Lemma 4. Suppose that the premises hold. Take any $F \in \mathcal{F}$. Consider the above construction of the sets specified by Condition μ^*. It is plain that F satisfies Condition μ^*(i). Take any $R, R^* \in \mathcal{R}^n$, with $x \in F(R)$, and any $i \in N$, and suppose that $y \in C_i(R,x) \subseteq L(R_i^*, y)$ and $y \in \max_{R_\ell^*} Y^F$ for all $\ell \in N\backslash\{i\}$. We proceed according to whether $R = R^*$ or not.

Suppose that $R = R^*$, so that $(x,y) \in I_i$. Suppose that there is a unique agent $j \in N\backslash\{i\}$ such that $\bar{O}_j(x,R,R) \neq \varnothing$. Then, $x \notin S_i(R;x,R)$, $S_i(R;x,R) = L(P_i,y)$, and $S_j(R;x,R) \equiv \{x\} \cup Q_j(x,R,R)$. Since the set $\bar{O}_i(x,R,R) = \varnothing$, (2) implies that $y \in F(R)$, which is consistent with Condition μ^*(ii).

Suppose that agent $j = i$. Then, $x \notin S_\ell(R;x,R)$ for all $\ell \in N\backslash\{i\}$ and $x \in S_i(R;x,R) \equiv \{x\} \cup Q_i(x,R,R)$. Suppose that $y \in \bar{O}_i(x,R,R)$. Then, $y \neq x$, $y \notin S_i(x,R,R)$ by (3), and $(y,x) \in I_i$, which is consistent with Condition μ^*(ii.a). Suppose that $y \notin \bar{O}_i(x,R,R)$. Then, (3) implies that $y \in S_i(x,R,R)$, which is consistent with Condition μ^*(ii.a). Note that the construction is consistent with Condition μ^*(ii.b) too.

Suppose that $\bar{O}_j(x,R,R) = \varnothing$ all $j \in N$. Then, $S_i(R;x,R) = \{x\} \cup Q_i(x,R,R)$. Since the set $\bar{O}_i(x,R,R) = \varnothing$, (2) and (3) imply $y \in Q_i(R;x;R)$, which is consistent with Condition μ^*(ii).

Suppose there are at least two agents $j,k \in N$, $j \neq k$, such that $\bar{O}_j(x,R,R) \neq \varnothing \neq \bar{O}_k(x,R,R)$. We observe that $y \notin \bar{O}_i(x,R,R)$, so that (2) and (3) imply that $y \in Q_i(x,R,R)$, which is consistent with Condition μ^*(ii).

Suppose that $R \neq R^*$. Suppose that $S_i(R^*;x,R) = Q_i(x,R,R^*) \neq \varnothing$. It is plain that Condition μ^*(ii.a) is satisfied if $y \in Q_i(x,R,R^*)$. Then, suppose that $y \in C_i(R,x)\backslash Q_i(x,R,R^*)$. Since $Q_i(x,R,R^*) \neq \varnothing$, (3) implies that there exists $z \in Q_i(x,R,R^*)$ such that $(z,y) \in I_i^*$, which is consistent with Condition μ^*(ii). Suppose that $Q_i(x,R,R^*) = \varnothing$, so that $y \in \bar{O}_i(x,R,R^*)$ and $S_i(R^*;x,R) = V_i(x,R,R^*) \neq \varnothing$. By definition (4), $y \notin V_i(x,R,R^*)$. Since $V_i(x,R,R^*) \neq \varnothing$, (4) implies that there exists $z \in V_i(x,R,R^*)$ such that $(z,y) \in I_i^*$, which is consistent with Condition μ^*(ii).

To check Condition μ^*(iii), take any $R, R^* \in \mathcal{R}^n$, with $x \in F(R)$, and suppose that $y \in \max_{R_\ell^*} Y^F$ for all $\ell \in N$. Then, by the supposition (ii) of the statement, $y \in F\left(R_{-i}^*, R_i^{*p}\right)$ for $i \in N$. Fix any $i \in N$, and let $\left(R_{-i}^*, R_i^{*p}\right) \equiv R^{*p}$ for the sake of brevity. (2) implies that $\bar{O}_\ell(y,R^{*p},R^{*p}) = \varnothing$ for all $\ell \in N$. Then, $C_i(R^{*p},y) = L\left(R_i^{*p},y\right) \cap Y^F \supseteq Y^F$. Therefore, either $S_i(R^*;y,R^{*p}) = Q_i(y,R^{*p},R^*)$ or $S_i(R^*;y,R^{*p}) = V_i(y,R^{*p},R^*)$ by the above construction. Moreover, $y \notin V_i(y,R^{*p},R^*)$. Condition μ^*(iii) is satisfied if $y \in F(R^*)$. Then, suppose that $y \notin F(R^*)$, so that $y \notin Q_i(y,R^{*p},R^*)$. Since $Q_i(y,R^{*p},R^*) \cup V_i(y,R^{*p},R^*) \neq \varnothing$, there exists $z \in Q_i(y,R^{*p},R^*) \cup V_i(y,R^{*p},R^*)$ such that $(y,z) \in I_i^*$. Since $S_i(R^*;y,R^{*p}) \neq \varnothing$, $z \in S_i(R^*;y,R^{*p}) \cap I_i^*\left(y,Y^F\right)$, which is consistent with Condition μ^*(iii). This completes the proof. ■

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1 Completion of the Proof of the Case 3 of Theorem

Case 3: m corresponds to *Rule 3.*

Let us show that for each $\ell \in N$, $Y^F \subseteq L(R_\ell, g(m))$. Take an arbitrary $i \in N$. We proceed according to whether $x^\ell = x$ for all $\ell \in N \backslash \{i\}$ or not. Before proceeding, a little more notation is needed. Let us denote the first component of the message m_ℓ as $m_{\ell 1} \in \mathcal{R}^n \cup \mathcal{S}$.

Sub-case 3.1: $x^\ell = x \in Y^F$ for all $\ell \in N \backslash \{i\}$.

Let us suppose that $x \notin F(R)$. Take any $x' \in Y^F$. By changing m_i into $m'_i = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3.* To attain x', agent i has only to adjust k^i so as to win the modulo game. We conclude that $Y^F \subseteq g(M_i, m_{-i})$. Therefore, in what follows, let us suppose that $x \in F(R)$.

Sub-case 3.1.a: $m_{\ell 1} = R$ (resp., $m_{\ell 1} = \phi(R)$) for each $\ell \in N \backslash \{i\}$.

Then, for the case $m_{\ell 1} = R$ for each $\ell \in N \backslash \{i\}$, Property I must be applied to $x \in F(R)$; otherwise, we fall into a contradiction. On the other hand, for the case $m_{\ell 1} = \phi(R)$ for each $\ell \in N \backslash \{i\}$, Property I can be applied to $x \in F(R)$, under the specification that agent k identified by this property differs from i; otherwise, we fall into a contradiction. Take any $x' \in Y^F \backslash \{x\}$.

For the case $m_{\ell 1} = R$ for each $\ell \in N \backslash \{i\}$, by changing m_i into $m'_i = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3.* To attain x', agent i has only to adjust k^i so as to win the modulo game. To attain x, agent i has only to adjust k^i so that agent $\ell \in N \backslash \{i\}$ wins the modulo game. We conclude that $Y^F \subseteq g(M_i, m_{-i})$. For the case $m_{\ell 1} = \phi(R)$ for each $\ell \in N \backslash \{i\}$, by changing m_i into $m'_i = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3.* To attain x', agent i has only to adjust k^i so as to win the modulo game. To attain x, agent i has only to adjust k^i so that agent $\ell \in N \backslash \{i\}$ wins the modulo game. We conclude that $Y^F \subseteq g(M_i, m_{-i})$.

Sub-case 3.1.b: $m_{\ell 1} \neq R$ (resp., $m_{\ell 1} \neq \phi(R)$) for some $\ell \in N \backslash \{i\}$.

Let us define the set N^R (resp., $N^{\phi(R)}$) as $N^R \equiv \{\ell \in N \backslash \{i\} \,|\, m_{\ell 1} \neq R\}$ (resp., $N^{\phi(R)} \equiv \{\ell \in N \backslash \{i\} \,|\, m_{\ell 1} \neq \phi(R)\}$). We proceed according to whether $\# N^R > 1$ (resp., $\# N^{\phi(R)} > 1$) or not.

Sub-case 3.1.b.1: $\# N^R = 1$ (resp., $\# N^{\phi(R)} = 1$).

Sub-case 3.1.b.1.1: $m_{j1} = \phi(R)$ (resp., $m_{j1} = R$) for $j \in N^R$ (resp., $j \in N^{\phi(R)}$).

Let us suppose that Property I applies to $x \in F(R)$. Then, for the case $m_{j1} = \phi(R)$, agent k identified by this property cannot coincides with agent $\ell \in N \backslash \left(N^R \cup \{i\}\right)$ if $\# N \backslash \left(N^R \cup \{i\}\right) = 1$; otherwise, we fall into a contradiction. Moreover, for the case $m_{j1} = R$, agent k identified by Property I cannot coincides with agent $j \in N^{\phi(R)}$; otherwise, we fall into a contradiction. Next, take any $x' \in Y^F \backslash \{x\}$. By changing m_i into $m'_i = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3.* To attain x', agent i has only to adjust k^i so as to win the modulo game. To attain x, agent i has only to adjust k^i so that agent $j \in N^R$ (resp., $j \in N^{\phi(R)}$) wins the modulo game. We conclude that $Y^F \subseteq g(M_i, m_{-i})$.

Let us suppose that Property II applies to $x \in F(R)$. Take any $x' \in Y^F \backslash \{x\}$. By changing m_i into $m'_i = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3.* To attain x', agent i has only to adjust k^i so as to win the modulo game. To attain x, agent i has only to adjust k^i so that agent $j \in N^R$ (resp., $j \in N^{\phi(R)}$) wins the modulo game. We conclude that $Y^F \subseteq g(M_i, m_{-i})$.

Sub-case 3.1.b.1.2: $m_{j1} \neq \phi(R)$ (resp., $m_{j1} \neq R$) for $j \in N^R$ (resp., $j \in N^{\phi(R)}$).

Take any $x' \in Y^F \backslash \{x\}$. By changing m_i into $m'_i = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3.* To attain $x' \in Y^F$, agent i has only to adjust k^i so as to win the modulo game. To attain x, agent i has only to adjust k^i so that agent $\ell \in N \backslash \left(N^R \cup \{i\}\right)$ (resp., $\ell \in N \backslash \left(N^{\phi(R)} \cup \{i\}\right)$) wins the modulo game. We conclude that $Y^F \subseteq g(M_i, m_{-i})$.

Sub-case 3.1.b.2: $\#N^{R}>1$ (resp., $\#N^{\phi(R)}>1$).

Sub-case 3.1.b.2.1: $m_{j1}=\phi(R)$ (resp., $m_{j1}=R$) for all $j\in N^{R}$ (resp., $j\in N^{\phi(R)}$).

The proof follows from *Sub-case 3.1.a* and *Sub-case 3.1.b.1.1* if $n-2\leq\#N^{R}\leq n-1$ (resp., $n-2\leq N^{\phi(R)}\leq n-1$). Therefore, let us suppose that $1<\#N^{R}<n-2$ (resp., $1<N^{\phi(R)}<n-2$). Therefore, $n\geq4$.

Let us suppose that Property I applies to $x\in F(R)$. Then, agent k identified by this property cannot coincides with agent $\ell\in N\backslash\left(N^{R}\cup\{i\}\right)$ if $\#N\backslash\left(N^{R}\cup\{i\}\right)=1$; otherwise, we fall into a contradiction. Next, take any $x'\in Y^{F}\backslash\{x\}$. By changing m_i into $m_i'=\left(\phi(R),x',k^{i}\right)$, agent i induces *Rule 3*. To attain x', agent i has only to adjust k^{i} so as to win the modulo game. To attain x, agent i has only to adjust k^{i} so that agent $j\in N^{R}$ (resp., $j\in N^{\phi(R)}$) wins the modulo game. We conclude that $Y^{F}\subseteq g(M_i,m_{-i})$.

Let us suppose that Property II applies to $x\in F(R)$. Take any $x'\in Y^{F}\backslash\{x\}$. By changing m_i into $m_i'=\left(\phi(R),x',k^{i}\right)$, agent i induces *Rule 3*. To attain x', agent i has only to adjust k^{i} so as to win the modulo game. To attain x, agent i has only to adjust k^{i} so that agent $j\in N^{R}$ (resp., $j\in N^{\phi(R)}$) wins the modulo game. We conclude that $Y^{F}\subseteq g(M_i,m_{-i})$.

Sub-case 3.1.b.2.2: $m_{j1}\neq\phi(R)$ (resp., $m_{j1}\neq R$) for some $j\in N^{R}$ (resp., $j\in N^{\phi(R)}$).

Then, there exists $j\in N^{R}$ (resp., $j\in N^{\phi(R)}$) such that $m_{j1}\notin\{R,\phi(R)\}$. Take any $x'\in Y^{F}\backslash\{x\}$. By changing m_i into $m_i'=\left(\phi(R),x',k^{i}\right)$, agent i induces *Rule 3*. To attain x', agent i has only to adjust k^{i} so as to win the modulo game.

To show that agent i can also attain x, let us proceed according to whether $\#N^{R}=n-1$ (resp., $\#N^{\phi(R)}=n-1$) or not. Let us suppose that $\#N^{R}\neq n-1$ (resp., $\#N^{\phi(R)}\neq n-1$). Then, there exists $\ell\in N$ such that $m_{\ell1}=R$ (resp., $m_{\ell1}=\phi(R)$). Take any $x'\in Y^{F}\backslash\{x\}$. Let us suppose that Property I applies to $x\in F(R)$. By changing m_i into $m_i'=\left(R,x',k^{i}\right)$ (resp., $m_i'=\left(\phi(R),x',k^{i}\right)$), agent i induces *Rule 3*. To attain x, agent i has only to adjust k^{i} so as agent ℓ wins the modulo game. Let us suppose that Property II applies to $x\in F(R)$. By changing m_i into $m_i'=\left(\phi(R),x,k^{i}\right)$, agent i induces *Rule 3*. To attain x, agent i has only to adjust k^{i} so as to win the modulo game. Therefore, let us suppose that $\#N^{R}=n-1$ (resp., $\#N^{\phi(R)}=n-1$). By changing m_i into $m_i'=\left(\phi(R),x,k^{i}\right)$, agent i induces *Rule 3*. To attain x, agent i has only to adjust k^{i} so as to win the modulo game. We conclude that $Y^{F}\subseteq g(M_i,m_{-i})$.

Sub-case 3.2: $x^{\ell}\neq x^{j}\in Y^{F}$ for some $j,\ell\in N\backslash\{i\}$, with $j\neq\ell$.

We proceed according to the following exhaustive cases: $\left\{x^{\ell},x^{j}\right\}\cap F(R)=\varnothing$, $\#\left\{\left\{x^{\ell},x^{j}\right\}\cap F(R)\right\}=1$, and $\left\{x^{\ell},x^{j}\right\}\subseteq F(R)$.

Sub-case 3.2.a: $\left\{x^{\ell},x^{j}\right\}\cap F(R)=\varnothing$.

Take any $x'\in Y^{F}$. By changing m_i into $m_i'=\left(\phi(R),x',k^{i}\right)$, agent i induces *Rule 3*. To attain x', agent i has only to adjust k^{i} so as to win the modulo game. We conclude that $Y^{F}\subseteq g(M_i,m_{-i})$.

Sub-case 3.2.b: $x^{\ell}\in F(R)$ and $x^{j}\notin F(R)$.[1]

Take any $x'\in Y^{F}\backslash\left\{x^{\ell}\right\}$. By changing m_i into $m_i'=\left(\phi(R),x',k^{i}\right)$, agent i induces *Rule 3*. To attain x', agent i has only to adjust k^{i} so as to win the modulo game. To show that agent i can also attain x^{ℓ}, let us proceed according to whether Property I applies to $x^{\ell}\in F(R)$ or not.

Let us suppose that Property I applies to $x^{\ell}\in F(R)$. Let us suppose that agent k identified by Property I coincides with agent i. Then, by changing m_i into $m_i'=\left(\phi(R),x^{\ell},k^{i}\right)$, agent i induces *Rule 3*. To attain x^{ℓ}, agent i has only to adjust k^{i} so as to win the modulo game. Let us suppose that agent k identified by Property I belongs to $N\backslash\{i\}$. Then, by changing m_i into

[1] The case $x^{j}\in F(R)$ and $x^{\ell}\notin F(R)$ is not explicitly considered, since it can be proved similarly to the case $x^{\ell}\in F(R)$ and $x^{j}\notin F(R)$.

$m_i' = \left(R, x^\ell, k^i\right)$, agent i induces *Rule 3*. To attain x^ℓ, agent i has only to adjust k^i so as to win the modulo game. We conclude that $Y^F \subseteq g\left(M_i, m_{-i}\right)$.

Let us suppose that PROPERTY II applies to $x^\ell \in F(R)$. Then, by changing m_i into $m_i' = \left(\phi(R), x^\ell, k^i\right)$, agent i induces *Rule 3*. To attain x^ℓ, agent i has only to adjust k^i so as to win the modulo game. We conclude that $Y^F \subseteq g\left(M_i, m_{-i}\right)$.

Sub-case 3.2.c: $\left\{x^\ell, x^j\right\} \subseteq F(R)$.

Take any $x' \in Y^F \setminus \left\{x^\ell, x^j\right\}$. By changing m_i into $m_i' = \left(\phi(R), x', k^i\right)$, agent i induces *Rule 3*. To attain x', agent i has only to adjust k^i so as to win the modulo game. To show that agent i can also attain x^ℓ and x^j, let us proceed according to whether or not PROPERTY I applies to $x^\ell \in F(R)$ and $x^j \in F(R)$. Before proceeding, a little more notation is needed. Let k_{x^ℓ} (resp., k_{x^j}) denote the agent k identified by PROPERTY I for $x^\ell \in F(R)$ (resp., $x^j \in F(R)$).

Sub-case 3.2.c.1: PROPERTY I applies to both $x^\ell \in F(R)$ and $x^j \in F(R)$.

We proceed according to whether or not $k_{x^\ell} = i$ and $k_{x^j} = i$.[2]

Let us suppose that $k_{x^\ell} = k_{x^j} = i$. Then, by changing m_i into $m_i' = \left(\phi(R), x^\ell, k^i\right)$ (resp., $m_i' = \left(\phi(R), x^j, k^i\right)$), agent i induces *Rule 3*. To attain x^ℓ (resp., x^j), agent i has only to adjust k^i so as to win the modulo game.

Let us suppose that $k_{x^\ell} = i$ and $k_{x^j} \neq i$. Then, by changing m_i into $m_i' = \left(\phi(R), x^\ell, k^i\right)$ (resp., $m_i' = \left(R, x^j, k^i\right)$), agent i induces *Rule 3*. To attain x^ℓ (resp., x^j), agent i has only to adjust k^i so as to win the modulo game.

Let us suppose that $k_{x^\ell} \neq i$ and $k_{x^j} \neq i$. Then, by changing m_i into $m_i' = \left(R, x^\ell, k^i\right)$ (resp., $m_i' = \left(R, x^j, k^i\right)$), agent i induces *Rule 3*. To attain x^ℓ (resp., x^j), agent i has only to adjust k^i so as to win the modulo game.

We conclude that $Y^F \subseteq g\left(M_i, m_{-i}\right)$.

Sub-case 3.2.c.2: PROPERTY II applies to both $x^\ell \in F(R)$ and $x^j \in F(R)$.

Then, by changing m_i into $m_i' = \left(\phi(R), x^\ell, k^i\right)$ (resp., $m_i' = \left(\phi(R), x^j, k^i\right)$), agent i induces *Rule 3*. To attain x^ℓ (resp., x^j), agent i has only to adjust k^i so as to win the modulo game. We conclude that $Y^F \subseteq g\left(M_i, m_{-i}\right)$.

Sub-case 3.2.c.3: PROPERTY I applies to $x^\ell \in F(R)$ and PROPERTY II applies to $x^j \in F(R)$.[3]

Let us suppose that $k_{x^\ell} = i$. Then, by changing m_i into $m_i' = \left(\phi(R), x^\ell, k^i\right)$ (resp., $m_i' = \left(\phi(R), x^j, k^i\right)$), agent i induces *Rule 3*. To attain x^ℓ (resp., x^j), agent i has only to adjust k^i so as to win the modulo game.

Let us suppose that $k_{x^\ell} \neq i$. Then, by changing m_i into $m_i' = \left(R, x^\ell, k^i\right)$ (resp., $m_i' = \left(\phi(R), x^j, k^i\right)$), agent i induces *Rule 3*. To attain x^ℓ (resp., x^j), agent i has only to adjust k^i so as to win the modulo game. We conclude that $Y^F \subseteq g\left(M_i, m_{-i}\right)$.

Since agent $i \in N$ is arbitrary, $Y^F \subseteq g\left(M_i, m_{-i}\right)$ for each $i \in N$. Furthermore, by our supposition that $m \in NE\left(\gamma, \succsim^{R,H}\right)$, it follows that $Y^F \subseteq L\left(R_i, g(m)\right)$ for each $i \in N$. ■

[2] The case $k_{x^\ell} \neq i$ and $k_{x^j} = i$ is not explicitly considered, since it can be proved similarly to the case $k_{x^\ell} = i$ and $k_{x^j} \neq i$.

[3] The case PROPERTY II applies to $x^\ell \in F(R)$ and PROPERTY I applies to $x^j \in F(R)$ is not explicitly considered, since it can be proved similarly to the case PROPERTY I applies to $x^\ell \in F(R)$ and PROPERTY II applies to $x^j \in F(R)$.