

Alternative Characterizations of Three Bargaining Solutions for Nonconvex Problems*

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Abstract

This paper studies compact and comprehensive bargaining problems for n players and axiomatically characterize the extensions of the three classical bargaining solutions to nonconvex bargaining problems: the Nash solution, the egalitarian solution and the Kalai-Smorodinsky solution. Our characterizing axioms are various extensions of Nash's original axioms.

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1 Introduction

This paper considers *nonconvex* bargaining problems for n players. Specifically, we study (normalized) bargaining problems that are *compact* and *comprehensive*, but are not necessarily convex. Nonconvex bargaining problems can arise in many economic contexts of resource allocations where, for example, due to ‘economies of scale’ in the production technology, the underlying set of feasible allocations is itself not convex, and randomization is unavailable due to the lack of correlating strategies among players. As a consequence, primitive nonconvex bargaining problems cannot be convexified. Nonconvex bargaining problems can also arise naturally in bargaining problems when individuals are not characterized by their utilities but by their *capability sets* à la Sen (1985) (see Xu and Yoshihara (2004) for such examples). In such cases, we would have to start with nonconvex bargaining problems as primitives and develop theories to deal with them.

The literature has some discussions on nonconvex bargaining problems. For example, there exists a number of characterizations of the Nash bargaining solution for the class of compact and comprehensive bargaining problems. However, in all the characterization results, either a type of continuity property is imposed (see, for example, Kaneko (1980), Herrero (1989), Conley and Wilkie (1996)), or the class of bargaining problems contains finite bargaining problems in addition to those that are compact and comprehensive (see, for example, Mariotti (1999)). The purpose of this paper is two-fold. First, we give a new characterization of the Nash bargaining solution for the class of compact and comprehensive bargaining problems by four axioms: Efficiency, Symmetry, Scale Invariance and Contraction Independence, and provide a simple proof that highlights the crucial role that Contraction Independence plays. Because of our proof method, it is interesting to note that we do not use any continuity type axiom in our characterization. The four axioms used in the characterization result of the Nash solution are natural extensions of Nash’s original four axioms (Nash (1950)) in our context. Viewed in this way, this characterization result reported in the paper is perhaps closer to Nash’s original program than those already existing in the literature. Secondly, we use variants of the four axioms used for characterizing the Nash solution to characterize the egalitarian solution (Kalai (1977)) and the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) for nonconvex bargaining problems. Our characterization results of the egalitarian and the Kalai-Smorodinsky solutions again highlights the crucial role that Con-

traction Independence or Weak Contraction Independence (see Section 3 for the formal definition) plays. It should be noted that our characterizations of the egalitarian as well as the Kalai-Smorodinsky solutions do not use the commonly used Monotonicity type axioms for characterizations of those two solutions.

The remainder of the paper is organized as follows. In Section 2, we lay down some basic notation and definitions. Section 3 presents our axioms and their discussions. The main results and their proofs are contained in Section 4. We conclude the paper with a few remarks in Section 5 comparing and contrasting the axioms used in characterizing the three solutions.

2 Notation and Definitions

$N = \{1, 2, \dots, n\}$ is the set of players with $n \geq 2$. \mathbf{R}_+ is the set of all non-negative real numbers and \mathbf{R}_{++} is the set of all positive numbers. \mathbf{R}_+^n (resp. \mathbf{R}_{++}^n) is the n -fold Cartesian product of \mathbf{R}_+ (resp. \mathbf{R}_{++}). For any $x, y \in \mathbf{R}_+^n$, we write $x > y$ to mean $[x_i \geq y_i \text{ for all } i \in N \text{ and } x \neq y]$, and $x \gg y$ to mean $[x_i > y_i \text{ for all } i \in N]$. For any $x \in \mathbf{R}_+^n$ and any non-negative number α , we write $z = (\alpha; \mathbf{x}_{-i}) \in \mathbf{R}_+^n$ to mean that $z_i = \alpha$ and $z_j = x_j$ for all $j \in N \setminus \{i\}$. For any subset $A \subseteq \mathbf{R}_+^n$, A is said to be (i) *non-trivial* if there exists $a \in A$ such that $a \gg 0$, and (ii) *comprehensive* if for all $x, y \in \mathbf{R}_+^n$, $[x > y \text{ and } x \in A] \Rightarrow y \in A$. For all $\{x^1, \dots, x^m\} \subseteq \mathbf{R}_+^n$, define the *comprehensive hull* of $\{x^1, \dots, x^m\}$, to be denoted by $comp(x^1, \dots, x^m)$, as follows:

$$comp(x^1, \dots, x^m) \equiv \{z \in \mathbf{R}_+^n : z \leq x \text{ for some } x \in \{x^1, \dots, x^m\}\}.$$

Let Σ be the set of all non-trivial, compact and comprehensive subsets of \mathbf{R}_+^n . Elements in Σ are interpreted as (normalized) bargaining problems. A bargaining solution F assigns a nonempty subset $F(A)$ of A for every bargaining problem $A \in \Sigma$.

Let π be a permutation of N . The set of all permutations of N is denoted by Π . For all $x = (x_i)_{i \in N} \in \mathbf{R}_+^n$, let $\pi(x) = (x_{\pi(i)})_{i \in N}$. For all $A \in \Sigma$ and any permutation $\pi \in \Pi$, let $\pi(A) = \{\pi(a) : a \in A\}$. For any $A \in \Sigma$, we say that A is *symmetric* if $A = \pi(A)$ for all $\pi \in \Pi$.

For all $A \in \Sigma$ and all $i \in N$, let $m_i(A) = \max\{a_i : (a_1, \dots, a_i, \dots, a_n) \in A\}$. Therefore, $m(A) \equiv (m_i(A))_{i \in N}$ is the *ideal point* of A .

Definition 1: A bargaining solution F over Σ is the Nash solution if for all $A \in \Sigma$, $F(A) = \{a \in A : \prod_{i \in N} a_i \geq \prod_{i \in N} x_i \text{ for all } x \in A\}$.

Definition 2: A bargaining solution F over Σ is the egalitarian solution if for all $A \in \Sigma$, $F(A) = \{a \in A : a_i = a_j \text{ for all } i, j \in N \text{ and there is no } x \in A \text{ such that } x \gg a\}$.

Definition 3: A bargaining solution F over Σ is the Kalai-Smorodinsky solution if for all $A \in \Sigma$, $F(A) = \{a \in A : m_i(A)/a_i = m_j(A)/a_j \text{ for all } i, j \in N \text{ and there is no } x \in A \text{ such that } x \gg a\}$.

Our notion of the Nash solution for nonconvex bargaining problems is identical to the one proposed by Kaneko (1980).¹ It should be noted that, given that Σ contains all non-trivial, compact and comprehensive bargaining problems, for any $A \in \Sigma$, the Nash solution $F(A)$ can contain more than one alternative, while both the egalitarian and the Kalai-Smorodinsky solutions are singletons.

3 Axioms

In this section, we present our axioms that are to be used for characterization results. We start with two efficiency type axioms which are commonly invoked in the literature.

Efficiency (E): For any $A \in \Sigma$ and any $a \in F(A)$, there is no $x \in A$ such that $x > a$.

Weak Efficiency (WE): For any $A \in \Sigma$ and any $a \in F(A)$, there is no $x \in A$ such that $x \gg a$.

The next two axioms are natural generalizations of Nash's original symmetry axiom in our context.

Symmetry (S): For any $A \in \Sigma$, if A is symmetric, then $[a \in F(A) \Rightarrow \pi(a) \in F(A) \text{ for all } \pi \in \Pi]$.

¹Mariotti (1999) also discusses axiomatic characterization of the Kaneko type of the Nash solution for nonconvex problems, although his domain is larger than ours in the sense that it includes "finite bargaining problems." In contrast, Herrero's proposal (Herrero (1989)) for the Nash extension solution constitutes a *superset* of the set of the Kaneko type solution outcomes on each nonconvex problem, and Conley and Wilkie (1996) proposes an extension of the Nash solution which is a single-valued mapping in that domain.

Strong Symmetry (SS): For any $A \in \Sigma$ and all $a \in A$, if A is symmetric and $a \in F(A)$, then $a_1 = \dots = a_n$.

Symmetry is a natural generalization of Nash’s original symmetry axiom to nonconvex problems and is also discussed in Mariotti (1999). Strong Symmetry is a stronger requirement than Symmetry in that (SS) demands $a_1 = \dots = a_n$ whenever A is symmetric and $a \in F(A)$, while (S) requires any permutation of a be in $F(A)$ whenever A is symmetric and $a \in F(A)$. It should be noted that, when restricted to convex bargaining problems, and bargaining solutions are required to be single-valued mappings, the two symmetry axioms coincide with and are identical to Nash’s original Symmetry axiom.

The next axiom is the familiar scale invariance property commonly used in both convex (see, for example, Nash (1950)) and nonconvex bargaining problems (see, for example, Conley and Wilkie (1996), Herrero (1989), Mariotti (1999)).

Scale Invariance (SI): For all $A \in \Sigma$ and all $\alpha \in \mathbf{R}_{++}^n$, if $\alpha A = \{(\alpha_i a_i)_{i \in N} : a \in A\}$ then $F(\alpha A) = \{(\alpha_i a_i)_{i \in N} : a \in F(A)\}$.

For convex bargaining problems, (SI) is often justified by appealing to the expected utility theory. In the current context, (SI) seems reasonable as long as players’ utilities are cardinally measurable, an implicit assumption made in many standard bargaining problems. The introduction of nonconvex bargaining problems does not seem to harm the attractiveness of (SI) in many economic contexts of resource allocations that give rise to such nonconvex problems. This is because, in such contexts, nonconvex problems arise from the nonconvexity of the underlying set of feasible allocations where the introduction of ‘lotteries’ over feasible allocations is unreasonable. We should also mention that, (SI) has at least two interpretations: “utility-unit invariance”, and “independence of utility intensities” as discussed in Yoshihara (2003). Both of these interpretations seem to be compatible with the nonconvexity of bargaining problems.

The final two axioms are extensions of Nash’s original Independence of Irrelevant Alternatives (IIA).

Contraction Independence (CI): For any $A, B \in \Sigma$, if $B \subseteq A$ and $B \cap F(A) \neq \emptyset$, then $F(B) = B \cap F(A)$.

Weak Contraction Independence (WCI): For any $A, B \in \Sigma$, if $m(A) = m(B)$, $B \subseteq A$ and $B \cap F(A) \neq \emptyset$, then $F(B) = B \cap F(A)$.

(CI) has been widely used in the literature of nonconvex bargaining problems, and is the usual (IIA) for correspondence. (WCI) extends Yu's (1973) axiom of "independence of irrelevant alternatives" to correspondence, and is formally weaker than (CI): it restricts contractions to those problems that have the same ideal point.

4 Extensions of the classical bargaining solutions and their characterizations

In this section, we provide axiomatic characterizations of the Nash solution, the egalitarian solution and the Kalai-Smorodinsky solution.

Theorem 1: A bargaining solution F over Σ is the Nash solution if and only if it satisfies (E), (S), (SI) and (CI).

Before proving Theorem 1, we prove the following lemma first.

Lemma 1: Suppose a bargaining solution F over Σ satisfies (E), (S), (SI) and (CI). Then, for all $x, y \in \mathbf{R}_+^n$,

$$(L.1.) \quad \prod_{i \in N} x_i = \prod_{i \in N} y_i > 0 \Rightarrow F(\text{comp}(x, y)) = \{x, y\},$$

and

$$(L.2.) \quad \prod_{i \in N} y_i < \prod_{i \in N} x_i \Rightarrow F(\text{comp}(x, y)) = \{x\}.$$

Proof. Let F over Σ satisfy (E), (S), (SI) and (CI). Let $x, y \in \mathbf{R}_+^n$.

Suppose first that $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0$. It then follows that $x_i > 0, y_i > 0$ for all $i \in N$. Let $\alpha \in \mathbf{R}_+^n$ be defined as follows: $\alpha_1 = 1, \alpha_2 = \frac{x_1}{y_2} \alpha_1, \dots, \alpha_i = \frac{x_{i-1}}{y_i} \alpha_{i-1}, \dots, \alpha_n = \frac{x_{n-1}}{y_n} \alpha_{n-1}$. Clearly, for all $i \in N$, α_i is well defined and $\alpha_i > 0$. Furthermore, $\alpha_2 y_2 = \frac{x_1}{y_2} y_2 = x_1 = \alpha_1 x_1, \alpha_3 y_3 = \frac{x_2}{y_3} \alpha_2 y_3 = \alpha_2 x_2, \dots, \alpha_n y_n = \frac{x_{n-1}}{y_n} \alpha_{n-1} y_n = \alpha_{n-1} x_{n-1}$, and $\alpha_n x_n = \frac{x_{n-1}}{y_n} \alpha_{n-1} x_n = \frac{x_n x_{n-1}}{y_n} \alpha_{n-1} = \dots = \frac{x_n \dots x_1}{y_n \dots y_2}$. Noting that $x_n \dots x_1 = y_n \dots y_1$, we then have $\alpha_n x_n = y_1 = \alpha_1 y_1$. Let the permutation π^0 be such that $\pi^0(i) = i + 1$ for $i = 1, \dots, n - 1$ and $\pi^0(n) = 1$. Then, $\pi^0(\alpha x) = \alpha y$. Let $A = \text{comp}(\alpha x, \alpha y)$. Consider the bargaining problem $B \in \Sigma$ defined as $B \equiv \cup_{\pi \in \Pi} \pi(A)$. From the construction, B is symmetric. Note that, since αx and αy are the only

efficient points in A , and αx and αy are permutations of each other, for every permutation $\pi \in \Pi$, the set $\pi(A)$ contains just two efficient points that are permutations of αx and αy . By (E) and the non-emptiness of F , $F(B)$ must contain at least one such point, say b . But b is a permutation of αx and of αy . By (S), it then follows that $\{\alpha x, \alpha y\} \subseteq F(B)$. By (CI) and noting that $A \subseteq B$, $\{\alpha x, \alpha y\} = F(A)$ follows easily. By (SI), we obtain $\{x, y\} = F(\text{comp}(x, y))$. Thus, (L.1.) is proved.

Next, suppose that $\Pi_{i \in N} x_i > \Pi_{i \in N} y_i$. Clearly, $x \neq y$. Noting that $\Pi_{i \in N} y_i \geq 0$, it then follows that $\Pi_{i \in N} x_i > 0$ and $x \gg 0$. We need to show that $F(\text{comp}(x, y)) = \{x\}$. Since $\Pi_{i \in N} x_i > \Pi_{i \in N} y_i \geq 0$, there exists $\epsilon \in \mathbf{R}_{++}^n$ such that $\Pi_{i \in N} (y_i + \epsilon_i) = \Pi_{i \in N} x_i$. Let $z = y + \epsilon$. From (L.1.), noting that $\Pi_{i \in N} x_i = \Pi_{i \in N} z_i > 0$, we must have $F(\text{comp}(x, z)) = \{x, z\}$. Since $z > y$, it must be true that $\text{comp}(x, y) \subseteq \text{comp}(x, z)$. By (CI), it then follows that $\{x\} = F(\text{comp}(x, y))$. This proves (L.2.). \diamond

Proof of Theorem 1. It can be checked that if F is the Nash solution over Σ then it satisfies the four axioms in Theorem 1. Thus, we need only to show that if a bargaining solution F over Σ satisfies (E), (S), (SI) and (CI), then it must be the Nash solution.

Let F over Σ satisfy the above four axioms. Given any bargaining problem $A \in \Sigma$, the non-emptiness of F implies that, for some $a \in A$, $a \in F(A)$. We note that a must be such that $\Pi_{i \in N} a_i \geq \Pi_{i \in N} x_i$ for all $x \in A$. For, otherwise, there would exist $b \in A$ with $\Pi_{i \in N} b_i > \Pi_{i \in N} a_i$. Noting that $\text{comp}(a, b) \subseteq A$, (CI) would then imply $a \in F(\text{comp}(a, b))$, a contradiction to Lemma 1(L.2.). Another straightforward application of (CI) and Lemma 1(L.1.) gives us that $F(A)$ contains all $x \in A$ with $\Pi_{i \in N} x_i = \Pi_{i \in N} a_i$. Therefore, $F(A)$ is the Nash solution. \diamond

Theorem 2: A bargaining solution F over Σ is the egalitarian solution if and only if it satisfies (WE), (SS) and (CI).

Proof. It can be checked that if F is the egalitarian solution over Σ then it satisfies the three axioms in Theorem 2. Thus, we need only to show that if a bargaining solution F over Σ satisfies (WE), (SS) and (CI), then it must be the egalitarian solution.

Let F over Σ satisfy the above three axioms. Given any bargaining problem $A \in \Sigma$, let $a \in A$ be such that it is weakly efficient in A and that $a_1 = \dots = a_n$. We need to show that $F(A) = \{a\}$. Define the real number γ

as $\gamma \equiv \max \{m_i(A) \mid i = 1, \dots, n\}$, and the vectors $x^i = (\gamma; a_{-i})$, $i = 1, \dots, n$. Consider the bargaining problem $B = \text{comp}(x^1, x^2, \dots, x^n)$. From the construction, B is symmetric and $A \subseteq B$. By (SS) and (WE), $F(B) = \{a\}$. Noting that $A \subseteq B$, by (CI), $F(A) = \{a\}$. This completes the proof of Theorem 2. \diamond

Theorem 3: A bargaining solution F over Σ is the Kalai-Smorodinsky solution if and only if it satisfies (WE), (SS), (SI) and (WCI).

Proof. It can be checked that if F is the Kalai-Smorodinsky solution over Σ then it satisfies the four axioms in Theorem 3. Thus, we need only to show that if a bargaining solution F over Σ satisfies (WE), (SS), (SI) and (WCI), then it must be the Kalai-Smorodinsky solution.

Let F over Σ satisfy the above four axioms. Given any bargaining problem $A \in \Sigma$, by (SI), without loss of generality, we take that $[m_i(A) = m_j(A)$ for all $i, j \in N]$. We need to show that if a is weakly efficient in A and $[a_i = a_j$ for all $i, j \in N]$, then $F(A) = \{a\}$. This is done by following a similar argument as for proving Theorem 2. Therefore, Theorem 3 is proved. \diamond

To conclude this section, we note that it is easy to check the independence of the axioms used in the three characterizations.

5 Conclusion

In this paper, we have presented a unified framework to provide axiomatic characterizations of the extensions of the three classical bargaining solutions for nonconvex bargaining problems. Our characterizations are simpler than those existing in the literature. Our axioms are various natural generalizations of the axioms used in Nash's original discussion of the bargaining problems for convex bargaining problems. The following table summarizes our findings:

Table 1

Axioms \ Solutions	NS	ES	KS
(E)	\oplus	\times	\times
(WE)	\circ	\oplus	\oplus
(S)	\oplus	\circ	\circ
(SS)	\times	\oplus	\oplus
(SI)	\oplus	\times	\oplus
(CI)	\oplus	\oplus	\times
(WCI)	\circ	\circ	\oplus

where

NS is for Nash Solution, ES for Egalitarian Solution, and KS for Kalai-Smorodinsky Solution

\oplus stands for that the axiom is used for the characterization,

\circ stands for that the axiom is satisfied by the solution,

\times stands for that the axiom is violated by the solution.

Clearly, (WE), (S) and (WCI) are satisfied by all three solutions. It is also clear that the Nash solution satisfies all the axioms but (SS), the egalitarian solution satisfies all the axioms but (E) and (SI), and the Kalai-Smorodinsky solution satisfies all but (E) and (CI). Note that Theorem 2 (*resp.* Theorem 3) constitutes a strengthening of the characterization of the egalitarian solution (*resp.* the Kalai-Smorodinsky solution) by Conley and Wilkie (1991), since (CI) (*resp.* (WCI)) is logically implied by the monotonicity axiom (*resp.* the *weak monotonicity* axiom) of Kalai (1977) in the presence of (WE).

It is also interesting to note that the Kalai-Smorodinsky solution has some constrained contraction property. It implies that once a partition of the class of bargaining problems is defined, where each equivalence class of this partition consists of the bargaining problems with the same ideal point, then the Kalai-Smorodinsky solution is rationalizable within each equivalence class of the problems. This fact gives us some insight on the rational choice property of this solution, although it was widely considered that it has no rational choice property.

We hope that our characterizations will shed some new light on the three solutions for nonconvex bargaining problems.

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