A New Insight into Three Bargaining Solutions in Convex Problems*

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Abstract

We reconsider the three well-known solutions: the Nash, the egalitarian and the Kalai-Smorodinsky solutions, to the classical domains of convex (bargaining) problems. A new proof for the Nash solution that highlights the crucial role the axiom Contraction Independence plays is provided. We also give new axiomatic characterizations for both the egalitarian and the Kalai-Smorodinsky solutions. Our results focus on both contraction and expansion independence properties of problems and, as a consequence, some new insights on the three solutions from the perspective of rational choice may be derived.

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1 Introduction

This paper reconsiders some well-known solutions to convex bargaining problems. Our purpose is two-fold. First, we provide a new proof for the Nash solution that highlights the crucial role the axiom Contraction Independence plays. Our proof method is proof-by-contradiction. Secondly, by employing similar proof methods as for the Nash solution, we provide new axiomatic characterizations for both the egalitarian and the Kalai-Smorodinsky solutions. Instead of using any monotonicity type axiom, which is commonly used in the literature for characterizing these two solutions (see, for example, Kalai (1977), Kalai and Smorodinsky (1975); see also Peters (1992) and Thomson (1994) for excellent surveys), we use variants of Contraction Independence and Expansion Independence to characterize the egalitarian and the Kalai-Smorodinsky solutions. Both Contraction Independence and Expansion Independence properties figure prominently in the theory of rational choice. Our new characterizations therefore may shed some new insights into the three well-known solutions to the problems.

The remainder of the paper is organized as follows. Section 2 provides a basic framework for the subsequent analysis. Section 3 presents the axioms. Our main results and their proofs are contained in Section 4. Section 5 makes several concluding remarks.

2 Basic Model

The set of players is to be denoted by $N = \{1, 2, \ldots, n\}$ where $n \geq 2$. We use $\mathbf{R}_+$ to denote the set of all non-negative real numbers, while $\mathbf{R}^n_+$ is used to denote the n-fold Cartesian product of $\mathbf{R}_+$. For each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, we write $\mathbf{x} > \mathbf{y}$ as $[x_i \geq y_i$ for each $i \in N$ and $\mathbf{x} \neq \mathbf{y}]$ and $\mathbf{x} \gg \mathbf{y}$ as $[x_i > y_i$ for each $i \in N]$.

Let π be a permutation of N. For each $\mathbf{x} = (x_i)_{i \in N} \in \mathbb{R}^n_+$, let $\pi(\mathbf{x}) = (x_{\pi(i)})_{i \in N}$. Let Π be the set of all permutations of N.

Let $\mathcal{B}$ be the set of all compact, convex, and comprehensive subsets of $\mathbb{R}^n_+$, each of which contains an interior point of $\mathbb{R}^n_+$. Elements in $\mathcal{B}$ are interpreted as *normalized (bargaining) problems.* For each $A \in \mathcal{B}$ and any $\pi \in \Pi$, let $\pi(A) = \{\pi(\mathbf{a}) \mid \mathbf{a} \in A\}$. For each $A \in \mathcal{B}$, A is a *symmetric problem* if $A = \pi(A)$ for all $\pi \in \Pi$.

For each $A \in \mathcal{B}$ and each $i \in N$, let $m_i(A) = \max\{a_i \mid (a_1, \cdots, a_i, \cdots, a_n) \in$

$A\}$. Therefore, $\mathbf{m}(A) \equiv (m_i(A))_{i \in N}$ is the *ideal point* of A.

For each $\mathbf{x} \in \mathbb{R}^n_+$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, let $\boldsymbol{\alpha}(\mathbf{x}) \equiv (\alpha_i x_i)_{i \in N}$. Given $A \in \mathcal{B}$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, let $\boldsymbol{\alpha}(A) \equiv \left\{\boldsymbol{\alpha}(\mathbf{x}) \in \mathbb{R}^n_+ \mid \mathbf{x} \in A\right\}$. For each A in $\mathbb{R}^n_+$, we define the *comprehensive hull of* A by

$$compA \equiv \left\{\mathbf{z} \in \mathbb{R}^n_+ \mid \exists \mathbf{x} \in A : \mathbf{z} \leq \mathbf{x}\right\}.$$

Let the *convex hull of* A be denoted by *conA*. The convex hull of *compA* will be called the *convex and comprehensive hull* of A, and will be denoted by *concompA*.

A *solution* F is a single-valued mapping from $\mathcal{B}$ to $\mathbb{R}^n_+$ such that for every problem $A \in \mathcal{B}$, $F(A) \in A$. For given $F(A) \in A$, let $F_i(A) \in \mathbb{R}_+$ be its i-th component. The following three are well-known solutions.

Nash Solution F^{NA}: *For every $A \in \mathcal{B}$, $F^{NA}(A) = \arg\max_{(a_i)_{i \in N} \in A} \prod_{i \in N} a_i$.*

Kalai-Smorodinsky Solution F^{KS}: *For every $A \in \mathcal{B}$, $F^{KS}(A) \in A$ implies that: (1) there is no other $\mathbf{a} \in A$ such that $\mathbf{a} \gg F^{KS}(A)$; and (2) there exists $\gamma \in (0,1)$ such that $F^{KS}(A) = \gamma \cdot \mathbf{m}(A)$.*

Egalitarian Solution F^E: *For every $A \in \mathcal{B}$, $F^E(A) \in A$ implies that: (1) there is no other $\mathbf{a} \in A$ such that $\mathbf{a} \gg F^E(A)$; and (2) $F^E_i(A) = F^E_j(A)$ for all $i, j \in N$.*

3 Axioms

We consider the following axioms:

Efficiency (E): *For each $A \in \mathcal{B}$, there is no $\mathbf{x} \in A$ such that $\mathbf{x} > F(A)$.*

Weak Efficiency (WE): *For each $A \in \mathcal{B}$, there is no $\mathbf{x} \in A$ such that $\mathbf{x} \gg F(A)$.*

Symmetry (S): *For each $A \in \mathcal{B}$, if A is symmetric, then $F_i(A) = F_j(A)$ for all $i, j \in N$.*

Scale Invariance (SI): *For each $A, B \in \mathcal{B}$, and each $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, if $B = \boldsymbol{\alpha}(A)$, then $F(B) = \boldsymbol{\alpha}(F(A))$.*

Contraction Independence (CI): *For each* $A, B \in \mathcal{B}$*, if* $A \supseteq B$ *and* $F(A) \in B$*, then* $F(B) = F(A)$*.*

Weak Contraction Independence (WCI): *For each* $A, B \in \mathcal{B}$ *such that* $\mathbf{m}(A) = \mathbf{m}(B)$*, if* $A \supseteq B$ *and* $F(A) \in B$*, then* $F(B) = F(A)$*.*

Expansion Independence (EI): *For each* $A, B \in \mathcal{B}$*, if* $A \subseteq B$ *and* $F(A)$ *is efficient on* B*, then* $F(B) = F(A)$*.*

Weak Expansion Independence (WEI): *For each* $A, B \in \mathcal{B}$ *such that* $\mathbf{m}(A) = \mathbf{m}(B)$*, if* $A \subseteq B$ *and* $F(A)$ *is efficient on* B*, then* $F(B) = F(A)$*.*

The first five axioms are standard ones discussed in the literature on convex problems (see, for example, Peters (1992) and Thomson (1994) for discussions). **WCI** is due to Yu (1973). It restricts its applicability to contraction situations in which the ideal point remains unchanged. **EI** requires that, when a problem A is enlarged to another problem B, if the solution $F(A)$ to A continues to be efficient on B, then $F(A)$ should continue to be the solution to the problem B. The idea is that, even though there is an enlargement of "opportunities" from A to B, given that $F(A)$ is both efficient on A and on B, and that $F(A)$ is already the solution to the original problem A, any movement away from $F(A)$ will hurt at least one player, and thus the solution to the enlarged problem B should continue to be $F(A)$. This requirement suggests a *solidarity* type property embedded in the solution. This can also be seen as stating a certain inertia of the choice process. **WEI** is weaker than **EI** in that it restricts its applicability to situations where the ideal point remains unchanged.

We note that **EI** is logically weaker than the following axiom, *Independence of Undominated Alternatives*, which is proposed in Thomson and Myerson (1980):

Independence of Undominated Alternatives (IUA): *For each* $A, B \in \mathcal{B}$*, if* $A \subseteq B$ *and* $F(A)$ *is weakly efficient on* B*, then* $F(B) = F(A)$*.*

It is worth noting that **CI** and **EI** are logically implied by the *Monotonicity* axiom, which is introduced by Kalai (1977), together with **WE**, but the converse relation does not hold. In fact, the *monotone path solution*, which is proposed by Thomson and Myerson (1980) and which is characterized by **WE** and the monotonicity axiom of Kalai (1977), satisfies both **CI** and **EI**.

On the other hand, we can construct a *non-monotone path solution* which satisfies **EI**, **CI** and **WE**, and which violates the monotonicity axiom of Kalai (1977).

4 Results and Their Proofs

This section presents our main results and their proofs follow.

Theorem 1: *F^{NA} is the unique solution satisfying* **E**, **S**, **SI**, *and* **CI**.

Proof. It can be checked that F^{NA} satisfies the four axioms of Theorem 1. We therefore show that if a solution satisfies the four axioms of Theorem 1, then it must be F^{NA}.

Let F be a solution satisfying the four axioms of Theorem 1. For each $A \in \mathcal{B}$, we first show that:

For each $\mathbf{x}$ and $\mathbf{a}$ that are both efficient in A, and if $\mathbf{x} \in A$ is such that $\prod_{i \in N} x_i < \prod_{i \in N} a_i$, then $\mathbf{x} \neq F(A)$.

Let $\mathbf{x}$ and $\mathbf{a}$ be such that both are efficient in A and $\prod_{i \in N} x_i < \prod_{i \in N} a_i$. We note that, since $\prod_{i \in N} x_i \geq 0$ and $\prod_{i \in N} x_i < \prod_{i \in N} a_i$, it follows that $\prod_{i \in N} a_i > 0$. It is therefore clear that $a_i > 0$ for each $i \in N$. Suppose to the contrary that $\mathbf{x} = F(A)$. Consider $B \equiv$ *concomp* $\{\mathbf{x}, \mathbf{a}\}$. By **CI**, it follows that $\mathbf{x} \in F(B)$.

Let $\beta \equiv \min_{i \in N}\{a_i\}$ and $\alpha \equiv (\beta/a_1, \ldots, \beta/a_i, \ldots, \beta/a_n)$. Note that each $a_i (i \in N)$ is positive. α is well-defined. Then, $\boldsymbol{\alpha}(\mathbf{a}) = (\beta, \ldots, \beta)$. Denote $B' \equiv \boldsymbol{\alpha}(B)$, $\mathbf{a}' \equiv \boldsymbol{\alpha}(\mathbf{a})$, and $\mathbf{x}' \equiv \boldsymbol{\alpha}(\mathbf{x})$. Note that $\mathbf{a}' \neq \mathbf{x}'$. By **SI**, $F(B') = \mathbf{x}'$.

Consider the set $[\cup_{\pi \in \Pi} \pi(B')]$, and denote it by C. By construction, noting that $\prod_{i \in N} x_i < \prod_{i \in N} a_i$, C is symmetric, convex, and $C \supseteq B'$. Moreover, by the construction of B', both $\mathbf{a}'$ and $\mathbf{x}'$ are efficient on C. By **E** and **S**, it follows that $F(C) = \mathbf{a}'$. By **CI**, $F(B') = \mathbf{a}'$, which is a contradiction. Therefore, $\mathbf{x} \neq F(A)$.

From the above, it follows that, for every $A \in \mathcal{B}$,

$$F(A) \subseteq \left\{ \mathbf{a} \in A \mid \forall \mathbf{x} \in A : \prod_{i \in N} a_i \geq \prod_{i \in N} x_i \right\}.$$

Since the right-hand set is a singleton, the non-emptiness of F implies that $F = F^{NA}$. ■

Theorem 2: *F^E is the unique solution satisfying* **WE**, **S**, **CI**, *and* **EI**.

Proof. It can be checked that if F^E satisfies the four axioms of Theorem 2. Therefore, we need only to show that if a solution satisfies the four axioms of Theorem 2, it must be F^E.

Let F be a solution satisfying the four axioms of Theorem 2. By nonemptiness of F and **WE**, we need only to show the following

For each $A \in \mathcal{B}$, each $\mathbf{x}$ and $\mathbf{a}$ that are weakly efficient in A, if $[a_i = a_j$ for all $i, j \in N]$, but $[x_i \neq x_j$ for some $i, j \in N]$, then $\mathbf{x} \neq F(A)$.

Let $\mathbf{x}$ and $\mathbf{a}$ be such that both are weakly efficient on A, $[a_i = a_j$ for all $i, j \in N]$, and $[x_i \neq x_j$ for some $i, j \in N]$. Suppose to the contrary that $\mathbf{x} = F(A)$. Consider $B \equiv comp\{\mathbf{x}\}$. Note that $B \subseteq A$. By **CI**, $\mathbf{x} = F(B)$.

Consider the set $con\left[\cup_{\pi \in \Pi} \pi(B)\right]$, and denote it by C. By construction, C is a symmetric convex set having $C \supseteq B$. By the construction of B and C, $\mathbf{x}$ is efficient on C. Therefore, noting that $\mathbf{x} = F(B)$, $B \subseteq C$ and $\mathbf{x}$ is efficient on C, $\mathbf{x} = F(C)$ follows from **EI**. Since C is symmetric, by **WE** and **S**, $F(C)$ must be weakly efficient and be the equal utility point, which is a contradiction. Therefore, $\mathbf{x} \neq F(A)$. This proves the above statement and thus Theorem 2. ■

Theorem 3: *F^{KS} is the unique solution satisfying* **WE**, **S**, **SI**, *and* **WCI**, *and* **WEI**.

Proof. It can be checked that F^{KS} satisfies the five axioms of Theorem 3. We need only to show that if a solution satisfies the five axioms of Theorem 3, it must be F^{KS}.

Let F satisfy the five axioms of Theorem 3. Given each problem $A \in \mathcal{B}$, by **SI**, without loss of generality, we take that $[m_i(A) = m_j(A)$ for all $i, j \in N]$. We need to show that if a is weakly efficient in A and $[a_i = a_j$ for all $i, j \in N]$, then $F(A) = \{a\}$. This is done by following a similar argument as for proving Theorem 2. Therefore, Theorem 3 is proved. ■

Remark 1: *It can be verified that F^E is also characterized by* **WE**, **S**, *and* **IUA**. *Note that if we use the axiom* **IUA**, *which is stronger than* **EI**, *in the characterization of F^E*, **CI** *becomes superfluous and thus can be dropped out.*

Remark 2: *If $\#N = 2$, then F^{KS} is characterized by* **E**, **S**, **SI**, *and* **WEI**. *Thus,* **WCI** *is no longer indispensable to characterize this solution in two person problems.*

To conclude this section, we note that the independence of the respective axioms used in Theorems 2 and 3 can be checked.

5 Concluding Remarks

Our results on the characterizations of the three solutions are summarized in the following table.

Table 1

Axioms\ Solutions	NA	ES	KS
E	$+^*$	$-$	$-$
WE	$+$	$+$	$+^*$
S	$+^*$	$+^*$	$+^*$
SI	$+^*$	$-$	$+^*$
CI	$+^*$	$+^*$	$-$
WCI	$+$	$+$	$+^*$
EI	$-$	$+^*$	$-$
WEI	$-$	$+$	$+^*$

where

$+^*$ stands for that the axiom is used for the characterization,

$+$ stands for that the axiom is satisfied by the solution,

$-$ stands for that the axiom is violated by the solution.

Clearly, all three solutions satisfy axioms **WE**, **S** and **WCI**. The Nash solution satisfies all but **EI** and **WEI**, the egalitarian solution satisfies all but **E** and **SI**, and the Kalai-Smorodinsky solution violates **E**, **CI** and **EI** while satisfies all the other axioms. It is also worth noting that Theorem 2 (*resp.* Theorem 3) constitutes a strengthening of the original characterization of the egalitarian solution (*resp.* the Kalai-Smorodinsky solution) by Kalai (1977) (*resp.* Kalai and Smorodinsky (1975)), since the combination of **CI** and **EI** (*resp.* **WCI** and **WEI**) is logically weaker than the monotonicity axiom (*resp.* the *weak monotonicity* axiom).

As far as contraction and expansion properties are concerned, it is interesting to note that the egalitarian solution satisfies all the contraction and expansion properties discussed in this paper, the Nash solution fails the two expansion properties while survives the two contraction properties, and the Kalai-Smorodinsky solution satisfies the weaker versions of contraction and expansion properties. The fact that the Kalai-Smorodinsky solution has some constrained contraction and expansion properties gives us some insights on the rational choice property of this solution.[1]

References

1. Kalai, E. (1977): "Proportional solutions to bargaining situations: interpersonal utility comparisons," *Econometrica* 45, 1623-1630.

2. Kalai, E. and M. Smorodinsky (1975): "Other solutions to Nash's bargaining problem," *Econometrica* 43, 513-518.

3. Nagahisa, R. and M. Tanaka (2002): "An axiomatization of the Kalai-Smorodinsky solution when the feasible sets can be finite," *Social Choice and Welfare* 19, 751-761.

4. Nash, J. F. (1950): "The bargaining problem," *Econometrica*, 18, 155-162.

5. Peters, H. J. M. (1992): *Axiomatic Bargaining Game Theory*, Kluwer Academic Press.

6. Thomson, W. (1994): "Cooperative Models of Bargaining," in *Handbook of Game Theory with Economic Applications*, Aumann, R. J. and S. Hart (eds.), Elsevier.

[1]Nagahisa and Tanaka (2002) observe that, in the context of a domain containing finite problems, the Kalai-Smorodinsky solution satisfies a similar constrained contraction property.

7. Thomson, W. and R. B. Myerson (1980): "Monotonicity and Independence Axioms," *International Journal of Game Theory* 9, 37-49.

8. Yu, P. L. (1973): "A class of solutions for group decision problems," *Management Science* 19, 936-946.