Proportional Nash solutions –A new and procedural analysis of nonconvex bargaining problems*

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This Version: 23 June 2011

* An earlier version of the paper was presented at the SEA meetings in Atlanta, Georgia, November 2010 and at the CEPET meeting in Udine, Italy, June 2011. We are grateful to M.A. Ballester, Youngsub Chun, Marco Mariotti, Hans Peters, Koichi Tadenuma, and William Thomson for helpful and encouraging comments.

Abstract. This paper studies the Nash solution to nonconvex bargaining problems. The Nash solution in such a context is typically multi-valued. We introduce a procedure to exclude some options recommended by the Nash solution. The procedure is based on the idea of the Kalai-Smorodinsky solution which has the same informational requirement on individual utilities as the Nash solution does and has an equity consideration as well. We then use this procedure to introduce two new solutions to nonconvex bargaining problems and study them axiomatically.
J.E.L. Classification Numbers: C71, C78, D6, D7

1 Introduction

In this paper, we study solutions to nonconvex bargaining problems from a *procedural* perspective by employing the Nash solution (Nash (1950)) and the Kalai-Smorodinsky (KS) solution (Kalai and Smorodinsky (1975)) sequentially. We know that the Nash solution has a greater efficiency consideration than the KS solution while the KS solution has a greater equity consideration than the Nash solution. The sequential applications of the two solutions thus combine both efficiency concerns and equity concerns in formulating a solution to nonconvex bargaining problems. More precisely, in this paper, we define and study solutions to nonconvex bargaining problems by first employing the Nash solution concept to make recommendations for nonconvex bargaining problems and —among the options recommended by the Nash solution— then we use a procedure based on the idea of the KS solution with reference to an *ideal point* to select final options for the given bargaining problem.

Our proposed solutions have several features. First, the solutions proposed in this paper are refinements of the Nash solution to nonconvex bargaining problems. In nonconvex bargaining problems, the Nash solution is typically multi-valued (see, among others, Kaneko (1980), Mariotti (1998, 1999), Wu (2006), Xu and Yoshihara (2006), and Peters and Vermeulen (2007))[1]: for a given nonconvex bargaining problem, the recommendation by the Nash solution may contain more than one option. This is because, as Mariotti (1998, 1999) shows, there is no single-valued solution satisfying the Nash axioms in nonconvex problems. Due to this multi-valuedness of the Nash solution, it would be desirable to have a *procedure* to exclude some options recommended by the Nash solution, if in particular such options are deemed to be *undesirable* from the viewpoint of distributive fairness. Given this, a natural way of proposing a refinement of the Nash solution would be to introduce a procedure based on an equity concept.[2] One such procedure

[1]Note that Kaneko (1980) proposes and characterizes the Nash solution in nonconvex problems as the set of Nash product maximizers, and Mariotti (1998, 1999), Xu and Yoshihara (2006) provide alternative characterizations of this solution, while Herrero (1989) defines a generalization of Nash solution on strictly comprehensive two-person domain as a superset of the Nash product maximizers.

[2]Note that, in nonconvex problems, Zhou (1997) investigates a single-valued selection from the asymmetric Nash solution by dropping the anonymity axiom, whereas Conley and Wilkie (1996) propose and characterize an extension of the Nash solution by keeping single-valuedness while dropping the efficiency axiom. In contrast to these works, this

is based on the idea of the KS solution: the KS solution has the same informational requirement on individual utilities as the Nash solution does, and has an equity consideration linking the utility gains of the players with respect to their maximum attainable utilities by equalizing proportional gains of individual utilities.

Secondly, the sequential procedure of our solutions is a natural method to reconcile the conflicting criteria such as efficiency and equity considerations. Indeed, Tadenuma (2002) has considered the sequential application of efficiency and fairness criteria to problems of resource allocations under exchange economies. Moreover, there is another recent literature on sequentially rationalizable choices and on bounded rational choice (see, for example, Apesteguia and Ballester (2008), Manzini and Mariotti (2006), and Manzini and Mariotti (2007)), which discuss similar issues as this paper does. In those papers, however, they study sequential rationalizability of choice functions. Specifically, they ask the following question: can the observed choices be explained by first employing an incomplete binary relation over the universal set to exclude some options and then employing another binary relation over the same universal set to select certain options from those surviving the elimination in the first round? At this point, we may note that neither of our solutions can respectively be explained as a specific sequentially rationalizable choice function. In our context, the observed solution outcomes can be explained by employing the strict part of the ordering based on the Nash product to exclude certain options in the first round, and then employing an ordering based on the idea of the KS solution to select certain options from those surviving the first round elimination. However, the second-round binary relation cannot be defined *independently* of option sets, and can only be defined *endogenously* for each option set,[3] which makes our solutions different from sequentially rationalizable choice functions studied in the above-mentioned literature. See Section 4 for a further elaboration on this feature of our solutions.

It is an attractive feature of our solutions not to be sequentially rationalizable. This is because fairness criteria are context dependent as they often use information about feasible options. The notion of fairness is more *pro-*

paper investigates a refinement of the Nash solution by explicitly introducing a procedure for the refinement while keeping most of Nash's axioms.

[3]This property is not the same as allowing the "menu effects", since there is a sequentially rationalizable solution to nonconvex bargaining problems which allows the "menu effects", as Manzini and Mariotti (2006) discuss.

cedurally based, and the important information to make a fairness judgment may come from what social outcomes being selected *from* which feasible option sets, rather than just from what social outcomes being chosen. In this case, the corresponding binary relation over options may not be defined independently of the information about the underlying option sets. The idea of fairness embedded in the KS solution that we are interested in this paper is procedurally based.

It may be noted that even if our solutions are not sequentially rationalizable, their first round choice procedures themselves are rationalizable. This property of our solutions seems reasonable whenever the first criterion is based on efficiency considerations. Efficiency considerations are often such that the underlying binary relations over options are independent of the sets of feasible options. If one is interested in employing a solution that is efficient as a first round choice procedure, then the question, which solution should be employed, arises naturally. As we shall discuss in Section 3, if the first round choice procedure is required to reflect the efficiency criterion and be rationalizable, then one is left with the Nash solution concept in a framework of cardinal and non-comparable individual utilities. This gives us a justification for employing the Nash solution as the first round choice procedure.

The two new solutions proposed and studied in the paper combine the Nash solution and the KS solution sequentially with different ideal points for nonconvex bargaining problems. In the first place, we choose the ideal point of the originally given problem as the *referenced ideal point* and define our first solution accordingly. In the second place, we choose the ideal point of the problem derived from the Nash solution to the original problem as the referenced ideal point and define our second solution based on this ideal point. It can be argued that our two solutions attempt to select those maximum-Nash-product points that are also "equitable".[4]

A chosen referenced ideal point may reflect players' perspectives about their maximum attainable utilities that are "feasible". For example, when the ideal point of the original problem is used as a reference point for making further recommendations, the players could be viewed as not updating the information and continuing to believe the attainability of the original

[4]Note that Conley and Wilkie (1996) propose an "equitable" Nash extension solution in nonconvex problems, which is taken as a hybrid of the Nash and the KS solutions. However, this solution is very different from the two solutions proposed in this paper, in the sense that their solution does not satisfy the efficiency criterion and is not a refinement of the Nash solution.

maximum attainable utilities. On the other hand, when the referenced ideal point is the ideal point of the problem derived from the Nash solution to the original problem, the players could be viewed as having updated the information and as realizing that the final chosen options must be from the set of the Nash solution to the original problem, and, as a consequence, they would re-adjust their views on the maximum attainable utilities. Putting this observation differently, our first solution seems to suggest no "learning" and no "updating" on the part of the players while our second solution seems to suggest that there is learning and updating about the perceived maximum attainable utilities from the players.

The remainder of the paper is organized as follows. In Section 2, we present notations and definitions. Section 3 introduces a set of our core axioms and studies their implications on nonconvex problems. Section 4 introduces the two new solutions. Section 5 is devoted to the study of our first solution, and Section 6 is concerned with our second solution. We conclude in Section 7.

2 Notation and definitions

Let $N = \{1, 2, \ldots, n\}$ be the set of players with $n \geq 2$. Let $\mathbf{R}_+$ be the set of all non-negative real numbers and $\mathbf{R}_{++}$ be the set of all positive numbers. Let $\mathbf{R}^n_+$ (resp. $\mathbf{R}^n_{++}$) be the n-fold Cartesian product of $\mathbf{R}_+$ (resp. $\mathbf{R}_{++}$). For any $x, y \in \mathbf{R}^n_+$, we write $x \geq y$ to mean [$x_i \geq y_i$ for all $i \in N$], $x > y$ to mean [$x_i \geq y_i$ for all $i \in N$ and $x \neq y$], and $x \gg y$ to mean [$x_i > y_i$ for all $i \in N$]. For any $x \in \mathbf{R}^n_+$ and any non-negative number α, we write $z = (\alpha; \mathbf{x}_{-i}) \in \mathbf{R}^n_+$ to mean that $z_i = \alpha$ and $z_j = x_j$ for all $j \in N \setminus \{i\}$. For any subset $A \subseteq \mathbf{R}^n_+$, A is said to be (i) *non-trivial* if there exists $a \in A$ such that $a \gg 0$, and (ii) *comprehensive* if for all $x, y \in \mathbf{R}^n_+$, [$x > y$ and $x \in A$] implies $y \in A$. For all $A \subseteq \mathbf{R}^n_+$, define the *comprehensive hull of* A, to be denoted by *compA*, as follows:

$$compA \equiv \left\{ z \in \mathbf{R}^n_+ \mid z \leq x \text{ for some } x \in A \right\}.$$

Let Σ be the set of all non-trivial, compact and comprehensive subsets of $\mathbf{R}^n_+$. Elements in Σ are interpreted as (normalized) bargaining problems. A *bargaining solution* F assigns a nonempty subset $F(A)$ of A for every bargaining problem $A \in \Sigma$.

Let π be a permutation of N. The set of all permutations of N is denoted by Π. For all $x = (x_i)_{i \in N} \in \mathbf{R}^n_+$, and any permutation π, let $\pi(x) = (x_{\pi(i)})_{i \in N}$. For all $A \in \Sigma$ and any permutation π, let $\pi(A) = \{\pi(a) \mid a \in A\}$. For any $A \in \Sigma$, we say that A is *symmetric* if $A = \pi(A)$ for all permutations π over N.

For all $A \in \Sigma$ and all $i \in N$, let $m_i(A) = \max\{a_i \mid (a_1, \cdots, a_i, \cdots, a_n) \in A\}$. Therefore, $m(A) \equiv (m_i(A))_{i \in N}$ is the *ideal point* of A.

Definition 1: A bargaining solution F over Σ is the *Nash solution* if for all $A \in \Sigma$, $F(A) = \{a \in A \mid \prod_{i \in N} a_i \geq \prod_{i \in N} x_i \text{ for all } x \in A\}$.

Denote the Nash solution by F^N. Note that, for nonconvex bargaining problems, the Nash solution is typically multi-valued.

3 Basic axioms and their implications

In this section, we present several standard axioms and examine their implications for solutions to bargaining problems. We begin by introducing three axioms, Efficiency, Anonymity and Scale Invariance, which are standard in the literature on Nash bargaining problems.

Efficiency (E): For any $A \in \Sigma$ and any $a \in F(A)$, there is no $x \in A$ such that $x > a$.

Anonymity (A): For any $A \in \Sigma$, if A is symmetric, then $[a \in F(A) \Rightarrow \pi(a) \in F(A)$ for any permutation π over $N]$.

Scale Invariance (SI): For all $A \in \Sigma$ and all $\alpha \in \mathbf{R}^n_{++}$, if $\alpha A = \{(\alpha_i a_i)_{i \in N} \mid a \in A\}$, then $F(\alpha A) = \{(\alpha_i a_i)_{i \in N} \mid a \in F(A)\}$.

The following axiom is a weaker version of Weak Axiom of Revealed Preference (WARP) used in the theory of revealed preference (Samuelson (1938, 1947)). The standard (WARP) such as defined in Sen (1971) requires that if an option z is "revealed to be worse" than another option x in the sense that there is a problem containing x and z from which x is chosen but z is not, then z should not be chosen from any problem as a solution as long as x is available. The following axiom is much weaker than the standard (WARP) in that its premise only takes a specific type of problem: the comprehensive hull of x and another point containing z, from which x is chosen but z is not. Note that an option set as a pair of two points is typically discussed in the rational choice theory. The comprehensive hull of two points is a natural generalization of a two-point set in rational choice theory for the domain of bargaining problems considered here. The axiom is stated as follows:

Binary Weak Axiom of Revealed Preference (BWARP): For all $x, y, z \in \mathbf{R}^n_+$ with $y > z$, if either $\{x, y\} = F(comp\{x, y\})$ or $\{x\} = F(comp\{x, z\})$, then $z \notin F(A)$ for any $A \in \Sigma$ with $x, z \in A$.

(BWARP) requires that, if an option z is "revealed to be worse" than another option x via a "pairwise comparison" involving x and z directly or through a "pairwise comparison" involving x and y where y vectorially dominates z, then z should not be chosen as a solution as long as x is available.

It may be noted that (BWARP) represents a weak property of rational choice: it can be checked that, if a solution satisfies (E) and (BWARP), then there is an acyclic relation P over $\mathbf{R}^n_+$ such that for any $A \in \Sigma$, $[x \in F(A) \Rightarrow \nexists y \in A$ s.t. $yPx]$.

We now explore the implications of the above axioms being imposed on a solution to nonconvex bargaining problems. Our first result shows that, when a solution satisfies (E), (A), (SI), and (BWARP), then, for any $x, y \in \mathbf{R}^n_+$, the solution to the problem $A = comp\{x, y\}$ must be such that $F(A) = \{x, y\}$ if $\Pi_{i \in N} x_i = \Pi_{i \in N} y_i > 0$ and $F(A) = \{x\}$ if $\Pi_{i \in N} x_i > \Pi_{i \in N} y_i$; that is, for a specific, simple problem given by A defined earlier, these axioms imply that the solution to A must be given by the Nash solution. After establishing our first result, we show, in Proposition 2, that, when a solution satisfies (E), (A), (SI), and (BWARP), then the solution to any bargaining problem must be a *subset* of the Nash solution.

Proposition 1. *Let a solution F satisfy* (E), (A), (SI), *and* (BWARP). *Then, for all $x, y \in \mathbf{R}^n_+$,*
(1.1) $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0 \Rightarrow F(comp\{x, y\}) = \{x, y\}$, and
(1.2) $\prod_{i \in N} x_i > \prod_{i \in N} y_i \geq 0 \Rightarrow F(comp\{x, y\}) = \{x\}$.

Proof. (1.1). Let $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0$. Consider an appropriate $\alpha \in \mathbf{R}^n_{++}$ such that αx and αy are permutations of each other (such α always exists because x and y have the same value of their respective Nash product). Let $S \equiv comp\{\alpha x, \alpha y\}$. Then, let $T \equiv \cup_{\pi \in \Pi} \pi(S)$. By construction, T is symmetric, and

$\{\pi(\alpha x), \pi(\alpha y) \mid \pi \in \Pi\} \subseteq T$ is the set of all efficient outcomes in T. Thus, $F(T) \subseteq \{\pi(\alpha x), \pi(\alpha y) \mid \pi \in \Pi\}$, and let $\alpha x \in F(T)$. Then, by (A), $\{\pi(\alpha x) \mid \pi \in \Pi\} \subseteq F(T)$. Also, since αx and αy are permutations of each other, $\alpha y \in F(T)$ by (A). Then, again by (A), $\{\pi(\alpha y) \mid \pi \in \Pi\} \subseteq F(T)$. Thus, $F(T) = \{\pi(\alpha x), \pi(\alpha y) \mid \pi \in \Pi\}$. Thus, $\alpha x, \alpha y \in F(T)$. Then, by (E) and (BWARP), $\{\alpha x, \alpha y\} = F(S)$. Thus, by (SI), $F(comp\{x, y\}) = \{x, y\}$.

(1.2). Let $\prod_{i \in N} x_i > \prod_{i \in N} y_i \geq 0$. Then, by choosing an appropriate $\varepsilon \in \mathbf{R}^n_+$ with $\epsilon > 0$, we can have $\prod_{i \in N} x_i = \prod_{i \in N} z_i$ for $z \equiv y + \varepsilon$. Then, by (1.1), $F(comp\{x, z\}) = \{x, z\}$. Then, by (BWARP), $y \notin F(comp\{x, y\})$. Thus, by (E), $F(comp\{x, y\}) = \{x\}$. $\diamond$

Proposition 2. *Let a solution F satisfy* (E), (A), (SI), and (BWARP). *Then, for any* $A \in \Sigma$, $F(A) \subseteq F^N(A)$.

Proof. Take any $A \in \Sigma$ and $x \in F(A)$. Suppose $x \notin F^N(A)$. Then, there exists $y \in F^N(A)$ such that $\prod_{i \in N} y_i > \prod_{i \in N} x_i \geq 0$. Then, by Proposition 1.2, $F(comp\{x, y\}) = \{y\}$. Then, by (BWARP), $x \notin F(A)$ since $y \in A$, which is a contradiction. $\diamond$

Our Proposition 2 can be regarded as a *partial* characterization of the Nash solution to nonconvex bargaining problems. It also shows that if we are interested in recommending from the naturally restricted domain an efficient solution having some (weak) property of rational choice such as (BWARP), we must choose from the set of options contained in the Nash solution.

It is also worth noting that Proposition 2 does not provide a full characterization of all selections from the Nash solution: by means of (E), (A), (SI), and (BWARP), the following (sub)class of *Nash-selections* is fully characterized:

$$\left\{ F \subseteq F^N \mid \forall x, y \in \mathbf{R}^n_+, \prod_{i \in N} x_i = \prod_{i \in N} y_i \geq 0 \Rightarrow F(comp\{x, y\}) = \{x, y\} \right\}.$$

Thus, there is a selection from the Nash solution, F', such that for some $x, y \in \mathbf{R}^n_+$ with $\prod_{i \in N} x_i = \prod_{i \in N} y_i \geq 0$, $F'(comp\{x, y\}) = \{x\}$ holds, which does not satisfy at least one of the four axioms in Proposition 2. In this paper, we are not interested in such a Nash-selection, since the solution like F' seems inappropriate from the point of efficiency and equity considerations.[5]

[5] This is because for any $x, y \in \mathbf{R}^n_+$ with $\prod_{i \in N} x_i = \prod_{i \in N} y_i \geq 0$ such that x and

4 Proposing Two Sequential Choice Procedures

Given Proposition 2 in the last section, if we are interested in solutions having certain properties like (E), (A), (SI) and (BWARP), then we are confined to considering alternatives recommended by the Nash solution. Given the multi-valuedness of the Nash solution, the Nash solution is not necessarily attractive as it typically contains many undesirable outcomes in terms of fairness. Thus, as alternative recommendations, we introduce two refinements of the Nash solution to nonconvex problems. Our first refinement of the Nash solution selects, for each bargaining problem A, all the points in $F^N(A)$ that also lie on the highest indifference surface given by $\min(a_1/m_1(A), \cdots, a_n/m_n(A))$ attainable in A. Specifically, it is defined as follows:

Definition 2: A bargaining solution over Σ is a *type 1 proportional Nash solution*, to be denoted by F^{1pN}, if for all $A \in \Sigma$, $F^{1pN}(A) = \{a \in F^N(A) \mid \min_{i \in N} \left\{\frac{a_i}{m_i(A)}\right\} \geq \min_{i \in N} \left\{\frac{x_i}{m_i(A)}\right\}$ for all $x \in F^N(A)\}$.

Therefore, for any given problem A, to obtain $F^{1pN}(A)$, we first find out the Nash solution, $F^N(A)$, to the bargaining problem A, and then, from the Nash solution set $F^N(A)$, we select all the points lying on the highest indifference surface given by $\min(a_1/m_1(A), \cdots, a_n/m_n(A))$. Note that the set $F^N(A)$ may not intersect the line segment linking the disagreement point $\mathbf{0}$ and the ideal point $m(A)$, and consequently, the KS solution with reference to the ideal point $m(A)$ may not exist for the problem $F^N(A)$. The highest indifference surface given by $\min(a_1/m_1(A), \cdots, a_n/m_n(A))$ is an alternative to the KS solution in our context (see, for example, Tanaka and Nagahisa (2002), Wu (2006), and Wu and Xu (2009), for this version of the KS solution to *discrete* bargaining problems). Viewed in this way, the solution F^{1pN} can be regarded as a *procedural rule* which first applies the Nash solution to *exclude* certain options in A and then applies the idea of the KS solution to *select* the final options.

Our next refinement of the Nash solution selects, for each bargaining problem, all the points in $F^N(A)$ that lie on the highest indifference surface given

y are permutations of each other, it seems inappropriate for any efficiency and equity considerations to exclude one of x and y from $F(comp\,\{x, y\})$. Thus, given (SI), for any $x', y' \in \mathbf{R}^n_+$ with $\prod_{i \in N} x'_i = \prod_{i \in N} y'_i \geq 0$, any efficiency and equity considerations may not require excluding one of x' and y' from $F(comp\,\{x', y'\})$.

by $\min(a_1/m_1(F^N(A)), \cdots, a_n/m_n(F^N(A)))$ attainable in A. Note that, in this case, the referenced ideal point is $m(F^N(A))$, where $m(F^N(A))$ is the ideal point of the bargaining problem $comp(F^N(A))$. More precisely, we introduce the following solution:

Definition 3: A bargaining solution over Σ is a *type 2 proportional Nash solution*, to be denoted by F^{2pN}, if for all $A \in \Sigma$, $F^{2pN}(A) = \{a \in F^N(A) \mid \min_{i \in N} \left\{ \frac{a_i}{m_i(compF^N(A))} \right\} \geq \min_{i \in N} \left\{ \frac{x_i}{m_i(compF^N(A))} \right\}$ for all $x \in F^N(A)\}$.

For any problem A, to obtain F^{2pN}, we again first find out the Nash solution, $F^N(A)$, to the problem A, and then from the Nash solution set $F^N(A)$, we select all the points lying on the highest indifference surface given by $\min(a_1/m_1(compF^N(A)), \cdots, a_n/m_n(compF^N(A)))$. Note that the referenced ideal point in the solution F^{2pN} is the ideal point of the problem $compF^N(A)$ which is different from the referenced ideal point for the solution F^{1pN}. Like F^{1pN}, the solution F^{2pN} can also be regarded as a procedural rule in which two procedures for selecting options are employed sequentially.

As noted in the Introduction, our two solution concepts are similar to the notion of sequential rationalizability of choice functions discussed in Manzini and Mariotti (2006, 2007) and in Tadenuma (2002). However, there is a significant difference between the two concepts. The difference is that F^{1pN} (to abuse the notion, we interpret F^{1pN} as a choice function here) cannot be a sequentially rationalizable choice function. To see this, for a given binary relation $R \subseteq \mathbf{R}^n_+ \times \mathbf{R}^n_+$ with $P(R)$ as its corresponding strict part, let $\max(A; R) \equiv \{x \in A \mid \forall y \in A : (y, x) \notin P(R)\}$. Then, a choice function C is sequentially rationalizable if there exist two binary relations R_1, R_2 over $\mathbf{R}^n_+$ such that, for any $A \in \Sigma$, $C(A) = \max(\max(A; R_1); R_2)$. Now, let $R^N \subseteq \mathbf{R}^n_+ \times \mathbf{R}^n_+$ be a binary relation such that for any $x, y \in \mathbf{R}^n_+$, $(x, y) \in R^N$ if and only if $\prod_{i \in N} x_i \geq \prod_{i \in N} y_i$. If F^{1pN} is a sequentially rationalizable choice function, then there must be another binary relation $R^{1p} \subseteq \mathbf{R}^n_+ \times \mathbf{R}^n_+$ such that for any $A \in \Sigma$, $F^{1pN}(A) = \max\left(\max\left(A; R^N\right); R^{1p}\right)$. Note, however, $F^{1pN}(A) = \max\left(\max\left(A; R^N\right); R^{1p}_A\right)$ for any $A \in \Sigma$, where $R^{1p}_A \subseteq \mathbf{R}^n_+ \times \mathbf{R}^n_+$ is such that for any $x, y \in \mathbf{R}^n_+$, $(x, y) \in R^{1p}_A$ if and only if $\min_{i \in N} \left\{ \frac{x_i}{m_i(A)} \right\} \geq \min_{i \in N} \left\{ \frac{y_i}{m_i(A)} \right\}$. This suggests that R^{1p}_A may vary according to the feasible set A, which implies that F^{1pN} is not sequentially rationalizable. In a similar fashion, we note that F^{2pN} is not sequentially rationalizable.

However, as also noted in the Introduction, the endogenous nature of the underlying second criterion R_A^{1p} is perhaps a desired feature of F^{1pN}. This is because R_A^{1p} represents an ethical principle in terms of procedural fairness stipulating what social outcomes should be selected from which feasible option set. In F^{1pN}, the necessary information about feasible option sets for identifying the ranking R_A^{1p} is of $m(A)$. A similar rationale behind F^{2pN} can be developed along the same line.

To see that F^{1pN} and F^{2pN} are (proper) refinements of the Nash solution, consider the following bargaining problem:

$$A = \mathit{comp}\{(0,0),(4,0),(2,2),(1,4),(3/2,8/3)\}$$

Then, $F^N(A) = \{(2,2),(1,4),(3/2,8/3)\}, F^{1pN}(A) = \{(2,2)\}, F^{2pN}(A) = \{(3/2,8/3)\}$. This example also suggests that, in general, F^{1pN} and F^{2pN} are two distinct solution concepts.

To conclude this section, we note that, for convex bargaining problems, both F^{1pN} and F^{2pN} coincide with the Nash solution F^N. This implies that each of F^{1pN} and F^{2pN} is, respectively, an extension of the standard Nash solution defined for convex problems to nonconvex problems.

5 Characterization of F^{1pN}

In this section, we study the behavior of the solution F^{1pN}. We begin by introducing a notation. For all $x \in \mathbf{R}_+^n$, let $\bar{x} = \max\{x_i \in \mathbf{R}_+ \mid i = 1, \cdots, n\}$ and $\underline{x} = \min\{x_i \in \mathbf{R}_+ \mid i = 1, \cdots, n\}$. Next, we note that F^{1pN} satisfies the following two axioms.

Equity Principle (EP): For all $A \in \Sigma$ such that A is symmetric and for all $x \in A$, if there exists $y \in A$ such that $\{x, y\} = F(\mathit{comp}\{x, y\})$ and $(\underline{x}, \cdots, \underline{x}) < y < (\bar{x}, \cdots, \bar{x})$, then $x \notin F(A)$.

Weak Contraction Independence (WCI): For any $A, B \in \Sigma$, if $m(A) = m(B)$, $A \subseteq B$ and $A \cap F(B) \neq \varnothing$, then $F(A) = F(B) \cap A$.

Axiom (EP) reflects an equity concern in making a solution recommendation for bargaining problems. It says that, for any symmetric problem A and for any two points, x and y, in A, if, in the simple bargaining problem formed by the comprehensive hull of x and y, they both are chosen as the solution to the problem $comp\{x, y\}$, but the maximum (*resp.* minimum) utility specified in x is greater (*resp.* less) than the maximum (*resp.* minimum) utility specified in y (so that the utility distribution under y is more evenly distributed than under x), then x should not be chosen from A. It may be noted that this equity idea is originated from the Hammond equity (Hammond (1976)) which is embedded in the requirements that $(\underline{x}, \cdots, \underline{x}) < y < (\bar{x}, \cdots, \bar{x})$ and that A is a symmetric problem. Axiom (WCI) is a familiar axiom used for characterizing the KS solution for nonconvex bargaining problems (see, for example, Xu and Yoshihara (2006)).

With the help of axioms (EP) and (WCI), we obtain the following result, which is a characterization of F^{1pN}.

Theorem 1. *A solution F satisfies* (E), (A), (SI), (BWARP), (WCI), *and* (EP) *if and only if $F = F^{1pN}$.*

Proof. It can be checked that F^{1pN} satisfies (E), (A), (SI), (BWARP), (WCI), and (EP). We now show that a solution F satisfying (E), (A), (SI), (BWARP), (WCI), and (EP) must be F^{1pN}. Let F satisfy (E), (A), (SI), (BWARP), (WCI), and (EP).

Take any $A \in \Sigma$ and $x^* \in F^{1pN}(A)$. By Proposition 2, $F(A) \subseteq F^N(A)$. Take any $y \in F^N(A) \backslash F^{1pN}(A)$, and suppose $y \in F(A)$. Consider $\alpha \in \mathbf{R}^n_{++}$ such that $m_i(A') = m_j(A')$ for any $i, j \in N$, where $A' \equiv \alpha A$. Then, by (SI), $y' \in F(A')$, where $y' \equiv \alpha y$ and $x^{*\prime} \equiv \alpha x^*$. Then, consider $S \equiv comp\left\{x^{*\prime}, y', \left(m_1\left(A'\right), \mathbf{0}_{-1}\right), \ldots, \left(m_n\left(A'\right), \mathbf{0}_{-n}\right)\right\} \subseteq A'$. Then, by (WCI), $y' \in F(S)$. Also consider $S' \equiv comp\left\{x^{*\prime}, y'\right\}$. Then, by Proposition 1.1., $\{x^{*\prime}, y'\} = F(S')$. Next, consider $T \equiv \cup_{\pi\in\Pi}\pi\left(S\right)$. Then, by (A), (WCI), and (E), $\left\{\pi\left(y'\right)\right\}_{\pi\in\Pi} \subseteq F(T)$. Thus, note that $S' \cup T = T$ is symmetric and $\{x^{*\prime}, y'\} = F(comp\{x^{*\prime}, y'\})$. Let $y^{*\prime} \equiv (\overline{y}', \cdots, \overline{y}')$ and $y^{**\prime} \equiv (\underline{y}', \cdots, \underline{y}')$. Then, since $x^{*\prime} \in F^{1pN}(A')$ and $y' \in F^N(A') \backslash F^{1pN}(A')$, we have $y^{*\prime} > x^{*\prime} > y^{**\prime}$. Thus, by (EP) and (A), $\left\{\pi\left(y'\right)\right\}_{\pi\in\Pi}$ cannot be a subset of $F(T)$, which is a contradiction. Thus, for any $y \in F^N(A) \backslash F^{1pN}(A)$, $y \notin F(A)$. Combining with Proposition 2, we conclude that $F(A) \subseteq F^{1pN}(A)$. Finally, we can see that $F(A) = F^{1pN}(A)$ by (SI) and (A). $\diamond$

From Theorem 1, we can see that F^{1pN} satisfies most of the axioms which KS solution to nonconvex problems satisfies.[6] Note that it can be checked

[6]The KS solution can be defined as usual: for all $A \in \Sigma$, $F^{KS}(A) = \{x \in A \mid$

that the KS solution to nonconvex problems is characterized by the properties of *weak efficiency* and *single-valuedness* together with (A), (SI), and (WCI) (see, for example, Xu and Yoshihara (2006) and Lombardi and Yoshihara (2010)). Moreover, this solution also satisfies (BWARP) and (EP). Thus, it is solely (E) among the axioms used in Theorem 1, which the KS solution does not satisfy. In contrast, F^{1pN} does not have the property of single-valuedness. These may suggest that, in the presence of the other axioms, there is a trade-off between efficiency and single-valuedness in nonconvex bargaining problems.

To conclude this section, we discuss the independence of the axioms figured in Theorem 1 by focusing on (BWARP), (EP) and (WCI). Note that F^{2pN} satisfies (E), (A), (SI), (BWARP) and (EP), but violates (WCI), F^N satisfies (E), (A),)(SI), (BWARP) and (WCI), but violates (EP). Finally, consider the following solution F^P: for all $A \in \Sigma$, let $P(A)$ be the Pareto set of A, and let $F^P(A) = \{a \in P(A) \mid \min_{i\in N}\left\{\frac{a_i}{m_i(A)}\right\} \geq \min_{i\in N}\left\{\frac{x_i}{m_i(A)}\right\}$ for all $x \in P(A)\}$. It can be checked that F^P satisfies (E), (A), (SI), (EP) and (WCI) but violates (BWARP).

6 Characterization of F^{2pN}

In this section, we study the behavior of the solution F^{2pN}. First, let us introduce a notation. We now introduce two axioms that are satisfied by F^{2pN}:

Weak WCI (WWCI) For all $A, B \in \Sigma$ with $m(A) = m(B)$ and $P(A) \subseteq P(B)$ such that $\{x, y\} = F(comp\{x, y\})$ for all $x, y \in P(B)$, if $F(B) \cap A \neq \varnothing$, then $F(A) = F(B) \cap A$.

Efficient Frontier Contraction Independence (EFCI): For all $A, B \in \Sigma$ such that $P(A) \subseteq P(B)$ and $[\{x, y\} = F(comp\,\{x, y\})$ for all $x, y \in P\,(A)]$, if $z \notin F(comp\,\{x, z\})$ for any $z \in P\,(B)\setminus P\,(A)$ and any $x \in P\,(A)$, then $[F\,(B) \cap A \neq \varnothing$ implies $F\,(A) = F\,(B) \cap A]$.

Axiom (WWCI) is a weaker version of axiom (WCI) in that it puts certain restrictions on the applicability of bargaining problems A and B specified under (WCI): it requires that the Pareto set of A is a subset of the Pareto set

$x_1/m_1(A) = \cdots = x_n/m_n(A)$, and there exists no $y \in A$ such that $y \gg x\}$.

of B and that every two points x and y in the Pareto set of B must be chosen as solutions to the problem $comp\{x, y\}$. Essentially, (WWCI) stipulates that, in situations involving two bargaining problems A and B, if they have the same ideal point and if the Pareto set of A is contained in the Pareto set of B, whenever every two efficient points in B are "informationally equivalent" in the sense of both being recommended as solutions to the bargaining problem formed by the comprehensive hull of the two points, it must be the case that the solution to A consists of all those points that are "chosen" in B and that continue to be available in A if the chosen ones in B overlap with A. This is built on the intuition that, in such cases, the information concerning "learning" and "updating" under A is identical to that under B (in particular, for each individual, the maximum attainable utility under A is exactly the same as the maximum attainable utility under B so that the updated "ideal points" for both problems are the same), and, as a consequence, any point recommended for B must continue to be recommended for A as long as this point is available in A.

Axiom (EFCI) is another type of Contraction Independence and requires that, for any two bargaining problems A and B, if the Pareto set of A is a subset of the Pareto set B, every two points x and y in $P(A)$ are both chosen from the problem $comp\{x, y\}$, and any point z in $P(B) \setminus P(A)$ is "worse" than any point $x \in P(A)$, then the solution to A must coincide with $F(B) \cap A$ as long as $F(B) \cap A$ is not empty. Under the situation where the premise of (EFCI) is satisfied, any point z in $P(B) \setminus P(A)$ seems irrelevant for the choice under B since z is "dominated" by *every* point in $P(A)$. This suggests that in such cases, the essential features of the bargaining problem B are exactly the same as those of the bargaining problem A. It is therefore reasonable to require the coincidence of $F(A)$ with $F(B) \cap A$ whenever $F(B) \cap A$ is not empty.

With the help of axioms (EFCI) and (WWCI), we are ready to state the following result which characterizes F^{2pN}.

Theorem 2. *A solution F satisfies* (E), (A), (SI), (BWARP), (EP), (EFCI), *and* (WWCI) *if and only if* $F = F^{2pN}$.

Proof. It can be checked that F^{2pN} satisfies (E), (A), (SI), (BWARP), (EP), (EFCI), and (WWCI). We now show that a solution F satisfying (E), (A), (SI), (BWARP), (EP), (EFCI), *and* (WWCI) must be F^{2pN}.

Let F be such a solution. Take any $A \in \Sigma$ and $x^* \in F^{2pN}(A)$. By

Proposition 2, $F(A) \subseteq F^N(A)$. Take any $y \in F^N(A) \backslash F^{2pN}(A)$, and suppose $y \in F(A)$. Let $compF^N(A) \equiv \cup_{x \in F^N(A)} comp\{x\}$, and denote $A' \equiv compF^N(A)$. Then, $F^N(A') = F^N(A) \supseteq F(A)$. Moreover, since $z \in P(A) \backslash P(A')$, $z \notin F(comp\{x, z\})$ holds for any $x \in P(A')$ by Proposition 1.2, thus $F(A') = F(A) \cap A'$ follows from (EFCI). Moreover, by (BWARP), $z \notin F(A)$ holds for any $z \in P(A) \backslash P(A')$, since $z \notin F(comp\{x, z\})$ holds for any $x \in P(A') \subseteq A$. Thus, the last property and (E) together imply $F(A') = F(A)$.

Let $\alpha \in \mathbf{R}^n_{++}$ be such that $m_i(S') = m_j(S')$ for any $i, j \in N$, where $S' \equiv \alpha A'$. Then, by (SI), $y' \in F(S')$ and $x^{*\prime} \in S'$, where $y' \equiv \alpha y$ and $x^{*\prime} \equiv \alpha x^*$. Next, consider $T \equiv \cup_{\pi \in \Pi} \pi(S')$. Then, by (A), (E), and (WWCI), $F(T) = \cup_{\pi \in \Pi} \pi(F(S'))$.

Also consider $S'' \equiv comp\{x^{*\prime}, y'\}$. Then, by Proposition 1.1., $\{x^{*\prime}, y'\} = F(S'')$. Note that $S'' \cup T = T$ is symmetric. Let $y^{*\prime} \equiv (\overline{y}', \cdots, \overline{y}')$ and $y^{**\prime} \equiv (\underline{y}', \cdots, \underline{y}')$. Then, since $x^{*\prime} \in F^{2pN}(S')$ and $y' \in F^N(S') \backslash F^{2pN}(S')$, we have $y^{*\prime} > x^{*\prime} > y^{**\prime}$. Thus, by (EP) and (A), $\{\pi(y')\}_{\pi \in \Pi} \nsubseteq F(T)$. However, since $\{\pi(y')\}_{\pi \in \Pi} \subseteq \{\pi(F(S'))\}_{\pi \in \Pi} = F(T)$, this is a contradiction. Thus, for any $y \in F^N(A) \backslash F^{2pN}(A)$, $y \notin F(A)$. This implies $F(A) \subseteq F^{2pN}(A)$.

Let $x^*, y \in F^{2pN}(A)$, and suppose $y \in F(A)$ and $x^* \notin F(A)$. Since $x^*, y \in F^{2pN}(A)$, x^* and y are permutations of each other. Let us consider $A' \equiv compF^N(A)$ and $S' \equiv \alpha A'$ as $\alpha \in \mathbf{R}^n_{++}$ such that $m_i(S') = m_j(S')$ for any $i, j \in N$. Moreover, $T \equiv \cup_{\pi \in \Pi} \pi(S')$. Then, by (SI), (A), (E), and (WWCI), $F(T) = \cup_{\pi \in \Pi} \pi(F(S'))$, so that $\{\pi(y')\}_{\pi \in \Pi} \subseteq F(T)$, where $y' \equiv \alpha y$. Then, since x^* and y are permutations of each other, $\alpha x^* \in \{\pi(y')\}_{\pi \in \Pi}$, thus $\alpha x^* \in F(S')$ by (WWCI), so that $x^* \in F(A')$ by (SI). Since $F(A') = F(A)$, $x^* \in F(A)$ holds, which is a contradiction. Thus, $F(A) = F^{2pN}(A)$ holds. $\diamond$

Again, noting that the KS solution does not satisfy (EFCI), Theorem 2 suggests that F^{2pN} has the same properties except (E) and (EFCI), as the KS solution to nonconvex problems.

As for the independence of the axioms figured in Theorem 2, we focus on (BWARP), (EP), (EFCI) and (WWCI). Note that F^N satisfies (E), (A), (SI), (BWARP), (EFCI), and (WWCI) but violates (EP), F^{1pN} satisfies (E), (A), (SI), (BWARP), (EP) and (WWCI) but violates (EFCI). Consider the following solution F^*: for all $A \in \Sigma$, if $F^N(A)$ is symmetric or if $\alpha(F^N(A))$ is symmetric for some $\alpha \in \mathbb{R}^n_{++}$, then $F^*(A) = F^{2pN}(A)$, and if otherwise, $F^*(A) = F^{1pN}(A)$. It can be checked that F^* satisfies (E), (A), (SI), (BWARP), (EP), (EFCI), but violates (WWCI).

Finally, to see the independence of (BWARP), let $\Sigma_{bi} \subset \Sigma$ be the class of problems such that for each $A \in \Sigma_{bi}$, there exist two alternatives $x, y \in P(A)$ such that $m(comp\{x,y\}) = m(A)$ and $F^P(A) \cap \{x,y\} \neq \varnothing$. Define F^{P2} as follows: for any $A \in \Sigma$, if $A \in \Sigma_{bi}$ with $m(comp\{x,y\}) = m(A)$ for some $x, y \in P(A)$, then $F^{P2}(A) = \{x,y\}$; if $A \in \Sigma \backslash \Sigma_{bi}$, then $F^{P2}(A) = F^P(A)$. Consider $\#N = 2$. Note that F^{P2} satisfies (E), (SI), (A), and (EP).

Let us check (WWCI). Take $B \in \Sigma \backslash \Sigma_{bi}$. Then, since $F^{P2}(B) = F^P(B)$. Let $A \in \Sigma$ be such that the premise of (WWCI) holds for A and B. Thus, $F^P(A) \cap \{x,y\} = \varnothing$ follows from $F^{P2}(B) \cap A \neq \varnothing$ and $F^P(B) \cap \{x,y\} = \varnothing$. Thus, $A \in \Sigma \backslash \Sigma_{bi}$, so that $F^{P2}(A) = F^P(A)$, which implies that $F^{P2}(A) = F^{P2}(B) \cap A$. Take $B \in \Sigma_{bi}$. Then, there exist $x, y \in P(B)$ such that $m(comp\{x,y\}) = m(B)$ and $F^P(B) \cap \{x,y\} \neq \varnothing$. Then, $F^{P2}(B) = \{x,y\}$. Let $A \in \Sigma$ be such that the premise of (WWCI) holds for A and B. Then, $m(comp\{x,y\}) = m(B) = m(A)$ and $P(A) \subseteq P(B)$ imply that $\{x,y\} \subseteq P(A)$. Thus, $F^P(A) \cap \{x,y\} \neq \varnothing$. Thus, $A \in \Sigma_{bi}$, so that $F^{P2}(A) = \{x,y\}$. In summary, F^{P2} satisfies (WWCI).

Let us check (EFCI). Take any $A, B \in \Sigma$ such that $P(A) \subseteq P(B)$ and $[\{x,y\} = F^{P2}(comp\{x,y\})$ for all $x, y \in P(A)]$. Note that for any $z \in P(B) \backslash P(A)$ and any $x \in P(A)$, $\{x,z\} = F^{P2}(comp\{x,z\})$ holds, since $comp\{x,z\} \in \Sigma_{bi}$. Thus, (EFCI) is vacuously satisfied.

Finally, let us see F^{P2} violates (BWARP). Take $x, y, z \in \mathbf{R}^n_+$ with $y > z$. Note that $comp\{x,y\}, comp\{x,z\} \in \Sigma_{bi}$. Then, $\{x,y\} = F^{P2}(comp\{x,y\})$ holds, but $\{x,z\} = F^{P2}(comp\{x,z\})$ also holds, which implies the violation of (BWARP).

7 Conclusion

In this paper, we have introduced two new solutions for nonconvex bargaining problems and studied them axiomatically. The two solutions that we propose are procedural: for any given problem A, we first apply the Nash criterion to exclude those points that give lower Nash product than the maximum Nash product in A, and then, we apply another procedure based on the idea of the KS solution for nonconvex bargaining problems to select those that are considered to be "equitable" among those maximum-Nash-product points. The procedure based on the KS solution requires the information about an ideal point. In the paper, we have studied two possibilities of selecting an ideal point: the first one is the ideal point of the underlying bargaining problem, and the second one is the ideal point of the bargaining problem consisting of the maximum Nash product points from the underlying bargaining problem. Our procedural approach to bargaining problems opens a new possibility of investigating bargaining solutions, namely, solutions that are based on sequential eliminations of certain alternatives in bargaining problems.

It may be of interest to note that, in the first stage, each of the two new solutions discussed in this paper applies the Nash solution to eliminate certain alternatives from a bargaining problem. One may wonder if there is any rationale for doing this and why one would not start with any other solution concept in the first stage. There seem at least two reasons for employing the Nash solution in the first stage. First, suppose that we are interested in employing, in the first stage, a solution that has some 'minimal' rationality property such as BWARP. Then, in the framework of cardinal, non-comparable individual utilities, as shown by Proposition 2, we do not have many choices: the Nash solution is the only candidate. Secondly, suppose that we would like to employ a solution that is efficient (so that our procedures for selecting a solution would be based on the efficiency-first principle). Then, the Nash solution seems a 'natural' candidate in the framework. Of course, if one would like to use an 'equity-oriented' solution in the first stage, it would then open for other possibilities. We leave these possibilities for another occasions.

There is a set of core axioms satisfied by both of the two solutions studied in this paper: (E), (A), (SI), (BWARP) and (EP). The two solutions differ with respect to specific axioms of contraction independence: F^{1pN} satisfies (WCI) and (WWCI) but violates (EFCI), whereas F^{2pN} satisfies (WWCI) and (EFCI) but violates (WCI). The following table summarizes the properties of the Nash, KS, F^{1pN} and F^{2pN} solutions discussed in the paper where $\circ$ indicates the specified solution satisfies the corresponding axiom while $\times$ indicates otherwise :

	N	KS	F^{1pN}	F^{2pN}
(E)	$\circ$	$\times$	$\circ$	$\circ$
(A)	$\circ$	$\circ$	$\circ$	$\circ$
(SI)	$\circ$	$\circ$	$\circ$	$\circ$
$(BWARP)$	$\circ$	$\circ$	$\circ$	$\circ$
(EP)	$\times$	$\circ$	$\circ$	$\circ$
(WCI)	$\circ$	$\circ$	$\circ$	$\times$
$(EFCI)$	$\circ$	$\times$	$\times$	$\circ$
$(WWCI)$	$\circ$	$\circ$	$\circ$	$\circ$

To conclude the paper, we note that our framework can be readily adapted to studying collective choice problems where each bargaining problem can be interpreted as a feasible set available to a fixed number of individuals in the society and a solution can be interpreted as a collective choice function. With this reinterpretation, the axioms employed in the paper can be reformulated and re-interpreted accordingly. This seems to open a new possibility of investigating and studying collective choice problems. It would be interesting to explore this possibility in various contexts.

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