A Full Characterization of Nash Implementation with Strategy Space Reduction*

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Abstract

The paper fully characterizes the class of Nash-implementable social choice correspondences (SCCs) by mechanisms endowed with Saijo's message space specification - *s-mechanisms*. This class of SCCs is equivalent to the class of Nash-implementable SCCs, though any game form involving 'one less' preference announcements breaks this equivalence relationship down.

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1 Introduction

In Nash implementation theory,[1] it is *Maskin's Theorem* (Maskin, 1999) which shows that when the planner faces at least three agents, a *social choice correspondence* (*SCC*) is implementable in (pure-strategy) Nash equilibria if it satisfies *Maskin monotonicity* and *no-veto power*; conversely, any implementable *SCC* is Maskin monotonic. Two issues pertaining to this theorem stand out. First, it does not provide a complete characterization of implementable *SCC*s, since no-veto power is not necessary for implementation. Second, the constructed mechanisms, called *canonical mechanisms*, employ large strategy spaces as each agent is required to report to the planner the entire profile of *all* agents' preferences, a feasible social outcome, and an integer. Therefore, the mechanism constructed in the proof of Maskin's Theorem is far from informationally efficient (on this point see, for instance, Hurwicz, 1960).

Moore and Repullo (1990) address the first issue by providing a necessary and sufficient condition, called *Condition* μ, for implementability of *SCC*s in societies with more than two agents.[2] On the other hand, the issue of informational efficiency is addressed by Saijo (1988), who devises mechanisms that implement *SCC*s by employing strategy spaces that are significantly smaller than those employed by canonical mechanisms. In particular, in Saijo's mechanisms (henceforth, *s-mechanisms*) each agent announces, in addition to a feasible social outcome and an integer, her own and her neighbor's preferences solely. Yet, while Moore and Repullo (1990) only employ canonical mechanisms for identifying the class of implementable *SCC*s in full, Saijo (1988) discusses only sufficient conditions for *SCC*s to be implemented by s-mechanisms. Therefore, it is left unclear not only whether Moore and Repullo's result indispensably relies on canonical mechanisms, but also whether s-mechanisms can implement any other *SCC* than those satisfying Maskin monotonicity and no-veto power.

[1] Henceforth, by implementation and its adjectives we mean Nash implementation and its adjectives, respectively.

[2] Note that, for two person societies, Moore and Repullo (1990) and Dutta and Sen (1991) independently provided necessary and sufficient conditions for implementation. Danilov (1992) and Yamato (1992) refined Maskin's Theorem by providing necessary and sufficient conditions for an *SCC* to be implementable under some domain restrictions. By devising mechanisms with awards, Sanver (2006) shows that a weaker variant of Maskin monotonicity constitutes a necessary and sufficient condition for implementation with awards.

In this paper, we address the issue of what constitutes the necessary and sufficient condition for implementation by s-mechanisms in the following two ways. Firstly, by directly employing the classical Condition μ, we resolve this issue. Secondly, the same issue is addressed by introducing an alternative condition, *Condition* M^s, which is relevant for incorporating behavioral economics into implementation theory (see Lombardi and Yoshihara, 2011a) and is similar to Condition M (Sjöstrom, 1991). Yet, while the class of implementable SCCs is equivalent to the class of SCCs that are implementable by s-mechanisms, this study also shows that Moore and Repullo (1990)'s characterization result no longer holds if the message conveyed by each participant to the planner involves the announcement of either her own preference or the preference of her neighbor - in addition to an outcome and an agent index.

Before turning to the formal arguments, it may be worth mentioning that, as is common in the mechanisms constructed in the literature on characterization theorems, the announcement of an agent index is also used to rule out undesirable equilibria in an s-mechanism. This feature renders the mechanism subject to information smuggling, that is, the announced index could be used as an encoding device to smuggle information about preferences of other participants. Therefore, for s-mechanisms to make sense, we introduce a regularity condition for an s-mechanism to prevent information smuggling, which is a variation of the *forthrightness* conditions introduced in economic environments by Dutta *et al.* (1995) and Saijo *et al.* (1996), and in abstract social choice contexts by Tatamitani (2001). Bolstered by this regularity condition, it is no longer rendered trivial to reduce the message space from the types of canonical mechanisms to those of s-mechanisms. In spite of this, the results reported herein demonstrate that, under implementability, an s-mechanism can be constructed from a canonical mechanism, and vice versa, even if the s-mechanism is required to satisfy forthrightness.

The paper is organized as follows. Section 2 describes the formal environment. Section 3 reports not only our main characterization result via Condition μ, but also an alternative characterization result via Condition M^s. Section 4 shows that the scope of implementation drastically reduces when the planner requires each participant to report, inter alia, either her own preference or that of her neighbor. Section 5 provides a short discussion of strategy space reduction for implementation with asymmetric information. Section 6 concludes briefly.

2 Preliminaries

The set of (*social choice) environments* is $(N, X, \mathcal{R}^n)$, where $N \equiv \{1, ..., n\}$ is a set of $n \geq 3$ *agents*, $X \equiv \{x, y, z, ...\}$ is the set of attainable *outcomes*, and $\mathcal{R}^n$ is the set of admissible *preference profiles* (or *states of the world*).[3] Henceforth, we assume that the cardinality of X is $\#X \geq 2$. Let $\mathcal{R}(X)$ be the set of all complete preorders on X.[4] We assume that $\mathcal{R}^n \equiv \mathcal{R}_1 \times ... \times \mathcal{R}_n$ is a non-empty subset of the n-fold Cartesian product $\mathcal{R}^n(X) \equiv \underbrace{\mathcal{R}(X) \times .... \times \mathcal{R}(X)}_{n\text{-times}}$. An element of $\mathcal{R}^n$ is denoted by $R \equiv (R_1, ..., R_n)$, where its ℓ-th component is $R_\ell \in \mathcal{R}_\ell$ for each $\ell \in N$. For any preference profile $R \in \mathcal{R}^n$ and any $\ell \in N$, let $R_{-\ell}$ be the list of elements of R for all agents except ℓ, i.e., $R_{-\ell} \equiv (R_1, ..., R_{\ell-1}, R_{\ell+1}, ..., R_n) \in \times_{i \in N \setminus \{\ell\}} \mathcal{R}_i$. Given a list $R_{-\ell}$ and $R_\ell \in \mathcal{R}_\ell$, we denote by $(R_{-\ell}, R_\ell) \in \mathcal{R}^n$ the preference profile consisting of these R_ℓ and $R_{-\ell}$. Similarly, for any preference profile $R \in \mathcal{R}^n$ and any $\varnothing \neq S \subset N$, let R_{-S} be the list of elements of R for all agents in $N \setminus S$.[5] Given a list R_{-S} and $R_S \in \times_{\ell \in S} \mathcal{R}_\ell$, we denote by (R_{-S}, R_S) the preference profile consisting of these R_S and R_{-S}. For any $(R_\ell, x) \in \mathcal{R}_\ell \times X$, agent ℓ's *weakly lower contour set* of R_ℓ at x is given by $L(R_\ell, x) \equiv \{y \in X | (x, y) \in R_\ell\}$. For each $\ell \in N$ and each $R_\ell \in \mathcal{R}_\ell$, $\max_{R_\ell} X \equiv \{x \in X | (x, y) \in R_\ell \text{ for all } y \in X\}$.

A *social choice correspondence* (*SCC*) is a correspondence $F : \mathcal{R}^n \twoheadrightarrow X$ with $F(R) \neq \varnothing$ for all $R \in \mathcal{R}^n$. An *SCC* F on $\mathcal{R}^n$ is (Maskin) *monotonic* if for all $R, R' \in \mathcal{R}^n$ with $x \in F(R)$, $x \in F(R')$ holds whenever $L(R_\ell, x) \subseteq L(R'_\ell, x)$ for all $\ell \in N$.[6] An *SCC* F on $\mathcal{R}^n$ satisfies *no-veto power* if for all $R \in \mathcal{R}^n$, $x \in F(R)$ holds whenever $x \in \max_{R_\ell} X$ for at least $n-1$ agents. Given an *SCC* F, an outcome x is F-optimal at a preference profile $R \in \mathcal{R}^n$ if $x \in F(R)$.

A mechanism (or game-form) is a pair $\gamma \equiv (M, g)$, where $M \equiv M_1 \times ... \times M_n$, and $g : M \rightarrow X$ is the *outcome function*. Denote a generic mes-

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[3] We assume that N and X are fixed throughout the following discussion, so that the set of environments is boiled down to $\mathcal{R}^n$.

[4] A complete preorder $R \in \mathcal{R}(X)$ is a complete and transitive binary relation. A relation R on X is *complete* if, for all $x, x' \in X$, $(x, x') \in R$ or $(x', x) \in R$; *transitive* if, for all $x, x', x'' \in X$, if $(x, x') \in R$ and $(x', x'') \in R$, then $(x, x'') \in R$.

[5] Weak set inclusion is denoted by $\subseteq$, while the strict set inclusion is denoted by $\subset$.

[6] Saijo (1987) shows that a monotonic social choice function is a constant function, provided that it is defined on the unrestricted domain of all profiles of complete preorders, $\mathcal{R}^n(X)$. Therefore, the problem under study is interesting only for multi-valued social choice solutions when $\mathcal{R}^n = \mathcal{R}^n(X)$.

sage (or strategy) for agent ℓ by $m_\ell \in M_\ell$ and a generic message profile by $m = (m_1, ..., m_n) \in M$. For any $m \in M$ and $\ell \in N$, let $m_{-\ell}$ be the list of elements of m for all agents except ℓ, i.e., $m_{-\ell} \equiv (m_1, ..., m_{\ell-1}, m_{\ell+1}, ..., m_n) \in \times_{i \in N \setminus \{\ell\}} M_i \equiv M_{-\ell}$. Given $m_{-\ell} \in M_{-\ell}$ and $m_\ell \in M_\ell$, denote by $(m_\ell, m_{-\ell})$ the message profile consisting of these m_ℓ and $m_{-\ell}$. Given an $R \in \mathcal{R}^n$ and a mechanism $\gamma = (M, g)$, the pair (γ, R) constitutes a (*non-cooperative*) *game*. Given a game (γ, R), $m \in M$ is a (pure strategy) *Nash equilibrium* of (γ, R) if and only if for all $\ell \in N$, $(g(m), g(m'_\ell, m_{-\ell})) \in R_\ell$ holds for all $m'_\ell \in M_\ell$. Let $NE(\gamma, R)$ denote the set of Nash equilibria of (γ, R), whereas the set of *Nash equilibrium outcomes* of (γ, R) is denoted by $NA(\gamma, R) \equiv g(NE(\gamma, R))$.

A mechanism $\gamma = (M, g)$ *implements* F in *Nash equilibria*, or simply *implements* F, if and only if $NA(\gamma, R) = F(R)$ for all $R \in \mathcal{R}^n$. An SCC F is (*Nash-*)*implementable* if there is such a mechanism.

Moore and Repullo (1990) show that, for a society with more than two agents, the following condition is the necessary and sufficient condition for any SCC to be implementable.

Condition μ (for short, μ). There exists a set $Y \subseteq X$; moreover, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R, x))_{\ell \in N}$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following conditions (i)-(iii) are satisfied:
(i) if $C_\ell(R, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for each $i \in N$, if $y \in C_i(R, x) \subseteq L(R^*_i, y)$ and $Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N$, then $y \in F(R^*)$.[7]

Condition μ(i) is equivalent to Maskin monotonicity, while Conditions μ(ii) and μ(iii) are weaker versions of no-veto power.

3 Main Result

The basic idea behind Saijo (1988)'s strategy space reduction is to cover each agent's preference twice. For example, agent ℓ's preference may be covered by her own announcement and by that of another agent involved in the mechanism. A way to proceed is to arrange agents in a circular fashion numerically

[7] We refer to the condition that requires only one of the conditions (i)–(iii) in Condition μ as Conditions μ(i)–μ(iii) respectively. Note that Condition μ implies Conditions μ(i)–μ(iii), but the converse is not true. We use similar conventions below.

clockwise facing inward, and require that each agent ℓ announces her own preference together with the preference of the agent standing immediately to her left, that is, of agent $\ell + 1$. Following Saijo (1988), an s-mechanism can be defined as follows.

Definition 1. A mechanism $\gamma = (M, g)$ is an s-mechanism if, for all $\ell \in N$, $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times Y \times N$, with $n + 1 = 1$ and $Y \subseteq X$.[8]

Thus, each agent ℓ announces her preference, R_ℓ^ℓ, the preference of her neighbor, $R_{\ell+1}^\ell$, an outcome, x^ℓ, and an agent index, k^ℓ. The reported indices in an s-mechanism are used to rule out undesired outcomes as equilibria of the mechanism. This type of device, common in the constructive proofs of the literature, is subject to criticism on several fronts.[9]

An s-mechanism is said to implement the *SCC* F if for each admissible profile of preferences, the set of equilibrium outcomes of the s-mechanism coincides with the set of F-optimal outcomes. The implementability of an *SCC* by an s-mechanism can be defined as follows.

Definition 2. An *SCC* F is implementable by an s-mechanism if there exists an s-mechanism $\gamma = (M, g)$ such that for all $R \in \mathcal{R}^n$, $F(R) = NA(\gamma, R)$.

In an s-mechanism, each agent is required to report her neighbor's preference and an agent index. This feature generally renders the mechanism subject to information smuggling, since there could be a case that the announced index or the announced preference can be used as an encoding device to smuggle the direct information about preferences of other participants.[10]

[8] For each agent $\ell \in N$, the message space $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times Y \times N$ can be replaced by $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times Y \times Q$, where Q is an arbitrary set. To deduce the necessary condition for implementation by s-mechanisms below, the announcement of an agent index is not needed.

[9] For a systematic criticism of the use of "modulo games" and "integer games" in the literature, see Jackson (1992).

[10] For instance, if $\#X = 2$ and $\#N = 3$, agent ℓ's announced index in N can be used to smuggle information about a preference of agent $\ell + 2$ within an s-mechanism. Also, we may consider a case that X is infinite and for each $\ell \in N$, there exists a bijection $\phi_\ell : \mathcal{R}_\ell \to [0, 1]$. Note that the model of abstract environments presented herein does not exclude such a case. Then, there also exists a bijection $\widetilde{\phi}_i : \times_{\ell \in N \setminus \{i\}} \mathcal{R}_\ell \to [0, 1]^{n-1}$ for each $i \in N$. Then, noting that there exists a bijection $\varphi : [0, 1]^{n-1} \to [0, 1]$, we can have a bijection $\Phi_i : \times_{\ell \in N \setminus \{i\}} \mathcal{R}_\ell \to \mathcal{R}_{i+1}$ for each $i \in N$, where $\Phi_i \equiv \phi_{i+1}^{-1} \circ \varphi \circ \widetilde{\phi}_i$. Then, using a similar reasoning in the proof of Dutta *et al.* (1995; Theorem 3.3), we can construct, from any F-implementing canonical mechanism with the message space $M_i = \mathcal{R}^n \times X \times N$ for any $i \in N$, an s-mechanism with the message space $M_i = \mathcal{R}_i \times \mathcal{R}_{i+1} \times X \times N$ for any

In this case, the problem of reducing the message space is immediately rendered trivial, thus the objective of the present paper is defeated by such mathematical tricks. Thus, for s-mechanisms to make sense, we require the following regularity condition to exclude this possibility.

Forthrightness for s-mechanisms, FR^s. For all $R \in \mathcal{R}^n$ and all $x \in F(R)$, if $m_\ell = \left(R_\ell, R_{\ell+1}, x, k^\ell\right) \in M_\ell$ for all $k^\ell \in N$ and all $\ell \in N$, with $n+1=1$, then $m \in NE(\gamma, R)$ and $g(m) = x$.

If an s-mechanism satisfies FR^s, we say that it is an s-mechanism with FR^s.

FR^s requires that if the outcome x is F-optimal at the state R, each agent announces truthfully her preference and that of her neighbor, and this x is unanimously announced, then the message profile should be a Nash equilibrium of an s-mechanism regardless of the profile of announced indices, and its equilibrium outcome should be the announced F-optimal outcome. Note that the requirement of truthful announcement of $(R_\ell, R_{\ell+1})$ for every $\ell \in N$ is to exclude the possibility of information smuggling via the announced preferences as an encoding device, while the irrelevance property of the profile of announced indices is to exclude the possibility of information smuggling via the profile of announced indices as an encoding device. To exclude information smuggling, requirements similar to ours are imposed in economic environments by Dutta *et al.* (1995) and Saijo *et al.* (1996), and in abstract social choice contexts by Tatamitani (2001). Finally, mechanisms satisfying these types of conditions are 'simple' in the sense that it is easy to compute the outcome of an equilibrium strategy profile.

Before turning to the main result of this section, it may be instructive to briefly discuss the class of s-mechanisms devised to guarantee that Condition μ fully identifies the class of implementable SCCs. As in Saijo (1988)'s mechanism, in our s-mechanisms agents make a cyclic announcement of strategies while the preference profile, especially the deviator's preference relation, is determined without relying upon the deviator's announcement.[11]

Formally, let $\gamma \equiv (M, g)$ be an s-mechanism. Fix any $m \in M$, $R \in \mathcal{R}^n$, and $x \in X$, and let $m_\ell = \left(R_\ell^\ell, R_{\ell+1}^\ell, x^\ell, k^\ell\right) \in M_\ell$, with $n+1=1$. Let the announcement of agent $\ell \in N$ about agent $\ell+1$'s preference be $R_{\ell+1}^\ell$. We say that the message profile $m \in M$ is:

$i \in N$, which also implements F.

[11] This type of construction is also used in Lombardi (in press) to prove that that any weakly unanimous SCC is Maskin monotonic if and only if it is Nash implementable via a simple stochastic mechanism endowed with Saijo's message space specification.

(i) *consistent* with R and x if for all $\ell \in N$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$ and $x^\ell = x$;
(ii) m_{-i} *quasi-consistent* with x and R, where $i \in N$, if for all $\ell \in N$, $x^\ell = x$, and for all $\ell \in N \backslash \{i, i+1\}$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$, $R_i^{i-1} = R_i$, $R_{i+1}^{i+1} = R_{i+1}$, and $[R_i^i \neq R_i$ or $R_{i+1}^i \neq R_{i+1}]$;
(iii) m_{-i} *consistent* with x and R, where $i \in N$, if for all $\ell \in N \backslash \{i\}$, $x^\ell = x \neq x^i$, and for all $\ell \in N \backslash \{i, i+1\}$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$, $R_i^{i-1} = R_i$, and $R_{i+1}^{i+1} = R_{i+1}$;
where $1 - 1 = n$ and $n + 1 = 1$.

In words, a message profile m is consistent with an outcome x and a preference profile R if there is no break in the cyclic announcement of preferences and all agents announce the outcome x. On the other hand, it is m_{-i} quasi-consistent with x and R if there are at most two consecutive breaks in the cyclic announcement of preferences such that these breaks happen in correspondence of the preferences announced by agent i, and x is unanimously announced. Finally, a message profile m is m_{-i} consistent with x and R if agent i announces an outcome different from the outcome x announced by the others, and there are no more than two consecutive breaks in the cyclic announcement of preferences such that these breaks (if any) happen in correspondence of the preferences announced by agent i.

Before defining the outcome function g of γ, some additional notation is required. For any $S \subseteq N$, any $R_{-S} \in \mathcal{R}^{n-\#S}$, and any $x \in X$, let $F^{-1}(R_{-S}, x) \equiv \{R'_S \in \mathcal{R}_S | x \in F(R'_S, R_{-S})\}$. Take any $r = 1, .., n-2$, and any *SCC* F which satisfies Condition μ. For any $\ell \in N$, any $R \in \mathcal{R}^n$, and any $x \in F(R)$, define the set $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$ as follows:

$$C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \equiv \bigcup_{R'_{\{\ell+1,\ldots,\ell+r\}} \in F^{-1}\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)} C_\ell\left(\left(R_{-\{\ell+1,\ldots,\ell+r\}}, R'_{\{\ell+1,\ldots,\ell+r\}}\right), x\right), \tag{1}$$

with the convention that $n + k = k$.

Notice that if an *SCC* F satisfies Condition μ, the following *properties* are assured by (1), for any $r = 1, ..., n-2$.

Property I: The set $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$ is well-defined and $x \in C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x) \cap Y$ for each $\ell \in N$;

Property II: For all $\ell \in N$, all $R, R' \in \mathcal{R}^n$, with $R'_{-\{\ell+1,\ldots,\ell+r\}} = R_{-\{\ell+1,\ldots,\ell+r\}}$, and all $x \in F(R) \cap F(R')$, it holds that

$$C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) = C_\ell\left(R'_{-\{\ell+1,\ldots,\ell+r\}}, x\right).$$

We are now in a position to define, for any $r = 1, ..., n-2$, the outcome function $g : M \rightarrow X$ of γ as follows. For any $m \in M$,

Rule 1: If m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, then $g(m) = x$.

Rule 2: If for some $i \in N$, m_{-i} is quasi-consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, then $g(m) = x$.

Rule 3: If for some $i \in N$, m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, and $C_i\left(\bar{R}_{-\{i+1,...,i+r\}}, x\right) \neq Y$, with $n + k = k$, then

$$g(m) = \begin{cases} x^i & \text{if} \quad x^i \in C_i\left(\bar{R}_{-\{i+1,...,i+r\}}, x\right) \\ x & \text{otherwise.} \end{cases}$$

Rule 4: Otherwise, $g(m) = x^{\ell^*(m)}$ where $\ell^*(m) \equiv \sum_{i \in N} k^i \,(\text{mod} \;\; n)$.[12]

The above mechanism satisfies *FR*^{s}. Moreover, in *Rules 2-3*, agent i is a *deviator*. However, in *Rule 2*, agent i is not necessarily the only deviator whenever there is exactly one break in the preference announcement profile between agent i's preference announcement and that of agent $i-1$, i.e., $R_i^i \neq R_i^{i-1} = \bar{R}_i$ and $R_{i+1}^i = R_{i+1}^{i+1} = \bar{R}_{i+1}$. Indeed, agents $i-1$ and i could both be deviators if

$$x \in F\left(\bar{R}\right) \cap F\left(\bar{R}_{-i}, R_i^i\right).$$

On the other hand, in *Rule 3*, since agent i is the only agent reporting an outcome different from that reported by all other participants, the mechanism identifies agent i as the only deviator.

The above mechanism is similar, but not identical, to that devised by Saijo (1988). The essential difference lies in the definition of the outcome function g when there is a single deviator. While in our mechanism the outcome selected by g lies in $C_i\left(\bar{R}_{-\{i+1,...,i+r\}}, x\right) \neq Y$, in Saijo's mechanism the outcome function picks an element of the weak lower contour set of the deviator i at x evaluated not by her own preference announcement R_i^i, but by her neighbor's announcement, that is, of $L\left(R_i^{i-1}, x\right)$.

Another feature of our mechanism worth commenting on is that in the definition of *Rule 3*, it is not possible to replace the set $C_i\left(\bar{R}_{-\{i+1,...,i+r\}}, x\right)$ with the set $C_i\left(\bar{R}, x\right)$ of outcomes specified in Condition μ. To explain this aspect,

[12] If the remainder is *zero*, the winner of the modulo game is agent n. Observe that in *Rule 4* if the modulo game is replaced with the integer game, the proof of Theorem 1 below will require only minor changes.

suppose that F satisfies Condition μ. Let us replace $C_i\left(\bar{R}_{-\{i+1,\dots,i+r\}}, x\right)$ with $C_i\left(\bar{R}, x\right)$ in the definition of *Rule 3*; suppose that the true state is R and let an equilibrium message profile m fall into *Rule 2*. Therefore, for a participant i, the message profile m is m_{-i} quasi-consistent with an outcome x and a profile $\bar{R}$, and $x \in F\left(\bar{R}\right)$. Moreover, suppose that m is such that there is exactly one break in the preference announcement profile between agent i's preference announcement and that of agent $i-1$, i.e., $R_i^i \neq R_i^{i-1} = \bar{R}_i$ and $R_{i+1}^i = R_{i+1}^{i+1} = \bar{R}_{i+1}$, and that $x \in F\left(\bar{R}\right) \cap F\left(\bar{R}_{-i}, R_i^i\right)$. Hence, under these specifications, there are two potential deviators, agent $i-1$ and agent i. To conclude that $x \in F(R)$, it must hold that $C_{i-1}\left(\left(\bar{R}_{-i}, R_i^i\right), x\right) = C_{i-1}\left(\bar{R}, x\right)$ to fulfil the premises of μ(i). However, $x \in F(R)$ is not assured in general by means of Condition μ, since $C_{i-1}\left(\left(\bar{R}_{-i}, R_i^i\right), x\right) = C_{i-1}\left(\bar{R}, x\right)$ does not necessarily hold in general due to $\left(\bar{R}_{-i}, R_i^i\right) \neq \bar{R}$. On the other hand, by Property II and the definition of g, the premises of μ(i) are fulfilled. To see this property, observe that the above s-mechanism and Property II assure that the set of outcomes that agent $i-1$ can attain is the same both in the case where the preference announcement $\bar{R}$ is taken as the true state of the world and in the case where the preference announcement $\left(\bar{R}_{-i}, R_i^i\right) \equiv \bar{R}'$ is taken as the true state of the world, i.e., $C_{i-1}\left(\bar{R}_{-\{i,\dots,i+r-1\}}, x\right) = C_{i-1}\left(\bar{R}'_{-\{i,\dots,i+r-1\}}, x\right)$ for any $r = 1, \dots, n-2$. Furthermore, definition (1) assures not only that $C_{i-1}\left(\bar{R}, x\right) \subseteq C_{i-1}\left(\bar{R}_{-\{i,\dots,i+r-1\}}, x\right)$, but also that $C_i\left(\bar{R}, x\right) \subseteq C_i\left(\bar{R}_{-\{i,\dots,i+r\}}, x\right)$. Therefore, since every other participant can induce *Rule 4* and obtain any outcome in Y, it is easy to see that the premises of μ(i) are met.

We are now in a position to prove that the class of *SCC*s that are implementable by an s-mechanism is fully identified by Condition μ.

Theorem 1. *An* SCC F defined on $\mathcal{R}^n$ *is* implementable by *an* s-mechanism *if and only if it satisfies* Condition μ.

Proof. For the necessary part, suppose that F defined on $\mathcal{R}^n$ is implementable by an s-mechanism. Then, it is implementable. By Moore and Repullo (1990)'s result, it follows that F satisfies Condition μ.

Next, we prove sufficiency. Suppose that F satisfies μ. Let $r = n-2$, and consider the corresponding s-mechanism γ constructed above. We show that $\gamma = (M, g)$ implements F. Take any $R \in \mathcal{R}^n$.

To show that $F(R) \subseteq NA(\gamma, R)$, let $x \in F(R)$ and suppose that for all $\ell \in N$, $m_\ell = (R_\ell, R_{\ell+1}, x, k)$, where $k \in N$ is an arbitrary agent index. *Rule*

1 implies that $g(m) = x$. By the definition of g, we have that any deviation of agent $\ell \in N$ will get her to an outcome in $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$, so that $g(M_\ell, m_{-\ell}) \subseteq C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$. Since $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x)$ by PROPERTY I, it follows that such deviations are not profitable, and so $m \in NE(\gamma, R)$.

Conversely, to show that $NA(\gamma, R) \subseteq F(R)$, let $m \in NE(\gamma, R)$. Consider the following cases.

Case 1: m falls into *Rule 1*.

Then, m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. Take any $\ell \in N$. Suppose that $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \neq Y$. For any $y \in C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \setminus \{x\}$, changing m_ℓ to $m_\ell^* = \left(R_\ell^\ell, R_{\ell+1}^\ell, y, k^\ell\right) \in M_\ell$, agent ℓ can obtain $y = g(m_\ell^*, m_{-\ell})$ via *Rule 3*. When $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) = Y$, agent ℓ can attain any $y \in Y$ via *Rule 4*, by appropriately choosing the agent index k^ℓ. Therefore, $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) = g(M_\ell, m_{-\ell})$ for all $\ell \in N$. Since $m \in NE(\gamma, R)$, we have that $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x)$ for all $\ell \in N$. By (1), it follows that $C_\ell\left(\bar{R}, x\right) \subseteq L(R_\ell, x)$ for all $\ell \in N$. Thus, μ(i) implies $x \in F(R)$.

Case 2: m falls into *Rule 2*.

Then, m is m_{-i} quasi-consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. We proceed according to the following sub-cases: 1) $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$; and 2) $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$.[13]

Sub-case 2.1. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$.

Since any $\ell \in N \setminus \{i\}$ can attain any $y \in Y \setminus \{x\}$ by inducing *Rule 4* and $m \in NE(\gamma, R)$, we have that $x \in \max_{R_\ell} Y$. Next, take any $y \in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \setminus \{x\}$. Suppose that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \neq Y$. By changing m_i to $m_i^* = \left(R_i^i, R_{i+1}^i, y, k^i\right) \in M_i$, agent i can obtain $y = g(m_i^*, m_{-i})$ via *Rule 3*. When $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) = Y$, agent i can attain $y = g(m_i^*, m_{-i})$ by changing m_i to $m_i^* = \left(R_i^i, R_{i+1}^i, y, k^i\right) \in M_i$ with the appropriate choice of k^i. It follows that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq g(M_i, m_{-i})$. Moreover, $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L(R_i, x)$ as $m \in NE(\gamma, R)$. By (1), it follows that $C_i\left(\bar{R}, x\right) \subseteq L(R_i, x)$. Therefore, we obtained that $x \in \max_{R_\ell} Y$ for each $\ell \in N \setminus \{i\}$ and $C_i\left(\bar{R}, x\right) \subseteq L(R_i, x)$. Thus, μ(ii) implies $x \in F(R)$.

Sub-case 2.2. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$.

[13] The sub-case $R_i^i = \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$ is not explicitly considered, since it can be proved similarly to the *sub-case 2.2* shown below.

Let $R_i^i = R_i'$ and $\bar{R}' \equiv \left(\bar{R}_{-i}, R_i'\right)$. We distinguish whether $x \in F\left(\bar{R}'\right)$ or not. Suppose that $x \notin F\left(\bar{R}'\right)$. Then, the same reasoning used above for *sub-case 2.1* carries over into *sub-case 2.2*, so that $x \in F(R)$. Otherwise, let $x \in F\left(\bar{R}'\right)$. Then, $i-1$ and i are both deviators. Agent $\ell \in N \backslash \{i-1, i\}$ can attain any $y \in Y \backslash \{x\}$ by inducing *Rule 4*, so that $x \in \max_{R_\ell} Y$ as $m \in NE(\gamma, R)$. Consider agent $i-1$. Note that, by Property II, $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right) = C_{i-1}\left(\bar{R}'_{-\{i,\ldots,i+r-1\}}, x\right)$ holds. Take any $y \in C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right) = C_{i-1}\left(\bar{R}'_{-\{i,\ldots,i+r-1\}}, x\right)$ with $y \neq x$. Suppose that $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right) \neq Y$. By changing m_{i-1} to $m_{i-1}^* = \left(R_{i-1}^{i-1}, R_i^{i-1}, y, k^{i-1}\right) \in M_{i-1}$, agent $i-1$ can obtain $y = g\left(m_{i-1}^*, m_{-(i-1)}\right)$ via *Rule 3*. When $y \in C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right) = Y$ with $y \neq x$, agent $i-1$ can attain $y = g\left(m_{i-1}^*, m_{-(i-1)}\right)$ by changing m_{i-1} to $m_{i-1}^* = \left(R_{i-1}^{i-1}, R_i^{i-1}, y, k^{i-1}\right) \in M_{i-1}$ with the appropriate choice of k^{i-1}. Since m is an equilibrium message profile of the game (γ, R), it follows that $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right) \subseteq g\left(M_{i-1}, m_{-(i-1)}\right) \subseteq L(R_{i-1}, x)$. Since a similar reasoning applies to agent i and $m \in NE(\gamma, R)$, it is easy to see that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq g(M_i, m_{-i}) \subseteq L(R_i, x)$. Therefore, $x \in \max_{R_\ell} Y$ for each $\ell \in N \backslash \{i-1, i\}$, $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right) \subseteq L(R_{i-1}, x)$, and $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L(R_i, x)$. Since $C_{i-1}\left(\bar{R}, x\right) \subseteq C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}}, x\right)$ and $C_i\left(\bar{R}, x\right) \subseteq C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right)$ by (1), μ(i) implies that $x \in F(R)$.

Case 3: m falls into Rule 3.

Then, m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Therefore, $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \neq Y$. First, we show that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq g(M_i, m_{-i})$. For any $x^i \in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \backslash \{x\}$, consider $m_i^* = \left(R_i^i, R_{i+1}^i, x^i, k^i\right)$. Then, *Rule 3* implies that $g(m_{-i}, m_i^*) = x^i$. On the other hand, to attain x, agent i can induce *Rule 1* by changing m_i to $m_i^* = \left(\bar{R}_i, \bar{R}_{i+1}, x, k^i\right)$. Hence, $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq g(M_i, m_{-i})$.

Next, we claim that $g(M_\ell, m_{-\ell}) = Y$ for any $\ell \in N \backslash \{i\}$. We proceed according to whether $\#Y = 2$ and $n = 3$ or not.

Sub-case 3.1. not[$\#Y = 2$ and $n = 3$].

Take any $\ell \in N \backslash \{i\}$. Suppose that $\#Y > 2$. By the definition of g, we have that $Y \subseteq g(M_\ell, m_{-\ell})$ for any $\ell \in N \backslash \{i\}$. Otherwise, let $\#Y = 2$. Then, $n > 3$. By replacing x with $x^\ell = x^i$, agent ℓ can make $\#\left\{\ell \in N | x^\ell = x\right\} \geq 2$ and $\#\left\{\ell \in N | x^\ell \neq x\right\} \geq 2$. Since the outcome is determined by *Rule 4*, agent ℓ can attain any outcome in Y by appropriately choosing k^ℓ. Therefore, $Y \subseteq g(M_\ell, m_{-\ell})$ for any $\ell \in N \backslash \{i\}$.

Sub-case 3.2. $\#Y = 2$ and $n = 3$.

Then, let $N = \{i-1, i, i+1\}$ with $n+1 = 1$ and $1-1 = n$. Since $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \neq Y$, it follows that $g(m) = x$. We proceed accordingly depending on whether or not there exist agents $\ell, \ell' \in N$ with $\ell \neq \ell'$ such that $\#\mathcal{R}_\ell \neq 1$ and $\#\mathcal{R}_{\ell'} \neq 1$ hold.

Sub-sub-case 3.2.1. For some $\ell, \ell' \in N$ with $\ell \neq \ell'$, $\#\mathcal{R}_\ell \neq 1$ and $\#\mathcal{R}_{\ell'} \neq 1$.

In this case, agent $i-1$ (resp., $i+1$) can always induce the modulo game by appropriately changing the announcement of her own preference or that of her successor, and by carefully choosing the outcome announcement. To attain x^i, agent $i-1$ (resp., $i+1$) has only to adjust the integer index.

Sub-sub-case 3.2.2. For all $\ell, \ell' \in N$ with $\ell \neq \ell'$, $\#\mathcal{R}_\ell = 1$ or $\#\mathcal{R}_{\ell'} = 1$.

Suppose that, for all $\ell^* \in \{i-1, i, i+1\}$, $\#\mathcal{R}_{\ell^*} = 1$. Since m falls into *Rule 3*, it follows that $x \in F(R) = F(\bar{R})$. Next, suppose that there exists $\ell^* \in \{i-1, i, i+1\}$ such that $\#\mathcal{R}_{\ell^*} \neq 1$. If either $\#\mathcal{R}_{i-1} > 1$ or $\#\mathcal{R}_i > 1$, then agent $i-1$ can induce the modulo game by changing m_{i-1} to either $m^*_{i-1} = \left(R^{i-1}_{i-1}, \bar{R}_i, x, k^{i-1}\right)$ with $R^{i-1}_{i-1} \neq \bar{R}_{i-1}$ (if $\#\mathcal{R}_{i-1} > 1$) or $m^*_{i-1} = \left(\bar{R}_{i-1}, R^{i-1}_i, x^i, k^{i-1}\right)$ with $R^{i-1}_i \neq R^i_i$ (if $\#\mathcal{R}_i > 1$). To attain x^i, agent $i-1$ has only to choose an appropriate k^{i-1} so that $i = \ell^*\left(m_{-(i-1)}, m^*_{i-1}\right)$. Therefore, $Y \subseteq g\left(M_{i-1}, m_{-(i-1)}\right)$. Then, let $\#\mathcal{R}_{i-1} = \#\mathcal{R}_i = 1$. Agent $i-1$ can change m_{i-1} to $m^*_{i-1} = \left(\bar{R}_{i-1}, \bar{R}_i, x^i, k^{i-1}\right)$. Suppose that $x^i \notin F\left(\bar{R}_{i-1}, \bar{R}_i, R^i_{i+1}\right)$. Then, *Rule 4* applies and agent $i-1$ can attain x^i by adjusting k^{i-1} so that $i-1 = \ell^*\left(m_{-(i-1)}, m^*_{i-1}\right)$. Next, suppose that $x^i \in F\left(\bar{R}_{i-1}, \bar{R}_i, R^i_{i+1}\right)$. If $C_{i+1}\left(\bar{R}_i, R^i_{i+1}, x^i\right) = \{x^i\}$, *Rule 3* implies $g\left(m_{-(i-1)}, m^*_{i-1}\right) = x^i$. When $C_{i+1}\left(\bar{R}_i, R^i_{i+1}, x^i\right) = Y$, the outcome is determined by *Rule 4*. In this case, by adjusting k^{i-1}, agent $i-1$ can attain x^i. By a similar reasoning, it can be shown that agent $i+1$ can attain $x^i \in Y$. Therefore, $Y \subseteq g(M_\ell, m_{-\ell})$ for $\ell \in \{i-1, i+1\}$.

From the above arguments, we obtained $Y \subseteq g(M_\ell, m_{-\ell})$ for all $\ell \in N\backslash\{i\}$ and $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq g(M_i, m_{-i})$. Since $m \in NE(\gamma, R)$, it follows that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L(R_i, g(m))$ and $g(m) \in \max_{R_\ell} Y$ for any $\ell \in N\backslash\{i\}$. Furthermore, since F satisfies μ, $C_i\left(\bar{R}, x\right) \subseteq L(R_i, g(m))$ holds by the definition of $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right)$ given by (1). Thus, μ(ii) implies that $g(m) \in F(R)$.

Case 4: m falls into Rule 4.

Then, $Y \subseteq g(M_\ell, m_{-\ell})$ for all $\ell \in N$. Since $m \in NE(\gamma, R)$, it follows that $g(m) \in \max_{R_\ell} Y$ for each $\ell \in N$. μ(iii) implies that $g(m) \in F(R)$. ■

From Theorem 1 and the definition of g provided above, the following corollaries are easily obtained.

Corollary 1. *An* SCC F defined on $\mathcal{R}^n$ *is* implementable by *an* s-mechanism with FR^s *if and only if it is* implementable.

Corollary 2. *An* SCC F defined on $\mathcal{R}^n$ *is* implementable by *an* s-mechanism with FR^s *if and only if it satisfies* Condition μ.

Note that it is also easy to see that implementation by a canonical mechanism satisfying forthrightness[14] is equivalent to Condition μ. Together with this fact, Corollary 2 gives us the following interesting observation: under implementability, (i) an s-mechanism satisfying FR^s can be constructed from a canonical mechanism satisfying forthrightness, and (ii) a canonical mechanism satisfying forthrightness can be constructed from an s-mechanism satisfying FR^s. Remember that without forthrightness, there could be a case that an s-mechanism can be constructed from a canonical mechanism, and vice versa, because information can be smuggled. On the other hand, the forthrightness condition makes the possibility of this equivalence quite unclear. Yet, Corollary 2 implies that this equivalence always holds even under the forthrightness condition.

Note that we can also show that for any intermediate strategy space reduction mechanism between a canonical mechanism and an s-mechanism, implementation by such an intermediate strategy space reduction mechanism is equivalent to implementation. Indeed, let us consider any intermediate strategy space reduction mechanism, say q-mechanism, with the strategy space $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times \ldots \times \mathcal{R}_{\ell+q} \times Y \times N$ for all $\ell \in N$, where $q = 2, \ldots, n-2$.[15] Then, by a similar way as in the proofs of Theorem 1, it can be shown that an *SCC* satisfies Condition μ if and only if it is implementable by a q-mechanism.

Before closing this section, it may be worth introducing an alternative implementing condition, the study of which seems relevant to characterizing the scope of implementation by s-mechanisms in a non-conventional framework. The condition can be stated as follows.

Condition M^s (for short, M^s). There exists a set $Z \subseteq X$; moreover, for all $R \in \mathcal{R}^n$ and for all $x \in F(R)$, there is a profile of sets $\left(C_\ell^*(R_\ell, x)\right)_{\ell \in N}$ such

[14] The definition of forthrightness for canonical mechanisms is analogous to the definition of FR^s, though there is a minor difference.

[15] According to this terminology, $q = 1$ corresponds to s-mechanisms while $q = n-1$ corresponds to canonical mechanisms. Both cases are excluded from the naming of q-mechanisms, since we are interested solely in intermediate strategy space reduction.

that $x \in C_\ell^* (R_\ell, x) \subseteq L(R_\ell, x) \cap Z$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following conditions (i)-(iii) are satisfied:
(i) if $C_\ell^* (R_\ell, x) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i^* (R_i, x) \subseteq L(R_i^*, y)$ and $Z \subseteq L(R_\ell^*, y)$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Z \subseteq L(R_\ell^*, y)$ for all $\ell \in N$, then $y \in F(R^*)$.

The above condition is simple, easy to check, and very similar to Condition M (Sjöström, 1991).[16] Moreover, it has a central role in identifying the class of SCCs that are implementable by an s-mechanism when some of the participants uphold an intrinsic motivation toward honesty. Indeed, a weaker variant of Condition M^s is the *unique* necessary and sufficient condition for implementation by s-mechanisms when some of the participants are partially honest (Lombardi and Yoshihara, 2011).[17] In this regard, Condition M^s is more interesting than Condition μ. Yet, Condition M^s is indeed equivalent to Condition μ in the standard set-up.

Lemma 1. Condition M^s *is equivalent to* Condition μ.

Proof. Let F on $\mathcal{R}^n$ be an SCC. First, suppose that F satisfies M^s. Then, it is obvious that F also satisfies Condition μ by taking $Y \equiv Z$ and $C_\ell(R, x) \equiv C_\ell^* (R_\ell, x)$ for any $\ell \in N$, any $R \in \mathcal{R}^n$, and any $x \in F(R)$. Conversely, suppose that F satisfies μ. For any $\ell \in N$, $R \in \mathcal{R}^n$, and $x \in F(R)$, let

$$F^{-1}(R_\ell, x) \equiv \left\{R'_{-\ell} \in \mathcal{R}^n_{-\ell} | x \in F\left(R_\ell, R'_{-\ell}\right)\right\}$$

where $\mathcal{R}^n_{-\ell} \equiv \mathcal{R}_1 \times ... \times \mathcal{R}_{\ell-1} \times \mathcal{R}_{\ell+1} \times ... \times \mathcal{R}_n$. For any $\ell \in N$, $R \in \mathcal{R}^n$ and $x \in F(R)$, define the set $C_\ell^* (R_\ell, x)$ as follows

$$C_\ell^* (R_\ell, x) \equiv \cup_{R'_{-\ell} \in F^{-1}(R_\ell, x)} C_\ell\left(\left(R_\ell, R'_{-\ell}\right), x\right). \tag{2}$$

We prove that F satisfies M^s. Let $Z \equiv Y$. Moreover, take any $R \in \mathcal{R}^n$ and $x \in F(R)$. From (2) and μ, it follows that for each $\ell \in N$, the set $C_\ell^* (R_\ell, x)$ is well-defined and $x \in C_\ell^* (R_\ell, x) \subseteq L(R_\ell, x) \cap Z$. Next, we show that F meets M^s(i)-M^s(iii). To do this, take any $R^* \in \mathcal{R}^n$.

[16] The above condition introduces the profile $(C_\ell^* (R_\ell, x))_{\ell \in N}$, which corresponds to the profile $\left(C_\ell\left(R_{-\{\ell+1,...,\ell+r\}}, x\right)\right)_{\ell \in N}$ with $r = n-1$. Note that the profile $(C_\ell^* (R_\ell, x))_{\ell \in N}$ is similar to the profile specified in Condition M.

[17] A partially honest agent is an agent who *strictly* prefers to tell the truth when lying has no better material consequences for her.

Let $C_\ell^*(R_\ell, x) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$. Since $x \in F(R)$, it follows from μ and (2) that $C_\ell(R, x) \subseteq C_\ell^*(R_\ell, x)$ for all $\ell \in N$. Then, μ(i) implies that $x \in F(R^*)$. Hence, M^s(i) holds. Let $y \in C_i^*(R_i, x) \subseteq L(R_i^*, y)$ for some $i \in N$ and $Z \subseteq L(R_\ell^*, y)$ for all $\ell \in N \backslash \{i\}$. As $y \in C_i^*(R_i, x)$, it follows from (2) and μ that $y \in C_i\left(R_i, R'_{-i}, x\right) \subseteq C_i^*(R_i, x)$ for some $R'_{-i} \in F^{-1}(R_i, x)$. Then, μ(ii) implies that $y \in F(R^*)$. Therefore, M^s(ii) is satisfied. Finally, let $y \in Z \subseteq L(R_\ell^*, y)$ for all $\ell \in N$. Then, μ(iii) implies that $y \in F(R^*)$, and so M^s(iii) holds. We conclude that F satisfies M^s if it satisfies μ. ■

From the above lemma, the next result is readily obtained.

Theorem 2. *An* SCC F defined on $\mathcal{R}^n$ *satisfies* Condition M^s *if and only if it is* implementable by an s-mechanism.

4 On 'one less' preference announcement mechanisms

The result of the previous section shows that the 'strategy space reduction' from the canonical mechanisms up to s-mechanisms does not have any effect on the class of implementable *SCC*s. The purpose of this section is to show that such a property can no longer hold if the message conveyed by each participant to the planner involves the announcement of either her own preference or the preference of her neighbor - in addition to an outcome and an agent index.[18] Indeed, if each participant only reports her own preference, an outcome, and an agent index (*self-relevant mechanisms*; Tatamitani, 2001), it can be shown that the class of implementable *SCC*s by self-relevant mechanisms is a proper subset of the class of implementable *SCC*s. Then, the aim of this section is to show that exactly the same conclusion can be drawn when a strategy space reduction action from an s-mechanism consists of removing

[18]Observe that exactly the same conclusion can be drawn when a strategy space reduction from an s-mechanism consists of removing either the set N or the set Y from each participant's strategy space. As far as the elimination of the set N is concerned, Jackson (1992) shows that there are environments where some *SCC*s can be implemented only if the integer games (or some essentially similar schemes) are employed. As far as the elimination of the set Y is concerned, if the domain of an *SCC* is such that Y is uncountable, this *SCC* cannot be implemented by a type of mechanism with an appropriate forthrightness condition, whose strategy space is reduced from an s-mechanism by eliminating Y.

the announcement of her own preference from each participant's message space. Therefore, in what follows, we only focus on *neighbor's preference mechanisms*.

Definition 3. A mechanism $\gamma = (M, g)$ is a neighbor's preference mechanism (*np*-mechanism) if for any $\ell \in N$, $M_\ell \equiv \mathcal{R}_{\ell+1} \times Y \times N$, where $Y \subseteq X$ and $n + 1 = 1$.

Definition 4. An *SCC* F is implementable by an *np*-mechanism if there exists an *np*-mechanism $\gamma = (M, g)$ such that for all $R \in \mathcal{R}^n$, $F(R) = NA(\gamma, R)$.

Forthrightness for np-mechanisms, FR^{np}. For all $R \in \mathcal{R}^n$ and all $x \in F(R)$, if $m_\ell = \left(R_{\ell+1}, x, k^\ell\right) \in M_\ell$ for all $k^\ell \in N$ and all $\ell \in N$, with $n+1 = 1$, then $m \in NE(\gamma, R)$ and $g(m) = x$.

If an *np*-mechanism satisfies FR^{np}, we say that it is an *np*-mechanism with FR^{np}.

Using the approach developed by Moore and Repullo (1990), we now introduce a condition, Condition μ^{np}, which turns out to be necessary for implementation by *np*-mechanisms.[19] Before describing the condition and proving its necessity, some additional notation is required.

Given $(R, x) \in \mathcal{R}^n \times X$, define $D^F(R, x) \equiv \left\{\ell \in N | F^{-1}\left(R_{-\{\ell+1\}}, x\right) \neq \varnothing\right\}$ as the set of potential deviators. The condition can be stated as follows.

Condition μ^{np} (for short, μ^{np}): There exists a set $Y \subseteq X$; moreover, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $\left(C_\ell\left(R_{-\{\ell+1\}}, x\right)\right)_{\ell \in N}$ such that $x \in C_\ell\left(R_{-\{\ell+1\}}, x\right) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$, with $n+1 = 1$; finally, for all $R^* \in \mathcal{R}^n$, the following conditions (i)-(iv) are satisfied:
(i) if $C_\ell\left(R_{-\{\ell+1\}}, x\right) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i\left(R_{-\{i+1\}}, x\right) \subseteq L(R^*_i, y)$ and $Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N$, then $y \in F(R^*)$,
(iv) if $x \notin F(R^*)$ and $D^F(R^*, x) \neq \varnothing$, then there exists an outcome $p(R^*, x) \in X$ such that:
(a) $p(R^*, x) \in C_\ell\left(R^*_{-\{\ell+1\}}, x\right)$ for any $\ell \in D^F(R^*, x)$; and
(b) for all $R^{**} \in \mathcal{R}^n$, if $C_i\left(R^*_{-\{i+1\}}, x\right) \subseteq L(R^{**}_i, p(R^*, x))$ for all $i \in D^F(R^*, x)$, and $Y \subseteq L(R^{**}_\ell, p(R^*, x))$ for all $\ell \in N \backslash D^F(R^*, x)$, then $p(R^*, x) \in F(R^{**})$.

[19] Indeed, Condition μ^{np} is also sufficient (details available on request).

The following proposition shows that Condition μ^{np} is a necessary condition for implementation by np-mechanisms with FR^{np}.

Proposition 1. *An* SCC F defined on $\mathcal{R}^n$ *satisfies* Condition μ^{np} *if it is* implementable *by an* np-mechanism with FR^{np}.

Proof. Let an *SCC* F on $\mathcal{R}^n$ be implementable by an np-mechanism with FR^{np}. Let $\gamma = (M, g)$ be such an np-mechanism with FR^{np}. Define $Y \equiv g(M)$. For all $R \in \mathcal{R}^n$ and $x \in F(R)$, there exists an $m \in NE(\gamma, R)$ such that $g(m) = x$ and $m_\ell = \left(R_{\ell+1}, x, k^\ell\right)$ for all $\ell \in N$ by FR^{np}. Take any $\ell \in N$. For any $R'_{\ell+1} \in F^{-1}\left(R_{-\{\ell+1\}}, x\right)$, there exists an $m'_\ell = \left(R'_{\ell+1}, x, k^\ell\right)$ such that $g(m'_\ell, m_{-\ell}) = x \in NA\left(\gamma, \left(R'_{\ell+1}, R_{-\{\ell+1\}}\right)\right)$ by FR^{np}. Thus, for any $R'_{\ell+1} \in F^{-1}\left(R_{-\{\ell+1\}}, x\right)$, $g(M_\ell, m_{-\ell}) \subseteq L(R_\ell, x)$. Moreover, $g(m'_\ell, m_{-\ell}) = x \in NA(\gamma, R)$ also holds for any $R'_{\ell+1} \in F^{-1}\left(R_{-\{\ell+1\}}, x\right)$ and any $m'_\ell = \left(R'_{\ell+1}, x, k^\ell\right)$.

Define $C_\ell\left(R_{-\{\ell+1\}}, x\right) \equiv g(M_\ell, m_{-\ell})$. Then, $x \in C_\ell\left(R_{-\{\ell+1\}}, x\right) \subseteq L(R_\ell, x) \cap Y$. Next, we show that F satisfies Conditions μ^{np}(i)-μ^{np}(iv). Take any $R^* \in \mathcal{R}^n$.

Suppose that $C_\ell\left(R_{-\{\ell+1\}}, x\right) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$. Then, from $C_\ell\left(R_{-\{\ell+1\}}, x\right) = g(M_\ell, m_{-\ell})$, it follows that $g(M_\ell, m_{-\ell}) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$. We conclude that $g(m) = x \in NA(\gamma, R^*) = F(R^*)$. Hence, μ^{np}(i) holds.

For each $i \in N$, let $y \in C_i\left(R_{-\{i+1\}}, x\right) \subseteq L(R^*_i, y)$ and $Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N \setminus \{i\}$. Then, $y = g(m'_i, m_{-i}) \in g(M_i, m_{-i}) \subseteq L(R^*_i, y)$ for some $m'_i \in M_i$. Moreover, $g(M) \subseteq L(R^*_\ell, y)$ for all $\ell \in N \setminus \{i\}$. We conclude that $y \in NA(\gamma, R^*) = F(R^*)$. Hence, F satisfies μ^{np}(ii).

If $y \in Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N$, then there exists $m^* \in M$ such that $y = g(m^*) \in g(M) \subseteq L(R^*_\ell, y)$ for each $\ell \in N$. Therefore, $y \in NA(\gamma, R^*) = F(R^*)$. Hence, μ^{np}(iii) holds.

Suppose that $x \notin F(R^*)$ and $D^F(R^*, x) \neq \varnothing$, and let us consider the strategy profile $m^*_\ell = \left(R^*_{\ell+1}, x, k^\ell\right) \in M_\ell$ for all $\ell \in N$. Let $p(R^*, x) \equiv g(m^*)$. Take any $i \in D^F(R^*, x)$ and any $R'_{i+1} \in F^{-1}\left(R^*_{-\{i+1\}}, x\right)$. Let $R' \equiv \left(R^*_{-\{i+1\}}, R'_{i+1}\right)$. Then, from the previous discussion, it follows that there exists a profile $\left(C_\ell\left(R'_{-\{\ell+1\}}, x\right)\right)_{\ell \in N}$ with $C_\ell\left(R'_{-\{\ell+1\}}, x\right) \equiv g\left(M_\ell, m'_{-\ell}\right)$ for each $\ell \in N$, where $m'_\ell = m^*_\ell$ for each $\ell \in N \setminus \{i\}$ and $m'_i = \left(R'_{i+1}, x, k^i\right)$. Since it holds for any $i \in D^F(R^*, x)$, it follows that $g(m^*) \in Y$ and $g(m^*) \in C_i\left(R^*_{-\{i+1\}}, x\right) \equiv g\left(M_i, m^*_{-i}\right)$ for any $i \in D^F(R^*, x)$. Therefore, μ^{np}(iv.a) is met. Finally, take any $R^{**} \in \mathcal{R}^n$, and suppose that $C_i\left(R^*_{-\{i+1\}}, x\right) \subseteq L\left(R^{**}_i, p\left(R^*, x\right)\right)$ for all $i \in D^F\left(R^*, x\right)$ and $Y \subseteq L\left(R^{**}_\ell, p\left(R^*, x\right)\right)$ for all $\ell \in N \backslash D^F\left(R^*, x\right)$. Then, since $g\left(M_i, m^*_{-i}\right) \subseteq L\left(R^{**}_i, g\left(m^*\right)\right)$ for all $i \in D^F\left(R^*, x\right)$ and $g\left(M\right) \subseteq L\left(R^{**}_\ell, g\left(m^*\right)\right)$ for all $\ell \in N \backslash D^F\left(R^*, x\right)$, it follows that $p\left(R^*, x\right) = g\left(m^*\right) \in NA\left(\gamma, R^{**}\right) = F\left(R^{**}\right)$. Thus, μ^{np}(iv.b) holds. We conclude that F satisfies μ^{np}. ■

The above proposition implies that implementation by an *np*-mechanism with FR^{np} is not equivalent to implementation, since the punishment condition, Condition μ^{np}(iv), makes Condition μ^{np} stronger than Condition μ. That is, Condition μ^{np} implies Condition μ, but the converse does not hold. Therefore, Proposition 1 implies that the class of implementable *SCC*s by *np*-mechanisms with FR^{np} is a proper subset of the class of implementable *SCC*s.[20] Finally, observe that the class of implementable *SCC*s by self-relevant mechanisms is *not* equivalent to the class of *SCC*s that are implementable by *np*-mechanisms with FR^{np}. This is because even though it can be easily seen that the former is contained by the latter, the converse relation does not hold. To see this, let us consider the standard problem of allocating an infinitely divisible commodity among a group of three or more agents with single-peaked preferences. Within this class of problems, it can be shown that the *individually rational from equal division and efficient correspondence* satisfies Condition μ^{np}, while it violates the self-relevant implementing condition devised by Tatamitani (2001).[21]

5 Toward message space reduction for implementation with asymmetric information: A discussion

The results of the previous sections rely on the assumption of complete information. That is, agent preferences and feasible outcomes were assumed to be common knowledge among the agents. However, the issue of devising informational efficient mechanisms pertains not only to implementation

[20] To see this, for instance, consider classical economic environments as the domain of *SCC*s. Then, as shown in Saijo *et al.* (1999), the no-envy and efficient correspondence does not satisfy μ^{np}(iv), though it satisfies μ.

[21] Details available on request.

problems with complete information, but also to problems with asymmetric information (see Jackson (1991; p. 473), for instance). For the latter problems, the present section briefly puts forward some ideas and guidelines needed to address this issue.

Implementation problems with asymmetric information have been studied in a series of papers including Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1987, 1989), Jackson (1991), and Hahn and Yannelis (2001). The basic model to study implementation problems with asymmetric information consists of a set, N, of n agents, a (fixed) set, X, of feasible outcomes, a set of types, Θ_i, for each agent $i \in N$, a collection of conditional probability distributions $\{p_i(\theta_{-i}|\theta_i)\}$, for each $i \in N$ and each $\theta_i \in \Theta_i$, and a von Neumann-Morgenstern utility function for each agent i, $u_i : \Theta \times X \rightarrow \mathbb{R}$, where $\Theta \equiv \Theta_1 \times \ldots \times \Theta_n$ denotes the set of states. Thus, one environment is given by a list $\langle N, X, (\Theta_i, p_i, u_i)_{i \in N} \rangle$. The classic assumption is that each agent knows her own type when she comes to participate in the mechanism. This is her sole informational advantage over the mechanism designer, because the latter does not know the types of all agents while the former does not know the types of the other agents. However, the mechanism designer knows the structure of the environment $\langle N, X, (\Theta_i, p_i, u_i)_{i \in N} \rangle$ as the agents do. Most implementation problems with asymmetric information discussed in the literature are organized around the solution concept of Bayesian equilibrium.

In what follows, our discussion of the Baysian approach is based on Jackson (1991), who devises an implementing mechanism which is finite in finite environments - the reader can consult that paper for more details. Moreover, the mechanism devised by this author differs considerably from mechanisms designed for Nash implementation. The main difference is that in the case of asymmetric information, where each agent's type is her *private* information, the mechanism designer must provide the agent with incentives to report that information, in contrast to the case of complete information in which the preference stated by an agent can be checked against another agent's report of that preference. Thus, while the construction of our s-mechanism under problems with complete information relies heavily on the possibility to devise an auditing scheme of announced preferences, which is due to the condition that participants stand in a circle and each of them states her own preference and that of her neighbor, such a construction of mechanism is impossible in problems with asymmetric information, where the agents' types constitute private information. Nonetheless, the idea of cyclic announcements of messages proposed in Hurwicz (1979) can still be useful in achieving a significant reduction in the size of the message space required for Bayesian implementation.

Before outlining how the strategy space reduction can be achieved, some additional notation is needed. An *outcome rule* is a function x from the set of states Θ to X, that is, $x(\theta) \in X$ for each $\theta \in \Theta$, while a *social choice set* (*SCS*) F specifies a set of optimal outcome rules $\{x \mid x : \Theta \to X\}$ rather than a collection of desirable outcomes.[22]

In Jackson's mechanism, each participant i is required to announce, *inter alia*, an element of F. Our suggestion is based on the following idea. As the first step of construction of a mechanism by the mechanism designer, she partitions the set of states Θ into n non-empty sets $\{\Theta^i\}_{i \in N}$, and then she assigns only one element Θ^i of this partition to each participant $i \in N$. Given an outcome rule x, let $x|\Theta^i$ denote the restriction of x to the set Θ^i. Similarly, given a *SCS* F, $F|\Theta^i$ denotes the set of optimal outcome rules restricted to the set Θ^i. Like in our s-mechanism, assume that all agents are put in a circle, facing toward its center. Then, assuming that agents are arranged clockwise, our idea is that part of the strategy for participant i is to announce an element of $F|\Theta^i$ and an element of $F|\Theta^{i+1}$, with $n+1=1$, rather than an element of $F(\Theta)$. Therefore, a typical message of participant i will be of the form

$$m_i = (\theta_i^i, x_i^i|\Theta^i, x_{i+1}^i|\Theta^{i+1}, s_i) \in \Theta_i \times F|\Theta^i \times F|\Theta^{i+1} \times S_i \equiv M_i,$$

where the announcement of participant i about participant $i+1$'s restriction of an F-optimal outcome rule x to Θ^{i+1} is $x_{i+1}^i|\Theta^{i+1}$, and S_i denotes the other components of Jackson's message space for participant i.

The above reduction of the message space poses a challenge in the construction of a Bayesian implementing mechanism. The main difficulty is the definition of the outcome function for the case in which there is exactly one break in the cyclic announcements of restricted outcome rules. In particular, we are referring to the case in which:

[1] the announcements of the restricted outcome rules are such that $x_j^j|\Theta^j = x_j^{j-1}|\Theta^j$ for every participant j except for participant i, and the announcement of participant i is such that $x_{i+1}^i|\Theta^{i+1} = x_{i+1}^{i+1}|\Theta^{i+1}$ and $x_i^{i-1}|\Theta^i \neq x_i^i|\Theta^i$;

[22] It is known that if the *closure condition* is satisfied and the information is complete, then an *SCC* is equivalent to an *SCS*.

[2] if we remove the restricted outcome rule announced by agent i and replace it with the announcement of her immediate neighbor $i-1$, we find that the outcome rule y defined as

$$y|\Theta^j = x_j^j|\Theta^j \text{ for all } j \neq i \text{ and } y|\Theta^i = x_i^{i-1}|\Theta^i$$

is F-optimal; and,

[3] if we remove the restricted outcome rule announced by agent $i-1$ and replace it with the announcement of her immediate neighbor $i-2$, we find that the outcome rule y' defined as

$$y'|\Theta^j = x_j^j|\Theta^j \text{ for all } j \neq i-1 \text{ and } y'|\Theta^{i-1} = x_{i-1}^{i-2}|\Theta^{i-1}$$

is F-optimal.[23]

In this case, agent i and agent $i-1$ can both be identified as potential deviators.

In Section 3, we show that the canonical mechanism can be suitably modified for the case in which part of the strategy for each participant i is to announce her own preference and that of her neighbor. We conjecture that analogous modifications can be made for the mechanism devised by Jackson (1991) when part of the strategy for each participant i is to announce an element of $(F|\Theta^i, F|\Theta^{i+1})$ rather than an optimal outcome rule in F.

It cannot be emphasized *enough* that our discussion so far is based on the assumption that agents adopt Bayesian-Nash behavior, which requires a great deal of ability from the part of agents[24]: Each agent forms Bayesian beliefs about the types of the other participants and seeks the maximization of her expected utility with respect to those beliefs. To the extent that agents themselves are not behaving as Bayesian with well-formed priors, the Bayesian approach to implementation with asymmetric information must be brought into question. Non-Bayesian models of behavior have received relatively little attention not only in implementation theory with asymmetric information, but, until recently, in resource allocation problems in general. In this regard, de Castro *et al.* (2011) show a quite surprising result that the maximin expected utility preferences introduced by Gilboa and Schmeidler (1989) guarantee that in economic environments with ambiguity, efficiency and incentive compatibility are consistent. Indeed, several interesting

[23] Note that the outcome rule y differs from the outcome rule y' only for the type profiles in Θ^i, that is, $y(\theta) = y'(\theta)$ for each $\theta \in \Theta \backslash \Theta^i$, and $y(\theta) \neq y'(\theta)$ for some $\theta \in \Theta^i$.

[24] In this regard, see Machina *et al.* (2011), for instance.

and surprising results are reported by a burgeoning literature which studies both experimentally and theoretically economic problems with asymmetric information when agents have maximin expected utility preferences. Some of these results can be found in Cerreia-Vioglio *et al.* (2011), Condie and Ganguli (2011), de Castro and Chateauneuf (2011), Dickhaut *et al.* (2011), Dominiak and Schnedler (2011), Eichberger and Kelsey (2011), Jungbauer and Ritzberger (2011), Klibanoff *et al.* (2011), Machina (2011), Nau (2011), Ozsoylev and Werner (2011), and Vergopoulos (2011). In line with this research, there may be important potential for a non-Bayesian approach to implementation in asymmetric information settings.

Indeed, analogical to the definition of maximin (individual) incentive compatibility given by de Castro *et al.* (2011), we may define the notion of maximin Nash equilibrium by using the maximin expected utility preferences. Then, we may consider abstract social choice environments with *ambiguity* and examine what types of SCCs are implementable in maximin Nash equilibria. Furthermore, we may ask a further question of whether it is possible to construct a mechanism with as small message space as possible in this context. For instance, given that the framework of de Castro *et al.* (2011) presumes that the information of a "differential information exchange economy" is common knowledge among agents, but this information is not shared by the mechanism designer, it would be interesting to devise an implementing mechanism in which part of the strategy for each participant i is to announce her information set and that of her neighbor $i+1$,[25] analogical to the strategy space reduction mechanism in Nash implementation developed in this paper. These are open questions that are worthwhile to examine in the future.

6 Concluding Remarks

In this paper, we deal with the informational efficiency issue pertaining to Maskin's Theorem (Maskin, 1999). We focus on s-mechanisms in which each agent reports to the planner solely her own preference and her neighbor's preference, in addition to a feasible social outcome and an integer. We show

[25] Each agent can announce her neighbor's information set after receiving a common signal from the nature, since Definition 5.14(i) of de Castro *et al.* (2011) implicitly presumes that a profile of each agent's *partition of the state space* is common knowledge among agents - otherwise, an agent who intends to misrepresent her observed state cannot be convinced of whether this type of behavior is beneficial to her or not.

that the class of SCCs that are implementable by s-mechanisms is fully identified by Condition μ of Moore and Repullo. We achieve it in two ways. Firstly, we show this result by directly employing the classical Condition μ. Secondly, in the framework developed by Moore and Repullo, we introduce a new condition, Condition M^s, which is shown to be equivalent to Condition μ.

With regard to the implementation theory, several implications of our analysis are worthwhile mentioning. First, the class of implementable SCCs is equivalent to the class of SCCs implementable by s-mechanisms with forthrightness. Second, even though our conditions are stated in terms of the existence of certain sets, they can easily be checked in practice by the algorithm provided by Sjöström (1991). Last but not least, under implementability, a canonical mechanism satisfying forthrightness can be constructed by an s-mechanism satisfying forthrightness, and vice versa.

Note that our results are in line with other well known results of implementation in economic environments. In particular, the equivalence relationship between implementation by s-mechanisms and implementation in general social choice environments is analogous to the equivalence relationship between implementation by natural allocation mechanisms and implementation by natural quantity[2] mechanisms (Saijo *et al.*, 1996). Moreover, Tatamitani (2001) provides a full characterization of implementation by self-relevant mechanisms. This result, together with our Proposition 1, indicates that *any* 'strategy space reduction' from s-mechanisms consisting of requiring each participant to announce a 'one less preference relation' drastically decreases the class of implementable SCCs. This is parallel to the case of natural implementation in economic environments, in which the class of SCCs implementable by natural quantity mechanisms is much smaller than that of SCCs implementable by natural allocation mechanisms.

Before closing the paper, we should make one last comment about the results presented herein. The results of the paper are built on the implicit assumption that agents participating in a mechanism are *perfectly rational.* Dissatisfaction with this classical assumption is mounting.[26] Attempts to replace it with alternative decision models as engines of inquiry into basic economic questions are growing.[27] In the light of this recent trend, the equivalence relationship reported herein may not necessarily hold when a small

[26] In this regard, see, for instance, Glycopantis *et al.* (2005).

[27] In this regard, see, for instance, de Castro *et al.* (2011).

departure from the "perfect rational man" paradigm is considered. With regard to implementation, Matsushima (2008) and Dutta and Sen (2012) introduce the notion of partial honesty in implementation theory and consider implementation problems with partially-honest agents. A partially-honest agent is an agent who has a preference over message profiles and displays concerns over two dimensions in lexicographic order: (1) her outcome and (2) her truth-telling behavior. In the presence of partially-honest agents, the equivalence relationship between implementation and implementation by s-mechanisms with forthrightness no longer holds, as Lombardi and Yoshihara (2011a) show.[28] This suggests that the equivalence relationship indispensably relies on the assumption that agents act purely on their own self-interest and are not inclined to attach (moral) rights and duties to their actions.

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