Extended Social Ordering Functions for Rationalizing Fair Allocation Rules as Game Forms in the Sense of Rawls and Sen*

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Abstract

We examine the possibility of constructing social ordering functions, each of which associates a social ordering over the feasible pairs of allocations and allocation rules with each simple production economy. Three axioms on the admissible class of social ordering functions are introduced, which embody the values of procedural fairness, non-welfaristic egalitarianism, and welfaristic consequentialism, respectively. The logical compatibility of these axioms and their lexicographic combinations subject to constraints are examined. Two social ordering functions which give priority to procedural values rather than to consequential values are identified, which can uniformly rationalize a nice allocation rule in terms of the values of procedural fairness, non-welfaristic egalitarianism, and Pareto efficiency.

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1 Introduction

Even in the developed market economies with matured market mechanisms and assured basic liberties, the issue of providing each individual with equitable standard of living without undue sacrifice of social efficiency and/or individual autonomy still remains largely unresolved. There is a wide social consensus that the uncompromising pursuit of economic efficiency, individual autonomy, and the equitable provision of decent living standard is a hardly sustainable target, but there remains a further issue of forming a social agreement on the priority to be assigned to the plural moral principles pursuing, respectively, social efficiency, individual autonomy, and equitable provision of decent living standard. This paper is devoted to the problem of combining these plural moral principles lexicographically as well as conditionally without falling into the impasse of logical inconsistency. In this context, it is worthwhile to recollect that John Rawls (1971, 1993) made an interesting proposal to combine plural moral principles lexicographically. However, his proposal had to be confronted with a criticism by Amartya Sen and Bernard Williams (1982) who pointed out that the lexicographic combination of plural moral principles may be logically inconsistent. Our present analysis represents an attempt to circumscribe the conditions under which the logical coherence of philosophical scenario articulated by Rawls and Sen can be rigorously ascertained.

To lend concreteness to the problem at hand, we focus on the following three moral principles in the context of defining a fair allocation rule as a game form in a class of simple production economies. The first principle is procedural in nature, and it requires that all individuals in the society should be assured of the minimal extent of autonomy in choosing his contribution to cooperative production. This principle is due originally to Lawrence Kranich (1994). Our articulation of allocation rules in terms of game forms seems to be appropriate from the viewpoint of individual autonomy. Indeed, individual autonomy has a natural expression in the game form articulation of allocation rules by means of the spontaneous strategic choice of individual actions. The second principle is consequential in nature, and it requires the Pareto efficiency of equilibrium social outcomes. The third principle is meant to capture an aspect of non-welfaristic egalitarianism along the line of Rawls and Sen, which is formally articulated in terms of the maximin assignment of individual capabilities rather than in terms of individual utilities. This principle was formulated and characterized by Reiko Gotoh and Naoki Yoshihara

(1999, 2003). The logical coherence of one lexicographic combination or the other of these moral principles, with or without further constraints on their applicability, can be verified by examining the existence, or the lack thereof, of a fair allocation rule as a game form thereby defined.

Two rather novel features of our analysis may deserve further clarifications. The first novel features is the capability maximin rule which is meant to give substance to our conception of equitable provision of decent living standard. Instead of using Rawls's own formulation of the difference principle articulated in terms of what he christened the social primary goods, or its social choice theoretic formulation in terms of interpersonally level comparable utilities along the line of Sen (1970, Chapter 6*; 1997), we are identifying the least advantaged individual by means of what Sen (1980, 1985) christened *capabilities*, which are meant to capture the freedom individuals can enjoy in pursuit of their own lives they have reasons to choose. The gist of this approach is to shift the focus of our attention from the *subjective* happiness or satisfaction enjoyed by individuals to the *objective* opportunities in the functioning space to which individuals can rightfully access. Note that the game form articulation of capability maximin allocation rule is useful, if not indispensable, in clarifying the role of strategic interactions among individuals in the social determination of individual opportunities in pursuit of their objective well-being. The second novel feature is the use we make of the extended social ordering function which associates a social ordering over the pairs of feasible resource allocations and allocation rules as game forms with each economic environment, capitalizing on the pioneering work by Prasanta Pattanaik and Kotaro Suzumura (1994; 1996). It is this analytical device that enables us to treat a procedural principle requiring individual autonomy, a welfaristic principle requiring the Pareto efficiency of social outcomes, and a non-welfaristic but consequential principle in the form of capability maximin rules simultaneously within a unified framework. It is also this analytical device which allows us to talk sensibly about rationality and uniform rationality of allocation rules as game forms.

Apart from this introduction, this paper consists of four sections and an appendix. In Section 2. we introduce a class of simple production economies, allocation rules as game forms, and extended social ordering functions. In Section 3, we formulate three basic axioms of fair allocation rules, and examine the existence of an allocation rule which is qualified to be fair in terms of these axioms. Section 4 defines the three basic axioms on extended social ordering functions and presents our possibility theorems. Section 5 concludes this paper with several final remarks. All the involved proofs are relegated into the Appendix at the end of the paper for the sake of simplicity of exposition.

2 The Basic Framework

2.1 Economic Environments and Allocation Rules

Consider an economy with the population $N = \{1, 2, \ldots, n\}$, where $2 \leq n < +\infty$. One good $y \in \mathbb{R}_+$ is produced from the vector of labor inputs $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$,[1] where x_i denotes the labor time supplied by $i \in N$. The production process of this economy is described by the production function $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$, which maps each $\mathbf{x} \in \mathbb{R}^n_+$ into $y = f(\mathbf{x}) \in \mathbb{R}_+$. It is assumed that f satisfies *continuity*, *strict increasingness*, *concavity*, and $f(\mathbf{0}) = 0$.

All individuals are assumed to have the common upper bound $\overline{x}$ of labor-leisure time, where $0 < \bar{x} < +\infty$. For each individual $i \in N$, his consumption vector is denoted by $z_i = (x_i, y_i) \in [0, \overline{x}] \times \mathbb{R}_+$, where x_i is his labor time and y_i is his share of output. Each $i \in N$ is characterized by his preference ordering on $[0, \overline{x}] \times \mathbb{R}_+$, which can be represented by a utility function $u_i : [0, \overline{x}] \times \mathbb{R}_+ \rightarrow \mathbb{R}$. We assume that u_i is *strictly monotonic* (decreasing in labor time and increasing in the share of output) on $[0, \overline{x}) \times \mathbb{R}_{++}$, *continuous* and *quasi-concave* on $[0, \overline{x}] \times \mathbb{R}_{++}$. It is also assumed that $u_i(z_i) > u_i(x_i, 0)$ for all $z_i \in [0, \overline{x}) \times \mathbb{R}_{++}$ and all $x_i \in [0, \overline{x}]$. We denote the class of utility functions satisfying these assumptions by $\mathcal{U}$.

Since the production function f is fixed throughout this paper, we may identify one economy simply by $\mathbf{u} \in \mathcal{U}^n$, where $\mathbf{u} = (u_1, \ldots, u_n)$. A *feasible allocation* in our economy is a vector $\mathbf{z} = (z_i)_{i \in N} = (x_i, y_i)_{i \in N} \in ([0, \overline{x}] \times \mathbb{R}_+)^n$ such that $f(\mathbf{x}) \geq \sum_N y_i$, where $\mathbf{x} = (x_1, \ldots, x_n)$. Let Z be the set of all feasible allocations.

To complete the description of how our simple economy functions, what remains is to specify an *allocation rule* which assigns, to each $i \in N$, how many hours he/she works, and how much share of output he/she receives in return. In this paper, an allocation rule is modelled as a *game form* which is a pair $\gamma = (M, g)$, where $M = M_1 \times \cdots \times M_n$ is the set of admissible profiles of individual *strategies*, and g is the *outcome function* which maps

[1] In what follows, $\mathbb{R}_+$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote, respectively, the set of non-negative real numbers, the non-negative orthant, and the positive orthant in the Euclidean n-space.

each strategy profile $\mathbf{m} \in M$ into a unique outcome $g(\mathbf{m}) \in Z$. For each $\mathbf{m} \in M$, $g(\mathbf{m}) = (g_i(\mathbf{m}))_{i \in N}$, where $g_i(\mathbf{m}) = (g_{i1}(\mathbf{m}), g_{i2}(\mathbf{m}))$ and $g_{i1}(\mathbf{m}) \in [0, \bar{x}]$ and $g_{i2}(\mathbf{m}) \in \mathbb{R}_+$ for each $i \in N$, represents a feasible allocation resulting from the strategic interactions among individuals represented by the strategy profile $\mathbf{m}$. Let Γ be the set of all game forms representing allocation rules of our economy. Given an allocation rule $\gamma = (M, g) \in \Gamma$ and an economy $\mathbf{u} \in \mathcal{U}^n$, we obtain a fully-fledged specification of a non-cooperative game $(N, \gamma, \mathbf{u})$. Since the set of players N is fixed throughout this paper, we may omit N and describe a game as $(\gamma, \mathbf{u}) \in \Gamma \times \mathcal{U}^n$ without ambiguity.

An important juncture in our analysis of the performance of game forms as social decision-making rules is the specification of the equilibrium concept. Throughout this paper, we will focus on the Nash equilibrium concept. To describe an equilibrium outcome of a game $(\gamma, \mathbf{u})$, where $\gamma = (M, g)$, define $\mathbf{m}_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n)$ for each $\mathbf{m} \in M$ and $i \in N$, which is an element of a set $M_{-i} := \times_{j \in N \setminus \{i\}} M_j$. Given an $\mathbf{m}_{-i} \in M_{-i}$ and an $m_i' \in M_i$, $(m_i'; \mathbf{m}_{-i})$ may be construed as an admissible strategy profile obtained from $\mathbf{m}$ by replacing m_i with m_i'. Given a game $(\gamma, \mathbf{u}) \in \Gamma \times \mathcal{U}^n$, an admissible strategy profile $\mathbf{m}^* \in M$ is a *pure strategy Nash equilibrium* if $u_i(g_i(\mathbf{m}^*)) \geq u_i(g_i(m_i; \mathbf{m}_{-i}^*))$ holds for all $i \in N$ and all $m_i \in M_i$. The set of all pure strategy Nash equilibria of the game $(\gamma, \mathbf{u})$ is denoted by $NE(\gamma, \mathbf{u})$. A feasible allocation $\mathbf{z}^* \in Z$ is a *pure strategy Nash equilibrium allocation* of the game $(\gamma, \mathbf{u})$ if $\mathbf{z}^* = g(\mathbf{m}^*)$ holds for some $\mathbf{m}^* \in NE(\gamma, \mathbf{u})$. The set of all pure strategy Nash equilibrium allocations of the game $(\gamma, \mathbf{u})$ is denoted by $A_{NE}(\gamma, \mathbf{u})$.

2.2 Extended Social Ordering Functions

In our comprehensive framework of analysis for social choice of allocation rules as game forms, a crucial role is played by the concept of *extended social ordering functions*, which are defined over the set of *extended social alternatives*, viz., pairs of feasible allocations and allocation rules as game forms. The intended interpretation of an extended social alternative, viz., a pair $(\mathbf{z}, \gamma) \in Z \times \Gamma$, is that a feasible allocation $\mathbf{z}$ is attained through an allocation rule γ.[2]

[2] The concept of an extended social alternative was introduced by Pattanaik and Suzumura (1994; 1996), capitalizing on the thought-provoking suggestion by Arrow (1951,

As we explained in section 1, the concept of extended social ordering functions enables us to treat the principle of individual autonomy, the Pareto principle on resource allocations, and the principle of equal provision of individual objective well-being in a unified framework. Indeed, the definition of social ordering by means of the standard binary relation over the set of allocations would not enable us to treat the axiom of individual autonomy in choice procedure appropriately, since the information of such an aspect is not contained in the description of allocations per se. Likewise, if we adopt the definition of social ordering by means of the binary relation over the set of game forms, such a framework would not provide us with the informational basis for discussing the Pareto principle as an outcome morality as well as the principle of equal opportunity, which is procedural in nature.

Note that a feasible allocation $\mathbf{z} \in Z$ may or may not be *realizable* through an allocation rule $\gamma \in \Gamma$. Indeed, an extended social alternative $(\mathbf{z}, \gamma) \in Z \times \Gamma$ is realizable only when an economy $\mathbf{u} \in \mathcal{U}^n$ is given and $\mathbf{z} \in A_{NE}(\gamma, \mathbf{u})$ holds. Thus, an extended social alternative can be judged realizable only when a profile of individual preference orderings over consequential outcomes is specified in sharp contrast with the traditional social choice framework. Let $\mathcal{R}(\mathbf{u})$ denote the set of realizable extended social alternatives under $\mathbf{u} \in \mathcal{U}^n$.

What we call an *extended social ordering function* (**ESOF**) is a function $Q : \mathcal{U}^n \to (Z \times \Gamma)^2$ such that $Q(\mathbf{u})$ is an ordering on $\mathcal{R}(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}^n$.[3] The intended interpretation of $Q(\mathbf{u})$ is that, for any extended social alternatives $(\mathbf{z}^1, \gamma^1)$, $(\mathbf{z}^2, \gamma^2) \in \mathcal{R}(\mathbf{u})$, $((\mathbf{z}^1, \gamma^1), (\mathbf{z}^2, \gamma^2)) \in Q(\mathbf{u})$ holds if and only if attaining a feasible allocation $\mathbf{z}^1$ through an allocation rule γ^1 is at least as good as attaining a feasible allocation $\mathbf{z}^2$ through an allocation rule γ^2 according to the social judgments embodied in $Q(\mathbf{u})$.[4] The asymmetric part

pp.89-91). See, also, Suzumura (1996; 1999; 2000) for further clarifications on the use and usefulness of this approach.

[3]A binary relation R on a universal set X is a *quasi-ordering* if it satisfies *reflexivity* and *transitivity*. An *ordering* is a quasi-ordering satisfying *completeness* as well.

[4]Note that this concept of an **ESOF** enables us to accommodate both consequential values and procedural values in the social evaluation of feasible allocations and allocation rules. If an **ESOF** Q_c is such that, for each $\mathbf{u} \in \mathcal{U}^n$, $((\mathbf{z}, \gamma^1), (\mathbf{z}, \gamma^2)) \in I(Q_c(\mathbf{u}))$ holds for all $(\mathbf{z}, \gamma^1), (\mathbf{z}, \gamma^2) \in \mathcal{R}(\mathbf{u})$, it represents a social evaluation that cares only about consequential outcomes of resource allocations. In this sense, Q_c may be christened the purely consequential **ESOF**. In contrast, an **ESOF** Q_p such that, for each $\mathbf{u} \in \mathcal{U}^n$, $((\mathbf{z}^1, \gamma), (\mathbf{z}^2, \gamma)) \in I(Q_p(\mathbf{u}))$ holds for all $(\mathbf{z}^1, \gamma), (\mathbf{z}^2, \gamma) \in \mathcal{R}(\mathbf{u})$ embodies a social evaluation that cares only about procedural features of resource allocations. In this sense, Q_p may be christened the purely procedural **ESOF**. In between these two polar extreme cases,

and the symmetric part of $Q(\mathbf{u})$ will be denoted by $P(Q(\mathbf{u}))$ and $I(Q(\mathbf{u}))$, respectively. The set of all **ESOF**s will be denoted by $\mathcal{Q}$.

Once an **ESOF** $Q \in \mathcal{Q}$ is specified, the set of best extended social alternatives is given, for each $\mathbf{u} \in \mathcal{U}^n$, by

$$B(\mathbf{u} : Q) \equiv \{(\mathbf{z}, \gamma) \in \mathcal{R}(\mathbf{u}) \mid \forall (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u}) : ((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u})\}.$$

The set of socially chosen allocation rules is then given by

$$D(\mathbf{u} : Q) \equiv \{\gamma \in \Gamma \mid \exists \mathbf{z} \in Z : (\mathbf{z}, \gamma) \in B(\mathbf{u} : Q)\}.$$

We say that an allocation rule $\gamma \in \Gamma$ is *uniformly rationalizable*[5] by means of the **ESOF** $Q \in \mathcal{Q}$ if and only if

$$\gamma \in \bigcap_{\mathbf{u} \in \mathcal{U}^n} D(\mathbf{u} : Q)$$

holds. By definition, such an allocation rule γ applies uniformly to each and every $\mathbf{u} \in \mathcal{U}^n$ without violating the values embodied in the **ESOF** Q. This implies that once $\gamma \in \bigcap_{\mathbf{u} \in \mathcal{U}^n} D(\mathbf{u} : Q)$ exists and is socially chosen, it will prevail no matter how frivolously the profile $\mathbf{u}$ undergoes a change. Since the allocation rule as a game form is nothing other than the formal method of specifying the legal structure prevailing in the society prior to the realization of the profile of individual utility functions, it seems desirable, if at all possible, to design **ESOF** $Q \in \mathcal{Q}$ satisfying the requirement of uniform rationalizability.

3 Fair Allocation Rules as Game Forms

In this section, we will discuss what properties qualify a game form to be a "fair" allocation rule, and examine the existence of such a rule. Let us begin

there is a wide range of **ESOF**s which embody both consequential values and procedural values.

[5]Recollect that a pair $(\mathbf{z}, \gamma) \in B(\mathbf{u} : Q)$ is said to be *rationalizable* by an ordering $Q(\mathbf{u})$ on $\mathcal{R}(\mathbf{u})$ if and only if $(\mathbf{z}, \gamma)$ is judged to be at least as good as any other pair in $\mathcal{R}(\mathbf{u})$ in terms of the ordering $Q(\mathbf{u})$. By a slight abuse of terminology, we may say in this case that $\gamma \in D(\mathbf{u} : Q)$ is rationalizable by $Q(\mathbf{u})$. Then an allocation rule $\gamma \in \bigcap_{\mathbf{u} \in \mathcal{U}^n} D(\mathbf{u} : Q)$ may be said to be uniformly rationalizable, as it is rationalizable by virtue of $Q(\mathbf{u})$ no matter which $\mathbf{u} \in \mathcal{U}^n$ may materialize.

with three conditions for allocation rules as game forms, which embody a value of individual autonomy, a value of economic efficiency, and a value of equal opportunity for individual objective well-being, respectively.

First, we introduce a condition for individual autonomy. If a game form satisfies this property, then every individual can secure whatever level of his labor time at the social outcome of strategic interactions among individuals by an appropriate choice of his own strategy.

Definition 1 [Kranich (1994)]: *An allocation rule* $\gamma = (M, g) \in \Gamma$ *is* labor-sovereign *if, for all* $i \in N$ *and all* $x_i \in [0, \overline{x}]$, *there exists* $m_i \in M_i$ *such that, for all* $\mathbf{m}_{-i} \in M_{-i}$, $g_{i1}(m_i, \mathbf{m}_{-i}) = x_i$.

Let Γ_{LS} denote the subclass of Γ which consists solely of allocation rules satisfying labor sovereignty.

The next condition requires the Pareto efficiency of equilibrium outcomes. That is, the Nash equilibrium allocations of the games defined by fair allocation rules as game forms should be Pareto efficient. For each $\mathbf{u} \in \mathcal{U}^n$, let[6]

$$PO(\mathbf{u}) \equiv \{\mathbf{z} \in Z \mid \forall \mathbf{z}' \in Z, \exists i \in N : u_i(z_i) \geq u_i(z_i')\}.$$

Definition 2: *An allocation rule* $\gamma = (M, g) \in \Gamma$ *is* efficient *if, for any* $\mathbf{u} \in \mathcal{U}^n$, $\mathbf{z} \in A_{NE}(\gamma, \mathbf{u})$ *implies* $\mathbf{z} \in PO(\mathbf{u})$.

Let Γ_{PE} denote the subclass of Γ which consists solely of efficient allocation rules.

3.1 Functioning and Capability

Our next condition for a fair allocation rule is meant to capture an aspect of non-welfaristic egalitarianism. It hinges on an objective concept of individual well-being in the spirit of the functioning and capability approach proposed by Sen (1980, 1985).

Suppose that there are s *basic functionings* in the economy, which are relevant for all individuals in describing their objective well-beings, such as being healthy and free from diseases, having enough longevity, being well-informed, being able to participate in community life. These functionings are

[6] Note that $PO(\mathbf{u})$ is the set of Pareto efficient allocations by virtue of the strict monotonicity of every utility function.

attainable by means of consumption vectors. We assume that these s functionings can be measured by means of adequate non-negative real numbers. Thus, an achievement of functioning k, where $k = 1, 2, \cdots, s$, by individual i is denoted by $b_{ik} \in \mathbb{R}_+$. Individual i's achievement of basic functionings is given by listing b_{ik}: $\mathbf{b}_i = (b_{i1}, \cdots, b_{is}) \in \mathbb{R}^s_+$. For each $i \in N$, i's *capability correspondence* is defined as $c_i : [0, \overline{x}] \times \mathbb{R}_+ \twoheadrightarrow \mathbb{R}^s_+$ which associates with each $z_i \in [0, \overline{x}] \times \mathbb{R}_+$ a non-empty subset $c_i(z_i)$ of $\mathbb{R}^s_+$. This $c_i(z_i)$ is called i's *capability at his consumption vector* z_i, which is the opportunity set of basic functionings when his consumption vector is z_i.

In what follows, we assume that the capability correspondences satisfy the following requirements:

(a) For all $z_i = (x_i, y_i)$, $z'_i = (x'_i, y'_i) \in [0, \overline{x}] \times \mathbb{R}_+$ such that $x_i = x'_i$ and $y_i \leq y'_i$ (resp. $y_i < y'_i$), $c_i(z_i) \subseteq c_i(z'_i)$ (resp. $c_i(z_i) \subsetneq c^o_i(z'_i)$) hold, where $c^o_i(z'_i)$ stands for the *interior* of $c_i(z'_i)$ in $\mathbb{R}^s_+$;[7]
(b) For all $z_i \in [0, \overline{x}] \times \mathbb{R}_+$, $c_i(z_i)$ is *compact* and *comprehensive* in $\mathbb{R}^s_+$; and
(c) c_i is *continuous* on $[0, \overline{x}] \times \mathbb{R}_+$.

Let us denote the universal class of capability correspondences which meet the above three requirements by $\mathfrak{C}$. Given a profile of individual capability correspondences $\mathbf{c} \in \mathfrak{C}^n$ and for each $\mathbf{z} = (z_i)_{i \in N} \in Z$, $\mathbf{c}(\mathbf{z}) = (c_i(z_i))_{i \in N}$ denotes a *feasible assignment of individual capabilities.*

3.2 J-Based Capability Maximin Allocations

We are now ready to introduce the third condition for a fair allocation rule as a game form, which requires that every Nash equilibrium allocation of the game defined by the allocation rule should guarantee a maximal level of capability to the least advantaged individual, where the identification of the least advantaged individual is made in terms of individual capabilities.

To be more precise, we introduce an ordering over capabilities, which represents an *evaluation on the impersonal well-ness of capabilities,* in order to identify who is the least advantaged in terms of capability assignments. Let the universal set of capabilities be

$$\mathcal{K} \equiv \left\{ C \subseteq \mathbb{R}^s_+ \mid \exists c \in \mathfrak{C} \ \& \ \exists z \in [0, \overline{x}] \times \mathbb{R}_+ : c(z) = C \right\}.$$

[7]For all vectors $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_p) \in \mathbb{R}^p$, $\mathbf{a} \geq \mathbf{b}$ if and only if $a_i \geq b_i$ $(i = 1, \ldots, p)$; $\mathbf{a} > \mathbf{b}$ if and only if $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$; $\mathbf{a} \gg \mathbf{b}$ if and only if $a_i > b_i$ $(i = 1, \ldots, p)$.

Suppose that the society has an *evaluation on the impersonal well-ness of capabilities*, which is represented by an ordering relation $J \subseteq \mathcal{K} \times \mathcal{K}$ satisfying completeness: [for all $C, C' \in \mathcal{K}$, $(C, C') \in J$ or $(C', C) \in J$] and transitivity: [for all $C, C', C'' \in \mathcal{K}$, if $(C, C') \in J$ & $(C', C'') \in J$, then $(C, C'') \in J$]. $P(J)$ and $I(J)$ denote, respectively, the asymmetric part and symmetric part of J.

At this juncture, let us introduce an appropriate topology into the space $\mathcal{K}$ in terms of the Hausdorff metric.[8] Equipped with this topology, we suppose that the ordering J satisfies the following intuitively plausible axioms:

(3.1.1) Monotonicity: For any $C, C' \in \mathcal{K}$, if $C \supseteq C'$ then $(C, C') \in J$, and if $C^0 \supseteq C'$, then $(C, C') \in P(J)$, where C^0 denotes the interior of C in $\mathbb{R}^s_+$.
(3.1.2) Dominance: For any $C, C', C'' \in \mathcal{K}$, if $[(C, C') \in P(J)$ and $(C, C'') \in P(J)]$, then $(C, C' \cup C'') \in P(J)$.

(3.1.3) Continuity: For any $C \in \mathcal{K}$ and any sequence of capabilities $\{C^r\}_{r=1}^{\infty}$ such that $C^r \in \mathcal{K}$ for all r and $C^* = \lim_{r\to\infty} C^r \in \mathcal{K}$, if $(C^r, C) \in J$ for all r, then $(C^*, C) \in J$.

The following characterization of the ordering $J \subseteq \mathcal{K} \times \mathcal{K}$ is due to Xu (2003).

Proposition 1 [Xu (2003)]: *If the ordering* $J \subseteq \mathcal{K} \times \mathcal{K}$ *satisfies* Monotonicity, Dominance, and Continuity, *then there exists a continuous and increasing function* $\eta : \mathbb{R}^s_+ \to \mathbb{R}$ *such that for all* $C, C' \in \mathcal{K}$,

$$(C, C') \in J \Leftrightarrow \left[\max_{\mathbf{b} \in C} \eta(\mathbf{b}) \geq \max_{\mathbf{b}' \in C'} \eta(\mathbf{b}')\right].$$

Denote the admissible class of evaluations which meet all of **(3.1.1)**, **(3.1.2)**, and **(3.1.3)** by $\mathcal{J}$.

Given the evaluation $J \in \mathcal{J}$ and the profile of capability correspondences $\mathbf{c} \in \mathfrak{C}^n$, let us define the subset $\mathcal{C}^J_{\min}(\mathbf{z}) \subseteq \mathcal{K}$ for each feasible allocation

[8] For any compact sets C, $C' \subseteq \mathbb{R}^s_+$, the Hausdorff metric between C and C' is defined by

$$d(C, C') \equiv \max\{\max\{\delta(\mathbf{b}, C) \mid \mathbf{b} \in C'\}, \max\{\delta(\mathbf{b}, C') \mid \mathbf{b} \in C\}\},$$

where $\delta(\mathbf{b}, C) \equiv \min_{\mathbf{b}' \in C} \parallel \mathbf{b}, \mathbf{b}' \parallel$, and $\parallel \mathbf{b}, \mathbf{b}' \parallel$ is the Euclidean distance between $\mathbf{b}$ and $\mathbf{b}'$.

$\mathbf{z} = (z_1, \ldots, z_n) \in Z$ by: $c_i(z_i) \in \mathcal{C}^J_{\min}(\mathbf{z})$ if and only if $(c_j(z_j), c_i(z_i)) \in J$ for all $j \in N$. For any $J \in \mathcal{J}$ and any $\mathbf{x} \in [0, \overline{x}]^n$, we now define[9]

$$Z(\mathbf{x}; J) \equiv \left\{ (\mathbf{x}, \mathbf{y}) \in Z \mid \forall (\mathbf{x}, \mathbf{y}') \in Z : \left(\mathcal{C}^J_{\min}(\mathbf{x}, \mathbf{y}), \mathcal{C}^J_{\min}(\mathbf{x}, \mathbf{y}') \right) \in J \right\}.$$

The set $Z(\mathbf{x}; J)$ consists of feasible allocations which are maximal in terms of the evaluation J for the given $\mathbf{x} \in [0, \overline{x}]^n$. In this sense, this set may be construed to consist of J-reference capability maximin allocations for the given $\mathbf{x} \in [0, \overline{x}]^n$. We are now ready to define the third condition as follows:

Definition 3: *An allocation rule* $\gamma = (M, g) \in \Gamma$ *is called the* J-reference capability maximin rule *if, for any* $\mathbf{u} \in \mathcal{U}^n$, $\mathbf{z} \in A_{NE}(\gamma, \mathbf{u})$ *implies* $\mathbf{z} \in Z(\mathbf{x}; J)$, *where* $\mathbf{z} = (\mathbf{x}, \mathbf{y})$.[10]

Let Γ_{JCM} denote the subclass of Γ which consists solely of J-reference capability maximin allocation rules.

3.3 Existence of Fair Allocation Rules as Game Forms

Let us now discuss the existence of a fair allocation rule as a game form which is labor sovereign, Pareto efficient, and J-reference capability maximin. A game form $\gamma = (M, g)$ is said to be *Nash-solvable* if $A_{NE}(\gamma, \mathbf{u}) \neq \varnothing$ for each and every $\mathbf{u} \in \mathcal{U}^n$. Denote the set of Nash solvable game forms by Γ_{NS}.

Assumption 1: The utility function u_i of each and every agent has the following property: $\forall z_i \in [0, \overline{x}) \times \mathbb{R}_{++}$, $u_i(z_i) > 0$, and $u_i(\overline{x}, 0) = 0$.

Assumption 2: The production function f is continuously differentiable.

Theorem 1: *Under* **Assumption 1** *and* **Assumption 2**, *and for any given evaluation* $J \in \mathcal{J}$, *there exists an allocation rule* $\gamma^* \in \Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}$.

In this theorem, **Assumption 1** can be weakened to the claim that u_i is bounded from below. Moreover, **Assumption 2** is not essential: it is introduced just to simplify the argument. Indeed, we can construct an allocation

[9] The use of the expression $\left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}') \right) \in J$ in this definition is slightly abusive. The rigorous expression should go as follows: for any $c_i(z_i) \in \mathcal{C}^J_{\min}(\mathbf{z})$ and any $c_j(z'_j) \in \mathcal{C}^J_{\min}(\mathbf{z}')$, $\left(c_i(z_i), c_j(z'_j) \right) \in J$ holds.

[10] This type of allocation rule originates in Gotoh and Yoshihara (1999, 2003).

rule having the property of **Theorem 1** even without **Assumption 2**, although the construction of such an allocation rule will be more complicated than the current method.

Although Gotoh and Yoshihara (1999, 2003) proposed and characterized the class of allocation rules $\Gamma_{LS} \cap \Gamma_{JCM}$, it was unclear whether or not there exists an element of this class which also belongs to Γ_{PE}. Here, however, **Theorem 1** asserts that if the method of ranking capability sets is constrained by the three plausible conditions **(3.1.1)**, **(3.1.2)**, and **(3.1.3)** introduced by Xu (2002, 2003), we can successfully find an allocation rule in $\Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}$.

4 ESOFs for Rationalizing Fair Allocation Rules

4.1 Three Basic Axioms on ESOFs and Their Compatibility

In what follows, we will examine the possibility of an **ESOF** embodying the three distinct values of individual autonomy, economic efficiency, and equal opportunity for individual objective well-being, along with uniform rationalizability of the allocation rule γ^*. To begin with, let us formulate three basic axioms of **ESOF**s, which go as follows:

Labor Sovereignty (LS): *For any* $\mathbf{u} \in \mathcal{U}^n$ *and any* $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$, *if* $\gamma \in \Gamma_{LS}$ *and* $\gamma' \in \Gamma \setminus \Gamma_{LS}$, *then* $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q(\mathbf{u}))$.

Respect of the J-Reference Least Advantaged (J-LA): *For any* $\mathbf{u} \in \mathcal{U}^n$ *and any* $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$, *if* $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, $\mathbf{z}' = (\mathbf{x}', \mathbf{y}')$ *and* $\mathbf{x} = \mathbf{x}'$, *then*:

$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u}) \Leftrightarrow \left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}')\right) \in J,$$
$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q(\mathbf{u})) \Leftrightarrow \left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}')\right) \in P(J).$$

Pareto in Allocations (PA): *For any* $\mathbf{u} \in \mathcal{U}^n$ *and any* $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$, *if* $u_i(z_i) > u_i(z'_i)$ *for all* $i \in N$, *then* $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q(\mathbf{u}))$, *and if* $u_i(z_i) = u_i(z'_i)$ *for all* $i \in N$, *then* $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in I(Q(\mathbf{u}))$.

Among the above three axiom, **LS** is the requirement of purely procedural fairness, as it imposes some constraints on the admissible class of **ESOF**s without having recourse to the nature of consequential outcomes. J-**LA** is the requirement of non-welfaristic egalitarianism, which attempts to enable the least advantaged individual, where being least advantaged is identified in terms of his capability through the social value judgements J, to secure as large capability as possible by choosing a feasible allocation appropriately. Finally, **PA**, being a variant of Paretianism, is based squarely on welfaristic consequentialism.

It may well be asked why J-**LA** imposes the premise $\mathbf{x} = \mathbf{x}'$. The reason is twofold. First, the choice of individual labor hours is a matter to be left to individual responsibility, and social value judgements should respect individual choices on a matter of this nature. Second, since J is a complete ordering, if the requirement of J-**LA** is applied to **ESOF**s without the premise $\mathbf{x} = \mathbf{x}'$, then it gives us a complete ordering by this axiom only, leaving no room for applying the Paretian axiom at all.

It is of little surprise that *for any* $J \in \mathcal{J}$, *there exists no social ordering function which satisfies any two of the basic* **LS**, J-**LA**, *and* **PA**. When two or more basic principles irrevocably conflict with each other and yet we do not want to discard any one of these principles altogether, a natural step to follow is to introduce a priority rule among these principles and define their lexicographic combinations. This idea has been explored repeatedly in the literature of normative economics, the most recent example being Koichi Tadenuma (2003).

To explore this intuitive idea systematically in our present context, take any distinct $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \{\mathbf{LS}, J\text{-}\mathbf{LA}, \mathbf{PA}\}$ and define a subclass $\mathcal{Q}^{\boldsymbol{\alpha} \vdash \boldsymbol{\beta} \vdash \boldsymbol{\gamma}}$ of **ESOF**s as follows: any $Q \in \mathcal{Q}^{\boldsymbol{\alpha} \vdash \boldsymbol{\beta} \vdash \boldsymbol{\gamma}}$ implies that, for any $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$, $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u})$ (resp. $P(Q(\mathbf{u}))$) if (1) the axiom $\boldsymbol{\alpha}$ requires it; (2) the axiom $\boldsymbol{\beta}$ requires it, given that the axiom $\boldsymbol{\alpha}$ keeps silence; or (3) the axiom $\boldsymbol{\gamma}$ requires it, given that the axioms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ keep silence.[11] Let us define $\mathcal{Q}^{lex} \equiv \cup \mathcal{Q}^{\boldsymbol{\alpha} \vdash \boldsymbol{\beta} \vdash \boldsymbol{\gamma}}$ over all distinct $\alpha, \beta, \gamma \in \{$**LS**, J-**LA**, **PA**$\}$, in

[11] Recollect that **LS**, J-**LA**, and **PA** are expressed in the conditional form of "if (A), then (B)" style. Thus, whenever the condition (A) is not satisfied for **Axiom** $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} \in \{\mathbf{LS}, J\text{-}\mathbf{LA}, \mathbf{PA}\}$, **Axiom** $\boldsymbol{\alpha}$ has nothing to offer and must keep silence. This being the case, an **ESOF** $Q \in \mathcal{Q}^{\boldsymbol{\alpha} \vdash \boldsymbol{\beta} \vdash \boldsymbol{\gamma}}$, where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \{\mathbf{LS}, J\text{-}\mathbf{LA}, \mathbf{PA}\}$, simply implies that **Axiom** $\boldsymbol{\beta}$ can have a bite only when the condition (A) is not satisfied for **Axiom** $\boldsymbol{\alpha}$, thereby forcing **Axiom** $\boldsymbol{\alpha}$ to keep silence; and **Axiom** $\boldsymbol{\gamma}$ can have a bite only when the condition (A) is not satisfied for **Axiom** $\boldsymbol{\alpha}$ as well as for **Axiom** $\boldsymbol{\beta}$, thereby forcing **Axiom** $\boldsymbol{\alpha}$ and **Axiom** $\boldsymbol{\beta}$ to keep silence. In other words, in $Q \in \mathcal{Q}^{\boldsymbol{\alpha} \vdash \boldsymbol{\beta} \vdash \boldsymbol{\gamma}}$, **Axiom** $\boldsymbol{\alpha}$ has a lexicographic priority to **Axiom** $\boldsymbol{\beta}$ and **Axiom** $\boldsymbol{\gamma}$, and **Axiom** $\boldsymbol{\beta}$ has a lexicographic priority to **Axiom** $\boldsymbol{\gamma}$.

which each $Q \in \mathcal{Q}^{lex}$ applies the three basic axioms lexicographically.

Even these lexicographic combinations of the basic three axioms are incompatible, as the following proposition holds.

Proposition 2: *$\mathcal{Q}^{lex}$ is empty.*

It is not difficult to check the consistency of the lexicographic combination of **LS** and J-**LA** as well as that of **LS** and **PA**. This being the case, the culprit of the impossibility result, viz., **Proposition 2**, should be attributed to the impossibility of lexicographically combining J-**LA** and **PA.** The essence of the proof of **Proposition 2** can be presented in terms of the following example, which illustrates the impossibility of lexicographically combining J-**LA** and **PA**.

Example 1: Let there be two types of relevant functionings, and let $N = \{1, 2\}$ and $\overline{x} = 3$. The production function is given by $f(x_1, x_2) = x_1 + x_2$ for all $(x_1, x_2) \in \mathbb{R}^2_+$. Individuals have the same capability correspondence c which is defined as follows: For any $z \in [0, \overline{x}] \times \mathbb{R}_+$

$$c(z) \equiv \left\{(b_1, b_2) \in \mathbb{R}^2_+ \mid \exists z^1, z^2 \in [0, \overline{x}] \times \mathbb{R}_+ \text{: } z^1 + z^2 \leq z,\ b_k = a_k(z^k)\ (k = 1, 2)\right\}$$

where $a_1(x, y) \equiv (\overline{x} - x)^{\frac{2}{3}} \cdot y^{\frac{1}{3}}$ and $a_2(x, y) \equiv (\overline{x} - x)^{\frac{1}{3}} \cdot y^{\frac{2}{3}}$ for any $(x, y) \in [0, \overline{x}] \times \mathbb{R}_+$. Note that the mapping $a_k(\cdot)$ assigns to each consumption vector an achievement of functioning k. Thus, $b_k = a_k(z^k)$ implies that if the consumption vector z^k is utilized for functioning k, then it is attained at the level of b_k.

Consider two feasible allocations $\mathbf{z}^* = ((1, 1), (1, 1))$ and $\mathbf{z}^{**} = ((2, 2), (2, 2))$. For some $\theta \in (0, 1)$, let $\mathbf{z}^*(\theta) = ((1, 1 + \theta), (1, 1 - \theta))$ and $\mathbf{z}^{**}(\theta) = ((2, 2 - \theta), (2, 2+\theta))$. It is easy to check that, for any $J \in \mathcal{J}$, $\left(\mathcal{C}^J_{\min}(\mathbf{z}^*), \mathcal{C}^J_{\min}(\mathbf{z}^*(\theta))\right) \in P(J)$ and $\left(\mathcal{C}^J_{\min}(\mathbf{z}^{**}), \mathcal{C}^J_{\min}(\mathbf{z}^{**}(\theta))\right) \in P(J)$, since J satisfies (3.1.1). Individual 1's utility function u_1 is defined for all $(x, y) \in [0, \overline{x}] \times \mathbb{R}_{++}$ by

$$u_1(x, y) = (1 - \theta) \cdot (\overline{x} - x) + y,$$

whereas individual 2's utility function u_2 is defined for all $(x, y) \in [0, \overline{x}] \times \mathbb{R}_{++}$ by

$$u_2(x, y) = \begin{cases} (1 - \theta) \cdot (\overline{x} - x) + y & \text{if } x \in [0, 1) \\ (1 + \theta) \cdot (\overline{x} - x) + y & \text{if } x \in [1, \overline{x}] \end{cases}.$$

This situation is described in the consumption space and in the functioning space in Figure 1.

Insert Figure 1 around here

Let γ^*, $\gamma^*(\theta)$, γ^{**}, and $\gamma^{**}(\theta)$ be the allocation rules in $\mathbf{\Gamma}\backslash\mathbf{\Gamma}_{LS}$ which generate the realizable allocations $\mathbf{z}^*$, $\mathbf{z}^*(\theta)$, $\mathbf{z}^{**}$, and $\mathbf{z}^{**}(\theta)$, respectively, when the economy is defined by $\mathbf{u} = (u_1, u_2) \in \mathcal{U}^2$.

Take any **ESOF** $Q \in \mathcal{Q}^{lex}$. Then, compare $(\mathbf{z}^*, \gamma^*)$ with $(\mathbf{z}^*(\theta), \gamma^*(\theta))$, and $(\mathbf{z}^{**}, \gamma^{**})$ with $(\mathbf{z}^{**}(\theta), \gamma^{**}(\theta))$. Since $\gamma^*, \gamma^*(\theta), \gamma^{**}, \gamma^{**}(\theta) \in \mathbf{\Gamma}\backslash\mathbf{\Gamma}_{LS}$, and $\mathbf{z}^*$ (resp. $\mathbf{z}^{**}$) and $\mathbf{z}^*(\theta)$ (resp. $\mathbf{z}^{**}(\theta)$) are Pareto non-comparable, **LS** and **PA** keep silence for any $Q \in \mathcal{Q}^{lex}$. In contrast, by J-**LA**, we have, for any $Q \in \mathcal{Q}^{lex}$,

$$((\mathbf{z}^*, \gamma^*), (\mathbf{z}^*(\theta), \gamma^*(\theta))) \in P(Q(\mathbf{u})),\ ((\mathbf{z}^{**}, \gamma^{**}), (\mathbf{z}^{**}(\theta), \gamma^{**}(\theta))) \in P(Q(\mathbf{u})).$$

Next, compare $(\mathbf{z}^{**}, \gamma^{**})$ with $(\mathbf{z}^*(\theta), \gamma^*(\theta))$, and $(\mathbf{z}^{**}(\theta), \gamma^{**}(\theta))$ with $(\mathbf{z}^*, \gamma^*)$. In this case, not only **LS** but also J-**LA** keep silence, since $\mathbf{x}^{**} \neq \mathbf{x}^*(\theta)$ and $\mathbf{x}^{**}(\theta) \neq \mathbf{x}^*$. In contrast, by **PA**, we have, for any $Q \in \mathcal{Q}^{lex}$,

$$((\mathbf{z}^*(\theta), \gamma^*(\theta)), (\mathbf{z}^{**}, \gamma^{**})) \in I(Q(\mathbf{u})),\ ((\mathbf{z}^{**}(\theta), \gamma^{**}(\theta)), (\mathbf{z}^*, \gamma^*)) \in I(Q(\mathbf{u})),$$

since $u_1(z_1^*(\theta)) = u_1(z_1^{**}) = 3 - \theta$, $u_2(z_2^*(\theta)) = u_2(z_2^{**}) = 3 + \theta$, $u_1(z_1^{**}(\theta)) = u_1(z_1^*) = 3 - 2\theta$, and $u_2(z_2^{**}(\theta)) = u_2(z_2^*) = 3 + 2\theta$. Thus, any $Q \in \mathcal{Q}^{lex}$ is not a consistent binary relation,[12] hence it cannot be an ordering. ■

4.2 Existence of ESOFs Uniformly Rationalizing γ^*

To secure the existence of a compatible lexicographic combination of our basic axioms, further concession seems to be indispensable in view of **Proposition 2**. For each given $J \in \mathcal{J}$, let us introduce the following conditional variants of J-**LA** and **PA,** respectively:

[12] A finite subset $\{x^1, \cdots, x^t\}$ of a universal set X, where $2 \leq t < +\infty$, satisfying $(x^1, x^2) \in P(R)$, $(x^2, x^3) \in R$, $\cdots$, $(x^t, x^1) \in R$ is called an *incoherent cycle of order* t of a binary relation R on X. R is said to be *consistent* if there exists no incoherent cycle of any order. A binary relation R^* is called an *extension* of R if and only if $R \subseteq R^*$ and $P(R) \subseteq P(R^*)$. It is shown in Suzumura (1983, Theorem A(5)) that *there exists an ordering extension of* R *if and only if* R *is consistent.*

J-**LA** $\cap$ **PO**: *For any* $\mathbf{u} \in \mathcal{U}^n$ *and any* $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$ *with* $\mathbf{z}, \mathbf{z}' \in PO(\mathbf{u})$*, if* $\mathbf{z} = (\mathbf{x}, \mathbf{y})$*,* $\mathbf{z}' = (\mathbf{x}', \mathbf{y}')$*, and* $\mathbf{x} = \mathbf{x}'$*, then:*

$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u}) \Leftrightarrow \left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}')\right) \in J,$$
$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q(\mathbf{u})) \Leftrightarrow \left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}')\right) \in P(J).$$

PA $\cap\, Z(J)$: *For any* $\mathbf{u} \in \mathcal{U}^n$ *and any* $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$ *such that* $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z(\mathbf{x}; J)$ *and* $\mathbf{z}' = (\mathbf{x}', \mathbf{y}') \in Z(\mathbf{x}'; J)$*, if* $u_i(z_i) > u_i(z_i')$ *for all* $i \in N$*, then* $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q(\mathbf{u}))$*, and if* $u_i(z_i) = u_i(z_i')$ *for all* $i \in N$*, then* $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in I(Q(\mathbf{u}))$.

Observe that $\mathbf{z}, \mathbf{z}' \in PO(\mathbf{u})$ implies that $\mathbf{z}$ and $\mathbf{z}'$ are Pareto noncomparable. This implies that J-**LA**$\cap$**PO** requires the applicability of J-**LA** in the special case where **PA** keeps silence. Also, observe that $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z(\mathbf{x}; J)$ and $\mathbf{z}' = (\mathbf{x}', \mathbf{y}') \in Z(\mathbf{x}'; J)$ imply $\mathbf{x} \neq \mathbf{x}'$ as we will show below. This implies that **PA** $\cap\, Z(J)$ requires the applicability of **PA** in the special case where J-**LA** keeps silence.

Now, for each given $J \in \mathcal{J}$, let us consider two subclasses of **ESOF**s, viz., $\mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash(\mathbf{PA}\cap Z(J))}$ and $\mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})}$. Note that $\mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash\mathbf{PA}} \subseteq \mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash(\mathbf{PA}\cap Z(J))}$ and $\mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash J\text{-}\mathbf{LA}} \subseteq \mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})}$. This is because $Q \in \mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash(\mathbf{PA}\cap Z(J))}$ implies that for any $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$, $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u})$ (resp. $P(Q(\mathbf{u}))$) holds if (1) **LS** requires it; or (2) J-**LA** requires it, given that **LS** keeps silence; or (3) **PA** requires it, given that not only **LS** and J-**LA** keep silence, but also $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z(\mathbf{x}; J)$ and $\mathbf{z}' = (\mathbf{x}', \mathbf{y}') \in Z(\mathbf{x}'; J)$ hold. Also, $Q \in \mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})}$ implies that for any $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$, $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u})$ (resp. $P(Q(\mathbf{u}))$) holds if (1) **LS** requires it; or (2) **PA** requires it, given that **LS** keeps silence; or (3) J-**LA** requires it, given that not only **LS** and **PA** keep silence, but also $\mathbf{z}, \mathbf{z}' \in PO(\mathbf{u})$ holds.

Thus, although both $\mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash\mathbf{PA}}$ and $\mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash J\text{-}\mathbf{LA}}$ are empty as is shown in **Proposition 2**, it may well be expected that $\mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash(\mathbf{PA}\cap Z(J))}$ and $\mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})}$ might be non-empty. The following theorems show that this is indeed the case.

Theorem 2: *For any* $J \in \mathcal{J}$*, there exists* $Q^*_J \in \mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash(\mathbf{PA}\cap Z(J))}$ *such that* $\bigcap_{\mathbf{u}\in\mathcal{U}^n} D(\mathbf{u} : Q^*_J) = \Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}$.

Theorem 3: *For any* $J \in \mathcal{J}$*, there exists* $Q^{**}_J \in \mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})}$ *such that* $\bigcap_{\mathbf{u}\in\mathcal{U}^n} D(\mathbf{u} : Q^{**}_J) = \Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}$.

According to **Theorem 2** and **Theorem 3,** there exist two types of **ESOF**s which not only combine lexicographically two conditional variants of our three basic moral principles, but also uniformly rationalize a fair allocation rule identified by **Theorem 1**. In other words, the three intrinsic values of individual autonomy, economic efficiency, and equitable provision of decent living standard can be made compatible, subject to constraints, in constructing a fair allocation rule as a game form, as well as in constructing a fair extended social ordering function.[13] Note that **Theorem 1** demonstrated the existence of a fair allocation rule as a game form which is identified by our three moral principles, together with the Nash solvability of the game form. In contrast, **Theorem 2** and **Theorem 3** demonstrated the existence of two types of **ESOF**s which enable us not only to identify the optimal allocation rule in the sense of **Theorem 1,** but also to identify the suboptimal allocation rules subject to some common constraints imposed on the set of labor sovereign game forms.

Another interesting feature of **Theorem 2** and **Theorem 3** seems to be worthwhile to point out. In general, the difference in the order of lexicographically combining various axioms should lead to the difference of rational choices thereby identified. In view of this fact, a conspicuous feature of **Theorem 2** and **Theorem 3** is that the **ESOF**s identified by these theorems can commonly yield a set of the first-best allocation rules as game forms including γ^* in **Theorem 1** as their uniformly rational game forms.

5 Conclusion

In the concrete context of a simple production economy, this paper identified three moral principles on the desirability of resource allocation rules as game forms. The first principle is procedural in nature, and it requires

[13] Q_J^* in **Theorem 2** and Q_J^{**} in **Theorem 3** commonly confer priority to **LS**. **LS** being an axiom of procedural fairness, Q_J^* and Q_J^{**} belong to the subclass of **ESOF**s which give priority to procedural considerations vis-à-vis consequential considerations. We can likewise consider the possible subclass of **ESOF**s which give priority to consequential considerations vis-à-vis procedural considerations. To be precise, it can be shown that:

Theorem*: *For any* $J \in \mathcal{J}$, $\mathcal{Q}^{J\text{-}\mathbf{LA}\vdash(\mathbf{PA}\cap Z(J))\vdash\mathbf{LS}}$ $(resp. \mathcal{Q}^{\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})\vdash\mathbf{LS}}) \neq \varnothing$.

However, there is no real parallelism between **Theorem 2** and **Theorem 3**, on the one hand, and **Theorem***, on the other. This is because an **ESOF** Q in **Theorem*** may well fail to assure the non-emptiness of $B(\mathbf{u} : Q)$.

that all individuals should be assured of the minimal extent of autonomy in choosing his contribution to cooperative production. The second principle is purely consequential in nature, and it requires that the consequential social outcomes should be Pareto efficient. The third principle requires that the consequential social outcome should warrant each and every individual of an equitable provision of decent living standard, which may be formally identified by means of the maximin assignment of individual capabilities in the sense of Sen.

It goes without saying that each one of these principles, in isolation, is a highly appealing moral claim. In combination, however, they represent a moral claim which is logically too demanding to be satisfied. A natural response to this logical impasse in the light of Rawls's (1971, 1993) thoughtful suggestion is to combine these principles lexicographically. In a related but distinct context of the equity-efficiency trade-off, Tadenuma (2002) successfully exploited this idea and has shown that the equity-first and efficiency-second lexicographic combination of these two moral principles can resolve the equity-efficiency trade-off, whereas the efficiency-first and equity-second lexicographic combination thereof cannot serve as a resolvent of the equity-efficiency trade-off. In our present context, however, we have shown that any one of the possible lexicographic combinations of our three moral principles still represent a moral claim which is logically too demanding to be satisfied. In this sense, Sen and Williams's (1982) acute warning is fully justified in our analytical setting. We are thus led to examine the conditional versions of the component moral principles and their lexicographic combinations. Theorem 2 and Theorem 3, which represent our possibility theorems in this paper, are on the workability of lexicographic combinations of *conditional* moral principles. It is true that the workable lexicographic combinations of conditional moral principles are far more complex than the simple lexicographic combinations of moral principles à la Rawls. This being the case, it is all the more noteworthy that these conditional moral principles can nevertheless uniformly rationalize in common the set of fair allocation rules as game forms, whose existence is established by Theorem 1.

Let us conclude by pointing out an open question. It is one thing to show the existence of a social ordering function which may identify a fair resource allocation rule as a game form, and it is quite another to show how such a social ordering function can be generated through democratic social decision-making procedures. However, an analysis to the latter effect requires altogether distinct conceptual development, which cannot but be left as one of our future agendas.

6 Appendix

6.1 Proof of Theorem 1

Let the set of feasible assignments of capabilities be defined by

$$\mathbf{C}(Z) \equiv \{\mathbf{c}(\mathbf{z}) \in \mathcal{K}^n \mid \mathbf{z} \in Z\},$$

which is compact in the Hausdorff topological space $\mathcal{K}^n$. Define a compact-valued and continuous correspondence $Y : [0, \overline{x}]^n \twoheadrightarrow \mathbb{R}^n_+$ by $Y(\mathbf{x}) \equiv \{\mathbf{y} = (y_i)_{i \in N} \in \mathbb{R}^n_+ \mid f(\mathbf{x}) \geq \sum_N y_i\}$ for each $\mathbf{x} \in [0, \overline{x}]^n$. Then, we can show, given $J \in \mathcal{J}$ and the profile of capability correspondences $\mathbf{c} \in \mathfrak{C}^n$, the following lemmas.

Lemma 1: *For each* $\mathbf{x} \in [0, \overline{x}]^n$*,* $Z(\mathbf{x}; J)$ *is non-empty and compact.*

Proof. For each $\mathbf{x} \in [0, \overline{x}]^n$, let $\mathbf{C}(\{\mathbf{x}\} \times Y(\mathbf{x})) \equiv \{\mathbf{c}(\mathbf{x}, \mathbf{y}) \in \mathbf{C}(Z) \mid \mathbf{y} \in Y(\mathbf{x})\}$ and $\mathbf{MC}(\mathbf{x}) \equiv \left\{\mathbf{c}(\mathbf{x}, \mathbf{y}) \in \mathbf{C}(Z) \mid \forall \mathbf{y}' \in Y(\mathbf{x}) : \left(\mathcal{C}^J_{\min}(\mathbf{x}, \mathbf{y}), \mathcal{C}^J_{\min}(\mathbf{x}, \mathbf{y}')\right) \in J\right\}$. Since J is continuous on $\mathcal{K}$ and $\mathbf{C}(\{\mathbf{x}\} \times Y(\mathbf{x}))$ is compact, we are assured that $\mathbf{MC}(\mathbf{x})$ is non-empty and compact. Thus, $Z(\mathbf{x}; J)$ is non-empty and compact. ■

Lemma 2: *For each* $\mathbf{x} \in [0, \overline{x}]^n$*,* $\mathbf{z} \in Z(\mathbf{x}; J)$ *implies* $(c_i(z_i), c_j(z_j)) \in I(J)$ *for all* $i, j \in N$*.*

Proof. Suppose there exist $\mathbf{x} \in [0, \overline{x}]^n$ and $\mathbf{z} \in Z(\mathbf{x}; J)$ such that $(c_i(z_i), c_j(z_j)) \in P(J)$ for some $i, j \in N$. Then, $\left(c_i(z_i), \mathcal{C}^J_{\min}(\mathbf{z})\right) \in P(J)$ holds. Consider an alternative allocation $\mathbf{z}' \in Z$ in which

$$\begin{aligned}
z'_i &= (x_i, y_i - \varepsilon) \text{ for some small enough } \varepsilon > 0, \\
z'_j &= \left(x_j, y_j + \frac{\varepsilon}{\#N\left(\mathcal{C}^J_{\min}(\mathbf{z})\right)}\right) \text{ for all } j \in N\left(\mathcal{C}^J_{\min}(\mathbf{z})\right), \text{ and} \\
z'_h &= (x_h, y_h) \text{ for any other } h \in N \backslash \left(N\left(\mathcal{C}^J_{\min}(\mathbf{z})\right) \cup \{i\}\right),
\end{aligned}$$

where $N\left(\mathcal{C}^J_{\min}(\mathbf{z})\right) \equiv \left\{j \in N \mid c_j(z_j) \in \mathcal{C}^J_{\min}(\mathbf{z})\right\}$. Then, by the conditions (a) and (c) of capability correspondences, we can see that $\left(c_j\left(z'_j\right), c_j(z_j)\right) \in$

$P(J)$ for all $j \in N\left(\mathcal{C}^J_{\min}(\mathbf{z})\right)$, and so that $\left(\mathcal{C}^J_{\min}(\mathbf{z}'), \mathcal{C}^J_{\min}(\mathbf{z})\right) \in P(J)$ holds, which is a contradiction. ■

Lemma 3: *For each* $\mathbf{x} \in [0, \overline{x}]^n$*,* $Z(\mathbf{x}; J)$ *is singleton.*

Proof. Suppose there exist $\mathbf{x} \in [0, \overline{x}]^n$ and $\mathbf{z}, \mathbf{z}' \in Z(\mathbf{x}; J)$ such that $\mathbf{z} \neq \mathbf{z}'$, which implies $\mathbf{y} \neq \mathbf{y}'$. Thus, there exists at least two individuals $i, j \in N$ such that $y_i > y_i'$ and $y_j < y_j'$. Then, by the condition (a) of capability correspondences, $c_i^o(z_i) \supsetneq c_i(z_i')$ and $c_j(z_j) \subsetneq c_j^o(z_j')$. By these set-inclusion relations, **(3.1.1)** of J, and Lemma 2, $(c_i(z_i), c_j(z_j)) \in I(J)$ and $\left(c_j(z_j'), c_i(z_i')\right) \in P(J)$ should hold. However, since $\mathbf{z}' \in Z(\mathbf{x}; J)$, $\left(c_i(z_i'), c_j(z_j')\right) \in I(J)$ should also hold by Lemma 2, which is a contradiction. ■

Let us define an ordering $R_J \subseteq Z \times Z$ as follows: for all $\mathbf{z}, \mathbf{z}' \in Z$, $(\mathbf{z}, \mathbf{z}') \in R_J$ (resp. $P(R_J)) \Leftrightarrow \left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}')\right) \in J$ (resp. $P(J)$). Since $\mathbf{c}$ is continuous, R_J is continuous on Z.

Lemma 4: $\underset{\mathbf{x}\in[0,\overline{x}]^n}{\cup} Z(\mathbf{x}; J)$ *has a closed graph in* Z.

Proof. Let a sequence $\{(\mathbf{x}^\lambda, \mathbf{y}^\lambda)\}_{\lambda=1}^{+\infty}$ be such that $(\mathbf{x}^\lambda, \mathbf{y}^\lambda) \to (\mathbf{x}, \mathbf{y})$ as $\lambda \to +\infty$, and $(\mathbf{x}^\lambda, \mathbf{y}^\lambda) \in Z(\mathbf{x}^\lambda; J)$ for every $\lambda = 1, \ldots,$ ad. inf. Suppose that $(\mathbf{x}, \mathbf{y}) \notin Z(\mathbf{x}; J)$. Then, there exists $(\mathbf{x}, \mathbf{y}') \in Z(\mathbf{x}; J)$ such that $((\mathbf{x}, \mathbf{y}'), (\mathbf{x}, \mathbf{y})) \in P(R_J)$, because $Z(\mathbf{x}; J)$ is the set of maximal element of R_J over $\{\mathbf{x}\} \times Y(\mathbf{x})$. Since Y is *l.h.c.*, there exists a sequence $\{(\mathbf{x}^\lambda, \mathbf{y}'^\lambda)\}_{\lambda=1}^{+\infty}$ such that $(\mathbf{x}^\lambda, \mathbf{y}'^\lambda) \in \{\mathbf{x}^\lambda\} \times Y(\mathbf{x}^\lambda)$ for every $\lambda = 1, \ldots,$ ad. inf., and $(\mathbf{x}^\lambda, \mathbf{y}'^\lambda) \to (\mathbf{x}, \mathbf{y}')$ $(\lambda \to +\infty)$. Then, for a large enough λ, $((\mathbf{x}^\lambda, \mathbf{y}'^\lambda), (\mathbf{x}^\lambda, \mathbf{y}^\lambda)) \in P(R_J)$ by continuity of R_J on $\{\mathbf{x}\} \times Y(\mathbf{x})$, which is a contradiction. It follows that $\underset{\mathbf{x}\in[0,\overline{x}]^n}{\cup} Z(\mathbf{x}; J)$ has a closed graph in Z. ■

Proposition 3 [Yoshihara (2000)][14]**:** *let* **Assumption 1** *hold. Let* $h : [0, \overline{x}]^n \to \mathbb{R}^n_+$ *be a continuous function such that, for each* $\mathbf{x} \in [0, \overline{x}]^n$*,* $h(\mathbf{x}) = \mathbf{y}$ *and* $f(\mathbf{x}) = \sum_N y_i$*. Then, for any* $\mathbf{u} \in \mathcal{U}^n$*, there exists* $\mathbf{x}^* \in [0, \overline{x}]^n$ *such that* $(\mathbf{x}^*, h(\mathbf{x}^*))$ *is a* Pareto efficient allocation for $\mathbf{u}$.

[14] Corchón and Puy (1998; Theorem 1) showed the same result under a stronger assumption than that in this proposition.

Proof. Given $\mathbf{u} \in \mathcal{U}^n$, let $S(\mathbf{u})$ be the utility possibility set of feasible allocations, and $\partial S(\mathbf{u})$ be its boundary. Since every utility function is strictly increasing, $\partial S(\mathbf{u})$ is the set of Pareto efficient utility allocations.

By **Assumption 1**, $\mathbf{0} \notin \partial S(\mathbf{u})$. Thus, $\Sigma \overline{u}_h > 0$ for every $\overline{\mathbf{u}} = (\overline{u}_i)_{i \in N} \in \partial S(\mathbf{u})$, and the mapping

$$\widehat{\mathbf{v}} : \partial S(\mathbf{u}) \to \triangle^{n-1} \text{ such that } \widehat{\mathbf{v}}(\mathbf{u}) = \frac{\overline{\mathbf{u}}}{\Sigma \overline{u}_h}$$

is well-defined and continuous on $\partial S(\mathbf{u})$, where $\triangle^{n-1}$ is an $n-1$-dimensional unit simplex. By Arrow and Hahn (1971; Lemma 5.3, p.114), $\widehat{\mathbf{v}}$ is a homeomorphism. Denote its inverse by $\widehat{\mathbf{u}}$. Define a correspondence

$$\widehat{W} : \triangle^{n-1} \twoheadrightarrow Z \text{ by } \widehat{W}(\widehat{\mathbf{u}}(\mathbf{v})) \equiv \{\mathbf{z} \in Z \mid u_i(z_i) \geq \widehat{u}_i(v_i)(\forall i \in N)\}.$$

By Arrow and Hahn (1971; Theorem 4.5, Corollary 5, p.99), $\widehat{W}$ is upper hemi-continuous with non-empty, compact and convex values.

Given a continuous function h and $\mathbf{z} = (x_i, y_i)_{i \in N} \in Z$, let $E_i(\mathbf{x}, y_i) \equiv h_i(\mathbf{x}) - y_i$. Then, we define the following optimization problem:

$$\max_{\mathbf{v} \in \triangle^{n-1}} \sum v_i \cdot E_i(\mathbf{x}, y_i).$$

By Berge's maximum theorem, we can define an upper hemi-continuous correspondence $\Theta : Z \twoheadrightarrow \triangle^{n-1}$ by

$$\Theta(\mathbf{z}) \equiv \{\mathbf{v}^* \in \triangle^{n-1} \mid \mathbf{v}^* \in \arg \max_{\mathbf{v} \in \triangle^{n-1}} \sum v_i \cdot E_i(\mathbf{x}, y_i)\}.$$

Note that Θ is non-empty, compact and convex-valued.

Now, we define a correspondence $\Phi : \triangle^{n-1} \times Z \twoheadrightarrow \triangle^{n-1} \times Z$ by

$$\Phi(\mathbf{v}, \mathbf{z}) \equiv \Theta(\mathbf{z}) \times \widehat{W}(\widehat{\mathbf{u}}(\mathbf{v})),$$

which is upper hemi-continuous with non-empty, compact and convex values. By Kakutani's fixed point theorem,

$$\exists (\mathbf{v}^*, \mathbf{z}^*) \in \triangle^{n-1} \times Z \text{ s.t. } (\mathbf{v}^*, \mathbf{z}^*) \in \Phi(\mathbf{v}^*, \mathbf{z}^*).$$

By definition, $\widehat{\mathbf{u}}(\mathbf{v}^*) = (u_i(z_i^*))_{i \in N}$, so that $\mathbf{z}^*$ is Pareto efficient for $\mathbf{u}$. Finally, we show that $\mathbf{z}^* = (\mathbf{x}^*, h(\mathbf{x}^*))$. To do this, it is sufficient to show $E_i(\mathbf{x}^*, y_i^*) = 0$ for all $i \in N$. Assume that there exists $j \in N$ such that $E_j(\mathbf{x}^*, y_j^*) > 0$.

Then, since $\mathbf{z}^*$ is Pareto efficient, there exists $l \in N$ such that $E_l(\mathbf{x}^*, y_l^*) < 0$. To maximize $\sum v_i \cdot E_i(\mathbf{x}, y_i)$, we obtain $v_l^* = 0$. Then, $u_l(z_l^*) = \widehat{u}_l(v_l^*) = 0$, so that, by the strict monotonicity of u_l and **Assumption 1**, we obtain either (1) $x_l^* = \overline{x}$ and $y_l^* > 0$, or (2) $x_l^* \leq \overline{x}$ and $y_l^* = 0$. First, (1) is impossible because $\mathbf{z}^*$ is Pareto efficient for $\mathbf{u}$. In fact, the vector $(1, 0)$ is a unique price which supports z_l^* of case (1) as an expenditure minimizing consumption. However, $(1, 0)$ cannot be consistent with any profit maximizing production except the origin. Second, (2) implies $E_l(\mathbf{x}^*, y_l^*) \geq 0$, which is a contradiction. Thus, $E_i(\mathbf{x}^*, y_i^*) = 0$ for all $i \in N$, as was to be verified. ■

Lemma 5: *Let* **Assumption 1** *hold. Then, for each* $\mathbf{u} \in \mathcal{U}^n$*, there exists a Pareto efficient allocation* $\mathbf{z}^* \in Z$ *such that* $\mathbf{z}^* \in \underset{\mathbf{x} \in [0,\overline{x}]^n}{\cup} Z(\mathbf{x}; J)$.

Proof. Let a correspondence $h^J : [0, \overline{x}]^n \twoheadrightarrow Y([0, \overline{x}]^n)$ be such that $\{\mathbf{x}\} \times h^J(\mathbf{x}) = Z(\mathbf{x}; J)$ for each $\mathbf{x} \in [0, \overline{x}]^n$. Since $Y([0, \overline{x}]^n)$ is compact, h^J is *u.h.c.* by **Lemma 4**. Moreover, for each $\mathbf{x} \in [0, \overline{x}]^n$, $h^J(\mathbf{x})$ is singleton, since $Z(\mathbf{x}; J)$ is singleton by **Lemma 3**. Thus, h^J is a continuous function. Then, under **Assumption 1**, we can obtain the desired result by the application of **Proposition 3**. ■

Proposition 4 [Yoshihara (2000)]**:** *Let* **Assumption 1** *and* **Assumption 2** *hold. Let* $h : [0, \overline{x}]^n \to \mathbb{R}_+^n$ *be a continuous function such that, for each* $\mathbf{x} \in [0, \overline{x}]^n$*,* $h(\mathbf{x}) = \mathbf{y}$ *and* $f(\mathbf{x}) = \sum_N y_i$*, and for any* $i, j \in N$ *with* $c_i = c_j$*,* $x_i = x_j$ *implies* $h_i(\mathbf{x}) = h_j(\mathbf{x})$*. Then, there exists a game form* $\gamma = (([0, \overline{x}] \times \mathbb{R}_+)^n, g) \in \Gamma_{LS}$ *such that, for any* $\mathbf{u} \in \mathcal{U}^n$*,* $\mathbf{z} \in A_{NE}(\gamma, \mathbf{u})$ *holds if and only if* $\mathbf{z} = (\mathbf{x}, h(\mathbf{x}))$*, and it is Pareto efficient.*

Proof. By **Proposition 3**, the continuous function h attains some Pareto efficient allocations for each $\mathbf{u} \in \mathcal{U}^n$. In other words, for any $\mathbf{u} \in \mathcal{U}^n$, there exists $\mathbf{x}^* \in [0, \overline{x}]^n$ such that $(\mathbf{x}^*, h(\mathbf{x}^*))$ is a Pareto efficient allocation for $\mathbf{u}$. Let us denote by $P(h : \mathbf{u})$ the set of all such Pareto efficient allocations which are attained by h under $\mathbf{u} \in \mathcal{U}^n$. Note that $([0, \overline{x}]^n, h) \in \Gamma_{LS}$. For h, we sometimes use notation like $h_i(\mathbf{x})$, which refers to the i-th component of the vector $h(\mathbf{x})$.

Step 1: *For each* $\widehat{\mathbf{z}} \in P(h : \mathbf{u})$*, we construct an outcome function* $h^{\widehat{\mathbf{z}}} : [0, \overline{x}]^n \to \mathbb{R}_+^n$ *such that* $\left([0, \overline{x}]^n, h^{\widehat{\mathbf{z}}}\right) \in \Gamma_{LE}$ *and* $\widehat{\mathbf{z}} \in A_{NE}\left(\left([0, \overline{x}]^n, h^{\widehat{\mathbf{z}}}\right), \mathbf{u}\right)$.

By **Assumption 2**, we can define a continuous function $f_i' : [0, \overline{x}]^n \to \mathbb{R}_+$ by $f_i'(\mathbf{x}) \equiv \frac{\partial f(\mathbf{x})}{\partial x_i}$ for all $\mathbf{x} \in [0, \overline{x}]^n$. Given $\mathbf{u} \in \mathcal{U}^n$, $\widehat{\mathbf{z}} = (\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in P(h : \mathbf{u})$, and $i, j \in N$, let

$$\lambda_j^i(\widehat{\mathbf{x}}) \equiv \begin{cases} \frac{\widehat{y}_i + \cdot f_i'(\widehat{\mathbf{x}}) \cdot (\widehat{x}_j - \widehat{x}_i) - h_i(\widehat{\mathbf{x}}_{-i}, \widehat{x}_j)}{(\widehat{x}_j - \widehat{x}_i)^2} & \text{if } \widehat{x}_j \neq \widehat{x}_i \\ 0 & \text{if } \widehat{x}_j = \widehat{x}_i \end{cases}.$$

Given $\mathbf{u} \in \mathcal{U}^n$, $\widehat{\mathbf{z}} \in P(h : \mathbf{u})$, and $\mathbf{x} \in [0, \overline{x}]^n$, define for each $i \in N$,

$$\Psi_i(\mathbf{x}) = \begin{cases} \widehat{y}_i + f_i'(\widehat{\mathbf{x}}) \cdot (x_i - \widehat{x}_i) & \text{if } x_i \in (\widehat{x}_i - \varepsilon_i(\widehat{\mathbf{x}}), \widehat{x}_i + \varepsilon_i(\widehat{\mathbf{x}})) \\ \widehat{y}_i + \left[f_i'(\widehat{\mathbf{x}}) \cdot (x_i - \widehat{x}_i) - \mu^i{}_{j^*(x_i)}(\widehat{\mathbf{x}}) \cdot (x_i - \widehat{x}_i)^2 \right] & \text{otherwise} \end{cases},$$

where

$$\varepsilon_i(\widehat{\mathbf{x}}) \equiv \min_{j \neq i,\ \widehat{x}_j \neq \widehat{x}_i} \| \widehat{x}_j, \widehat{x}_i \|, \quad j^*(x_i) = \max_{j \neq i} \left\{ \arg\min_{j \neq i} \| \widehat{x}_j, x_i \| \right\},$$

and

$$\mu^i{}_{j^*(x_i)}(\widehat{\mathbf{x}}) = \begin{cases} \lambda^i_{j^*(x_i)}(\widehat{\mathbf{x}}) & \text{if } 0 \leq \lambda^i_{j^*(x_i)}(\widehat{\mathbf{x}}) \\ 0 & \text{otherwise} \end{cases}.$$

By construction of $\Psi_i(\mathbf{x})$, we have (i) $\Psi_i(\mathbf{x}) = \widehat{y}_i$ if $x_i = \widehat{x}_i$; (ii) $\Psi_i(\mathbf{x}) \leq \widehat{y}_i + f_i'(\widehat{\mathbf{x}}) \cdot (x_i - \widehat{x}_i)$ if $x_i \neq \widehat{x}_i$; and (iii) $\Psi_i(\mathbf{x}) = \min\{h_i(\widehat{\mathbf{x}}_{-i}, x_i), \widehat{y}_i + f_i'(\widehat{\mathbf{x}}) \cdot (x_i - \widehat{x}_i)\}$ if $x_i = \widehat{x}_j$ for some $j \neq i$.

For each $i \in N$, define

$$\zeta_i(\mathbf{x}) = \min\left\{\max\left(0, \Psi_i(\mathbf{x})\right), f(\mathbf{x})\right\}.$$

Moreover, for each $i \in N$ and each $\mathbf{x} \in [0, \overline{x}]^n$, define

$$n(\mathbf{x}, x_i) \equiv \#\{j \in N \mid x_j = x_i\}.$$

Then, given $\mathbf{u} \in \mathcal{U}^n$ and $\widehat{\mathbf{z}} \in P(h : \mathbf{u})$, define a function $h^{\widehat{\mathbf{z}}} : [0, \overline{x}]^n \to \mathbb{R}_+^n$ as follows: for each $\mathbf{x} \in [0, \overline{x}]^n$, and for each $i \in N$,

$$h_i^{\widehat{\mathbf{z}}}(\mathbf{x}) = \begin{cases} \zeta_i(\mathbf{x}) & \text{if } \forall j \neq i,\ x_j = \widehat{x}_j, \\ [f(\mathbf{x}) - n(\mathbf{x}, x_j) \cdot \zeta_j(\mathbf{x})] \cdot \frac{1}{n - n(\mathbf{x}, x_j)} & \text{if } \exists j \neq i,\ \forall l \neq j,\ x_l = \widehat{x}_k,\ \&\ x_j \neq \widehat{x}_i \\ \zeta_j(\mathbf{x}) & \text{if } \exists j \neq i,\ \forall l \neq j,\ x_l = \widehat{x}_l,\ \&\ x_j = \widehat{x}_i \\ h_i(\mathbf{x}) & \text{otherwise.} \end{cases}$$

This $h^{\widehat{\mathbf{z}}}$ has the following properties: (I) $\left([0,\overline{x}]^n, h^{\widehat{\mathbf{z}}}\right) \in \Gamma_{LE}$; and (II) $\widehat{\mathbf{z}} \in A_{NE}\left(\left([0,\overline{x}]^n, h^{\widehat{\mathbf{z}}}\right), \mathbf{u}\right)$ whenever $\widehat{\mathbf{z}} \in P(h:\mathbf{u})$. The property (II) follows from the property (ii) of $(\Psi_i)_{i\in N}$.

Step 2: *We construct two outcome functions h^0 and h^m such that $([0,\overline{x}]^n, h^0)$, $\left([0,\overline{x}]^n \times \mathbb{R}^n_+, h^m\right) \in \Gamma_{LS}$.*

Let us introduce two functions h^0 and h^m as follows:

(1) $h^0 : [0,\overline{x}]^n \to \mathbb{R}^n_+$ by $h^0(\mathbf{x}) = \mathbf{0}$ for each $(\mathbf{x},\mathbf{y}) \in [0,\overline{x}]^n \times \mathbb{R}^n_+$,
and
(2) $h^m : [0,\overline{x}]^n \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by,

$$h^m_i(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{f(\mathbf{x})}{\#(\max N(\mathbf{y}))} \text{ if } i \in N^m(\mathbf{y}) \\ 0 \text{ if } k \notin N^m(\mathbf{y}) \end{cases},$$

for each $(\mathbf{x},\mathbf{y}) \in [0,\overline{x}]^n \times \mathbb{R}^n_+$, and for all $i \in N$, where $N^m(\mathbf{y}) \equiv \{i \in N \mid \forall j \in N : y_i \geq y_j\}$.

It is clear that $([0,\overline{x}]^n, h^0), \left([0,\overline{x}]^n \times \mathbb{R}^n_+, h^m\right) \in \Gamma_{LS}$. Note that for any $\mathbf{u} \in \mathcal{U}^n$, there is no Nash equilibrium for the game defined by $\left([0,\overline{x}]^n \times \mathbb{R}^n_+, h^m\right)$.

Step 3: *We construct a game form $\gamma^* = (([0,\overline{x}] \times \mathbb{R}_+)^n, g^*)$, in which g^* is defined by using $\{h^{\widehat{\mathbf{z}}}\}_{\widehat{\mathbf{z}} \in \underset{\mathbf{u}\in\mathcal{U}^n}{\cup} P(h:\mathbf{u})}$, h^0, and h^m.*

Given $\mathbf{x} \in [0,\overline{x}]^n$ and $\mathbf{y} \in \mathbb{R}^n_+$, let $\rho(\mathbf{x},\mathbf{y}:h) \equiv \{\mathbf{u} \in \mathcal{U}^n \mid (\mathbf{x}, h(\mathbf{x})) \in P(h:\mathbf{u})\ \&\ h(\mathbf{x}) = \mathbf{y}\}$. Let us call $(\mathbf{x},\mathbf{y}) \in Z$ a *potential P^h-allocation if* $\rho(\mathbf{x},\mathbf{y}:h) \neq \varnothing$. Given $\mathbf{x} \in [0,\overline{x}]^n$ and $\mathbf{y} \in \mathbb{R}^n_+$, let

$$N(\mathbf{x},\mathbf{y}) \equiv \{l \in N \mid \exists (x'_l, y'_l)(\neq (x_l, y_l)) \in [0,\overline{x}]\times\mathbb{R}_+ : \rho((x'_l, \mathbf{x}_{-l}), (y'_l, \mathbf{y}_{-l}) : h) \neq \varnothing\}.$$

The set $N(\mathbf{x},\mathbf{y})$ will be used in defining γ^* below as the set of *potential deviators.* That is, if $\rho(\mathbf{x},\mathbf{y}:h) = \varnothing$, and there is some $j \in N(\mathbf{x},\mathbf{y})$, then this j may be interpreted as deviating from $P(h:\mathbf{u})$ for some $\mathbf{u} \in \mathcal{U}^n$.

Given $j \in N$, $\mathbf{x} \in [0,\overline{x}]^n$, and $\mathbf{y} \in \mathbb{R}^n_+$, let

$$\begin{aligned} X_j(\mathbf{x}_{-j},\mathbf{y}_{-j}) &\equiv \left\{x'_j \in [0,\overline{x}] \mid \rho\left(\left(x'_j, h_j(x'_j,\mathbf{x}_{-j})\right), (\mathbf{x}_{-j},\mathbf{y}_{-j}) : h\right) \neq \varnothing\right\}, \text{ and} \\ Z(x_j,\mathbf{x}_{-j},\mathbf{y}_{-j}) &\equiv \left\{\left(\left(x'_j, h_j(x'_j,\mathbf{x}_{-j})\right), (\mathbf{x}_{-j},\mathbf{y}_{-j})\right) \in Z \mid x'_j \in X_j(\mathbf{x}_{-j},\mathbf{y}_{-j})\right\}. \end{aligned}$$

Moreover, given $j \in N$, $\mathbf{x} \in [0,\overline{x}]^n$, and $\mathbf{y} \in \mathbb{R}^n_+$, let

$$\widehat{\mathbf{z}}(x_j,\mathbf{x}_{-j},\mathbf{y}_{-j}) \equiv \arg \min_{((x'_j,y'_j),(\mathbf{x}_{-j},\mathbf{y}_{-j}))\in Z(x_j,\mathbf{x}_{-j},\mathbf{y}_{-j})} y'_j + f'_j(x'_j,\mathbf{x}_{-j}) \cdot (x_j - x'_j).$$

Note that the above three notations will be used in defining γ^* below to punish a unique potential deviator. If $\rho(\mathbf{x}, \mathbf{y} : h) = \varnothing$ and $\{j\} = N(\mathbf{x}, \mathbf{y})$, then we can identify j as the unique potential deviator. Then, by definition of $N(\mathbf{x}, \mathbf{y})$, $X_j(\mathbf{x}_{-j}, \mathbf{y}_{-j})$ is non-empty, so that $Z(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})$ is non-empty. Note that $Z(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})$ is the set of potential P^h-allocations which would be implemented if j were not to deviate. Then, by selecting $\widehat{\mathbf{z}}(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})$ from this set, we will consider the outcome function g^* in order to punish j in such a situation.

Given $\mathbf{x} \in [0, \overline{x}]^n$, let $N^o(\mathbf{x}) \equiv \{i \in N \mid x_i \in [0, \overline{x})\}$, $\max N^o(\mathbf{x}) \equiv \{i \in N^o(\mathbf{x}) \mid \nexists j \in N^o(\mathbf{x}) \text{ s.t. } x_j > x_i\}$, and $\max N(\mathbf{x}) \equiv \{i \in N \mid \nexists j \in N \text{ s.t. } x_j > x_i\}$. Now, let us define a labor sovereign and equal treatment of equals rule $\gamma^* = (([0, \overline{x}] \times \mathbb{R}_+)^n, g^*)$ in the following way: given a strategy profile $(\mathbf{x}, \mathbf{y}) \in ([0, \overline{x}] \times \mathbb{R}_+)^n$,

Rule 1: if $\rho(\mathbf{x}, \mathbf{y} : h) \neq \varnothing$, then $g^*(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h^{\widehat{\mathbf{z}}}(\mathbf{x}))$, where $\widehat{\mathbf{z}} = (\mathbf{x}, h(\mathbf{x}))$.

Rule 2: if $\rho(\mathbf{x}, \mathbf{y} : h) = \varnothing$ and there exists a non-empty $N(\mathbf{x}, \mathbf{y})$, then
2-1: if $\# N(\mathbf{x}, \mathbf{y}) > 1$, then $g^*(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h^0(\mathbf{x}))$,
2-2: if $N(\mathbf{x}, \mathbf{y}) = \{j\}$, then $g_j^*(\mathbf{x}, \mathbf{y}) = (x_j, h_j^{\widehat{\mathbf{z}}(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})}(\mathbf{x}))$ and for all $i \neq j$,

$$g_i^*(\mathbf{x}, \mathbf{y}) = \begin{cases} (x_i, h_i^0(\mathbf{x})) & \text{if } \{j\} = \max N(\mathbf{x}) \cap \max N^o(\mathbf{x}), \\ (x_i, h_i^{\widehat{\mathbf{z}}(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})}(\mathbf{x})) & \text{otherwise.} \end{cases}$$

Rule 3: in all other cases, $g^*(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h^m(\mathbf{x}, \mathbf{y}))$.

In this γ^*, if a strategy profile $(\mathbf{x}, \mathbf{y})$ is consistent with a potential P^h-allocation, then **Rule 1** applies, and $(\mathbf{x}, \mathbf{y})$ becomes the outcome; if $(\mathbf{x}, \mathbf{y})$ is inconsistent with any potential P^h-allocation, and a unique potential deviator j is identified, then **Rule 2-2** applies and identifies some potential P^h-allocation $\widehat{\mathbf{z}}(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j}) = (\widehat{z}_j, (\mathbf{x}_{-j}, \mathbf{y}_{-j}))$, which would be the outcome if j were not to deviate. Thus, j is punished by $h_j^{\widehat{\mathbf{z}}(x_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})}(\mathbf{x})$ under **Rule 2-2**. If $(\mathbf{x}, \mathbf{y})$ corresponds to neither of the above two cases, then h^m or h^0 is applied in order to punish all potential deviators. It is clear that $\gamma^* \in \Gamma_{LS}$, since in every case of strategy profile, the value of g^* is that of either h^m, h^m, or $h^{\widehat{\mathbf{z}}}$-types.

Step 4: *We show that $A_{NE}(\gamma^*, \mathbf{u}) = P(h : \mathbf{u})$ for all $\mathbf{u} \in \mathcal{U}^n$.*

(1) First, we show that $A_{NE}(\gamma^*, \mathbf{u}) \supseteq P(h : \mathbf{u})$ for all $\mathbf{u} \in \mathcal{U}^n$. Let $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in P(h : \mathbf{u})$. Then, if a strategy profile of every agent is $(\mathbf{x}, \mathbf{y}) = (x_i, y_i)_{i \in N} \in [0, \overline{x}]^n \times \mathbb{R}^n_+$, then $g^*(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h^{\mathbf{z}}(\mathbf{x})) = (\mathbf{x}, h(\mathbf{x})) = (\mathbf{x}, \mathbf{y})$ by **Rule 1**. Suppose that individual $j \in N$ deviates from (x_i, y_i) to (x'_j, y'_j). Then, if j induces **Rule 2-1**, then $g^*_j((x'_j, \mathbf{x}_{-j}), (y'_j, \mathbf{y}_{-j})) = 0$. If j induces **Rule 1**, he cannot be better off, since $y'_j = f(x'_j, \mathbf{x}_{-j}) - \sum_{l \neq j} y_l$ holds true, and $\mathbf{z}$ is Pareto efficient for $\mathbf{u}$. If j induces **Rule 2-2**, then $g^*_j((x'_j, \mathbf{x}_{-j}), (y'_j, \mathbf{y}_{-j})) = (x'_j, h_j^{\widehat{\mathbf{z}}(x'_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})}(x'_j, \mathbf{x}_{-j}))$. Since $\mathbf{z} \in Z(x'_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})$ holds true, it follows that if $\widehat{\mathbf{z}}(x'_j, \mathbf{x}_{-j}, \mathbf{y}_{-j}) = ((\widehat{x}_j, \mathbf{x}_{-j}), (\widehat{y}_j, \mathbf{y}_{-j}))$, then

$$h_j^{\widehat{\mathbf{z}}(x'_j, \mathbf{x}_{-j}, \mathbf{y}_{-j})}(x'_j, \mathbf{x}_{-j}) \leq \widehat{y}_j + f'_j((\widehat{x}_j, \mathbf{x}_{-j})) \cdot (x'_j - \widehat{x}_j) \leq y_j + f'_j(\mathbf{x}) \cdot (x'_j - x_j).$$

This implies that j cannot be better off by this deviation. Note that j cannot induce **Rule 3**. Thus, $A_{NE}(\gamma^*, \mathbf{u}) \supseteq P(h : \mathbf{u})$ holds.

(2) Second, we show that $A_{NE}(\gamma^*, \mathbf{u}) \subseteq P(h : \mathbf{u})$. Let $(\mathbf{x}, \mathbf{y}) = (x_i, y_i)_{i \in N}$ be a Nash equilibrium of the game $(\gamma^*, \mathbf{u})$. Note that $(\mathbf{x}, \mathbf{y})$ cannot correspond to **Rule 3**. This is because every agent j can get everything in **Rule 3** by changing from y_j to large enough $y'_j > \max\left\{\max\{y_i\}_{i \neq j}, f(\mathbf{x})\right\}$. Also, $(\mathbf{x}, \mathbf{y})$ cannot correspond to **Rule 2-2**, since, in **Rule 2-2**, there is an agent $j \in N \backslash N(\mathbf{x}, \mathbf{y})$ who can induce **Rule 3** by changing from y_j to large enough $y'_j > \max\left\{\max\{y_i\}_{i \neq j}, f(\mathbf{x})\right\}$, thereby getting everything. Finally, $(\mathbf{x}, \mathbf{y})$ cannot correspond to **Rule 2-1** either, since every agent $l \in N(\mathbf{x}, \mathbf{y})$ can induce **Rule 2-2** by changing from y_l to $y'_l = f(\mathbf{x}) + \varepsilon$, so that l can obtain positive output.

Suppose that $(\mathbf{x}, \mathbf{y})$ corresponds to **Rule 1**. Then, for $\mathbf{z} = (\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h(\mathbf{x}))$, $g^*(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h^{\mathbf{z}}(\mathbf{x})) = \mathbf{z} \in A_{NE}(\gamma^*, \mathbf{u})$. Suppose that $\mathbf{z}$ is not Pareto efficient. Then, there is at least one individual $i \in N$ who changes slightly from x_i to x'_i, so that $u_i(x_i, h_i^{\mathbf{z}}(\mathbf{x})) < u_i(x'_i, h_i^{\mathbf{z}}(x'_i, \mathbf{x}_{-i}))$. Note that if $x'_i = x_i + \epsilon$ where the value $\mid \epsilon \mid$ is small enough, then $h_i^{\mathbf{z}}(x'_i, \mathbf{x}_{-i}) = y_i + f'_i(\mathbf{x}) \cdot (x'_i - x_i)$, and $\widehat{\mathbf{z}}(x'_i, \mathbf{x}_{-i}, \mathbf{y}_{-i}) = \mathbf{z}$ also holds by the concavity of f. Thus, by changing from (x_i, y_i) to (x'_i, y'_i), where $y'_i = f(x'_i, \mathbf{x}_{-i}) + \varepsilon$, i can induce **Rule 2-2** and obtain $g^*_i((x'_i, \mathbf{x}_{-j}), (y'_i, \mathbf{y}_{-i})) = (x'_i, h_i^{\mathbf{z}}(x'_i, \mathbf{x}_{-i}))$. This is a contradiction, since $\mathbf{z} \in A_{NE}(\gamma^*, \mathbf{u})$. Thus, $\mathbf{z}$ is Pareto efficient for $\mathbf{u}$. ■

Proof of Theorem 1: Given $J \in \mathcal{J}$, for each $\mathbf{u} \in \mathcal{U}^n$, let $PM^J(\mathbf{u}) \equiv PO(\mathbf{u}) \cap \left[\bigcup_{\mathbf{x} \in [0, \overline{x}]^n} Z(\mathbf{x}; J)\right]$. Note that if $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in PM^J(\mathbf{u})$, then $h^J(\mathbf{x}) =$

$\mathbf{y}$. Moreover, $\left([0,\overline{x}]^n, h^J\right) \in \Gamma_{LS}$ holds true. It follows from the property of $Z(\cdot;J)$ that $\{(\mathbf{x},\mathbf{y})\} = Z(\mathbf{x};J)$, $c_i = c_j$, and $x_i = x_j$ for some $i,j \in N$ imply $y_i = y_j$. Thus, by **Proposition 4**, there exists an allocation rule $\gamma^* = (([0,\overline{x}] \times \mathbb{R}_+)^n, g^*) \in \Gamma_{LS}$ such that $A_{NE}(\gamma^*, \mathbf{u}) = PM^J(\mathbf{u})$ holds for all $\mathbf{u} \in \mathcal{U}^n$. ■

6.2 Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2: Given $\mathbf{u} \in \mathcal{U}^n$, let $S(\mathbf{u})$ be the utility possibility set of feasible allocations, and $\partial S(\mathbf{u})$ be its boundary. Since every utility function is strictly increasing, $\partial S(\mathbf{u})$ is the set of Pareto efficient utility allocations.

Now we define an ordering $V(\mathbf{u})$ over $S(\mathbf{u})$ as follows:
1) if $\overline{\mathbf{u}}, \overline{\mathbf{u}}' \in \partial S(\mathbf{u})$, then $(\overline{\mathbf{u}}, \overline{\mathbf{u}}') \in I\left(V(\mathbf{u})\right)$,
2) for any $\overline{\mathbf{u}}, \overline{\mathbf{u}}' \in S(\mathbf{u})$, there exist $\mu, \mu' \in [1, +\infty)$ such that $\mu \cdot \overline{\mathbf{u}}, \mu' \cdot \overline{\mathbf{u}}' \in \partial S(\mathbf{u})$ and $(\overline{\mathbf{u}}, \overline{\mathbf{u}}') \in V(\mathbf{u})$ if and only if $\mu \leq \mu'$. This ordering $V(\mathbf{u})$ is continuous over $S(\mathbf{u})$.

Given $J \in \mathcal{J}$ and $\mathbf{u} \in \mathcal{U}^n$, let us define a complete ordering $R_{\mathbf{u},J}$ over $\bigcup_{\mathbf{x} \in [0,\overline{x}]^n} Z\left(\mathbf{x};J\right)$ as follows: for any $\mathbf{z}, \mathbf{z}' \in \bigcup_{\mathbf{x} \in [0,\overline{x}]^n} Z\left(\mathbf{x};J\right)$, $(\mathbf{z}, \mathbf{z}') \in R_{\mathbf{u},J} \Leftrightarrow (\mathbf{u}(\mathbf{z}), \mathbf{u}(\mathbf{z}')) \in V(\mathbf{u})$. This ordering $R_{\mathbf{u},J}$ is continuous on $\bigcup_{\mathbf{x} \in [0,\overline{x}]^n} Z\left(\mathbf{x};J\right)$, and its maximal element consists of $\mathbf{z} \in PM^J(\mathbf{u})$, where $PM^J(\mathbf{u}) = PO(\mathbf{u}) \cap \left[\bigcup_{\mathbf{x} \in [0,\overline{x}]^n} Z(\mathbf{x};J)\right]$, which is non-empty by **Lemma 5**. Given $J \in \mathcal{J}$, let $R_J\left(\mathbf{x}\right)$ be the restriction of R_J into $\left(\{\mathbf{x}\} \times Y(\mathbf{x})\right)^2$.

Consider a binary relation $R_{\mathbf{u},J} \cup \left[\bigcup_{\mathbf{x} \in [0,\overline{x}]^n} R_J(\mathbf{x})\right]$ over Z. It is easy to see that this binary relation is consistent, so that there exists an ordering extension $R^*_{\mathbf{u},J}$ of $R_{\mathbf{u},J} \cup \left[\bigcup_{\mathbf{x} \in [0,\overline{x}]^n} R_J(\mathbf{x})\right]$ by Suzumura's (1983) extension theorem. Based upon this $R^*_{\mathbf{u},J}$, let us consider an ordering function Q^*_J as follows: for each $\mathbf{u} \in \mathcal{U}^n$ and all $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$,
1) if $\gamma \in \mathbf{\Gamma}_{LS}$ and $\gamma' \in \mathbf{\Gamma} \setminus \mathbf{\Gamma}_{LS}$, then $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q^*_J(\mathbf{u}))$;
2) if either $\gamma, \gamma' \in \mathbf{\Gamma}_{LS}$ or $\gamma, \gamma' \in \mathbf{\Gamma} \setminus \mathbf{\Gamma}_{LS}$, then

$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q^*_J(\mathbf{u}) \Leftrightarrow (\mathbf{z}, \mathbf{z}') \in R^*_{\mathbf{u},J},$$
$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q^*_J(\mathbf{u})) \Leftrightarrow (\mathbf{z}, \mathbf{z}') \in P(R^*_{\mathbf{u},J}).$$

Note that $Q^*_J(\mathbf{u})$ is complete and transitive, and $Q^*_J \in \mathcal{Q}^{\mathbf{LS}\vdash J\text{-}\mathbf{LA}\vdash(\mathbf{PA} \cap Z(J))}$ by the definition. Finally, we can see that Q^*_J uniformly rationalizes $\gamma^* \in \mathbf{\Gamma}_{LS}$ as well as any $\gamma^{**} \in \mathbf{\Gamma}_{LS} \cap \mathbf{\Gamma}_{NS}$ whose every Nash equilibrium allocation always belongs to $PM^J(\mathbf{u})$ for any $\mathbf{u} \in \mathcal{U}^n$. ■

Proof of Theorem 3: Given $\mathbf{u} \in \mathcal{U}^n$ and $\underset{\mathbf{x}\in[0,\overline{x}]^n}{\cup} Z(\mathbf{x}; J)$, let us define an ordering $R^0_{\mathbf{u},J}$ over $PO(\mathbf{u})$ as follows: for all $\mathbf{z}, \mathbf{z}' \in PO(\mathbf{u})$,
1) if $\mathbf{x} = \mathbf{x}'$, then $(\mathbf{z}, \mathbf{z}') \in R^0_{\mathbf{u},J}$ (resp. $P\left(R^0_{\mathbf{u},J}\right)$) $\Leftrightarrow \left(\mathcal{C}^J_{\min}(\mathbf{z}), \mathcal{C}^J_{\min}(\mathbf{z}')\right) \in J$ (resp. $P(J)$); and
2) if $\mathbf{x} \neq \mathbf{x}'$, then $(\mathbf{z}, \mathbf{z}') \in R^0_{\mathbf{u},J}$ (resp. $P\left(R^0_{\mathbf{u},J}\right)$) $\Leftrightarrow \frac{\max_{\mathbf{b}\in\mathcal{C}^J_{\min}(\mathbf{z})} \eta(\mathbf{b})}{\max_{\mathbf{b}\in\mathcal{C}^J_{\min}(Z(\mathbf{x};J))} \eta(\mathbf{b})} \geq$ (resp. $>$) $\frac{\max_{\mathbf{b}\in\mathcal{C}^J_{\min}(\mathbf{z}')} \eta(\mathbf{b})}{\max_{\mathbf{b}\in\mathcal{C}^J_{\min}(Z(\mathbf{x}';J))} \eta(\mathbf{b})}$.
Note that the set of maximal elements of the ordering $R^0_{\mathbf{u},J}$ over $PO(\mathbf{u})$ coincides with $PM^J(\mathbf{u})$.

Next, given $\mathbf{u} \in \mathcal{U}^n$, let us define the strict Pareto preference relation (resp. the Pareto indifference relation) $SP_{\mathbf{u}} \subseteq Z \times Z$ (resp. $IP_{\mathbf{u}} \subseteq Z \times Z$) by $(\mathbf{z}, \mathbf{z}') \in SP_{\mathbf{u}} \Leftrightarrow u_i(z_i) > u_i(z'_i)$ for all $i \in N$ (resp. $(\mathbf{z}, \mathbf{z}') \in IP_{\mathbf{u}} \Leftrightarrow u_i(z_i) = u_i(z'_i)$ for all $i \in N$). Then, define a quasi-ordering $P_{\mathbf{u}} \subseteq Z \times Z$ as $P_{\mathbf{u}} \equiv SP_{\mathbf{u}} \cup IP_{\mathbf{u}}$.

Consider a binary relation $P_{\mathbf{u}} \cup R^0_{\mathbf{u},J}$ on Z. It is easy to see that this binary relation is consistent, so that there exists an ordering extension $R^{**}_{\mathbf{u},J}$ of $P_{\mathbf{u}} \cup R^0_{\mathbf{u},J}$ by Suzumura's (1983) extension theorem. Based upon this $R^{**}_{\mathbf{u},J}$, let us consider an ordering function Q^{**}_J as follows: for each $\mathbf{u} \in \mathcal{U}^n$ and all $(\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})$,
1) if $\gamma \in \mathbf{\Gamma}_{LS}$ and $\gamma' \in \mathbf{\Gamma} \setminus \mathbf{\Gamma}_{LS}$, then $((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q^{**}_J(\mathbf{u}))$;
2) if either $\gamma, \gamma' \in \mathbf{\Gamma}_{LS}$ or $\gamma, \gamma' \in \mathbf{\Gamma} \setminus \mathbf{\Gamma}_{LS}$, then

$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q^{**}_J(\mathbf{u}) \Leftrightarrow (\mathbf{z}, \mathbf{z}') \in R^{**}_{\mathbf{u},J},$$
$$((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q^{**}_J(\mathbf{u})) \Leftrightarrow (\mathbf{z}, \mathbf{z}') \in P(R^{**}_{\mathbf{u},J}).$$

Note that $Q^{**}_J(\mathbf{u})$ is complete and transitive, and $Q^{**}_J \in \mathcal{Q}^{\mathbf{LS}\vdash\mathbf{PA}\vdash(J\text{-}\mathbf{LA}\cap\mathbf{PO})}$ by the definition. Finally, we can see that Q^{**}_J uniformly rationalizes $\gamma^* \in \mathbf{\Gamma}_{LS}$ as well as any $\gamma^{**} \in \mathbf{\Gamma}_{LS} \cap \mathbf{\Gamma}_{NS}$ whose every Nash equilibrium allocation always belongs to $PM^J(\mathbf{u})$ for any $\mathbf{u} \in \mathcal{U}^n$. ■

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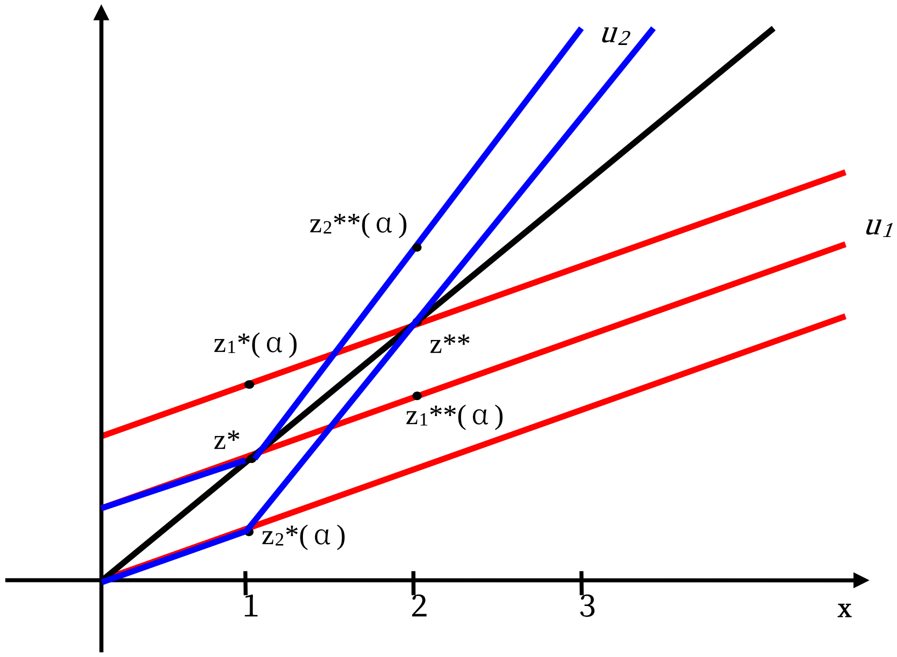
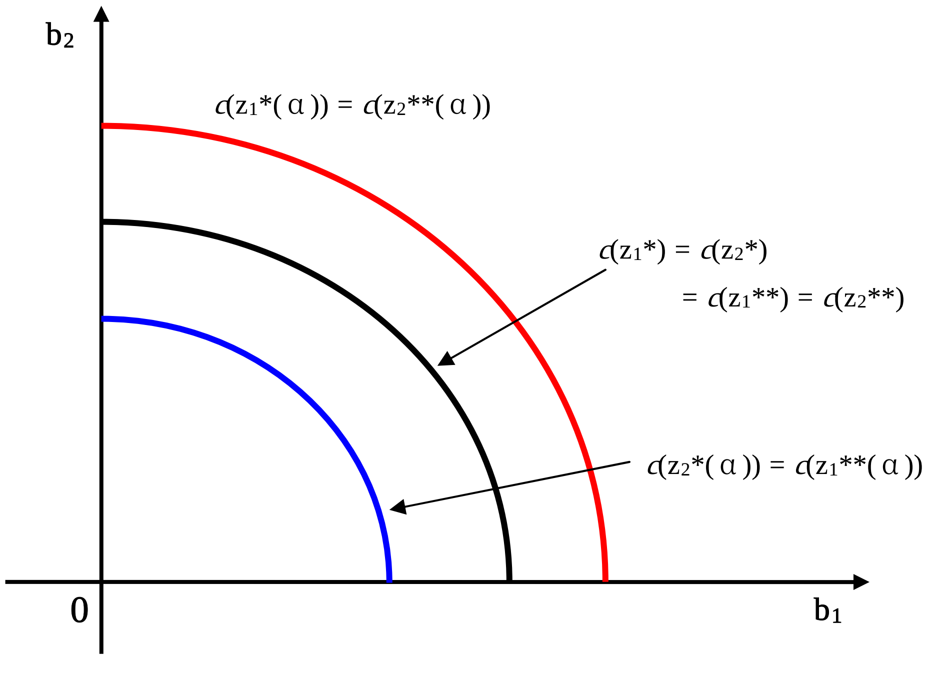


Example 1 in the consumption space



Example 1 in the functioning space

Figure 1