Existence of Social Ordering Functions Which Embody Procedural Values and Consequential Values*

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Abstract

We examine the possibility of constructing social ordering functions, each of which associates a social ordering over the feasible pairs of allocations and allocation rules with each and every simple production economy. Three axioms on the admissible class of social ordering functions are introduced, which embody the values of procedural fairness, non-welfaristic egalitarianism, and welfaristic consequentialism, respectively. The logical compatibility of these axioms as well as of their lexicographic combinations are examined. Two social ordering functions which give priority to procedural values rather than to consequential values are identified, which can uniformly rationalize a nice allocation rule in terms of the values of procedural fairness, non-welfaristic egalitarianism, and Pareto efficiency.

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1 Introduction

There are two contrasting approaches to making social evaluations of resource allocations and resource allocation rules. The first approach starts from an explicit outcome morality defined in the space of resource allocations; an allocation rule is then defined to be fair, or socially satisfactory, if it brings about resource allocations which satisfy the specified outcome morality. The crucial feature of this first approach is that it confers priority to consequential values over procedural values. Thus, this approach may be called the priority-to-consequence approach. In contrast, the second approach starts from an explicit requirement of procedural fairness of resource allocation rules; a resource allocation is then defined to be fair, or socially satisfactory, if it is brought about via the due application of a procedurally fair allocation rule. The crucial feature of this second approach is that it gives priority to procedural values over consequential values. Thus, this approach may be called the priority-to-procedure approach. It should be clear that standard welfare economics is, more often than not, based on the priority-to-consequence approach, where the outcome moralities typically invoked include Pareto efficiency, equity-as-no-envy à la Foley (1967) and Kolm (1972), and egalitarian equivalence introduced by Pazner and Schmeidler (1978).1 On the other hand, the priority-to-procedure approach is seldom adopted in the standard welfare economics with a few salient exceptions, such as Kranich (1994) and Moulin and Schenker (1994). Note, however, that there are strong proponents of this approach in political philosophy. It is sufficient to quote Rawls (1971) and Nozick (1974) as the two most influential proponents.2

In this paper, we propose a comprehensive framework of analysis which enables us to accommodate both consequential values and procedural values in a simple model of production economies. The crucial concept in our analysis is that of a social ordering function, which associates a social order-
ing over the set of pairs of feasible allocations and allocation rules with each admissible production economy. By using this conceptual framework, we can analyze the social desirability of resource allocations and resource allocation rules without going to either of the two extreme polar cases of the priority-to-consequence approach and the priority-to-procedure approach.\(^3\) To illustrate the use and usefulness of this general framework, we impose three axioms on the admissible class of social ordering functions which embody the values of procedural fairness, non-welfaristic egalitarianism, and welfaristic consequentialism. Although these three axioms per se are logically incompatible, we can devise appropriate lexicographic combinations of some variants thereof which can be demonstrated to be logically compatible. The focus of our analysis, then, is to examine the existence of a social ordering function which can rationalize a fair allocation rule.

The first axiom of procedural fairness capitalizes on the principles of labor sovereignty and equal treatment of equals introduced by Kranich (1994). The second axiom of non-welfaristic egalitarianism is formulated in terms of a reference functioning vector in the Sen (1980; 1985) space of functionings. Apart from the difference in the space in which the axiom is formulated, this axiom is reminiscent of the notion of egalitarian equivalence introduced by Pazner and Schmeidler (1978). The third axiom of welfaristic consequentialism is the familiar Pareto principle.

Apart from this introduction, this paper consists of four sections and an appendix. In section 2, we introduce our basic production economies and allocation rules as game forms. In section 3, we discuss the basic concepts of extended social alternatives, social ordering functions, the three basic axioms, and their lexicographic variations. Section 4 expounds our possibility theorems. All the involved proofs are relegated into the appendix for the sake of simplicity of exposition.

\(^3\)If we decide to go beyond the traditional framework in which we focus exclusively on the intrinsic value of consequences, there are alternative routes which deserve to be explored. One route, which we are exploring in the present paper, is to admit the intrinsic value of procedures along with that of consequences. Another route is to admit the intrinsic value of opportunities of choice along with that of consequences, which was recently explored by Suzumura and Xu (2001; 2002).
2 The Basic Model

2.1 Simple Production Economies and Feasible Allocations

Consider an economy with the population $N = \{1, 2, \ldots, n\}$, where $2 \leq n < +\infty$. One good $y \in \mathbb{R}_+$ is produced from the vector of labor inputs $x = (x_1, \ldots, x_n) \in \mathbb{R}_n^n$ where $x_i$ denotes the labor time supplied by $i \in N$. The production process of this economy is described by the production function $f : \mathbb{R}_n^n \to \mathbb{R}_+$, which maps each $x \in \mathbb{R}_n^n$ into $y = f(x) \in \mathbb{R}_+$. It is assumed that $f$ satisfies continuity, strict increasingness, concavity, and $f(0) = 0$.

All individuals are assumed to have the common upper bound $\bar{x}$ of labor-leisure time, where $0 < \bar{x} < +\infty$. For each individual $i \in N$, his consumption vector is denoted by $z_i = (x_i, y_i) \in [0, \bar{x}] \times \mathbb{R}_+$ where $x_i$ is his labor time and $y_i$ is his share of output. Each $i \in N$ is characterized by his preference ordering on $[0, \bar{x}] \times \mathbb{R}_+$, which can be represented by a utility function $u_i : [0, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}$. We assume that $u_i$ is strictly monotonic (decreasing in labor time and increasing in the share of output) on $[0, \bar{x}) \times \mathbb{R}_+$, continuous and quasi-concave on $[0, \bar{x}] \times \mathbb{R}_+$. It is also assumed that $u_i(z_i) > u_i(x_i, 0)$ for all $z_i \in [0, \bar{x}] \times \mathbb{R}_+$ and all $x_i \in [0, \bar{x}]$. We denote the class of utility functions satisfying these assumptions by $\mathcal{U}$.

Along with the subjective characteristics of the economy, which are specified in terms of the profiles of individual utility functions $u = (u_1, \ldots, u_n) \in \mathcal{U}^n$, the objective characteristics of the economy are described in terms of the following two fixed data. The first datum is the aforementioned production function, and the second datum is the profile of individual capability correspondences $C = (C_1, \ldots, C_n)$ to be defined as follows. To begin with, we assume that there are $m$ basic functionings in the economy, which are relevant for all individuals for the purpose of describing their objective well-beings attainable by means of their consumption vectors. We assume that these $m$ functionings can be measured by means of adequate non-negative real numbers. Thus, an achievement of functioning $k$, where $k = 1, 2, \ldots, m$, by individual $i$ is denoted by $b_{ik} \in \mathbb{R}_+$. Individual $i$’s achievement of basic functionings is given by listing $b_{ik}$: $b_i = (b_{i1}, \ldots, b_{im}) \in \mathbb{R}_+^m$. Then, for each $i \in N$, $i$’s capability correspondence is defined as $C_i : [0, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}_+^m$ which

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4In what follows, $\mathbb{R}_+$, $\mathbb{R}_n^+$ and $\mathbb{R}_+^n$ denote, respectively, the set of non-negative real numbers, the non-negative orthant and the positive orthant in the Euclidean $n$-space.
associates to every $z_i \in [0, \overline{x}] \times \mathbb{R}_+$ a non-empty subset $C_i(z_i)$ of $\mathbb{R}_+^n$. The intended interpretation is that $i$ is able to have access to each functioning vector $b_i \in C_i(z_i)$ by means of his consumption vector $z_i$.

In what follows, we assume that the capability correspondences satisfy the following requirements:
(a) For all $z_i = (x_i, y_i)$, $z'_i = (x'_i, y'_i) \in [0, \overline{x}] \times \mathbb{R}_+$ such that $x_i = x'_i$ and $y_i \leq y'_i$ (resp. $y_i < y'_i$), $C_i(z_i) \subseteq C_i(z'_i)$ (resp. $C_i(z_i) \subsetneq C_i(z'_i)$) hold, where $C_i(z'_i)$ stands for the interior of $C_i(z'_i)$ in $\mathbb{R}_+^n$.
(b) For all $z_i \in [0, \overline{x}] \times \mathbb{R}_+$, $C_i(z_i)$ is compact and comprehensive in $\mathbb{R}_+^n$; and
(c) $C_i$ is continuous on $[0, \overline{x}] \times \mathbb{R}_+$.

A feasible allocation in our economy is a vector $z = (z_i)_{i \in N} = (x_i, y_i)_{i \in N} \in ([0, \overline{x}] \times \mathbb{R}_+)^n$ such that $f(x) \geq \sum N y_i$, where $x = (x_1, \ldots, x_n)$. Let $Z$ be the set of all feasible allocations. For each $z = (z_i)_{i \in N} \in Z$, $C(z) = (C_i(z_i))_{i \in N}$ denotes a feasible assignment of individual capabilities.

Our economy is characterized by the pair of subjective characteristics denoted by $u \in U^n$ and objective characteristics denoted by $f$ and $C$. Since the objective characteristics are fixed throughout this paper, however, we may identify our economy simply by $u \in U^n$.

### 2.2 Allocation Rules as Game Forms

To complete the description of how our simple economy functions, what remains to be specified is an allocation rule which assigns, to each $i \in N$, how many hours he/she works, and how great share of output he/she receives in return. In this paper, an allocation rule is modelled as a game form which is a pair $\gamma = (M, g)$, where $M = M_1 \times \cdots \times M_n$ is the set of admissible profiles of individual strategies, and $g$ is the outcome function which maps each strategy profile $m \in M$ into a unique outcome $g(m) \in Z$. For each $m \in M$, $g(m) = (g_i(m))_{i \in N}$, where $g_i(m) = (g_{i1}(m), g_{i2}(m)) \in [0, \overline{x}] \times \mathbb{R}_+$ for each $i \in N$, represents a feasible allocation resulting from the strategic interactions among individuals represented by the strategy profile $m$. Let $\Gamma$ be the set of all game forms representing allocation rules of our economy. Given an allocation rule $\gamma = (M, g) \in \Gamma$ and an economy $u \in U^n$, we obtain a fully-fledged specification of a non-cooperative game $(N, \gamma, u)$. Since the

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5For all vectors $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p) \in \mathbb{R}_+^p$, $a \geq b$ if and only if $a_i \geq b_i$ ($i = 1, \ldots, p$); $a > b$ if and only if $a \geq b$ and $a \neq b$; $a \gg b$ if and only if $a_i > b_i$ ($i = 1, \ldots, p$).
set of players $N$ is fixed throughout this paper, we may omit $N$ and describe a game as $(\gamma, u) \in \Gamma \times \mathcal{U}^n$ without ambiguity.

To describe an equilibrium outcome of a game $(\gamma, u)$, where $\gamma = (M, g)$, define $m_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n)$ for each $m \in M$ and $i \in N$, which is an element of a set $M_{-i} := \times_{j \in N \setminus \{i\}} M_j$. Given an $m_{-i} \in M_{-i}$ and an $m'_i \in M_i$, $(m'_i; m_{-i})$ may be construed as an admissible strategy profile obtained from $m$ by replacing $m_i$ with $m'_i$. Given a game $(\gamma, u) \in \Gamma \times \mathcal{U}^n$, an admissible strategy profile $m^* \in M$ is a pure strategy Nash equilibrium if $u_i(g_i(m^*)) \geq u_i(g_i(m_i; m^*_{-i}))$ holds for each $i \in N$ and each $m_i \in M_i$.

The set of all pure strategy Nash equilibria of the game $(\gamma, u)$ is denoted by $\text{NE}(\gamma, u)$. A feasible allocation $z \in Z$ is a pure strategy Nash equilibrium allocation of the game $(\gamma, u)$ if $z = g(m^*)$ holds for some $m^* \in \text{NE}(\gamma, u)$. The set of all pure strategy Nash equilibrium allocations of the game $(\gamma, u)$ is denoted by $A_{\text{NE}}(\gamma, u)$.

Throughout this paper, we will focus on the pure strategy Nash equilibrium concept, which we will abbreviate as the Nash equilibrium concept for short.

3 Consequential Values, Procedural Values, and Social Ordering Functions

We are now ready to introduce the comprehensive framework of analysis on the social choice of allocation rules as game forms. To accommodate not only the consequential values of resource allocations and allocation rules, but also the procedural values thereof, the analysis is based on social ordering functions, which are defined over the set of extended social alternatives, viz. pairs of feasible allocations and allocation rules. The intended interpretation of an extended social alternative, viz. a pair $(z, \gamma) \in Z \times \Gamma$, is that a feasible allocation $z$ is attained through an allocation rule $\gamma$. Note that a feasible allocation $z \in Z$ may or may not be realizable through an allocation rule $\gamma \in \Gamma$. Indeed, an extended social alternative $(z, \gamma) \in Z \times \Gamma$ is realizable only when an economy $u \in \mathcal{U}^n$ is given and satisfies $z \in A_{\text{NE}}(\gamma, u)$. Let $\mathcal{R}(u)$ denote the set of realizable extended social alternatives under $u \in \mathcal{U}^n$.

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6The concept of an extended social alternative was introduced by Pattanaik and Suzumura (1994; 1996) capitalizing on the thought-provoking suggestion by Arrow (1951, pp.89-91). See, also, Suzumura (1996; 1999; 2000).
feasible allocations and allocation rules. If an allocation rule embodies both consequential values and procedural values in the social evaluation of alternatives, it represents a social evaluation that cares only about consequential outcomes of resource allocations. In this sense, it may be christened the purely consequential SOF. In contrast, an SOF that embodies a social evaluation that cares only about procedural features of resource allocations. In this sense, it may be christened the purely procedural SOF. In between these two polar extreme cases, there is a wide range of SOFs which embody both consequential values and procedural values.

Once an SOF $Q \in \mathcal{Q}$ is specified, the set of best extended social alternatives is given, for each $u \in \mathcal{U}^n$, by

$$B(u : Q) \equiv \{(z, \gamma) \in \mathcal{R}(u) \mid \forall(z', \gamma') \in \mathcal{R}(u) : ((z, \gamma), (z', \gamma')) \in Q(u)\}.$$  

The set of socially chosen allocation rules is then given by

$$D(u : Q) \equiv \{\gamma \in \Gamma \mid \exists z \in Z : (z, \gamma) \in B(u : Q)\}.$$  

We say that an allocation rule $\gamma \in \Gamma$ is uniformly rationalizable via the SOF $Q \in \mathcal{Q}$ if and only if

$$\gamma \in \bigcap_{u \in \mathcal{U}^n} D(u : Q)$$

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7 A binary relation $R$ on a universal set $X$ is a quasi-ordering if it satisfies reflexivity and transitivity. An ordering is a quasi-ordering satisfying completeness as well.

8 Recollect that a pair $(z, \gamma) \in B(u : Q)$ is said to be rationalizable by an ordering $Q(u)$ on $\mathcal{R}(u)$ if and only if $(z, \gamma)$ is judged to be at least as good as any other pair in $\mathcal{R}(u)$ in terms of the ordering $Q(u)$. By a slight abuse of terminology, we may say in this case that $\gamma \in D(u : Q)$ is rationalizable by $Q(u)$. An allocation rule $\gamma \in \bigcap_{u \in \mathcal{U}^n} D(u : Q)$ is said to be uniformly rationalizable, as it is rationalizable by $Q(u)$ for each and every $u \in \mathcal{U}^n$. 

7 The intended interpretation of the ordering $Q(u)$ is that, for any extended social alternatives $(z^1, \gamma^1), (z^2, \gamma^2) \in \mathcal{R}(u)$, $(z^1, \gamma^1), (z^2, \gamma^2) \in Q(u)$ holds if and only if attaining a feasible allocation $z^1$ through an allocation rule $\gamma^1$ is at least as good as attaining a feasible allocation $z^2$ through an allocation rule $\gamma^2$ according to the social judgment embodied in $Q(u)$. The asymmetric part and the symmetric part of $Q(u)$ will be denoted by $P(Q(u))$ and $I(Q(u))$, respectively. The set of all SOFs will be denoted by $\mathcal{Q}$. 

What we call a social ordering function (SOF) is a function $Q : \mathcal{U}^n \to (Z \times \Gamma)^2$ such that $Q(u)$ is an ordering on $\mathcal{R}(u)$ for every $u \in \mathcal{U}^n$.

We say that an allocation rule $\gamma \in \Gamma \in (Z \times \Gamma)^2$, such that for each $u \in \mathcal{U}^n$, $(z^1, \gamma^1), (z^2, \gamma^2) \in \mathcal{R}(u)$, $(z^1, \gamma^1), (z^2, \gamma^2) \in Q(u)$ holds if and only if attaining a feasible allocation $z^1$ through an allocation rule $\gamma^1$ is at least as good as attaining a feasible allocation $z^2$ through an allocation rule $\gamma^2$ according to the social judgment embodied in $Q(u)$. The asymmetric part and the symmetric part of $Q(u)$ will be denoted by $P(Q(u))$ and $I(Q(u))$, respectively. The set of all SOFs will be denoted by $\mathcal{Q}$.

Note that it is this concept of an SOF that enables us to accommodate both consequential values and procedural values in the social evaluation of feasible allocations and allocation rules. If an SOF $Q_c$ is such that, for each $u \in \mathcal{U}^n$, $((z, \gamma^1), (z, \gamma^2)) \in I(Q_c(u))$ holds for all $(z, \gamma^1), (z, \gamma^2) \in \mathcal{R}(u)$, it represents a social evaluation that cares only about consequential outcomes of resource allocations. In this sense, $Q_c$ may be christened the purely consequential SOF. In contrast, an SOF $Q_p$ such that, for each $u \in \mathcal{U}^n$, $((z^1, \gamma), (z^2, \gamma)) \in I(Q_p(u))$ holds for all $(z^1, \gamma), (z^2, \gamma) \in \mathcal{R}(u)$ embodies a social evaluation that cares only about procedural features of resource allocations. In this sense, $Q_p$ may be christened the purely procedural SOF. In between these two polar extreme cases, there is a wide range of SOFs which embody both consequential values and procedural values.

Once an SOF $Q \in \mathcal{Q}$ is specified, the set of best extended social alternatives is given, for each $u \in \mathcal{U}^n$, by

$$B(u : Q) \equiv \{(z, \gamma) \in \mathcal{R}(u) \mid \forall(z', \gamma') \in \mathcal{R}(u) : ((z, \gamma), (z', \gamma')) \in Q(u)\}.$$  

The set of socially chosen allocation rules is then given by

$$D(u : Q) \equiv \{\gamma \in \Gamma \mid \exists z \in Z : (z, \gamma) \in B(u : Q)\}.$$  

We say that an allocation rule $\gamma \in \Gamma$ is uniformly rationalizable via the SOF $Q \in \mathcal{Q}$ if and only if

$$\gamma \in \bigcap_{u \in \mathcal{U}^n} D(u : Q)$$

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7 A binary relation $R$ on a universal set $X$ is a quasi-ordering if it satisfies reflexivity and transitivity. An ordering is a quasi-ordering satisfying completeness as well.

8 Recollect that a pair $(z, \gamma) \in B(u : Q)$ is said to be rationalizable by an ordering $Q(u)$ on $\mathcal{R}(u)$ if and only if $(z, \gamma)$ is judged to be at least as good as any other pair in $\mathcal{R}(u)$ in terms of the ordering $Q(u)$. By a slight abuse of terminology, we may say in this case that $\gamma \in D(u : Q)$ is rationalizable by $Q(u)$. An allocation rule $\gamma \in \bigcap_{u \in \mathcal{U}^n} D(u : Q)$ is said to be uniformly rationalizable, as it is rationalizable by $Q(u)$ for each and every $u \in \mathcal{U}^n$. 

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holds. By definition, such an allocation rule $\gamma$ can be uniformly applied to each and every $u \in U^n$ without violating the values embodied in the SOF $Q$.

The uniform rationalizability is a natural requirement for social choice of allocation rules in this model. This is because if $\cap_{u \in U^n} D(u : Q) = \emptyset$, it implies that any socially chosen allocation rule by $Q$ must be varied according to the variation in economic environments. Such a situation is, however, unnatural since an allocation rule represents an institution or a legal system of the society which is to be applicable independently of the change in economic environments.

In what follows, we will examine the possibility of an SOF embodying consequential values as well as procedural values along with rationalizing an allocation rule with some nice properties.

### 3.1 Basic Axioms

To begin with, let us formulate three basic axioms on SOFs which embody the viewpoints of procedural fairness, non-welfaristic egalitarianism, and welfaristic efficiency.

The first axiom, which is inspired by Kranich’s (1994) concept of *procedural fairness of sharing rules*, is meant to capture one aspect of procedural fairness of allocation rules. As an auxiliary step, let us introduce the following:

**Definition 1** [Kranich (1994)]:

(a) An allocation rule $\gamma = (M,g) \in \Gamma$ is labor-sovereign if, for all $i \in N$ and all $x_i \in [0,\bar{x}]$, there exists $m_i \in M_i$ such that, for all $m_{-i} \in M_{-i}$, $g_{i1}(m_i, m_{-i}) = x_i$;

(b) An allocation rule $\gamma = (M,g) \in \Gamma$ satisfies equal treatment of equals if, for all $i, j \in N$ and all $m \in M$, $C_i = C_j$ ensures that $g_{i1}(m) = g_{j1}(m)$ implies $g_{i2}(m) = g_{j2}(m)$.

Let $\Gamma_{LE}$ denote the subclass of $\Gamma$ which consists solely of allocation rules satisfying labor sovereignty as well as equal treatment of equals. Note that an allocation rule $\gamma \in \Gamma_{LE}$ embodies the basic idea of equality in treatment; it allows each and every individual to choose his/her labor-leisure portfolio

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9This condition is a weaker variant of *equal division for equal work* provided by Kranich (1994), in which the constraint of $C_i = C_j$ is not imposed.
as he/she sees fit; it also allows individuals having the same capability correspondence to have equal access to the share of output as long as their labor contributions are the same.

The first axiom we impose on the admissible class of SOFs is meant to capture one aspect of procedural fairness:

**Axiom 1 (Equality in Treatment):** For all \( u \in U^n \) and all \((z, \gamma), (z', \gamma') \in \mathcal{R}(u)\), if \( \gamma \in \Gamma_{LE} \) and \( \gamma' \in \Gamma \setminus \Gamma_{LE} \), then \(((z, \gamma), (z', \gamma')) \in P(Q(u))\).

It is clear that **Axiom 1** is a requirement of purely procedural nature, as it imposes some constraints on the admissible class of SOFs without having recourse to consequential outcomes. It also deserves notice that **Axiom 1** is not vacuous, since there exists an allocation rule \( \gamma \in \Gamma_{LE} \) such that \((z, \gamma) \in \mathcal{R}(u)\) for all \( u \in U^n \).

The second axiom, which is meant to capture one aspect of non-welfaristic egalitarianism, hinges on the concept of a reference functioning vector. As an auxiliary step, let \( \Delta \equiv \left\{ \delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}_+^m \mid \sum_{k=1}^m \delta_k = 1 \right\} \) be the unit simplex in \( \mathbb{R}^m \). Pick any \( \delta \in \Delta \) as a reference functioning vector. Suppose that there exist a feasible allocation \( z = (z_1, \ldots, z_n) \in Z \) and a scalar \( \pi \in \mathbb{R}_+ \) such that \( \pi \cdot \delta \in C_i(z_i) \) and \( \pi \cdot \delta \notin C_j(z_j) \) for some \( i, j \in N \). In this situation, we may assert that person \( i \) is better off than person \( j \) at \( z \) with reference to \( \delta \in \Delta \). Noting this fact, let us define

\[
\pi_i(z; \delta) \equiv \sup \{ \pi_i \in \mathbb{R}_+ \mid \pi_i \cdot \delta \in C_i(z_i) \}
\]

for each \( i \in N \), and let

\[
\pi(z; \delta) \equiv \inf \{ \pi_i(z; \delta) \}_{i \in N}.
\]

**Insert Figure 1 around here**

For each given \( \delta \in \Delta \), let us introduce the following axiom which is meant to capture one aspect of non-welfaristic egalitarianism:

\[\text{Consider the equal division sharing rule } \gamma^{ED} = (M^{ED}, g^{ED}), \text{ where } g^{ED} : [0, \pi]^n \to Z \text{ is such that } g^{ED}(x) = \left( x_i \cdot \frac{f_i(x)}{\sum_{j \in N} f_j(x)} \right)_{i \in N} \text{ for all } x \in [0, \pi]^n. \text{ Not only is } \gamma^{ED} \text{ labor-sovereign and treats equals equally, but it also has a Nash equilibrium allocation for all } u \in U^n.\]
Axiom 2 (δ-Reference Functioning Maximin): For all \( u \in U^n \) and all \((z, \gamma), (z', \gamma') \in R(u)\), if \( z = (x, y)\), \( z' = (x', y') \) and \( x = x' \), then:

\[
((z, \gamma), (z', \gamma')) \in Q(u) \iff \pi(z; \delta) \geq \pi(z'; \delta),
\]
\[
((z, \gamma), (z', \gamma')) \in P(Q(u)) \iff \pi(z; \delta) > \pi(z'; \delta).
\]

Remark 1: The ordering function which is constrained by Axiom 2 hinges squarely on the choice of a reference functioning vector \( \delta \in \Delta \) unless there is only one basic functioning, viz. \( m = 1 \), where there is no choice of \( \delta \) at all.

A salient feature of this version of non-welfaristic egalitarianism deserves emphasis. Instead of equalizing utilities among individuals, it attempts to enable the least advantaged individual, whoever he may be, to attain as large a functioning vector as possible along the ray determined by a given reference vector in the space of functionings.

The third axiom, which is welfaristic in nature, is meant to capture one aspect of welfaristic consequentialism in terms of a variant of the well-known Pareto principle:

Axiom 3 (Pareto Principle in Allocations): For all \( u \in U^n \) and all \((z, \gamma), (z', \gamma') \in R(u)\), if \( u_i(z_i) > u_i(z'_i) \) for all \( i \in N \), then \((z, \gamma), (z', \gamma')) \in P(Q(u))\), and if \( u_i(z_i) = u_i(z'_i) \) for all \( i \in N \), then \((z, \gamma), (z', \gamma')) \in I(Q(u))\).

3.2 Logical Incompatibility of the Basic Axioms

The three basic axioms introduced above represent the requirements on the admissible class of SOFs from diametrically different angles. Indeed, Axiom 1 requires equality in treatment from a purely procedural point of view, whereas Axiom 2 represents an egalitarian requirement which is rooted in non-welfaristic consequentialism. Finally, Axiom 3, being a variant of Paretoism, is based squarely on welfaristic consequentialism. It is no surprise, therefore, that the following negative result cannot but hold if we adhere to these basic axioms without any compromise.

Proposition 1: There exists no social ordering function which satisfies any two of the basic Axiom 1, Axiom 2, and Axiom 3.

To verify this negative assertion, we have only to check the following:
Example 1: Let there be two types of relevant functionings, and let \( N = \{1, 2\} \) and \( \pi = 3 \). The production function is given by \( f(x_1, x_2) = x_1 + x_2 \) for all \((x_1, x_2) \in \mathbb{R}_+^2\). Individuals have the same capability correspondence \( C \) which is defined as follows: For any \( z \in [0, \pi] \times \mathbb{R}_+\),

\[
C(z) \equiv \{(b_1, b_2) \in \mathbb{R}_+^2 \mid \exists z^1, z^2 \in [0, \pi] \times \mathbb{R}_+: z^1 + z^2 \leq z, b_k = c_k(z^k) \ (k = 1, 2)\},
\]

where \( c_1(x, y) \equiv (\pi - x)^{\frac{1}{3}} \cdot y^{\frac{2}{3}} \) and \( c_2(x, y) \equiv (\pi - x)^{\frac{1}{3}} \cdot y^{\frac{2}{3}} \) for any \((x, y) \in [0, \pi] \times \mathbb{R}_+\). Note that the mapping \( c_k(\cdot) \) assigns to each consumption vector an achievement of functioning \( k \). Thus, \( b_k = c_k(z^k) \) implies that if the consumption vector \( z^k \) is utilized for functioning \( k \), then the functioning \( k \) is attained at the level of \( b_k \).

Consider two feasible allocations \( z^* = ((1, 1), (1, 1)) \) and \( z^{**} = ((2, 2), (2, 2)) \). For some \( \alpha \in (0, 1) \), let \( z^*(\alpha) = ((1 + \alpha, 1 - \alpha)) \) and \( z^{**}(\alpha) = ((2 - \alpha, 2 + \alpha)) \). It is easy to check that, for all \( \delta \in \Delta, \pi(z^*(\alpha); \delta) > \pi(z^{**}(\alpha); \delta) \) and \( \pi(z^{**}; \delta) > \pi(z^{**}(\alpha); \delta) \). Individual 1’s utility function \( u_1 \) is defined, for all \((x, y) \in [0, \pi] \times \mathbb{R}_+\), by

\[
u_1(x, y) = (1 - \alpha) \cdot (\pi - x) + y,
\]

whereas individual 2’s utility function \( u_2 \) is defined, for all \((x, y) \in [0, \pi] \times \mathbb{R}_+\), by

\[
u_2(x, y) = \begin{cases} 
(1 - \alpha) \cdot (\pi - x) + y & \text{if } x \in [0, 1) \\
(1 + \alpha) \cdot (\pi - x) + y & \text{if } x \in [1, \pi].
\end{cases}
\]

This situation is described in the consumption space and in the functioning space in Figure 2.

[Insert Figure 2 around here]

Let \( \gamma^*, \gamma^*(\alpha), \gamma^{**}, \) and \( \gamma^{**}(\alpha) \) be the allocation rules in \( \Gamma \setminus \Gamma_{LE} \) which generate the realizable allocations \( z^*, z^*(\alpha), z^{**}, \) and \( z^{**}(\alpha) \), respectively, when the economy is defined by \( u = (u_1, u_2) \in U^2 \).

Suppose that an SOF \( Q \) satisfies both Axiom 2 and Axiom 3. Then, by Axiom 2,

\[
((z^*, \gamma^*), (z^*(\alpha), \gamma^*(\alpha))) \in P(Q(u)), \ ((z^{**}, \gamma^{**}), (z^{**}(\alpha), \gamma^{**}(\alpha))) \in P(Q(u)),
\]

whereas, by Axiom 3,

\[
((z^*(\alpha), \gamma^*(\alpha)), (z^{**}, \gamma^{**})) \in I(Q(u)), \ ((z^{**}(\alpha), \gamma^{**}(\alpha)), (z^*, \gamma^*)) \in I(Q(u)),
\]
since \( u_1(z_1^*(\alpha)) = u_1(z_1^{**}) = 3 - \alpha \), \( u_2(z_2^*(\alpha)) = u_2(z_2^{**}) = 3 + \alpha \), \( u_1(z_1^{**}(\alpha)) = u_1(z_1^*) = 3 - 2\alpha \), and \( u_2(z_2^{**}(\alpha)) = u_2(z_2^*) = 3 + 2\alpha \). Thus, \( Q(u) \) is not a consistent binary relation,\(^{11}\) hence it cannot be an ordering. ■

According to Example 1, there exists no SOF which satisfies Axiom 2 and Axiom 3. The incompatibility between Axiom 1 and Axiom 2, on the one hand, and Axiom 1 and Axiom 3, on the other hand, should be clear in view of the purely procedural nature of Axiom 1 which pays no attention whatsoever to consequences.

### 3.3 Weaker Variants of the Basic Axioms

When two or more basic principles irrevocably conflict with each other and yet we do not want to discard some of these principles altogether, a natural step to follow is to introduce a priority rule among these principles so as to define their lexical combinations. This idea has been explored repeatedly in the literature of normative economics, the most recent example being Tadenuma (2001).

To explore this intuitive idea systematically in the present context of our basic Axiom 1, Axiom 2, and Axiom 3, take any distinct \( \alpha, \beta \in \{1, 2, 3\} \) and define Axiom \( \alpha \ast \beta \) by invoking Axiom \( \beta \) only in the situation where Axiom \( \alpha \) cannot but keep silence. Likewise, Axiom \((\alpha \ast \beta) \ast \gamma \) is defined by invoking Axiom \( \gamma \) only in the situation where Axiom \( \alpha \ast \beta \) cannot but keep silence.\(^{12}\) In what follows, our attention will be focussed, on the one hand, upon the weaker variants of Axiom 2, viz. Axiom 1 \( \ast \) 2 and Axiom \((1 \ast 3) \ast 2\), and upon the weaker variants of Axiom 3, viz. Axiom 1 \( \ast \) 3 and Axiom \((1 \ast 2) \ast 3\), on the other.

---

\(^{11}\)A finite subset \( \{x_1, \cdots, x_t\} \) of a universal set \( X \), where \( 2 \leq t < +\infty \), satisfying \( (x_1, x_2^t) \in P(R), (x^2, x^3) \in R, \cdots, (x^t, x^1) \in R \) is called an incoherent cycle of order \( t \) of a binary relation \( R \) on \( X \). \( R \) is said to be consistent if there exists no incoherent cycle of any order. A binary relation \( R^* \) is called an extension of \( R \) if and only if \( R \subseteq R^* \) and \( P(R) \subseteq P(R^*) \). It is shown in Suzumura (1983, Theorem A(5)) that there exists an ordering extension of \( R \) if and only if \( R \) is consistent.

\(^{12}\)Recollect that Axiom 1, Axiom 2, and Axiom 3 are expressed in the conditional form of “if (A), then (B)” style. Thus, whenever the condition (A) is not satisfied for Axiom \( \alpha \), where \( \alpha \in \{1, 2, 3\} \), Axiom \( \alpha \) has nothing to offer and must keep silence. This being the case, Axiom \( \alpha \ast \beta \), where \( \alpha, \beta \in \{1, 2, 3\} \), simply implies that Axiom \( \beta \) can have bite only when the condition (A) is not satisfied for Axiom \( \alpha \), thereby forcing Axiom \( \alpha \) to keep silence. In other words, the simultaneous requirement of Axiom \( \alpha \) and Axiom \( \alpha \ast \beta \) means that Axiom \( \alpha \) has a lexicographic priority to Axiom \( \beta \).
Even these weaker variants of Axiom 2 and Axiom 3 are still incompatible in the presence of the basic Axiom 1, as the following proposition shows.

**Proposition 2:** There exists no social ordering function which satisfies any combination of conditions chosen from the following three categories, one condition from each category:

(C1) Axiom 1;
(C2) Axiom 2, Axiom 1*2, Axiom (1*3)*2;
(C3) Axiom 3, Axiom 1*3, Axiom (1*2)*3.

**Proof:** Among the assertions of Proposition 2, the incompatibility of Axiom 1, Axiom 2, and Axiom 3 simply repeats the assertion of Proposition 1. The other assertions of Proposition 2 can be verified by having another look at Example 1. Consider the same four extended alternatives as in Example 1. Then, since \( z^* \) and \( z^*(\alpha) \) (resp. \( z^{**} \) and \( z^{**}(\alpha) \)) are comparable by Axiom (1*3)*2, but non-comparable by Axiom (1*2)*3, whereas \( z^*(\alpha) \) and \( z^{**}(\alpha) \) are comparable by Axiom (1*2)*3, but non-comparable by Axiom (1*3)*2, we may conclude that any \( Q \) satisfying Axiom (1*3)*2 and Axiom (1*2)*3 is not a consistent relation, hence it cannot be an ordering. Since Axiom 1*2 (resp. Axiom 1*3) is stronger than Axiom (1*3)*2 (resp. Axiom (1*2)*3), the other assertions of Proposition 2 follow immediately. ■

Although Proposition 2 only considers the case that Axiom 1 is given a lexicographic priority over Axiom 2 and Axiom 3, we can also verify that the same types of impossibility results are obtained even in the case that either Axiom 2 or Axiom 3 is given a lexicographic priority over Axiom 1, by using a similar argument as in the proof of Proposition 2.

To aspire for a compatible lexicographic combination of our basic axioms, further concession seems to be indispensable in view of Proposition 2. As an auxiliary step, given each \( u \in U^n \), let

\[
PO(u) \equiv \{ z \in Z \mid \forall z' \in Z, \exists i \in N : u_i(z_i) \geq u_i(z'_i) \}.
\]

Note that \( PO(u) \) is the set of Pareto efficient allocations by the strict monotonicity of every utility function.

For each given \( \delta \in \Delta \), let us introduce the following two conditional variants of Axiom 2:
Axiom 2 ∩ PO: For all $u \in U^n$ and all $(z, \gamma), (z', \gamma') \in R(u)$ with $z, z' \in PO(u)$, if $z = (x, y)$, $z' = (x', y')$, and $x = x'$, then:

$((z, \gamma), (z', \gamma')) \in Q(u) \iff \pi(z; \delta) \geq \pi(z'; \delta),$

$((z, \gamma), (z', \gamma')) \in P(Q(u)) \iff \pi(z; \delta) > \pi(z'; \delta).$

Axiom (1 * 2) ∩ PO: For all $u \in U^n$ and all $(z, \gamma), (z', \gamma') \in R(u)$ such that

(1) either $\gamma, \gamma' \in \Gamma_{LE}$ or $\gamma, \gamma' \in \Gamma \setminus \Gamma_{LE}$ and

(2) $z, z' \in PO(u)$, if $z = (x, y)$, $z' = (x', y')$, and $x = x'$, then:

$((z, \gamma), (z', \gamma')) \in Q(u) \iff \pi(z; \delta) \geq \pi(z'; \delta),$

$((z, \gamma), (z', \gamma')) \in P(Q(u)) \iff \pi(z; \delta) > \pi(z'; \delta).$

Axiom (1 * 2) ∩ PO is a weaker variant of Axiom 2 than Axiom 2 ∩ PO, since the former is contingent on the fact that Axiom 1 keeps silence, whereas the latter is not. Moreover, Axiom (1 * 2) ∩ PO (resp. Axiom 2 ∩ PO) is a weaker variant of Axiom 2 than Axiom (1 * 3) * 2 (resp. Axiom 3 * 2), since $z, z' \in PO(u)$ for $(z, \gamma), (z', \gamma') \in R(u)$ implies that Axiom 3 keeps silence on these $(z, \gamma), (z', \gamma') \in R(u)$.

We should also introduce some conditional variants of Axiom 1 and Axiom 3. As an auxiliary step, take any $\delta \in \Delta$ and $x \in [0, \pi]^n$, and define

$Z(x; \delta) = \{(x, y) \in Z \mid \forall(x, y') \in Z : \pi(x, y; \delta) \geq \pi(x, y'; \delta)\}.$

The set $Z(x; \delta)$ consists of feasible allocations which attain a maximal value of $\pi(x, :, \delta)$ for the given $\delta \in \Delta$ and $x \in [0, \pi]^n$. So, this is the set of $\delta$-reference functioning maximin allocations for the given $x \in [0, \pi]^n$.

Given a reference functioning vector $\delta \in \Delta$, let us define the following two conditional variants of Axiom 1:

Axiom (2 * 1) ∩ PO: For all $u \in U^n$ and all $(z, \gamma), (z', \gamma') \in R(u)$ such that

(1) $z = (x, y)$ and $z' = (x', y')$ are $x \neq x'$, and

(2) $z, z' \in PO(u)$, if $\gamma \in \Gamma_{LE}$ and $\gamma' \in \Gamma \setminus \Gamma_{LE}$, then $((z, \gamma), (z', \gamma')) \in P(Q(u)).$

Axiom (3 * 1) ∩ $Z(\delta)$: For all $u \in U^n$ and all $(z, \gamma), (z', \gamma') \in R(u)$ such that

(1) $z$ and $z'$ are Pareto non-comparable in $u$, and

(2) $z = (x, y) \in PO(u)$.

Two feasible allocations $z, z' \in Z$ are said to be Pareto non-comparable in $u \in U^n$ if and only if any one of (1) $u_i(z_i) > u_i(z'_i)$ for all $i \in N$, (2) $u_i(z_i) = u_i(z'_i)$ for all $i \in N$, and (3) $u_i(z_i) < u_i(z'_i)$ for all $i \in N$, fails to hold. Note that, if $u \in U^n$ and $(z, \gamma), (z', \gamma') \in R(u)$ are such that $z$ and $z'$ are Pareto non-comparable in $u$, Axiom 3 keeps silence on these $(z, \gamma), (z', \gamma') \in R(u)$.
Axiom 3 and \( z' = (x',y') \in Z(x';\delta) \), if \( \gamma \in \Gamma_{LE} \) and \( \gamma' \in \Gamma \setminus \Gamma_{LE} \), then \((z,\gamma),(z',\gamma') \in P(Q(u))\).

Note, in passing, that the fact \( z \) and \( z' \) are Pareto efficient trivially implies that \( z \) and \( z' \) being Pareto non-comparable, but not vice versa. Thus, Axiom \((2 \ast 1) \cap \text{PO}\) is weaker than Axiom \((3 \ast 2) \ast 1\).

Turning now to Axiom 3, and given a reference functioning vector \( \delta \in \Delta \), let us introduce the following two conditional variants thereof:

Axiom 3 \( \cap Z(\delta) \): For all \( u \in U^n \) and all \((z,\gamma),(z',\gamma') \in \mathcal{R}(u)\) such that \( z = (x,y) \in Z(x;\delta) \) and \( z' = (x',y') \in Z(x';\delta) \), if \( u_i(z_i) > u_i(z'_i) \) for all \( i \in N \), then \((z,\gamma),(z',\gamma') \in P(Q(u))\), and if \( u_i(z_i) = u_i(z'_i) \) for all \( i \in N \), then \((z,\gamma),(z',\gamma') \in I(Q(u))\).

Axiom \((1 \ast 3) \cap Z(\delta) \): For all \( u \in U^n \) and all \((z,\gamma),(z',\gamma') \in \mathcal{R}(u)\) such that (1) either \( \gamma,\gamma' \in \Gamma_{LE} \), or \( \gamma,\gamma' \in \Gamma \setminus \Gamma_{LE} \), and (2) \( z = (x,y) \in Z(x;\delta) \) and \( z' = (x',y') \in Z(x';\delta) \), if \( u_i(z_i) > u_i(z'_i) \) for all \( i \in N \), then \((z,\gamma),(z',\gamma') \in P(Q(u))\), and if \( u_i(z_i) = u_i(z'_i) \) for all \( i \in N \), then \((z,\gamma),(z',\gamma') \in I(Q(u))\).

Clearly, Axiom 3 \( \cap Z(\delta) \) is stronger than Axiom \((1 \ast 3) \cap Z(\delta) \), since the former axiom is not contingent on the fact that Axiom 1 keeps silence, whereas the latter is. Note that we can also see that Axiom 2 \( \ast 3 \) is stronger than Axiom 3 \( \cap Z(\delta) \). This follows from the fact that \( Z(x;\delta) \) is singleton for each given \( \delta \in \Delta \) and \( x \in [0,\pi]^n \), which will be discussed in Lemma 3 later, since when \( z \in Z(x;\delta) \) and \( z' \in Z(x';\delta) \) are different allocations, it implies that \( x \neq x' \). Thus, Axiom 2 keeps silence on \((z,\gamma),(z',\gamma') \in \mathcal{R}(u)\) with \( z \in Z(x;\delta) \) and \( z' \in Z(x';\delta) \).

[Insert Figure 3 around here]

Figure 3 summarizes the logical implications which hold among the axioms introduced so far. Our task is to identify a compatible set of axioms, one axiom from each cluster emanating from the basic Axiom 1, Axiom 2, and Axiom 3, by proving the existence of a social ordering function satisfying the identified set of axioms. In particular, the focus of our analysis is on the existence of a social ordering function which uniformly rationalizes a “fair” allocation rule.
4 Existence of Social Ordering Functions Satisfying Weaker Variants of Basic Axioms

Among several possibilities of combining weaker variants of the basic Axiom 1 (Equality of Treatment), Axiom 2 (δ-Reference Functioning Maximin) and Axiom 3 (Pareto Principle in Allocations), the following two possibilities stand out: (a) Axiom 1, Axiom 1∗2, and Axiom (1∗3) ∩ Z(δ), and (b) Axiom (3∗1) ∩ Z(δ), Axiom 2, and Axiom 3 ∩ Z(δ). Clearly, the former set of axioms confers a lexicographic priority on the requirement of procedural fairness over the requirement of outcome morality, whereas the latter set of axioms confers the reverse logical priority. In this sense, it is of particular interest to pursue how these two sets of axioms fare in the arena of the existence of an SOF.

4.1 Reference Functioning Maximin and Capability Maximin

As a preliminary step, let us characterize the set $Z(x; δ)$ of best feasible allocations $z = (x, y) ∈ Z$ when the labor input vector $x ∈ [0, π]^n$ and the reference functioning vector $δ ∈ Δ$ are specified. For that purpose, we must introduce the crucial concept of common capability.\(^{14}\) For each feasible allocation $z = (z_i)_{i ∈ N} ∈ Z$, an assignment of capabilities to individuals is determined by $C(z) ≡ (C_i(z_i))_{i ∈ N}$. Then

$$CC(z) ≡ ∩_{i ∈ N} C_i(z_i)$$

represents the common capability under $z$, which allows us to define the family of common capabilities by $CC ≡ \{CC(z) | z ∈ Z\}$. Note that, for each feasible allocation $z = (z_i)_{i ∈ N} ∈ Z$, all individuals are minimally entitled to functioning vectors in the common capability $CC(z)$, no matter how they differ in their capability correspondences. Since this warranted set of functioning vectors hinges squarely on the choice of a feasible allocation, a deliberate choice of an allocation rule, so as to make the common capability as large as possible, can serve as a social contrivance through which the

\(^{14}\)The notion and formulation of common capability comes from Gotoh and Yoshihara (1999; 1999a).
well-being of the least privileged individual, whoever he/she may be, can be improved as much as possible.

For each $x \in [0, \bar{x}]^n$, let $Y(x) \equiv \{ y = (y_i)_{i \in N} \in \mathbb{R}^n_+ \mid f(x) \geq \sum N y_i \}$ be the set of all feasible distributions of produced goods when the labor input vector $x$ is specified. Then, given $x \in [0, \bar{x}]^n$, let $CC(x) \equiv \{ CC(x, y) \mid y \in Y(x) \}$ be the family of common capabilities under $x$.\(^{15}\) At this juncture, let us introduce an appropriate topology into the space of compact sets in $\mathbb{R}^n_+$ in terms of the Hausdorff metric.\(^{16}\) Equipped with this topology and given $x \in [0, \bar{x}]^n$, the common capability $CC(x, y) \in CC(x)$, where $y \in Y(x)$, is said to be undominated under $x$ if there exists no $y' \in Y(x)$ such that $CC(x, y') \supset CC(x, y)$. Let $UC(x)$ denote the set of undominated common capabilities under $x$.

**Lemma 1:** For all $x \in [0, \bar{x}]^n$, $UC(x)$ is non-empty and compact.

Since $Y(x)$ is clearly compact, and $\pi(x, y; \delta)$ constitutes a continuous mapping on $\{x\} \times Y(x)$ for each fixed $\delta \in \Delta$, the set $Z(x; \delta)$ is surely non-empty and compact. We can characterize $Z(x; \delta)$ by the following two lemmas:

**Lemma 2:** For all $\delta \in \Delta$ and all $x \in [0, \bar{x}]^n$, if $(x, y) \in Z(x; \delta)$, then $CC(x, y) \in UC(x)$.

**Lemma 3:** For all $\delta \in \Delta$ and all $x \in [0, \bar{x}]^n$, $Z(x; \delta)$ is a singleton set. Moreover, $(x, y) \in Z(x; \delta)$ implies $\pi(x, y; \delta) \cdot \delta \in \partial C_i(x_i, y_i)$ for all $i \in N$, where $\partial C_i(x_i, y_i)$ is the boundary of $C_i(x_i, y_i)$.

According to Lemma 2, if $(x, y)$ is a maximin feasible allocation with respect to a reference functioning vector $\delta$ under $x$, then the common capability $CC(x, y)$ is undominated under $x$. On the other hand, according to Lemma 3, for any reference functioning vector $\delta$ and any vector $x$ of labor inputs, the corresponding maximin allocation is uniquely determined, under which each and every individual has equal access to the same maximal

\(^{15}\) For the sake of notational convenience, we sometimes denote the common capability by $CC(x, y)$ instead of $CC(z)$, where $z = (x, y) = (x_i, y_i)_{i \in N} \in Z$.

\(^{16}\) For any compact sets $C, C' \subseteq \mathbb{R}^n$, the Hausdorff metric between $C$ and $C'$ is defined by

$$d(C, C') \equiv \max\{\max\{\delta(b, C) \mid b \in C'\}, \max\{\delta(b, C') \mid b \in C\}\},$$

where $\delta(b, C) \equiv \min_{b \in C} \| b, b' \|$, and $\| b, b' \|$ is the Euclidean distance between $b$ and $b'$.
functioning vector along the ray determined by \( \delta \). Figure 4 describes the capability assignment corresponding to \((x, y) \in Z(x; \delta)\).

Remark 2: The definition of \( Z(x; \cdot) \) depends upon the choice of the reference unit functioning vector whenever the number of basic functionings is \( m \geq 2 \). Moreover, the capability assignment corresponding to \((x, y) \in Z(x; \delta)\) does not necessarily imply an equal assignment of capabilities among individuals as Figure 4 exemplifies. However, whenever \( m = 1 \), \( Z(x; \cdot) \) is trivially defined without reference to any unit functioning vector, and it clearly corresponds to an equal assignment of capabilities.

It can be verified by using Lemma 2 and Lemma 3 that:

**Lemma 4:** For all \( \delta \in \Delta \), there exists a social ordering function satisfying Axiom 2 and Axiom 3 \( \cap Z(\delta) \).

### 4.2 Priority-to-Procedure Social Ordering Function and Priority-to-Consequence Social Ordering Function

We are now ready to present the first two main results of this paper.

**Theorem 1:** For any given reference functioning vector \( \delta \in \Delta \), there exists a social ordering function satisfying Axiom 1, Axiom \( 1 * 2 \), and Axiom \((1 * 3) \cap Z(\delta)\).

Let the set of SOF’s satisfying Axiom 1, Axiom \( 1 * 2 \), and Axiom \((1 * 3) \cap Z(\delta)\) for some \( \delta \in \Delta \) be denoted by \( Q^1(\Delta) \). The SOF \( Q^1 \in Q^1(\Delta) \) constructed in the proof of Theorem 1, which is relegated to Appendix at the end of the paper, satisfies \( B(u : Q^1) \neq \emptyset \), hence \( D(u : Q^1) \neq \emptyset \) for all \( u \in U^n \).

**Theorem 2:** For any given reference functioning vector \( \delta \in \Delta \), there exists a social ordering function satisfying Axiom \((3 * 1) \cap Z(\delta)\), Axiom 2, and Axiom \( 3 \cap Z(\delta) \).

Let the set of SOF’s satisfying Axiom \((3 * 1) \cap Z(\delta)\), Axiom 2, and Axiom \( 3 \cap Z(\delta) \) for some \( \delta \in \Delta \) be denoted by \( Q^2(\Delta) \). In contrast to \( Q^1(\Delta) \), the
SOF $Q^2 \in Q^2(\Delta)$ constructed in the proof of Theorem 2, which is also relegated to Appendix at the end of the paper, does not necessarily satisfy $B(u : Q^2) \neq \emptyset$. This is because even the class $\Gamma_{LE}$ contains infinitely many types of allocation rules, and is not guaranteed to be a “compact” subset of $\Gamma$.\footnote{Note that in addition to the compactness of $\Gamma_{LE}$, $Q^2$ should assign an “upper semi-continuous” ordering over $\Gamma_{LE}$ to guarantee the non-emptiness of $B(u : Q^2)$.} In fact, there is no natural way to introduce a topology into the set $\Gamma$ of allocation rules as game forms, so that we could not guarantee the compactness of $\Gamma_{LE}$.

This contrast between a priority-to-procedure $Q^1 \in Q^1(\Delta)$ and a priority-to-consequence $Q^2 \in Q^2(\Delta)$ seems worth emphasis; it may be more difficult task to identify the best allocation rule $\gamma_u \in D(u : Q)$ for each and every economy $u \in U^n$ with the latter SOF in $Q^2(\Delta)$ than with the former SOF in $Q^1(\Delta)$. Although this argument may be effective in favoring the priority-to-procedure approach vis-à-vis the priority-to-consequence approach, the score of the former approach may still not be high enough, as the best allocation rule $\gamma_u \in D(u : Q^1)$ identified by $Q^1 \in Q^1(\Delta)$ may well differ in response to the different realizations of the profile of utility functions $u \in U^n$. It is to the problem of uniform rationalizability of an allocation rule via the priority-to-procedure SOF that we now turn.

4.3 Uniform Rationalizability of Allocation Rules

Note that the above Theorem 1 and Theorem 2 do not exhaust interesting lexicographic combinations of the basic axioms. Indeed, we can show the existence of an SOF satisfying, for any given $\delta \in \Delta$, $\text{Axiom 1}$, $\text{Axiom (1 \ast 2)} \cap \text{PO}$, and $\text{Axiom 1 \ast 3}$. Likewise, we can show the existence of an SOF satisfying, for any given $\delta \in \Delta$, $\text{Axiom (2 \ast 1)} \cap \text{PO}$, $\text{Axiom 2 \cap PO}$, and $\text{Axiom 3}$. The possible lexicographic combinations of the basic axioms are summarized in Figure 5.

\[ \text{Insert Figure 5 around here} \]

The above two SOFs may have the merit of rationalizing allocation rules which implement Pareto efficient allocations in Nash equilibria, but the rationalized allocation rules may not necessarily guarantee undominated common capabilities in Nash equilibria. In contrast, although the SOF $Q^1 \in Q^1(\Delta)$
rationalizes undominated common capability rules, the Nash equilibrium allocations thereof may not be Pareto efficient. However, we can identify a set of mild sufficient conditions under which there exists an SOF $Q^1 \in Q_1^2(\Delta)$ which uniformly rationalizes allocation rules whose Nash equilibrium allocations are Pareto efficient.

**Assumption 1:** The utility function $u_i$ of each and every agent has the following property: $\forall z_i \in [0, \bar{x}] \times \mathbb{R}_+, u_i(z_i) > 0$, and $u_i(\bar{x}, 0) = 0$.

**Assumption 2:** The production function $f$ is continuously differentiable.

We are now ready to present the following:

**Theorem 3:** Under Assumption 1 and Assumption 2, and for any given reference functioning vector $\delta \in \Delta$, there exists an allocation rule $\gamma^* \in \Gamma_{LE}$ which is Nash solvable.\(^{18}\) Furthermore, every $z = (x, y) \in A_{NE}(\gamma^*, u)$ is not only Pareto efficient, but also $z = (x, y) \in Z(x; \delta)$ (hence it is associated with $CC(x, y) \in UC(x)$) for every $u \in U^n$.

In the above theorem, Assumption 1 can be weakened to the claim that $u_i$ is bounded from below. Moreover, the imposition of Assumption 2 is not essential: it is just to simplify the argument. Indeed, we can construct an allocation rule having the property of Theorem 3 even without Assumption 2, although the construction of such an allocation rule will be more complicated than the current one.

The second two main results of this paper are on the existence of an SOF which uniformly rationalizes the allocation rule $\gamma^*$.

**Theorem 4:** Under Assumption 1 and Assumption 2, and for any given reference functioning vector $\delta \in \Delta$, there exists a social ordering function $Q^*_{\delta}$ satisfying Axiom 1, Axiom $(1 \ast 2)$, and Axiom $(1 \ast 3) \cap Z(\delta)$, which uniformly rationalizes the allocation rule $\gamma^*$.

**Theorem 5:** Under Assumption 1 and Assumption 2, and for any given reference functioning vector $\delta \in \Delta$, there exists a social ordering function $Q^*_{\delta}$ satisfying Axiom 1, Axiom $(1 \ast 2) \cap PO$, and Axiom $1 \ast 3$, which uniformly rationalizes the allocation rule $\gamma^*$.

\(^{18}\) A game form $\gamma = (M, g)$ is said to be Nash-solvable if $NE(\gamma, u) \neq \emptyset$ for each and every $u \in U^n$. 

20
We have thus shown that there exist two SOFs which, in their respective ways, lexicographically combine the requirements of procedural fairness, non-welfaristic egalitarianism, and welfaristic consequentialism, with the common conferment of priority to procedural fairness. Although both $Q_\delta^*$ and $Q_{\delta}^{**}$ respectively satisfy only weaker variants of the three basic axioms, they can uniformly rationalize the allocation rule $\gamma^*$ which is strongly desirable in terms of labor sovereignty, equal treatment of equals, non-welfaristic egalitarianism, and Pareto efficiency. In this sense, both $Q_\delta^*$ and $Q_{\delta}^{**}$ seem to be appealing.

5 Discussion

1. Note that the ordering over feasible allocations based upon $\pi(\cdot; \delta)$ with reference to any unit functioning vector $\delta$, which is used in defining Axiom 2 and its weaker variants, can be regarded as an ordering over capability assignments, which is defined as an ordering over common capabilities. This is because each feasible allocation $z \in Z$ uniquely corresponds to a capability assignment $C(z)$, and each capability assignment $C(z)$ uniquely corresponds to a common capability $CC(z)$. Such an ordering over capability assignments is defined as an ordering extension of set-inclusion relation over common capabilities. In fact, as shown in Lemma 2, for any $\delta \in \Delta$ and any $x \in [0, \overline{x}]$, each $\delta$-reference maximin allocation in $Z(x; \delta)$ guarantees a maximal intersection of all individuals’ capability sets in terms of set-inclusion. Thus, the ordering over feasible allocations based upon $\pi(\cdot; \delta)$ can be characterized by using the axioms of ranking over profiles of opportunity sets introduced by Herrero et al. (1998) and Bossert et al. (1999). For example, the ordering over capability assignments derived from $\pi(\cdot; \delta)$ satisfies the Hammond equity axiom [Hammond (1976)] in terms of set-inclusion.

Note that there are many other orderings over capability assignments which also meet the Hammond equity axiom and evaluate the “wellness of opportunity assignments” with no reference to any $\delta \in \Delta$, as Herrero et al. (1998) and Bossert et al. (1999) did. However, an ordering with no reference to any $\delta \in \Delta$ is, in general, no longer compatible with the Pareto principle.

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19 In fact, Herrero et al. (1998) and Bossert et al. (1999) provided axiomatic characterizations of ranking over opportunity assignments based on common opportunity.

20 The opportunity set ranking version of the Hammond equity axiom was introduced and discussed by Herrero et al. (1998) and Bossert et al. (1999).
even in the weakest lexical combination, such as the combination of **Axiom 2** and **Axiom 3** $\cap Z(\delta)$.

2. Throughout this paper, we fixed a profile of capability correspondences $C$ as an objective datum in discussing SOFs and fair allocation rules. Moreover, we discussed the fair allocation rule $\gamma^*$ in Theorem 3, given the data of capability correspondences $C$. This is because our main purpose in defining allocation rules as game forms was not for solving incentive problems, but for designing allocation rules having the normative properties of procedural fairness and outcome morality. If we were interested in incentive problems in resource allocations, we would have considered the issue of information revelation about $C$, by assuming that $C$ is private information. We have eliminated this kind of incentive issue from our concern in this paper. However, if we considered this issue, we would be able to construct a game form which could solve the issue with the same desirable property as $\gamma^*$; for example, every individual’s strategy in such a game form would be composed of not only choosing his labor time and announcing his demand for output, but also announcing a profile of capability correspondences.

3. Although we interpret the range of $C$ as the Sen space of basic functionings, we need not adhere to this interpretation; since $C$ are defined in a rather general way with only a few conditions like (a), (b), and (c), we can start from another type of well-being index instead of basic functionings, and interpret $C$ as opportunity set mappings of this new index. For example, we can interpret the output $y$ as the money good, and $m$ as indicating the number of consumption goods except money which exist outside our economic model. Then, we may interpret $\mathbb{R}^m_+$ as the space of consumption goods and $C_i(x, y)$ as $i$’s opportunity set of the consumption goods which he can purchase with $y$. Thus, the implications obtained by introducing $C$ in this paper about non-welfaristic egalitarianism need not be limited to the case of Sen’s theory of functionings and capability.

4. We may consider further variants of the three basic axioms other than the ones focussed in this paper. For example, despite the incompatibility result among the three basic axioms, one may argue that the notions of procedural fairness and non-welfaristic egalitarianism behind **Axiom 1** and **Axiom 2** are still moderate: there is no libertarian requirement for allocation rules relevant to *self-ownership* in **Axiom 1**, while the $\delta$-reference functioning maximin criterion in **Axiom 2** only applies to the *problem of distribution of output* under the constraint of $x = x'$. We may define another variant of **Axiom 1** by replacing the equal treatment of equals condition
with some claim on self-ownership, as well as a stronger variant of Axiom 2 without the constraint of $x = x'$. However, using these variants of basic axioms cannot lead us to any consistent reconciliation: such a variant of Axiom 1 and its lexicographic variants would be incompatible with any lexicographic variants of Axiom 2. Also, such a stronger variant of Axiom 2 and its lexicographic variants would never be compatible with the Pareto principle.

6 Conclusion

This paper analyzed the possibility of constructing a social ordering function (SOF) over the set of extended alternatives, viz. ordered pairs of resource allocations and allocation rules, which satisfies procedural values as well as consequential values. The procedural value we tried to capture is concerned with the procedural fairness of allocation rules, that is, equal freedom in choosing labor time; while the consequential values in our initial ambitious set of axioms are two-fold: one is of non-welfaristic egalitarianism in the form of the reference functioning maximin criterion; and the other is of welfaristic consequentialism in the form of the Pareto principle in allocations. Since this initial set of axioms consists of diametrically contrasting requirements, it is all too natural that there exists no SOF which satisfies all of them. However, this paper identified some lexicographic combinations of these basic axioms which are demonstrably consistent with each other. In particular, priority-to-procedure SOFs exist if the axiom of procedural fairness is conferred lexicographic priority over the axioms of (constrained) non-welfaristic egalitarianism and (constrained) welfaristic consequentialism, while priority-to-consequence SOFs exist if the axioms of consequential values are conferred lexicographic priority over the axiom of (constrained) procedural fairness. Moreover, there exist at least two priority-to-procedure SOFs which can successfully identify a procedurally-fair allocation rule that may be applied uniformly no matter what economic environment materializes, and implements Pareto efficient and functionings-maximin allocations.

Thus, “desirable” pairs of allocations and allocation rules in terms of procedural fairness, non-welfaristic egalitarianism, and allocative efficiency can exist and have rational choice foundations through the two priority-to-procedure SOFs. In particular, the rather strong constraints on the two consequentialist axioms induce neither any loss in equity, nor any loss in ef-
ficiency, at the “best” allocations which are rationalized by the SOFs. In conclusion, it is hoped that these results exemplify the use and usefulness of the analytical framework we developed in this paper with the purpose of accommodating the procedural fairness considerations as well as the considerations of outcome morality.

7 Appendix

7.1 Proof of Theorem 1 and Theorem 2

Proof of Lemma 1: To begin with, for each \( x \in [0, \pi]^n \), the set \( Y(x) \) is well-defined and compact, which can be used to define a compact-valued and continuous correspondence \( Y : [0, \pi]^n \rightarrow \mathbb{R}^n_+ \). Let \( \partial Y(x) \equiv \{ y = (y_i)_{i \in N} \in \mathbb{R}^n_+ \mid f(x) = \sum_{i \in N} y_i \} \).

Given \( x \in [0, \pi]^n \), let \( \zeta_x : Y(x) \rightarrow \mathbb{R}^n_+ \) be such that, for all \( y \in Y(x) \), \( \zeta_x(y) \equiv CC(x, y) \). By the definition, \( \zeta_x \) is compact-valued. Since \( C_i \) is upper hemi-continuous (u.h.c.) for each \( i \in N \), \( \zeta_x \) is u.h.c. We show that \( \zeta_x \) is also lower hemi-continuous (l.h.c.). Let \( \eta^i_x : Y(x) \rightarrow \mathbb{R}^n_+ \) be such that, \( \eta^i_x(y) \equiv C^o_i(x_i, y_i) \), for all \( y \in Y(x) \). Since \( \eta^i_x \) is l.h.c. and \( \eta^i_x \) can be shown to have open graph, \( \eta^i_x \cap \eta^j_x \) is l.h.c. Similarly, we can show that \( \cap_{i \in N} C^o_i \) is l.h.c. Since the closure of \( \cap_{i \in N} \eta^i_x(y) \) is \( \zeta_x(y) \) for all \( y \in Y(x) \), we can see that \( \zeta_x \) is l.h.c. Thus, \( \zeta_x \) is continuous. Since, by \((\alpha)-(\gamma)\), \( CC(x) \) is a family of compact sets, \( \zeta_x \) is a continuous function from \( Y(x) \) to \( CC(x) \). Since \( Y(x) \) is compact, \( \zeta_x(Y(x)) = CC(x) \) is compact for each \( x \in [0, \pi]^n \).

Given \( CC(x, y) \in CC(x) \), let

\[
\mathcal{L}_x (CC(x, y), \supseteq) \equiv \{ CC(x, y') \in CC(x) \mid CC^o(x, y) \supseteq CC(x, y') \}.
\]

For any \( CC(x, y') \in \mathcal{L}_x (CC(x, y), \supseteq) \), there exists an \( \epsilon > 0 \) such that \( \epsilon = \min \{ \delta(b, CC(x, y')) \mid b \in \partial CC(x, y) \} \), where \( \partial CC(x, y) \) is the boundary of the set \( CC(x, y) \). Define

\[
\mathcal{O}(CC(x, y'), \frac{1}{2} \epsilon) \equiv \{ CC(x, y'') \in CC(x) \mid d(CC(x, y'), CC(x, y'')) < \frac{1}{2} \epsilon \}.
\]

By the definition of the Hausdorff metric \( d \), \( CC^o(x, y) \supseteq CC(x, y'') \) for all \( CC(x, y'') \in \mathcal{O}(CC(x, y'), \frac{1}{2} \epsilon) \). Thus, \( \mathcal{L}_x (CC(x, y), \supseteq) \) is open in the Hausdorff topology on \( CC(x) \). Then, since the relation \( \supseteq \) is transitive, hence
acyclic, there exists a $CC(x, y^*) \in CC(x)$ such that $CC(x, y') \not\supseteq CC(x, y^*)$ for all $CC(x, y') \in CC(x)$. Denote the set of such $CC(x, y')$ by $UIC(x)$ for each $x \in [0, \pi]^n$. Note that the non-empty set $UIC(x)$ is closed in $CC(x)$, and $UIC(x) \supseteq UIC(x)$. Since $CC(x)$ is compact, $UIC(x)$ is compact.

For $CC(x, y) \in UIC(x)$, if $y \in Y(x) \setminus \partial Y(x)$, then there exists $y^* \in \partial Y(x)$ such that $CC(x, y^*) \supseteq CC(x, y)$. Moreover, it is shown that $CC(x, y^*) \in UIC(x)$. Since $CC(x, y^*) \in UIC(x)$, there exists $b_i \in \partial C_i(x_i, y_i^*)$ for all $i \in N$ such that $b_i \in \partial CC(x, y^*)$. Since $y^* \in \partial Y(x)$, for any $y^{**}(\neq y^*) \in Y(x)$, there exists at least one individual $j \in N$ such that $C^o_j(x_j, y_j^*) \supseteq C^o_j(x_j, y_j^{**})$ by the property (a) of the capability correspondence. It follows that there is no $CC(x, y^{**}) \in CC(x)$ such that $CC(x, y^{**}) \supseteq CC(x, y^*)$. Thus, $UIC(x) \neq \emptyset$ and $UIC(x) \cap \zeta_x(\partial Y(x)) = UIC(x)$. Since $\zeta_x(\partial Y(x))$ is compact, $UIC(x)$ is compact. 

**Proof of Lemma 2:** Let $(x, y) \in Z(x; \delta)$, and suppose that $CC(x, y) \notin UIC(x)$. Then, there exists a proper subset $N' \subseteq N$ such that $\bigcap_{i \in N'} C_i(x_i, y_i) = CC(x, y) \subset \bigcup_{j \in N \setminus N'} C^o_j(x_j, y_j)$. Since $\pi(x, y; \delta) \cdot \delta \in \bigcap_{i \in N'} \partial C_i(x_i, y_i)$, we can consider another distribution $y' \in \partial Y(x)$ with $CC(x, y') \in UIC(x)$ such that for all $i \in N'$, $y_i < y_i'$, and for all $j \in N \setminus N'$, $y_j > y_j'$. By continuity of the capability correspondences, we can find such a distribution. Then, since $CC^o(x, y') \supseteq CC(x, y)$, we have $\pi(x, y'; \delta) \cdot \delta \gg \pi(x, y; \delta) \cdot \delta$, which is a desired contradiction. 

**Proof of Lemma 3:** (1) Let $(x, y) \in Z(x; \delta)$, and $\pi(x, y; \delta) \cdot \delta \in \partial C_i(x_i, y_i)$ for all $i \in N$. We will show that for any other $y' \in Y(x)$, $\pi(x, y'; \delta) < \pi(x, y; \delta)$. Since $y \neq y'$ and $y \in \partial Y(x)$, there exists at least one individual $j \in N$ such that $y_j > y_j'$. By monotonicity of $C_j$, $\partial C_j(x_j, y_j') \subset C^o_j(x_j, y_j)$. Thus, $\partial CC(x, y') \subset \partial C_j(x_j, y_j') \subset C^o_j(x_j, y_j)$. Since $\pi(x, y; \delta) \cdot \delta \in \partial C_j(x_j, y_j)$ and $\pi(x, y'; \delta) \cdot \delta \in \partial CC(x, y')$, we obtain the desired result.

(2) Suppose that $(x, y), (x', y') \in Z(x; \delta)$, hence $\pi(x, y; \delta) = \pi(x, y'; \delta)$. By the case (1) of the proof of this lemma, there exist proper subsets $N(x, y; \delta)$ and $N(x, y'; \delta)$ of $N$ such that $\pi(x, y; \delta) \cdot \delta \in \bigcap_{j \in N(x, y; \delta)} \partial C_j(x_i, y_i)$ (resp. $\bigcap_{j \in N \setminus N(x, y; \delta)} C_j^o(x_j, y_j')$) and $\pi(x, y'; \delta) \cdot \delta \in \bigcap_{i \in N(y; \delta)} \partial C_i(x_i, y_i)$ (resp. $\bigcap_{j \in N \setminus N(x, y; \delta)} C_j^o(x_j, y_j')$). By Lemma 2, $CC(x, y), CC(x, y') \in UIC(x)$, so that $y, y' \in \partial Y(x)$. Let us consider $y'' \in \partial Y(x)$ such that $CC(x, y'') \in UIC(x)$, $y_i < y_i''$ for all $i \in N(x, y; \delta)$, and $y_j > y_j''$ for all $j \in N \setminus N(x, y; \delta)$, and moreover $N(x, y''; \delta) \subseteq N(x, y; \delta)$. By continuity of the capability correspondences,
we can find such a distribution. Thus, by monotonicity of the capability correspondences, \( \pi(x, y'\| \delta) > \pi(x, y; \delta) \), which is a desired contradiction. Thus, \( Z(x; \delta) \) must be singleton.

(3) Suppose that \( \{ (x, y) \} = Z(x; \delta) \), but there exists \( j \in N \) such that \( \pi(x, y; \delta) \cdot \delta \in C_j(x_j, y_j) \). Then, we can consider another distribution \( y' \in \partial Y(x) \) such that for all \( i \in N(x, y; \delta) \), \( y_i < y'_j \) and \( y_j > y'_j \). Thus, we can find another \( \pi(x, y'; \delta) \) with \( \pi(x, y'; \delta) > \pi(x, y; \delta) \), which is a desired contradiction. \( \blacksquare \)

**Proof of Lemma 4:** Given \( \delta \in \Delta \), let us define an ordering \( R_\delta \subseteq Z \times Z \) as follows: for all \( z, z' \in Z \),

\[
(\mathbf{z}, \mathbf{z}') \in R_\delta \iff \pi(z; \delta) \geq \pi(z'; \delta)
\]

and \( (\mathbf{z}, \mathbf{z}') \in P(R_\delta) \iff \pi(z; \delta) > \pi(z'; \delta) \),

where \( P(R_\delta) \) is the asymmetric part of \( R_\delta \). Based upon this ordering, let \( R_\delta(x) \subseteq \{ \{x\} \times Y(x) \}^2 \) be defined by \( R_\delta(x) \equiv R_\delta \cap \{ \{x\} \times Y(x) \}^2 \) for each \( x \in [0, \pi]^n \). Note that the set of best elements for \( R_\delta(x) \) over \( \{ x \} \times Y(x) \) coincides with \( Z(x; \delta) \). Hence, without loss of generality, we use \( Z(x; \delta) \) to denote this set.

Next, given \( u \in U^n \), let us define the strict Pareto preference relation (resp. the Pareto indifference relation) \( SP_u \subseteq Z \times Z \) (resp. \( IP_u \subseteq Z \times Z \)) by \( (\mathbf{z}, \mathbf{z}') \in SP_u \iff u_i(z_i) > u_i(z'_i) \) for all \( i \in N \) (resp. \( (\mathbf{z}, \mathbf{z}') \in IP_u \iff u_i(z_i) = u_i(z'_i) \) for all \( i \in N \)). Then, define a quasi-ordering \( P_u \subseteq Z \times Z \) as \( P_u \equiv SP_u \cup IP_u \).

Given \( u \in U^n \), let us define \( R_u \subseteq Z \times Z \) as

\[
R_u \equiv \left[ P_u \cap \left( \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \right)^2 \right] \cup \left[ \bigcup_{x \in [0, \pi]^n} R_\delta(x) \right].
\]

We will show that \( R_u \) has an ordering extension. Note that for all \( x \in [0, \pi]^n \), \( Z(x; \delta) \) is singleton, so that \( P_u \cap \left( \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \right)^2 \) is an upper semi-continuous quasi-ordering over \( \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \). Thus, there exists an upper semi-continuous ordering extension \( P_u^* \subseteq \left( \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \right)^2 \) of
by the Bossert-Sprumont-Suzumura (2002; Theorem 3) extension theorem. It is easy to see that

\[
P_u \cap \left( \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \right)^2
\]

is a consistent relation, since \( R_\delta(x) \) is a quasi-ordering for each \( x \in [0, \pi]^n \). Thus, by Suzumura’s (1983) extension theorem, there exists an ordering extension \( R^*_u \subseteq Z \times Z \) of \( P_u \cap \left( \bigcup_{x \in [0, \pi]^n} R_\delta(x) \right) \).

Now, we will construct \( Q_\delta \) satisfying the two axioms on the basis of the ordering \( R^*_u \) over \( Z \). Let us define a social ordering function \( Q_\delta \) as follows: for each \( u \in U^n \) and all \( (z, \gamma), (z', \gamma') \in R(u) \),

\[
((z, \gamma), (z', \gamma')) \in Q_\delta(u) \iff (z, z') \in R^*_u,
\]

\[
((z, \gamma), (z', \gamma')) \in P(Q_\delta(u)) \iff (z, z') \in P(R^*_u).
\]

By virtue of the definition of \( R^*_u \), \( Q_\delta(u) \) is compatible with the claims of Axiom 2 and Axiom 3 \( \cap Z(\delta) \) for each \( u \in U^n \). Moreover, \( Q_\delta(u) \) is an ordering over \( Z \times \Gamma \) for each \( u \in U^n \). This completes the proof. ■

**Proof of Theorem 1:** Given \( \delta \in \Delta \), let us define an ordering function \( Q^1_\delta \) as follows: for each \( u \in U^n \) and all \( (z, \gamma), (z', \gamma') \in R(u) \),

1) if \( \gamma \in \Gamma_{LE} \) and \( \gamma' \in \Gamma \setminus \Gamma_{LE} \), then \((z, \gamma), (z', \gamma') \in P(Q^1_\delta(u))\); 2) if either \( \gamma, \gamma' \in \Gamma_{LE} \) or \( \gamma, \gamma' \in \Gamma \setminus \Gamma_{LE} \), then

\[
((z, \gamma), (z', \gamma')) \in Q^1_\delta(u) \iff (z, z') \in R^*_u,
\]

\[
((z, \gamma), (z', \gamma')) \in P(Q^1_\delta(u)) \iff (z, z') \in P(R^*_u),
\]

where \( R^*_u \) is defined in the proof of Lemma 4. Note that \( Q^1_\delta(u) \) is complete and transitive, and \( Q^1_\delta \) satisfies Axiom 1, Axiom 1 * 2, and Axiom (1 * 3) \( \cap Z(\delta) \). ■

**Proof of Theorem 2:** Let us define, for each \( u \in U^n \), a binary relation \( Q_\delta(u) \) as follows: for all \( (z, \gamma), (z', \gamma') \in R(u) \),

1) if \( z, z' \in \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \), and \( [(z, z') \notin P_u \) and \( (z', z) \notin P_u \) \], then

\[
((z, \gamma), (z', \gamma')) \in P(Q_\delta(u)) \iff [\gamma \in \Gamma_{LE} \) and \( \gamma' \in \Gamma \setminus \Gamma_{LE} \];
\]

2) otherwise,

\[
((z, \gamma), (z', \gamma')) \in Q_\delta(u) \iff (z, z') \in R^*_u,
\]

\[
((z, \gamma), (z', \gamma')) \in P(Q_\delta(u)) \iff (z, z') \in P(R^*_u).\]
Then, $Q_{\delta}(u)$ constitutes a quasi-ordering, so that there exists an ordering extension $Q^2_{\delta}(u)$ thereof, which satisfies Axiom (3 * 1) $\cap$ $Z(\delta)$, Axiom 2, and Axiom 3 $\cap$ $Z(\delta)$. $\blacksquare$

7.2 Proof of Theorem 3

Lemma 5: $\bigcup_{x \in [0, \pi]^n} Z(x; \delta)$ constitutes a closed graph in $Z$.

Proof. Let a sequence $\{(x^\mu, y^\mu)\}_{\mu=1}^{+\infty}$ be such that $(x^\mu, y^\mu) \to (x, y)$ as $\mu \to +\infty$, and $(x^\mu, y^\mu) \in Z(x^\mu; \delta)$ for every $\mu = 1, \ldots$, ad. sup. Suppose that $(x, y) \notin Z(x; \delta)$. Then, there exists $(x, y') \in Z(x; \delta)$ such that $((x, y'), (x, y)) \in P(R_{\delta})$, because $Z(x; \delta)$ is the set of maximal element of $R_{\delta}$ over $\{x\} \times \partial Y(x)$. Since $Y$ is l.h.c., there exists a sequence $\{(x^\mu, y^\mu)\}_{\mu=1}^{+\infty}$ such that $(x^\mu, y^\mu) \in \{x^\mu\} \times Y(x^\mu)$ for every $\mu = 1, \ldots$, ad. inf., and $(x^\mu, y^\mu) \to (x, y') (\mu \to +\infty)$. Then, for a large enough $\mu$, $(x^\mu, y^\mu), (x^\mu, y^\mu) \in P(R_{\delta})$ by continuity of $R_{\delta}$, which is a contradiction. Thus, $\bigcup_{x \in [0, \pi]^n} Z(x; \delta)$ is closed in $Z$. $\blacksquare$

Proposition 3 [Yoshihara (2000)]: Let Assumption 1 hold. Let $h : [0, \pi]^n \to \mathbb{R}^n_+$ be a continuous function such that, for each $x \in [0, \pi]^n$, $h(x) = y$ and $y \in \partial Y(x)$. Then, for any $u \in U^*$, there exists $x^* \in [0, \pi]^n$ such that $(x^*, h(x^*))$ is a Pareto efficient allocation for $u$.

Proof. Given $u \in U^*$, let $S(u)$ be the utility possibility set of feasible allocations, and $\partial S(u)$ be its boundary. Since every utility function is strictly increasing, $\partial S(u)$ is the set of Pareto efficient utility allocations.

By Assumption 1, $0 \notin \partial S(u)$, so for every $\pi = (\pi_i)_{i \in N} \in \partial S(u)$, $\Sigma \pi_h > 0$. Then, the mapping

$$\hat{\nu} : \partial S(u) \to \Delta^{n-1} \text{ via } \hat{\nu}(u) = \frac{\pi}{\Sigma \pi_h}$$

is well-defined and continuous on $\partial S(u)$, where $\Delta^{n-1}$ is an $n-1$-dimensional unit simplex. By [Arrow and Hahn (1971); Lemma 5.3., p.114], $\hat{\nu}$ is a homeomorphism. Denote its inverse by $\hat{u}$. Define a correspondence

$$\widehat{W} : \Delta^{n-1} \ni Z \text{ by } \widehat{W}(\hat{\nu}(v)) \equiv \{z \in Z \mid u_i(z_i) \geq \hat{u}_i(v_i) (\forall i \in N)\}.$$ 

By [Arrow and Hahn (1971); Theorem 4.5., Corollary 5., p.99], $\widehat{W}$ is upper hemi-continuous with non-empty compact convex values.
Given a continuous function \( h \) and \( z = (x_i, y_i)_{i \in N} \in Z \), let \( E_i(x, y_i) = h_i(x) - y_i \). Then, we define an optimization problem as:

\[
\max_{v \in \Delta^{n-1}} \sum_{i \in N} v_i \cdot E_i(x, y_i).
\]

By Berge’s maximum theorem, we can define an upper hemi-continuous correspondence \( \Theta : Z^3 \to \Delta^{n-1} \) via

\[
\Theta(z) \equiv \{ v^* \in \Delta^{n-1} \mid v^* \in \arg \max_{v \in \Delta^{n-1}} \sum_{i \in N} v_i \cdot E_i(x, y_i) \}.
\]

Note that \( \Theta \) is non-empty compact and convex-valued.

Now, we define a correspondence \( \Phi : \Delta^{n-1} \times Z^3 \to \Delta^{n-1} \times Z \) via

\[
\Phi(v, z) \equiv \Theta(z) \times \widehat{W}(\hat{u}(v)),
\]

which is upper hemi-continuous with non-empty compact convex values. By Kakutani’s fixed point theorem,

\[
\exists (v^*, z^*) \in \Delta^{n-1} \times Z \text{ s.t. } (v^*, z^*) \in \Phi(v^*, z^*).
\]

By definition, \( \hat{u}(v^*) = (u_i(z^*_i))_{i \in N} \), so that \( z^* \) is Pareto efficient for \( u \). Finally, we will show that \( z^* = (x^*, h(x^*)) \). To do this, it is sufficient to show \( E_i(x^*, y^*_i) = 0 \) for all \( i \in N \). Assume that there exists \( j \in N \) such that \( E_j(x^*, y^*_j) > 0 \). Then, since \( x^* \) is Pareto efficient, there exists \( l \in N \) such that \( E_l(x^*, y^*_l) < 0 \). To maximize \( \sum v_i \cdot E_i(x, y_i) \), we obtain \( v^*_i = 0 \). Then, \( u_i(z^*_i) = \hat{u}_i(v^*_i) = 0 \), so that, by the strict monotonicity of \( u_i \) and Assumption 1, we obtain either (1) \( x^*_i = \text{I} \) and \( y^*_i > 0 \), or (2) \( x^*_i \leq \text{I} \) and \( y^*_i = 0 \).

First, (1) is impossible because \( z^* \) is Pareto efficient for \( u \). In fact, the vector (1, 0) is a unique price which supports \( z^*_i \) of case (1) as an expenditure minimizing consumption. However, (1, 0) cannot be consistent with any profit maximizing production except the origin. Second, (2) implies \( E_l(x^*, y^*_l) \geq 0 \). This is a contradiction, so \( E_i(x^*, y^*_i) = 0 \) for all \( i \in N \). 

\[\square\]

**Lemma 6:** Let Assumption 1 hold. Then, for each \( u \in U^n \), there exists a Pareto efficient allocation \( z^* \in Z \) such that \( z^* \in \bigcup_{x \in [0, x^n]} Z(x; \delta) \).

**Proof.** Let us define a correspondence \( h_\delta : [0, \overrightarrow{x^n}]^3 \to Y([0, \overrightarrow{x^n}]) \) such that for each \( x \in [0, \overrightarrow{x^n}, \{x \} \times h_\delta(x) = Z(x; \delta) \). Since \( Y([0, \overrightarrow{x^n}) \) is compact, \( h_\delta \) is
**Proposition 3**

Under **Assumption 1**, we can obtain the desired result by the application of Proposition 3.

**Proof of Theorem 3:** Given $\delta \in \Delta$, for each $u \in U^i$, let $PM(\delta : u) \equiv PO(u) \cap \left\{ \bigcup_{x \in [0, \pi]^n} Z(x; \delta) \right\}$. Note that if $z = (x, y) \in PM(\delta : u)$, then $h_\delta(x) = y$. Moreover, it follows that $([0, \pi]^n, h_\delta) \in \Gamma_{LE}$. The equal treatment property of this game form follows from the property of $Z(\cdot; \delta)$: if $\{(x, y)\} = Z(x; \delta)$, and $C_i = C_j$ and $x_i = x_j$ for some $i, j \in N$, then $y_i = y_j$ must follow. For $h_\delta$, we sometimes use notation like $h_{\delta,i}(x)$, which refers to $i$-th component of the vector $h_\delta(x)$.

**Step 1:** For each $\hat{z} \in PM(\delta : u)$, we will construct an outcome function $h^2 : [0, \pi]^n \to \mathbb{R}_+$, such that $([0, \pi]^n, h^2) \in \Gamma_{LE}$ and $\hat{z} \in A_{NE} \left( ([0, \pi]^n, h^2) , u \right)$.

By **Assumption 2**, we can define a continuous function $f'_i : [0, \pi]^n \to \mathbb{R}_+$ by $f'_i(x) \equiv \frac{\partial f_i(x)}{\partial x}$ for all $x \in [0, \pi]^n$. Given $u \in U^n$, $\hat{z} = (\hat{x}, \hat{y}) \in PM(\delta : u)$, and $i, j \in N$, let

$$\lambda^i_j(\hat{x}) \equiv \begin{cases} \frac{\hat{y}_i + f'_i(\hat{x}) \cdot (x_i - \hat{x}_i) - h_\delta,i(\hat{x}^{-i}, \hat{x}_i)}{(x_i - \hat{x}_i)^2} & \text{if } \hat{x}_j \neq \hat{x}_i, \\ 0 & \text{if } \hat{x}_j = \hat{x}_i. \end{cases}$$

Given $u \in U^n$, $\hat{z} \in PM(\delta : u)$, and $x \in [0, \pi]^n$, let, for each $i \in N$,

$$\Psi_i(x) = \begin{cases} \hat{y}_i + f'_i(\hat{x}) \cdot (x_i - \hat{x}_i) & \text{if } x_i \in (\hat{x}_i - \varepsilon_i(\hat{x}), \hat{x}_i + \varepsilon_i(\hat{x})) \\ \hat{y}_i + \left[ f'_i(\hat{x}) \cdot (x_i - \hat{x}_i) - \mu^i_j(x_i)(\hat{x}) \cdot (x_i - \hat{x}_i)^2 \right] & \text{otherwise}, \end{cases}$$

where

$$\varepsilon_i(\hat{x}) \equiv \min_{j \neq i, \hat{x}_j \neq \hat{x}_i} \| \hat{x}_j - \hat{x}_i \|, \quad j^*(x_i) = \max_{j \neq i} \left\{ \arg \min_{j \neq i} \| \hat{x}_j, x_i \| \right\},$$

and

$$\mu^i_j(x_i)(\hat{x}) = \begin{cases} \lambda^i_j(x_i)(\hat{x}) & \text{if } 0 \leq \lambda^i_j(x_i)(\hat{x}) \\ 0 & \text{otherwise}. \end{cases}$$

By construction of $\Psi_i(x)$, we have (i) $\Psi_i(x) = \hat{y}_i$ if $x_i = \hat{x}_i$; (ii) $\Psi_i(x) \leq \hat{y}_i + f'_i(\hat{x}) \cdot (x_i - \hat{x}_i)$ if $x_i \neq \hat{x}_i$; and (iii) $\Psi_i(x) = \min\{ h_\delta,i(\hat{x}^{-i}, x_i), \hat{y}_i + f'_i(\hat{x}) \cdot (x_i - \hat{x}_i) \}$ if $x_i = \hat{x}_i$ for some $j \neq i$.

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21 The way to construct $\gamma^*$ below comes from Yoshihara (2000).
Let, for each \( i \in N \),
\[
\zeta_i(x) = \min \{ \max \{ 0, \Psi_i(x) \}, f(x) \}.
\]
Moreover, let, for each \( i \in N \) and each \( x \in [0, \overline{x}]^n \),
\[
n(x, x_i) \equiv \# \{ j \in N \mid x_j = x_i \}.
\]
That is, \( n(x, x_i) \) is the number of individuals who provide the same labor supply as \( x_i \) under \( x \in [0, \overline{x}]^n \). Then, given \( u \in U^n \) and \( \hat{z} \in PM(\delta : u) \), define a function \( h^2 : [0, \overline{x}]^n \to \mathbb{R}_+^n \) as follows: for each \( x \in [0, \overline{x}]^n \), for each \( i \in N \),
\[
h^2_i(x) = \begin{cases} 
\zeta_i(x) & \text{if } \forall j \neq i, x_j = \hat{x}_j, \\
[f(x) - n(x, x_j) \cdot \zeta_j(x)] \cdot \frac{1}{n-n(x, x_j)} & \text{if } \exists j \neq i, \forall l \neq j, x_l = \hat{x}_k, \& x_j \neq \hat{x}_i \\
\zeta_j(x) & \text{if } \exists j \neq i, \forall l \neq j, x_l = \hat{x}_l, \& x_j = \hat{x}_i \\
h_{\delta,i}(x) & \text{otherwise}.
\end{cases}
\]

This \( h^2 \) has the following properties: (I) \( ([0, \overline{x}]^n, h^2) \in \Gamma_{LE} \); and (II) \( \hat{z} \in A_{NE}([0, \overline{x}]^n, h^2, u) \) whenever \( \hat{z} \in PM(\delta : u) \). The equal treatment property of \( ([0, \overline{x}]^n, h^2) \) follows from the property (iii) of \( (\Psi_i)_{i \in N} \) and \( h_\delta \). The property (II) follows from the property (ii) of \( (\Psi_i)_{i \in N} \).

**Step 2:** We will construct two outcome functions \( h^0 \) and \( h^m \) such that \( ([0, \overline{x}]^n, h^0), ([0, \overline{x}]^n \times \mathbb{R}_+^n, h^m) \in \Gamma_{LE} \).

Let us introduce two functions \( h^0 \) and \( h^m \) as follows:

(1) \( h^0 : [0, \overline{x}]^n \to \mathbb{R}_+^n \) via, for each \( (x, y) \in [0, \overline{x}]^n \times \mathbb{R}_+^n \), \( h^0(x) = 0 \), and

(2) \( h^m : [0, \overline{x}]^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n \) via, for each \( (x, y) \in [0, \overline{x}]^n \times \mathbb{R}_+^n \), and for all \( i \in N \),
\[
h^m_i(x, y) = \begin{cases} 
\frac{f(x)}{\#(\max N^m(x,y))} & \text{if } i \in N^m(y) \\
0 & \text{if } i \notin N^m(y)
\end{cases},
\]
where \( N^m(y) \equiv \{ i \in N \mid \forall j \in N : y_i \geq y_j \} \).

It is clear that \( ([0, \overline{x}]^n, h^0), ([0, \overline{x}]^n \times \mathbb{R}_+^n, h^m) \in \Gamma_{LE} \). Note that for any \( u \in U^n \), there is no Nash equilibrium for the game defined by \( ([0, \overline{x}]^n \times \mathbb{R}_+^n, h^m) \).
Step 3: We will construct a game form \( \gamma^* = ([0, \overline{\pi}] \times \mathbb{R}^+)^n, g^* \), in which \( g^* \) is defined by using \( \{h^2\}_{\mu \in \bigcup \mathcal{PM}(\delta, \mu)} \), \( h^0 \), and \( h^m \).

Given \( x \in [0, \overline{\pi}]^n \) and \( y \in \mathbb{R}_+^n \), let \( \rho(x, y : \delta) \equiv \{ u \in \mathcal{U}^n : (x, h_\delta(x)) \in PM(\delta : u) \& h_\delta(x) = y \} \). Let us call \( (x, y) \in Z \) a potential PM\( \delta \)-allocation if \( \rho(x, y : \delta) \neq \emptyset \). Given \( x \in [0, \overline{\pi}]^n \) and \( y \in \mathbb{R}_+^n \), let

\[
N(x, y) \equiv \{ l \in N | \exists (x'_l, y'_l)(\neq (x_l, y_l)) \in [0, \overline{\pi}] \times \mathbb{R}_+^n : \rho((x'_l, x_{-l}), (y'_l, y_{-l}) : \delta) \neq \emptyset \}.
\]

The set \( N(x, y) \) will be used in defining \( \gamma^* \) below as the set of “potential deviators.” That is, if \( \rho(x, y : \delta) = \emptyset \), and there is some \( j \in N(x, y) \), then this \( j \) may be interpreted as deviating from \( PM(\delta : u) \) for some \( u \in \mathcal{U}^n \).

Given \( j \in N, x \in [0, \overline{\pi}]^n \), and \( y \in \mathbb{R}_+^n \), let

\[
X_j(x_{-j}, y_{-j}) \equiv \{ x'_j \in [0, \overline{\pi}] \mid \rho((x'_j, h_{\delta,j}(x'_j, x_{-j})), (x_{-j}, y_{-j}) : \delta) \neq \emptyset \}, \text{ and}
\]

\[
Z(x_j, x_{-j}, y_{-j}) \equiv \{ ((x'_j, h_{\delta,j}(x'_j, x_{-j})), (x_{-j}, y_{-j})) \in Z \mid x'_j \in X_j(x_{-j}, y_{-j}) \}.
\]

Moreover, given \( j \in N, x \in [0, \overline{\pi}]^n \), and \( y \in \mathbb{R}_+^n \), let

\[
\tilde{\zeta}(x_j, x_{-j}, y_{-j}) \equiv \arg \min_{(x'_j, y'_j) \in Z(x_j, x_{-j}, y_{-j})} y'_j + f'_j(x'_j, x_{-j}) \cdot (x_j - x'_j).
\]

Note that the above three notations will be used in defining \( \gamma^* \) below to punish a unique potential deviator. If \( \rho(x, y : \delta) = \emptyset \) and \( \{ j \} = N(x, y) \), then we can identify \( j \) as the unique potential deviator. Then, by definition of \( N(x, y) \), \( X_j(x_{-j}, y_{-j}) \) is non-empty, so that \( Z(x_j, x_{-j}, y_{-j}) \) is non-empty. Note that \( Z(x_j, x_{-j}, y_{-j}) \) is the set of potential PM\( \delta \)-allocations which would be implemented if \( j \) were not to deviate. Then, by selecting \( \tilde{\zeta}(x_j, x_{-j}, y_{-j}) \) from this set, we will consider the outcome function \( g^* \) in order to punish \( j \) in such a situation.

Given \( x \in [0, \overline{\pi}]^n \), let \( N^0(x) \equiv \{ i \in N \mid x_i \in [0, \overline{\pi}] \} \), max \( N^0(x) \equiv \{ i \in N^0(x) \mid \exists j \in N^0(x) \text{ s.t. } x_j > x_i \} \), and max \( N(x) \equiv \{ i \in N' \mid \exists j \in N' \text{ s.t. } x_j > x_i \} \). Now, let us define a labor sovereign and equal treatment of equals rule \( \gamma^* = \{ ([0, \overline{\pi}] \times \mathbb{R}_+)^n, g^* \} \) in the following way: given \( \delta \in \Delta \) and a strategy profile \( (x, y) \in ([0, \overline{\pi}] \times \mathbb{R}_+)^n \),

**Rule 1:** If \( \rho(x, y : \delta) \neq \emptyset \), then \( g^*(x, y) = (x, h^2(x)) \), where \( \tilde{\zeta} = (x, h_\delta(x)) \).

**Rule 2:** If \( \rho(x, y : \delta) = \emptyset \) and there exists a non-empty \( N(x, y) \), then

2-1: if \( |N(x, y)| > 1 \), then \( g^*(x, y) = (x, h^0(x)) \),

32
2-2: if \( N(x, y) = \{j\} \), then \( g^*_j(x, y) = (x_j, h^*_j(x)\) \( \times -i, y-i)\) \( (x) \) \( \) and for all \( i \neq j \),

\[
g^*_i(x, y) = \begin{cases} 
(x_i, h^*_i(x)) & \text{if } \{j\} = \max N(x) \cap \max N^0(x) \\
(x_i, h^*_j(x)\) \( \times -i, y-i)\) \( (x) \) & \text{otherwise.}
\end{cases}
\]

**Rule 3:** In all other cases, \( g^*(x, y) = (x, h^m(x, y)) \).

In this \( \gamma^* \), if a strategy profile \( (x, y) \) is consistent with a potential \( PM^\delta \)-allocation, then Rule 1 applies, and \( (x, y) \) becomes the outcome; if \( (x, y) \) is inconsistent with any potential \( PM^\delta \)-allocation, and a unique potential deviator \( j \) is identified, then Rule 2-2 applies, and sets some potential \( PM^\delta \)-allocation \( \tilde{z}(x_j, x_{-j}, y_{-j}) = (\tilde{z}_j, (x_{-j}, y_{-j})) \), which would be the outcome if \( j \) were not to deviate. Thus, \( j \) is punished by \( h^*_{\gamma_j}(x) \) under Rule 2-2. If \( (x, y) \) corresponds to neither of the above two cases, then \( h^m \) or \( h^0 \) is applied in order to punish all potential deviators. It is clear that \( \gamma^* \in \Gamma_{LE} \), since in every case of strategy profile, the value of \( g^* \) is that of either \( h^m \), \( h^m \), or \( h^2 \)-types.

**Step 4:** We will show that \( A_{NE}(\gamma^*, u) = PM(\delta : u) \) for all \( u \in \mathcal{U}^n \).

(1) First, we show that \( A_{NE}(\gamma^*, u) \supseteq PM(\delta : u) \) for all \( u \in \mathcal{U}^n \). Let \( z = (x, y) \in PM(\delta : u) \). Then, if a strategy profile of every agent is \( (x_i, y_i) \in \mathfrak{N} \in [0, \mathfrak{M}]^n \times \mathbb{R}^n_+ \), then \( g^*(x, y) = (x, h^\delta(x)) = (x, h^\delta(x)) = (x, y) \) by Rule 1. Suppose that individual \( j \in N \) deviates from \( (x_i, y_i) \) to \( (x'_j, y'_j) \). Then, if \( j \) induces Rule 2-1, then \( g^*_j((x'_j, x_{-j})), (y'_j, y_{-j})) = 0 \). If \( j \) induces Rule 1, he cannot be better off, since \( y'_j = f(x'_j, x_{-j}) - \sum_{i \neq j} y_i \), and \( z \) is Pareto efficient for \( u \). If \( j \) induces Rule 2-2, then \( g^*_j((x'_j, x_{-j})), (y'_j, y_{-j})) = (x'_j, h^\delta_j(x'_j, x_{-j})), (y'_j, y_{-j})) \), since \( z \in Z(x'_j, x_{-j}, y_{-j}) \), it follows that if \( \tilde{z}(x'_j, x_{-j}, y_{-j}) = ((\tilde{x}_j, x_{-j})), (y_j, y_{-j})) \), then

\[
h^\delta_j(x'_j, x_{-j}))(x'_j, x_{-j}) \leq \tilde{y}_j + f'_j((\tilde{x}_j, x_{-j}))) \cdot (x'_j - \tilde{x}_j) \leq y_j + f'_j(x) \cdot (x'_j - x_j).
\]

This implies that \( j \) cannot be better off by this deviation. Note that \( j \) cannot induce Rule 3. Thus, \( A_{NE}(\gamma^*, u) \supseteq PM(\delta : u) \) holds.

(2) Second, show \( A_{NE}(\gamma^*, u) \subseteq PM(\delta : u) \). Let \( (x_i, y_i) \in \mathcal{N} \) be a Nash equilibrium of the game \( (\gamma^*, u) \). Note that \( (x, y) \) cannot correspond to Rule 3. This is because every agent \( j \) can get everything in Rule 3 by changing from \( y_j \) to a large enough \( y'_j > \max \{\max \{y_i\}_{i \neq j} \}, f(x) \} \). Also, \( (x, y) \) cannot correspond to Rule 2-2, since, in Rule 2-2, there is an agent
\( j \in N \setminus N(x, y) \) who can induce Rule 3 by changing from \( y_j \) to a large enough \( y_j' > \max \{ \max \{ y_i \}_{i \neq j}, f(x) \} \), thereby he can get everything. Finally, \((x, y)\) cannot also correspond to Rule 2-1, since every agent \( l \in N(x, y) \) can induce Rule 2-2 by changing from \( y_l \) to \( y_l' = f(x) + \varepsilon \), so that \( l \) can obtain positive output.

Suppose that \((x, y)\) corresponds to Rule 1. Then, for \( z = (x, y) = (x, h_\delta(x)) \), $g^*(x, y) = (x, h_\delta(x)) = z \in A_{NE}(\gamma^*, u)$. Suppose that \( z \) is not Pareto efficient. Then, there is at least one individual \( i \in N \) who changes slightly from \( x_i \) to \( x_i' \), so that $u_i(x_i, h_\delta^2(x)) < u_i(x_i', h_\delta^2(x_i', x_{-i}))$. Note that if $x_i' = x_i + \varepsilon$ where the value $| \varepsilon |$ is small enough, then $h_i^2(x_i', x_{-i}) = y_i + f_i'(x) \cdot (x_i' - x_i)$, and $\hat{z}(x_i', x_{-i}, y_{-i}) = z$ also holds by the concavity of \( f \). Thus, by changing from \((x_i, y_i)\) to \((x_i', y_i')\), where $y_i' = f(x_i', x_{-i}) + \varepsilon$, \( i \) can induce Rule 2-2 and obtain $g_i^*((x_i', x_{-i})), (y_i', y_{-i})) = (x_i', h_\delta^2(x_i', x_{-i}))$. This is a contradiction, since $z \in A_{NE}(\gamma^*, u)$. Thus, \( z \) is Pareto efficient for \( u \). \( \blacksquare \)

### 7.3 Proof of Theorem 4 and Theorem 5

**Proof of Theorem 4:** Given $u \in \mathcal{U}^n$ and $\bigcup_{x \in \mathbb{R}^n} Z(x; \delta)$, since $R_\delta(x)$ is consistent with Axiom 1 \& 2, we have $PM(\delta : u) \neq \emptyset$ by the proof of Theorem 3.

Now we define an ordering $V(u)$ over $S(u)$ as follows:

1) if $\mathbf{u}, \mathbf{u}' \in \partial S(u)$, then $(\mathbf{u}, \mathbf{u}') \in I(V(u))$,
2) for any $\mathbf{u}, \mathbf{u}' \in S(u)$, there exist $\alpha, \alpha' \in [0, 1]$ such that $\alpha \cdot \mathbf{u}, \alpha' \cdot \mathbf{u}' \in \partial S(u)$ and $(\mathbf{u}, \mathbf{u}') \in V(u)$ if and only if $\alpha \geq \alpha'$. This ordering $V(u)$ is continuous over $S(u)$.

Let us define a complete ordering $R_{u, \delta}$ over $\bigcup_{x \in \mathbb{R}^n} Z(x; \delta)$ as follows: for any $z, z' \in \bigcup_{x \in \mathbb{R}^n} Z(x; \delta)$,

\[(z, z') \in R_{u, \delta} \iff (u(z), u(z')) \in V(u)\]

This ordering is continuous by the continuity of utility functions, and is consistent with Axiom (1 \& 3) \& $Z(\delta)$ by the definition of $V(u)$.

Thus, it is easy to see that $R_{u, \delta} \cup \left[ \bigcup_{x \in \mathbb{R}^n} R_\delta(x) \right]$ is consistent over $Z$, so that there exists an ordering extension $R_{u, \delta}^*$ of $R_{u, \delta} \cup \left[ \bigcup_{x \in \mathbb{R}^n} R_\delta(x) \right]$ by
Suzumura’s (1983) extension theorem. Based upon the ordering $R^*_u$ over $Z$, we can construct a social ordering function $Q^*_u$ satisfying Axiom 1, Axiom 1 * 2, and Axiom (1 * 3) $\cap Z(\delta)$ as in the proof of Theorem 1.

As $(\bigcup_{x \in [0,T]} Z(x; \delta))$ is compact and $R^*_u$ is continuous over $(\bigcup_{x \in [0,T]} Z(x; \delta))$, there exists a non-empty best element set $B(R^*_u, \delta)$ of $(\bigcup_{x \in [0,T]} Z(x; \delta))$ with respect to $R^*_u, \delta$ for each $u \in \mathcal{U}$. Moreover, by definition, $B(R^*_u, \delta) = PM(\delta : u)$. As shown in the proof of Theorem 3, there exists a labor sovereign and equal treatment of equals allocation rule $\gamma^*$ such that for each $u \in \mathcal{U}$, $A_{NE}(\gamma^*, u) = PM(\delta : u)$. Thus, by the construction of $Q^*_u$, for all $u \in \mathcal{U}$ and all $z \in B(R^*_u, \delta), (z, \gamma^*)$ is a best element of $Q^*_u(u)$, which implies that $Q^*_u(u)$ uniformly rationalizes $\gamma^*$.

**Proof of Theorem 5:** Given $u \in \mathcal{U}$ and $(\mathcal{U} \cap Z(\delta))$, where $R^*_u(x)$ is consistent with Axiom (1 * 2) $\cap PO$, we have $PM(\delta : u) \neq \emptyset$, by the proof of Theorem 3. Let us define a quasi-ordering $R^*_u \subseteq Z \times Z$ as follows:

1): $P_u \subseteq R^*_u$,

2): for all $z, z' \in PO(u)$, if $x = x'$, then $(z, z') \in R^*_u \iff \pi(z : \delta) \geq \pi(z' : \delta)$ and $(z, z') \in P(R^*_u) \iff \pi(z : \delta) > \pi(z' : \delta)$,

3): for all $z, z' \in PO(u)$, if $z \in PM(\delta : u)$ and $z' \notin PM(\delta : u)$, then $(z, z') \in P(R^*_u)$,

4): for all $z, z' \in PM(\delta : u)$, $(z, z') \in I(R^*_u)$.

Note that 1), 2), 3), and 4) are mutually compatible cases. Thus, it is easy to see that $R^*_u$ is transitive, so that there exists an ordering extension $R^*_u$ of $R^*_u$.

Based upon the ordering $R^*_u$ over $Z$, we can construct an ordering function $Q^*_u$ satisfying Axiom 1, Axiom (1 * 2) $\cap PO$, and Axiom 1 * 3 as in the proof of Theorem 1. Since $PM(\delta : u) \neq \emptyset$, we can varify that there exists a non-empty best element set $B(R^*_u, \delta) = PM(\delta : u)$ of $(\bigcup_{x \in [0,T]} Z(x; \delta))$ with respect to $R^*_u$ for each $u \in \mathcal{U}$. As shown in the proof of Theorem 3, there exists a labor sovereign and equal treatment of equals allocation rule $\gamma^*$ such that for each $u \in \mathcal{U}$, $A_{NE}(\gamma^*, u) = PM(\delta : u)$. Thus, by the construction of $Q^*_u$, for all $u \in \mathcal{U}$ and all $z \in B(R^*_u, \delta), (z, \gamma^*)$ is a best element of $Q^*_u(u)$, which implies that $Q^*_u(u)$ uniformly rationalizes $\gamma^*$.

**References**

35


Figure 3: Logical implications among Axioms

Axiom 1 ⇒ Axiom (2 * 3) * 1 ⇒ Axiom (3 * 1) ∩ Z(δ)
Axiom 1 ⇒ Axiom (3 * 2) * 1 ⇒ Axiom (2 * 1) ∩ PO

Axiom 2 ⇒ Axiom 1 * 2 ⇒ Axiom (1 * 3) * 2 ⇒ Axiom (1 * 2) ∩ PO
Axiom 2 ⇒ Axiom 3 * 2 ⇒ Axiom 2 ∩ PO ⇒ Axiom (1 * 2) ∩ PO

Axiom 3 ⇒ Axiom 1 * 3 ⇒ Axiom (1 * 2) * 3 ⇒ Axiom (1 * 3) ∩ Z(δ)
Axiom 3 ⇒ Axiom 2 * 3 ⇒ Axiom 3 ∩ Z(δ) ⇒ Axiom (1 * 3) ∩ Z(δ)