On the General Existence of Pure Strategy Political Competition Equilibrium in Multi-dimensional Party-Faction Models

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Abstract

In this paper, we consider political competition games of two parties in multi-dimensional policy spaces, where the two parties have two factions, opportunists and militants, that *intra*-party bargain with each other. In such a game, we adopt the *party-unanimity Nash equilibrium* (**PUNE**) [Roemer (1998; 1999; 2001)] as an appropriate solution concept, and examine the general existence problem of this. In particular, we suppose that any faction of each party does not necessarily have dictatorial power. We then provide a general existence theorem for **PUNE** in this class of games.

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1 Introduction

Until recently, most formal political analyses of party competition have assumed that both parties are Downsian [Downs (1957)]—that is, their objective is to maximize the probability of winning office. As an alternative, there exists a growing body of literature concerning the competition between *partisan* parties (those that have policy preferences)—see, e.g., Wittman (1973) and Roemer (1997). Almost all of these analyses, however, have assumed that the policy space is uni-dimensional. Moreover, it is well-known that neither Downsian nor partisan party models can reliably produce a Nash equilibrium (in pure strategies) whenever, as discussed in Roemer (2001), the policy space is multi-dimensional. Importantly, in many real political competition contexts, we may naturally assume more than a single policy issue, and need to address the issue of non-existence. Given this, there are perhaps at least three possible ways forward: the first is to allow for mixed strategies, the second is to change the game into a stage game and use some variant of a subgame perfect equilibrium, and the third possibility is to adopt the Besley-Coate-Osborne-Slivinski notion of citizen-candidate equilibrium [Besley and Coate (1997); Osborne and Slivinski (1996)]. However, in the case of competing political parties, playing mixed strategies is difficult to interpret, and it is not always the case that a policy contest takes place as a stage game between a challenger and an incumbent. Moreover, in the model of citizen-candidate equilibrium, the "citizen candidates" cannot commit to any policy but their own ideal policy: there are essentially no parties in that model. Thus, in multi-dimensional party competition games with simultaneous moves, it is still important to investigate another solution for the non-existence of pure strategy equilibrium.

It is Roemer (1998; 1999; 2001) who proposed a new equilibrium concept, known as *party-unanimity Nash equilibrium* (**PUNE**) for these political games. This is where the notion of a Nash equilibrium in a simultaneous move game between the parties is retained, but their preferences are replaced with *incomplete preferences*: put differently, each party's preference is a quasi-ordering. The model introduces the idea that the decision makers in parties have different interests. In this approach, the activists in each party are divided into one of three *factions*: the *Opportunist*, the *Militant*, and the *Reformist*. The Opportunist is solely concerned with winning office, the Militant is only concerned with publicizing the party's view, and the Reformist is concerned with the expected welfare of the party's members. Given the structure of the three factions within a party, how then does the party make policy decisions in the electoral context? Roemer (1998; 1999; 2001) proposed the following scenario. The three factions of each party should bargain on the policy proposal, given a policy proposal by its opponent party, and if a policy proposal agreed on in this party is Pareto efficient for the three factions, this is the solution for the bargaining problem within the party. A **PUNE** is then a pair of policy proposals, each component of which is the result of *intra*-party bargaining when facing the other party's proposal.[1] [2]

In this paper, we consider a general existence problem of **PUNE**s in multi-dimensional political competition games. It is worth noting that there are a few studies, such as Roemer (1998; 1999) and Yoshihara (2008), that show the existence of **PUNE**s in some specific types of multi-dimensional political games. Moreover, Roemer (2001: Section 13.7) discussed the existence of **PUNE**s in general multi-dimensional political games. However, this existence theorem refers only to a specific type of **PUNE** in which the Militants are assumed to have dictatorial powers in both parties. It is easy to see the general existence of this sort of **PUNE**, because the pair of Militants' *ideal policies* of both parties constitutes a **PUNE** (which we call an M-**PUNE** below).

Thus, we still have the following open question concerning the general existence of **PUNE**s: If the Militants of both parties are not assumed to have dictatorial powers, under what general conditions is the existence of **PUNE**s guaranteed? This problem is worth investigating, because this premise appears to be more natural and general as far as real politics is concerned. At the same time, however, the premise makes the problem more difficult.[3]

[1] A possible criticism against the notion of **PUNE** would be the following one: the condition that every Pareto efficient outcome with respect to the factions within a party could be an equilibrium strategy is not persuasive: a better approach would be to determine relative bargaining powers endogenously in a non-cooperative bargaining framework, so that an appropriate Pareto efficient outcome within the party is selected as an equilibrium strategy. That approach, however, should be more difficult to guarantee the existence of equilibrium than the case of the Wittman model. Note that the Wittman equilibrium is a refinement of **PUNE**, in which the relative bargaining power is exogenously given as the symmetric one.

[2] Another justification for **PUNE** is that there are a number of applications, as in Roemer and Silvestre (2002) and in Roemer, Lee, and Van der Straeten (2007), in which **PUNE**s are computed with real data, and those computations suggest that the **PUNE** model can well explain real politics. For detail, see Roemer (2004).

[3] In fact, Roemer (2001; Section 13.7, pp. 277-279) also wrote: "is there an interesting

One reason is that, proving the existence of a **PUNE** is not enough, because we already know, as discussed above, that an M-**PUNE** exists. So, we need to construct an argument which proves the existence of *another* **PUNE** (which we will call C-**PUNE** below) besides the M-**PUNE**. The second reason is that we cannot adopt the strategy of finding a sufficient condition of the model parameters to straightforwardly apply Kakutani's fixed point theorem. This is because if the Militants of both parties are not dictators, then each party's best response strategy should be a compromise among the factions, particularly reflecting the Opportunists' objectives. However, the objective function of the Opportunist—the probability of winning the election—is neither generally continuous nor quasi-concave.[4]

In this paper, we provide sufficient conditions for the existence of C-**PUNE**s. In this existence problem, we apply the *Urai-Hayashi fixed point theorem* [Urai and Hayashi (2000)], as it does not require the convex valuedness of correspondences. The sufficient condition contains the following: Each party's preference is represented by a continuous and strictly quasi-concave function, and aggregate uncertainty over voters' behavior is sufficiently large. Such a condition appears natural and would be satisfied in many political environments.

The paper is structured as follows. Section 2 defines a basic model of multi-dimensional political games, and introduces the **PUNE** and its refinements (M-**PUNE** and C-**PUNE**). Section 3 discusses the existence of C-**PUNE**s. Finally, section 4 provides some concluding remarks.

2 Model

Let the continuum of voter types be $H \subseteq \mathbb{R}^k$, the policy space be $T \subseteq \mathbb{R}^n$, a continuous probability measure of voter types in the polity be $\mathbb{F}$ on H, and the utility function of type $h \in H$ over policies be $v(\cdot, h)$. Let $v(\cdot, h)$ be a non-negative real valued function, which is continuous, concave, and

general existence theorem for party-unanimity Nash equilibrium? I conjecture there is not. $\cdots$ What we really desire is a theorem asserting the existence of a **PUNE** in which no faction is at its ideal point. But that appears to be hard to come by. $\cdots$ It is probably very difficult to find interesting sufficient conditions for the existence of (non-trivial) **PUNE**s."

[4] This is a problem similar to the existence of Wittman equilibrium. There are two existence theorems of Wittman equilibrium in uni-dimensional policy spaces, Roemer (1997) and Terai (2006), but both of them depend upon unsatisfactory specific assumptions for the distribution of voter types.

strictly quasi-concave on T, for any $h \in H$. Let $(t^1, t^2) \in T \times T$ be a pair of policies. The set of voters who prefer t^1 to t^2 is denoted by $\Omega(t^1, t^2) \equiv \{h \in H \mid v(t^1, h) > v(t^2, h)\}$. Now, we impose the following assumption:

Assumption 1 (A1): *The measure $\mathbb{F}$ is equivalent to Lebesgue measure on H.*

Assumption 2 (A2): *For any $t, t' \in T$ with $t \neq t'$, the set of voters who are indifferent between t and t' is of $\mathbb{F}$-measure zero.*

Following Roemer (2001: Section 2.3; 2005), the *fraction* of the vote going to policy t^1 would be $\mathbb{F}(\Omega(t^1, t^2))$. We also assume that there is some *aggregate uncertainty* in how people will vote, so that the probability of victory depends on the fraction of the vote, and on a noise parameter ε which is uniformly distributed over $[-\beta, \beta]$, where $\beta \in \left(0, \frac{1}{2}\right)$. Thus, the probability that t^1 defeats t^2 is:

$$\pi(t^1, t^2) = \begin{cases} 0 & \text{if } \mathbb{F}(\Omega(t^1, t^2)) + \beta \leq \frac{1}{2} \\ \frac{\mathbb{F}\left(\Omega(t^1, t^2)\right) + \beta - \frac{1}{2}}{2\beta} & \text{if } \frac{1}{2} \in (\mathbb{F}(\Omega(t^1, t^2)) - \beta, \mathbb{F}(\Omega(t^1, t^2)) + \beta) \\ 1 & \text{if } \mathbb{F}(\Omega(t^1, t^2)) - \beta \geq \frac{1}{2} \end{cases},$$

whenever $t^1 \neq t^2$, and $\pi(t^1, t^2) = \frac{1}{2}$ whenever $t^1 = t^2$. Then, one *political environment* is specified by a tuple $\langle H, \mathbb{F}, T, v, \beta \rangle$.

Let us suppose that exactly two parties will form. The two parties will each represent a coalition of voter types such that $A \cup B \subseteq H$ and $A \cap B = \varnothing$. Moreover, we define the parties' utility functions on T by $V^A : T \to \mathbb{R}$ and $V^B : T \to \mathbb{R}$. It may sometimes be assumed that V^A and V^B are given by:

$$V^A(t) = \left[\int_{h \in A} v(t, h) \mathrm{d}\mathbb{F}(h) \right]^a, \text{ and } V^B(t) = \left[\int_{h \in B} v(t, h) \mathrm{d}\mathbb{F}(h) \right]^b,$$

where $0 < a < 1$ and $0 < b < 1$.

We are now ready to define an equilibrium notion of this multi-dimensional political competition game, *party-unanimity Nash equilibria* (**PUNEs**), as introduced by Roemer (1998; 1999; 2001).

Definition 1: *Given a pair of parties, A and B, a pair of policies $(t^A, t^B) \in T \times T$ constitutes a* party-unanimity Nash equilibrium (**PUNE**) *if*:
(t^A, t^B) *satisfies the following*:

(a) *given* t^B*, there is no policy* $t \in T$ *such that*

$\pi(t, t^B) \geq \pi(t^A, t^B)$*and* $V^A(t) \geq V^A(t^A)$, with at least one strict inequality;

(b) *given* t^A*, there is no policy* $t \in T$ *such that*

$\pi(t^A, t) \leq \pi(t^A, t^B)$ *and* $V^B(t) \geq V^B(t^B)$, with at least one strict inequality.

In Definition 1, condition (a) states that, facing the opponent's proposal t^B, there is no policy in T that can improve the payoffs of all three factions in party A, and condition (b) makes an analogous statement for the factions of party B. Note that in this definition, there is no statement for the Reformists' payoffs, because (a) and (b) describe the conditions for the Opportunists' payoffs, $\pi(\cdot, \cdot)$ and $1 - \pi(\cdot, \cdot)$, and the Militants' payoffs, $V^A(\cdot)$ and $V^B(\cdot)$, only. However, as Roemer (2001: Chapter 8; Theorem 8.1(3)) showed, the equilibrium set corresponding to this simpler definition of **PUNE** is equivalent to that of the original definition of **PUNE** given in Roemer (2001: Chapter 8; Definition 8.1).

This general definition admits the case in which $t^A = t^B$. To eliminate such a case, let us define the following: given a pair of parties, A and B, a pair of policies $(t^A, t^B) \in T \times T$ is called a *non-trivial* **PUNE** if it is a **PUNE** such that $t^A \neq t^B$.

Among the various **PUNE**s, a polar case is where both parties only care about satisfying their own preferences, such that both parties never care about their probability of winning an election. In other words, the Militants are assumed to be dictators in both parties. Such a specific **PUNE** is given by the following:

Definition 2: *Given a pair of parties,* A *and* B*, a pair of policies* $(t^A, t^B) \in T \times T$ *is a* Militant-dictatorial **PUNE** (M-**PUNE**) *if this is a* **PUNE** *such that* $t^A = \arg\max_{t \in T} V^A(t)$ *and* $t^B = \arg\max_{t \in T} V^B(t)$.

Let $\bar{t}^A \equiv \arg\max_{t \in T} V^A(t)$ and $\bar{t}^B \equiv \arg\max_{t \in T} V^B(t)$. W. l. o. g., we assume that $\bar{t}^A \neq \bar{t}^B$ throughout the following discussion.

Finally, the most realistic and interesting type of **PUNE** is the case where both parties offer different policies, and where no faction in either party is assumed to have a dictatorial power. Such a **PUNE** is given by the following:

Definition 3: *Given a pair of parties, A and B, a pair of policies $(t^A, t^B) \in T \times T$ is a* pure-compromise **PUNE** (C-**PUNE**) *if this is a* **PUNE** *with (i) $t^A \neq t^B$, (ii) $t^A \neq \bar{t}^A$, and (iii) $t^B \neq \bar{t}^B$.*

This definition assumes the existence of a *real* inter-faction bargaining process within each party, so that the Opportunists in each party have a chance to influence the party's decision making.

3 Existence Theorem for C-PUNE

In the following discussion, we provide a rather general and reasonable condition under which the existence of C-**PUNE**s is shown in multi-dimensional political competition games. Note that, if the two parties' utility functions V^A and V^B are strictly quasi-concave, then the pair $\left(\bar{t}^A, \bar{t}^B\right)$ constitutes the unique M-**PUNE**.

As the first preliminary step, let us introduce a generalization of the Kakutani fixed point theorem, which was first discussed by Urai and Hayashi (2000), and is useful where the correspondence is nonconvex-valued.

Lemma 1 (**Urai and Hayashi fixed point theorem:** Urai and Hayashi (2000)):[5] *Consider, for each player $i \in N = \{1, \ldots, n\}$, the set X_i is a compact convex subset of $\mathbb{R}^l$. Suppose that there is a family of non-empty valued correspondences $\phi_i : X \twoheadrightarrow X_i$, where $i \in N$ and $X \equiv \prod_{j \in N} X_j$, satisfying the following condition*:

(**LDV1**) *For each $x \in X$ such that $x \notin \prod_{j \in N} \phi_j(x)$, there exist a player i, a vector $p_i(x) \in \mathbb{R}^l$ and an open neighborhood $N(x)$ of x such that $p_i(x) \cdot (w_i - z_i) > 0$ for all $z \in N(x)$ and $w_i \in \phi_i(z)$.*

Then, $\prod_{j \in N} \phi_j$ has a fixed point $x^* \in X$ such that $x_i^* \in \phi_i(x^*)$ for all $i \in N$.

Note that in the Urai and Hayashi fixed point theorem, the correspondence $\prod_{j \in N} \phi_j$ need not be convex-valued nor upper hemi-continuous. In the existence problems of our **PUNE** discussed below, the best response correspondence will not be necessarily convex-valued. Thus, **Lemma 1** will play a crucial role.

[5] The original statement of the Urai-Hayashi fixed point is more general than this version: the domain of the correspondence need not be the product space as here. To discuss our own issue, however, a simplier version as this lemma is sufficient.

As the second preliminary step, let us introduce the following equilibrium notion:

Definition 1': *Given a pair of parties, A and B, a pair of policies* $(t^A, t^B) \in T \times T$ *constitutes a* quasi-**PUNE** (q-**PUNE**) *if*:
(1) (t^A, t^B) *satisfies the following*:
(a) *given* t^B*, there is no policy* $t \in T$ *such that*

$\mathbb{F}\left(\Omega(t, t^B)\right) \geq \mathbb{F}\left(\Omega(t^A, t^B)\right)$ *and* $V^A(t) \geq V^A(t^A)$, with at least one strict inequality;

(b) *given* t^A*, there is no policy* $t \in T$ *such that*

$\mathbb{F}\left(\Omega(t^A, t)\right) \leq \mathbb{F}\left(\Omega(t^A, t^B)\right)$ *and* $V^B(t) \geq V^B(t^B)$, with at least one strict inequality.

Definition 3': *Given a pair of parties, A and B, a pair of policies* $(t^A, t^B) \in T \times T$ *constitutes a* quasi-C-**PUNE** (q-C-**PUNE**) *if it is a* q-**PUNE** *with (i)* $t^A \neq t^B$*, (ii)* $t^A \neq \overline{t}^A$*, and (iii)* $t^B \neq \overline{t}^B$.

The notion of q-**PUNE** has another interesting implication regarding the real political competition. The difference of q-**PUNE** from **PUNE** is that the Opportunists are modeled to maximize *vote share* instead of maximizing the probability of victory. So, this notion can be relevant to a Proportional representation system.

In the following discussion, we will first show an existence theorem of q-C-**PUNE**, as a corollary of which, an existence theorem of C-**PUNE** will be discussed later. First, we impose the following additional assumptions:

Assumption 3 (A3): *For any* $t^B \in T$*, there exists* $t' \in T$ *such that* $\Omega(t', t^B)$ *is non-empty and* $\mathbb{F}\left(\Omega(t', t^B)\right) > 0$*. Also, for any* $t^A \in T$*, there exists* $t'' \in T$ *with* $t'' \neq t^A$ *such that* $H \backslash \Omega(t^A, t'')$ *is non-empty and* $\mathbb{F}\left(H \backslash \Omega(t^A, t'')\right) > 0$.

A3 eliminates a situation that party B (resp. A) has a dominant strategy in terms of its opportunist objective. In fact, if **A3** fails, it implies that party B (*resp.* A) has a policy by which the party wins the election with certainty regardless of what strategy party A (*resp.* B) employs. In such a type of game, there may exist only the trivial **PUNE** (t^A, t^B) with $t^A = t^B$ other than M-**PUNE**.

Given $\varepsilon > 0$ and $t^i \in T$ for $i = A, B$, let $\overline{U}_i\left(t^i, \varepsilon\right) \equiv \{t \in T \mid 0 \leqq V^i(t^i) - V^i(t) \leqq \varepsilon\}$. Let us use the notation coX as the *convex hull* of X for any set X. Then:

Assumption 4 (A4): *For any non-trivial* M-**PUNE**, $\left(\bar{t}^A, \bar{t}^B\right)$, *there exists sufficiently small* $\varepsilon_A > 0$ *(resp.* $\varepsilon_B > 0$*) such that for any* $t'^A \in \overline{U}_A\left(\bar{t}^A, \varepsilon_A\right)$ *(resp.* $t'^B \in \overline{U}_B\left(\bar{t}^B, \varepsilon_B\right)$*), any* $t^B \in T \backslash \overline{U}_A\left(\bar{t}^A, \varepsilon_A\right)$ *(resp.* $t^A \in T \backslash \overline{U}_B\left(\bar{t}^B, \varepsilon_B\right)$*), and any* $t \in co\{t'^A, t^B\}$ *with* $t \neq t'^A, t^B$ *(resp.* $t \in co\{t^A, t'^B\}$ *with* $t \neq t^A, t'^B$*), we have* $\Omega(t, t^B) \supsetneq \Omega(t'^A, t^B)$ *and* $\mathbb{F}\left(\Omega\left(t, t^B\right)\right) > \mathbb{F}\left(\Omega\left(t'^A, t^B\right)\right)$ (resp. $H\backslash\Omega(t^A, t) \supsetneq H\backslash\Omega(t^A, t'^B)$ *and* $\mathbb{F}\left(H\backslash\Omega(t^A, t)\right) > \mathbb{F}\left(H\backslash\Omega(t^A, t'^B)\right)$*).*

A4 is reasonable, if every voter's utility function is concave. In fact, for any $t, t' \in T$ and any $t'' \in co\{t, t'\}$ with $t'' \neq t$ and $t'' \neq t'$, it follows that $\Omega(t, t') \subseteq \Omega(t'', t')$, since any $h \in \Omega(t, t')$ has $v(t, h) > v(t', h)$, and also $v(t'', h) > v(t', h)$ by the concavity of v. In addition to this property, there may be another type of voter $h' \in H\backslash\Omega(t, t')$ such that $v(t, h') < v(t', h')$ and $v(t'', h') > v(t', h')$. Note that these two inequalities are compatible with the concavity of v. **A4** only requires that, if $t \in \overline{U}_A\left(\bar{t}^A, \varepsilon_A\right)$ and $t' \in T\backslash\overline{U}_A\left(\bar{t}^A, \varepsilon_A\right)$, the voters such as h' exist in $H\backslash\Omega(t, t')$, and the measure of such voters is non-negligible. In other words, **A4** requires a variety of voter types. This assumption eliminates a trivial case that $\bar{t}^A$ is the best strategy against $\bar{t}^B$ or another t'^B, not only for the Militant, but also for the Opportunist of party A. In fact, if such a trivial case holds in some political game, then there may be only the types of **PUNE**s $\left\{(t^A, t^B)\right\}$ with $t^A = \bar{t}^A$ besides the trivial **PUNE** (t^A, t^B) with $t^A = t^B$ in this game.

For each $(t^A, t^B) \in T \times T$, let

$$U^A(t^A) \equiv \left\{\widetilde{t}^A \in T \mid V^A(\widetilde{t}^A) \geq V^A(t^A)\right\}$$

and

$$U^B(t^B) \equiv \left\{\widetilde{t}^B \in T \mid V^B(\widetilde{t}^B) \geq V^B(t^B)\right\}.$$

Note that $U^A(t^A)$ (*resp.* $U^B(t^B)$) is convex. Let us define a mapping $F : T \times T \to [0, 1]$ as: for any $(t^A, t^B) \in T \times T$, $F(t^A, t^B) = \mathbb{F}\left(\Omega\left(t^A, t^B\right)\right)$. Given $(t^A, t^B) \in T \times T$, the collection of $t'^A \in T$ such that $F(t'^A, t^B) = F(t^A, t^B)$ constitutes the *iso-fraction curve of* $F(\cdot, t^B)$ *at* t^A. Then, the last assumption requires a kind of 'local quasi-concavity' of F.

Assumption 5 (A5): *For any non-trivial* M-**PUNE**, $\left(\bar{t}^A, \bar{t}^B\right)$, *there exists sufficiently small* $\varepsilon_A^* > 0$ *(resp.* $\varepsilon_B^* > 0$*) such that for each* $t^B \in$

$T \backslash \overline{U}_A \left(\overline{t}^A, \varepsilon_A^* \right)$ *(resp.* $t^A \in T \backslash \overline{U}_B \left(\overline{t}^B, \varepsilon_B^* \right)$*), and for each* $t^A \in \overline{U}_A \left(\overline{t}^A, \varepsilon_A^* \right)$ *(resp.* $t^B \in \overline{U}_B \left(\overline{t}^B, \varepsilon_B^* \right)$*), there is a sufficiently small neighborhood of* t^A *(resp.* t^B*), on which* $F(\cdot, t^B)$ *is quasi-concave (resp.* $1 - F(t^A, \cdot)$ *is quasi-concave).*

This assumption also seems reasonable. This is because, though the function $F(\cdot, t^B)$ is not *globally* quasi-concave in general, it seems reasonable to assume that it is *locally* quasi-concave if the 'locality' is taken as a *sufficiently small subset* of the domain.[6] In fact, as we argue below, if $F(\cdot, t^B)$ is differentiable at each $t^A \in T \backslash \{t^B\}$, and each V^i is also differentiable and its Hessian matrix is non-singular, **A5** is supported.[7] Then, if parties' utility functions are strictly quasi-concave, since $\overline{U}_A \left(\overline{t}^A, \varepsilon_A^* \right)$ is a sufficiently small, strict convex set, it follows from **A5** that, for each $t^B \in T \backslash \overline{U}_A \left(\overline{t}^A, \varepsilon_A^* \right)$ and each $t^A \in \overline{U}_A \left(\overline{t}^A, \varepsilon_A^* \right)$, there is a unique point $t'^A \in U^A(t^A)$ *at which* the *iso-fraction curve of* $F(\cdot, t^B)$ is *tangent to* $U^A(t^A)$. The same argument is also applied to $1 - F(t^A, \cdot)$.

Then:

Theorem 1: *Suppose* T *is a compact and convex subset in* $\mathbb{R}^n$*, and each party's utility function* V^i*, where* $i \in \{A, B\}$*, is continuous and strictly quasi-concave on* T*. Let* **A1**, **A2**, **A3**, **A4**, *and* **A5** *hold. Then, there exists a* q-C-**PUNE** *for the environment* $\langle H, \mathbb{F}, T, v, \beta \rangle$.

Proof. Given $\left(\overline{t}^A, \overline{t}^B \right) \in T \times T$ with $\overline{t}^A \neq \overline{t}^B$, we can appropriately choose $(t^{A*}, t^{B*}) \in T \times T \backslash \left\{ \left(\overline{t}^A, \overline{t}^B \right) \right\}$ with $t^{A*} \neq t^{B*}$ such that $U^A(t^{A*}) \cap U^B(t^{B*}) = \varnothing$. Note that $\overline{t}^A \in U^A(t^{A*})$ and $\overline{t}^B \in U^B(t^{B*})$. Note that for any pair $(t^A, t^B) \in U^A(t^{A*}) \times U^B(t^{B*})$, $t^A \neq t^B$ holds. For each $i = A, B$, let $U^i(t^i; \overline{t}^i, \varepsilon_i) \equiv U^i(t^i) \cup \overline{U}_i \left(\overline{t}^i, \varepsilon_i \right)$, where $\overline{U}_i \left(\overline{t}^i, \varepsilon_i \right)$ is given by **A4**. Then, for each $\mathbf{t} = (t^A, t^B) \in U^A(t^{A*}) \times U^B(t^{B*})$, let $\left(t'^A, t'^B \right) \in U^A(t^{A*}) \times U^B(t^{B*})$ have the following property:

[6] In fact, all of the works on PUNEs by John Roemer and others have implicitly used this assumption.

[7] Note that the differentiability of $F(\cdot, t^B)$ is not so strong an assumption, since $\mathbb{F}$ is equivalent to the Lebesgue measure by **A1**.

(1-a) t'^A is a solution of the following problem:

$$\max_{t \in U^A(t^A;\bar{t}^A,\varepsilon_A)} F\left(t, t^B\right) - F\left(\bar{t}^A, t^B\right), \quad (1)$$

(1-b) t'^B is a solution of the following problem:

$$\max_{t \in U^B(t^B;\bar{t}^B,\varepsilon_B)} F\left(t^A, \bar{t}^B\right) - F\left(t^A, t\right). \quad (2)$$

Denote the set of such t'^A by $G^A\left(t^A, t^B\right)$. In the same way, denote the set of such t'^B by $G^B\left(t^A, t^B\right)$. Note that either $U^i(t^i;\bar{t}^i,\varepsilon_i) = U^i(t^i)$ or $U^i(t^i;\bar{t}^i,\varepsilon_i) = \overline{U}_i\left(\bar{t}^i,\varepsilon_i\right)$ holds for each $i = A, B$. This implies that $U^i(t^i;\bar{t}^i,\varepsilon_i)$ is continuous at each $t^i \in T$ for each $i = A, B$. Since F is continuous at each $(t^A, t^B) \in T \times T$ with $t^A \neq t^B$ by **A1**, $\left(G^A \times G^B\right)$ is upper hemi-continuous on $U^A(t^{A*}) \times U^B(t^{B*})$, by Berge's maximum theorem.

Let *intX* (resp. ∂X) be the *interior* (resp. *boundary*) of the set X. Define a correspondence $\varphi_A : U^A(t^{A*}) \times U^B(t^{B*}) \twoheadrightarrow U^A(t^{A*})$ as follows: for each $(t^A, t^B) \in U^A(t^{A*}) \times U^B(t^{B*})$, $\varphi_A(t^A, t^B) = G^A\left(t^A, t^B\right)$. Also, define $\varphi_B : U^A(t^{A*}) \times U^B(t^{B*}) \twoheadrightarrow U^B(t^{B*})$ as: for each $(t^A, t^B) \in U^A(t^{A*}) \times U^B(t^{B*})$, $\varphi_B(t^A, t^B) = G^B\left(t^A, t^B\right)$. Finally, let $\varphi \equiv \varphi_A \times \varphi_B$. It is clear that $\varphi(t^A, t^B) \neq \varnothing$ for every $(t^A, t^B) \in U^A(t^{A*}) \times U^B(t^{B*})$.

We show below that φ also satisfies **LDV1** of **Lemma 1**. Let $(t^A, t^B) \notin \varphi(t^A, t^B)$. This implies

$$\varphi(t^A, t^B) \subseteq \left(U^A(t^A;\bar{t}^A,\varepsilon_A) \times U^B(t^B;\bar{t}^B,\varepsilon_B)\right) \setminus \left\{(t^A, t^B)\right\}.$$

Insert Figure 1 around here.

By **LDV1**, it suffices to discuss the case of $t^A \notin \varphi_A(t^A, t^B)$. Suppose that $t^A \notin \overline{U}_A\left(\bar{t}^A, \varepsilon_A\right)$. Take $co\left[\varphi_A(t^A, t^B)\right]$. Since $U^A(t^A;\bar{t}^A,\varepsilon_A)$ is convex, $co\left[\varphi_A(t^A, t^B)\right] \subseteq U^A(t^A;\bar{t}^A,\varepsilon_A)$. Moreover, since $U^A(t^A;\bar{t}^A,\varepsilon_A)$ is strictly convex by the strict quasi-concavity of V^A, we can guarantee that $t^A \notin co\left[\varphi_A(t^A, t^B)\right]$ holds. This implies that, by the separating hyperplane theorem, there exists $p_A\left(t^A\right) \in \mathbb{R}^n$ such that $p_A\left(t^A\right) \cdot \left(t'^A - t^A\right) > 0$ holds for any $t'^A \in \varphi_A\left(t^A, t^B\right)$.

Insert Figure 2 around here.

Consider a neighborhood $N\left(\varphi_A(t^A,t^B)\right)$ of $\varphi_A(t^A,t^B)$, which is small enough and satisfies $co\left[\varphi_A(t^A,t^B)\right] \subseteq N\left(\varphi_A(t^A,t^B)\right)$ and $t^A \notin N\left(\varphi_A(t^A,t^B)\right)$. Since φ_A is upper hemi-continuous at (t^A,t^B), there is a neighborhood $N(t^A,t^B)$ of (t^A,t^B) such that, for this $N\left(\varphi_A(t^A,t^B)\right)$, we have $\varphi_A(t'^A,t'^B) \subseteq N\left(\varphi_A(t^A,t^B)\right)$ for any $(t'^A,t'^B) \in N(t^A,t^B)$. Consider $\overline{N}(t^A,t^B) \subsetneq N(t^A,t^B)$, which is small enough, so that for any $\left(t^{A\prime},t^{B\prime}\right) \in \overline{N}(t^A,t^B)$, $t^{A\prime} \notin N\left(\varphi_A(t^A,t^B)\right)$. Since $N\left(\varphi_A(t^A,t^B)\right) \supseteq co\left[\varphi_A(t^A,t^B)\right]$, and $t^A \notin co\left[\varphi_A(t^A,t^B)\right]$, we can find such a small set $\overline{N}(t^A,t^B)$. Thus, we have $\varphi_A(t'^A,t'^B) \subseteq N\left(\varphi_A(t^A,t^B)\right)$ for any $(t'^A,t'^B) \in \overline{N}(t^A,t^B)$.

By the construction of $N\left(\varphi_A(t^A,t^B)\right)$ and $\overline{N}(t^A,t^B)$, we can see that for any $(t'^A,t'^B) \in \overline{N}(t^A,t^B)$ and any $t''^A \in N\left(\varphi_A(t^A,t^B)\right)$, $p_A\left(t^A\right)\cdot\left(t''^A - t'^A\right) > 0$. This implies that for any $(t'^A,t'^B) \in \overline{N}(t^A,t^B)$ and any $t''^A \in \varphi_A(t'^A,t'^B)$, we have $p_A\left(t^A\right)\cdot\left(t''^A - t'^A\right) > 0$.

Next, let $t^A \in \overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$. Then, $U^A(t^A;\bar{t}^A,\varepsilon_A)$ is reduced to $\overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$. Note that for a given t^B, $G^A\left(t^A,t^B\right)$ is invariable for any $t^A \in \overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$. This means that, for any $t^A, t'^A \in \overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$, $G^A\left(t^A,t^B\right) = G^A\left(t'^A,t^B\right)$. Also, by **A4**, for any $t'^A \in G^A\left(t^A,t^B\right)$ and any $t \in co\left\{t'^A,t^B\right\}$ with $t \neq t'^A$, $t \notin \overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$, which implies $G^A\left(t^A,t^B\right) \subseteq \partial\overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$. Moreover, we can assume by **A5** and the strong quasi-concavity of V^A that $G^A(t^A,t^B) = \left\{\widetilde{t}^A\right\}$ holds without loss of generality. Thus, if $t^A \in \partial\overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$, then we can apply the same argument as in the case of $t^A \notin \overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$, so that there exist a neighborhood $\overline{N}(t^A,t^B)$ of (t^A,t^B) and $p_A\left(t^A\right) \in \mathbb{R}^n$ such that for any $(t'^A,t'^B) \in \overline{N}(t^A,t^B)$ and any $t''^A \in \varphi_A(t'^A,t'^B)$, we have $p_A\left(t^A\right)\cdot\left(t''^A - t'^A\right) > 0$. If $t^A \in int\overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$, there exist $p_A\left(t^A\right) \in \mathbb{R}^n$ and a sufficiently small neighborhood $\overline{N}(t^A) \subseteq int\overline{U}_A\left(\bar{t}^A,\varepsilon_A\right)$ of t^A such that $p_A\left(t^A\right)\cdot\left(\widetilde{t}^A - t'^A\right) > 0$ holds for any $t'^A \in \overline{N}(t^A)$. Moreover, for a sufficiently small neighborhood $\overline{N}(t^B)$ of t^B, for any $t'^B \in \overline{N}(t^B)$, $G^A(t^A,t'^B)$ is also single-valued by **A5** and the strong quasi-concavity of V^A, so that we can find $\left\{\widetilde{t'}^A\right\} = \varphi_A(t^A,t'^B)$. Then, by the upper hemi-continuity of G^A, we can see that, for some sufficiently small neighborhood $\overline{N}(t^A,t^B) \equiv \overline{N}(t^A)\times\overline{N}(t^B)$ of (t^A,t^B), $p_A\left(t^A\right)\cdot\left(\widetilde{t'}^A - t'^A\right) > 0$ holds for any $(t'^A,t'^B) \in \overline{N}(t^A,t^B)$.

In summary, φ satisfies **LDV1** of **Lemma 1**. Thus, by **Lemma 1**, φ has a fixed point $(\widehat{t}^A, \widehat{t}^B) \in \varphi(\widehat{t}^A, \widehat{t}^B)$. By the definition of φ, $\widehat{t}^A \neq \widehat{t}^B$ holds.

For the fixed point $(\widehat{t}^A, \widehat{t}^B) \in \varphi(\widehat{t}^A, \widehat{t}^B)$, we have:

$$\begin{aligned}
\mathbb{F}\left(\Omega\left(\widehat{t}^A, \widehat{t}^B\right)\right) &\geq \mathbb{F}\left(\Omega\left(t^A, \widehat{t}^B\right)\right) \text{ for any } t^A \in U^A(\widehat{t}^A; \overline{t}^A, \varepsilon_A), \quad (3)\\
\mathbb{F}\left(\Omega\left(\widehat{t}^A, t^B\right)\right) &\geq \mathbb{F}\left(\Omega\left(\widehat{t}^A, \widehat{t}^B\right)\right) \text{ for any } t^B \in U^B(\widehat{t}^B; \overline{t}^B, \varepsilon_B). \quad (4)
\end{aligned}$$

Thus, $\left(\widehat{t}^A, \widehat{t}^B\right)$ is a q-**PUNE**. Moreover, by **A3**, $0 < \mathbb{F}\left(\Omega\left(\widehat{t}^A, \widehat{t}^B\right)\right) < 1$.

Suppose that $\left(\widehat{t}^A, \widehat{t}^B\right)$ is a q-**PUNE** with $\widehat{t}^A = \overline{t}^A$ or $\widehat{t}^B = \overline{t}^B$. Let $\widehat{t}^A = \overline{t}^A$. Since $\overline{U}_A\left(\overline{t}^A, \varepsilon_A\right) \supsetneqq \left\{\overline{t}^A\right\}$, there exists $t^A \in \overline{U}_A\left(\overline{t}^A, \varepsilon_A\right)$ such that $\mathbb{F}\left(\Omega\left(t^A, \overline{t}^B\right)\right) > \mathbb{F}\left(\Omega\left(\overline{t}^A, \overline{t}^B\right)\right)$ by **A4**, which is a contradiction. The same argument is applied to $\widehat{t}^B = \overline{t}^B$. Thus, the above fixed point $(\widehat{t}^A, \widehat{t}^B)$ has $\widehat{t}^A \neq \overline{t}^A$ and $\widehat{t}^B \neq \overline{t}^B$. This implies that $(\widehat{t}^A, \widehat{t}^B)$ is a q-C-**PUNE**. ■

Corollary 1: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and each party's utility function V^i, where $i \in \{A, B\}$, is continuous and strictly quasi-concave on T. Let the error term $\beta \in \left(0, \frac{1}{2}\right)$ be close enough to $\frac{1}{2}$, and let* **A1**, **A2**, **A3**, **A4**, *and* **A5** *hold. Then, there exists a* C-**PUNE** *for the environment $\langle H, \mathbb{F}, T, v, \beta \rangle$.*

Proof. By **Theorem 1**, there exists a q-C-**PUNE** $(\widehat{t}^A, \widehat{t}^B)$. Let us take an error term $\beta \in \left(0, \frac{1}{2}\right)$, which is sufficiently close to $\frac{1}{2}$. Then, since $0 < \mathbb{F}\left(\Omega\left(\widehat{t}^A, \widehat{t}^B\right)\right) < 1$, $\mathbb{F}\left(\Omega\left(\widehat{t}^A, \widehat{t}^B\right)\right) - \beta < \frac{1}{2} < \mathbb{F}\left(\Omega\left(\widehat{t}^A, \widehat{t}^B\right)\right) + \beta$ holds. Then, by the definition, $0 < \pi\left(\widehat{t}^A, \widehat{t}^B\right) < 1$ holds, so that $\left(\widehat{t}^A, \widehat{t}^B\right)$ is a C-**PUNE** for $\langle H, \mathbb{F}, T, v, \beta \rangle$. ■

We may provide a characterization of a subset of 'local' **PUNE**s. Note that if $\left(t^A, t^B\right)$ is a **PUNE**, it implies that, under the assumption that V^A, V^B, and F are differentiable, the following condition holds: there exist appropriate numbers $\left(\alpha^A, \alpha^B\right) \in [0,1] \times [0,1]$ such that for this (t^A, t^B),

$$\alpha^A \left(\frac{V^A\left(t^A\right) - V^A\left(t^B\right)}{F\left(t^A, t^B\right) - F\left(\overline{t}^A, t^B\right)}\right) \nabla_A F\left(t^A, t^B\right) + \left(1 - \alpha^A\right) \nabla_A V^A\left(t^A\right) = \mathbf{0}, \quad (5)$$

and

$$-\alpha^B \left(\frac{V^B\left(t^B\right) - V^B\left(t^A\right)}{F\left(t^A, \overline{t}^B\right) - F\left(t^A, t^B\right)}\right) \nabla_B F\left(t^A, t^B\right) + \left(1 - \alpha^B\right) \nabla_B V^A\left(t^B\right) = \mathbf{0}, \quad (6)$$

where $\nabla_i F\left(t^A, t^B\right) \equiv \left(\frac{\partial F\left(t^A,t^B\right)}{\partial t_1^i}, \ldots, \frac{\partial F\left(t^A,t^B\right)}{\partial t_n^i}\right)$ and $\nabla_i V^i\left(t^i\right) \equiv \left(\frac{\partial V^i\left(t^i\right)}{\partial t_1^i}, \ldots, \frac{\partial V^i\left(t^i\right)}{\partial t_n^i}\right)$ for $i = A, B$. That is, the above $2 \times n$ equations (5) and (6) with $\left(\alpha^A, \alpha^B\right)$ constitute the necessary condition for (t^A, t^B) to be a **PUNE**. Note that these (5) and (6) are of the form

$$\begin{aligned} G^A\left(t^A, t^B; \alpha^A, \alpha^B\right) &= \mathbf{0}, \text{ (5a)} \\ G^B\left(t^A, t^B; \alpha^A, \alpha^B\right) &= \mathbf{0}. \text{ (6a)} \end{aligned}$$

Now, denote the *Jacobian matrix* of these $2 \times n$ equations (5a) and (6a) by $J\left[G^A, G^B; \left(t^A, t^B; \alpha^A, \alpha^B\right)\right]$.

We know that a M-**PUNE** $\left(\bar{t}^A, \bar{t}^B\right)$ exists, so that (5a) and (6a) hold at $\left(\alpha^A, \alpha^B\right) = (0, 0)$. Evaluate the Jacobian of (5a) and (6a) at $\left(\alpha^A, \alpha^B\right) = (0, 0)$, so that we have

$$J\left[G^A, G^B; \left(\bar{t}^A, \bar{t}^B; 0, 0\right)\right] = \begin{bmatrix} J\left[\nabla_A V^A\left(\bar{t}^A\right)\right] & \mathbf{0} \\ \mathbf{0} & J\left[\nabla_B V^B\left(\bar{t}^B\right)\right] \end{bmatrix},$$

where $J\left[\nabla_i V^i\left(\bar{t}^i\right)\right] \equiv \begin{bmatrix} \frac{\partial^2 V^i\left(\bar{t}^i\right)}{\partial t_1^i \partial t_1^i} & \cdots & \frac{\partial^2 V^i\left(\bar{t}^i\right)}{\partial t_n^i \partial t_1^i} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V^i\left(\bar{t}^i\right)}{\partial t_1^i \partial t_n^i} & \cdots & \frac{\partial^2 V^i\left(\bar{t}^i\right)}{\partial t_n^i \partial t_n^i} \end{bmatrix}$ for each $i = A, B$. Then, if $\det\left(J\left[G^A, G^B; \left(\bar{t}^A, \bar{t}^B; 0, 0\right)\right]\right) \neq 0$, then by the implicit function theorem, there is a neighborhood $N^A \times N^B \subseteq [0 - \varepsilon, 1] \times [0 - \varepsilon, 1]$ of $(0, 0)$ such that for each $\left(\alpha^A, \alpha^B\right) \in N^A \times N^B$, there is a unique value $\left(t^A\left(\alpha^A\right), t^B\left(\alpha^B\right)\right) \in T \times T$ which constitutes $G^A\left(t^A\left(\alpha^A\right), t^B\left(\alpha^B\right); \alpha^A, \alpha^B\right) = \mathbf{0}$ and $G^B\left(t^A\left(\alpha^A\right), t^B\left(\alpha^B\right); \alpha^A, \alpha^B\right) = \mathbf{0}$. This implies that if $J\left[G^A, G^B; \left(\bar{t}^A, \bar{t}^B; 0, 0\right)\right]$ is non-singular, there will be a two-dimensional manifold of the solutions

$$\left\{\left(t^A\left(\alpha^A\right), t^B\left(\alpha^B\right)\right) \mid \left(\alpha^A, \alpha^B\right) \in N^A \times N^B\right\}$$

of the equations (5a) and (6a). The solutions do not necessarily constitute **PUNE**s, but they are at least 'local' **PUNE**s, since (5a) and (6a) constitute the necessary condition of **PUNE**, but not the sufficient condition.

Therefore, our concern is under what natural conditions, the non-singular of $J\left[G^A, G^B; \left(\bar{t}^A, \bar{t}^B; 0, 0\right)\right]$ is guaranteed. We have the following:

Proposition 1: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and each party's utility function V^i, where $i \in \{A, B\}$, is continuously differentiable and strictly quasi-concave on T. Let* **A1**, **A2**, **A3**, *and* **A4** *hold. Let F be continuously differentiable. If* $\det\left(J\left[\nabla_i V^i\left(\bar{t}^i\right)\right]\right) \neq 0$ *for $i = A, B$, then* $J\left[G^A, G^B; \left(\bar{t}^A, \bar{t}^B; 0, 0\right)\right]$ *is* non-singular.

Note that $\det\left(J\left[\nabla_i V^i\left(\bar{t}^i\right)\right]\right)$ is equal to $\det\left(H\left[V^i\left(\bar{t}^i\right)\right]\right)$, the *determinant of the Hessian matrix of* $V^i\left(t^i\right)$ evaluated at $\bar{t}^i$. So, there are many cases such that $\det\left(J\left[\nabla_i V^i\left(\bar{t}^i\right)\right]\right) \neq 0$ holds when V^i is strictly concave. For instance, if for both $i = A, B$, $\frac{\partial^2 V^i\left(\bar{t}^i\right)}{\partial t_j^i \partial t_j^i} < 0$ and $\frac{\partial^2 V^i\left(\bar{t}^i\right)}{\partial t_j^i \partial t_k^i} = 0$ for any $j, k = 1, \ldots, n$ with $j \neq k$, then $\det\left(J\left[\nabla_i V^i\left(\bar{t}^i\right)\right]\right) \neq 0$ for both $i = A, B$. Such a condition is satisfied if every party's utility function is separable and strictly concave. In most concrete examples of political competition games, parties' utility functions have these properties, such as quasi-linear utility function or Euclidean preference. Thus, the sufficient condition of **Proposition 1** is reasonable.

Proposition 1 also indicates that, if V^A, V^B, and F are differentiable, and the Hessian matrices of V^A and V^B are non-singular, then **A5** can be supported. This is because, by the implicit function theorem, the solution $\left(t^A\left(\alpha^A\right), t^B\left(\alpha^B\right)\right)$ of the equations (5) and (6) is uniquely determined within a neighborhood of $\left(\bar{t}^A, \bar{t}^B\right)$, which implies that F should be at least locally quasi-concave within a neighborhood of $\left(\bar{t}^A, \bar{t}^B\right)$. Also, for any given $t'^B \in T \backslash \overline{U}_A\left(\bar{t}^A, \varepsilon_A\right)$, the implicit function theorem can be applied to the equation (5), so that the solution $t^A\left(\alpha^A\right)$ of (5) is uniquely determined within a neighborhood of $\bar{t}^A$. These arguments imply that **A5** holds. Thus:

Corollary 2: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and each party's utility function V^i, where $i \in \{A, B\}$, is continuously differentiable and strictly quasi-concave on T. Let the error term $\beta \in \left(0, \frac{1}{2}\right)$ be close enough to $\frac{1}{2}$, and* **A1**, **A2**, **A3**, *and* **A4** *hold. Let F be continuously differentiable, and* $\det\left(H\left[V^i\left(\bar{t}^i\right)\right]\right) \neq 0$ *for $i = A, B$. Then, there exists a* C-**PUNE** *for the environment $\langle H, \mathbb{F}, T, v, \beta \rangle$.*

Note that the combination of **A1**, **A2**, **A3**, **A4**, and **A5** with β close enough to $\frac{1}{2}$ is nonvacuous, and the condition, β close enough to $\frac{1}{2}$, is indispensable for the existence of C-**PUNE**. Note also, though Berinsky and Lewis (2007) empirically found that voters' utility functions are close to *linear* rather than strictly concave, **Corollary 1** relies not on strict concavity, but only on strict quasi-concavity of parties' utility functions.

To illustrate the above first point, for instance, consider the *Euclidean model with a two-dimensional policy space* given by Roemer (2001: Section 8.7), where the two-dimensional policy space is a *disc* and the probability distribution defined on the disc is the *uniform* distribution. We can see that, in this model, **A1**, **A2**, **A3**, **A4**, and **A5** with β close enough to $\frac{1}{2}$ are satisfied, so that a C-**PUNE** exists. There is yet another example of multi-dimensional political games, which meets **A1**, **A2**, **A3**, **A4**, and **A5** with β close enough to $\frac{1}{2}$. This model is based on Roemer (1998), though the modeling of uncertainty differs from Roemer (1998), which is given as follows:

Example 1: Consider a political environment $\langle H, \mathbb{F}, T, v, \beta \rangle$ such that $H = \{(w, a) \in W \times \mathcal{A} \mid W \equiv [\underline{w}, \overline{w}] \subsetneq \mathbb{R}_+ \ \& \ \mathcal{A} \equiv [\underline{a}, \overline{a}] \subsetneq \mathbb{R}\}$, where W is the set of income levels, and $\mathcal{A}$ is the set of religious views, $T = \{(\tau, z) \mid \tau \in [0, 1] \text{ and } z \in \mathcal{A}\}$, where τ is a uniform tax rate on income, and z is a religious position of the government, and $v(\tau, z; w, a) = (1 - \gamma)[(1 - \tau) w + \tau\mu] - \frac{\gamma}{2}(z - a)^2$, where μ is the mean income of this society. Moreover, $\mathbb{F}$ has its associated density function $f(w, a) = g(w) r(a; w)$ such that $\widehat{F}(a') \equiv \int_W \int_{\underline{a}}^{a'} g(w) r(a; w) \mathrm{d}a\mathrm{d}w$ is strictly increasing at every $a' \in \mathcal{A}$. Finally, $\mathbb{F}$ is assumed to satisfy **A1** and **A2**, and β is close enough to $\frac{1}{2}$.

Note that for any $h \in H$, if his or her income $w_h > \mu$, then $\tau = 0$ is the ideal tax rate for him or her, whereas if $w_h \leq \mu$, then $\tau = 1$ is the ideal tax rate for him or her. Let $A \equiv \{h \in H \mid w_h \leq \mu\}$ and $B \equiv \{h \in H \mid w_h > \mu\}$. Let a_A be the ideal religious view of A and a_B be the median ideal view of B. Moreover, let a_H be the median religious view over H. Assume $a_A \neq a_B$. Then, $\mathbb{F}$ also satisfies **A3**. Let $\left(\overline{t}^A(\gamma), \overline{t}^B(\gamma)\right)$ be a non-trivial M-**PUNE** for each $\gamma \in [0, 1]$.

If $\gamma = 1$, then a non-trivial M-**PUNE** for $\gamma = 1$ implies $\overline{z}^A(1) \neq \overline{z}^B(1)$. Thus, if $\gamma < 1$ is close to one, then $\overline{z}^A(\gamma) \neq \overline{z}^B(\gamma)$ holds. Then, for such $\gamma < 1$ close to one, **A4** holds. This is because for any $a \in \left[\overline{z}^A(\gamma), \overline{z}^B(\gamma)\right]$, there are some voters whose ideal religious policies are identical to a, and the $\mathbb{F}$-measure of those voters is positive. The last condition follows from the strictly increasing $\widehat{F}(a')$. Note in the case where $\gamma < 1$ is sufficiently close to one, the effect of the tax policy τ on the voters' welfare is negligible relative to that of the religious policy z. Finally, since each party's utility function is given by $V^i(\tau, z) = (1-\gamma)\left[(1-\tau)w_i + \tau\mu\right]^{\frac{1}{2}} - \frac{\gamma}{2}(z - a_i)^2$ for each $i = A, B$, and $\mathbb{F}\left(\Omega\left(t^A, t^B\right)\right)$ is differentiable by assumption, **A5** also holds.

Thus, **Corollary 2** tells us that there exists a C-**PUNE**. In fact, for $\gamma = 1$, there exists a non-trivial M-**PUNE** $\left(\bar{t}^A(1), \bar{t}^B(1)\right)$ such that $\bar{t}^A(1) = (0, a_A)$ and $\bar{t}^B(1) = (0, a_B)$. Then, for $\gamma = 1$, consider any profile

$$\left(\widehat{t}^A(1), \widehat{t}^B(1)\right) = \left(\left(0, \widehat{z}_A\right), \left(0, \widehat{z}_B\right)\right)$$

such that $a_A < \widehat{z}_A < a_H < \widehat{z}_B < a_B$ with $\frac{\widehat{z}_A + \widehat{z}_B}{2} = a_H$. This profile constitutes a C-**PUNE** when β is close enough to $\frac{1}{2}$. Moreover, assume that the mean income of the cohort of voters with the median religious view a_H is higher than the mean income, μ, of the population. Then, for any $\gamma < 1$ close enough to one, any profile $\left(\widehat{t}^A(\gamma), \widehat{t}^B(\gamma)\right) = \left(\widehat{t}^A(1), \widehat{t}^B(1)\right)$ still constitutes a C-**PUNE** when β is close enough to $\frac{1}{2}$. ■

To see that the condition, β close enough to $\frac{1}{2}$, is indispensable for the existence of C-**PUNE**, let us consider the following example of political games in which there only exist one non-trivial M-**PUNE** and one trivial **PUNE** under β close enough to 0:

Example 2: Consider a political environment $\langle H, \mathbb{F}, T, v, \beta\rangle$ such that $H = W \equiv [\underline{w}, \overline{w}] \subsetneq \mathbb{R}_{++}$, where W is the set of income levels, $T = \{(\tau, \alpha) \mid \tau \in [0,1] \text{ and } \alpha \in [0,1]\}$, where τ is a uniform tax rate on income, and α is the ratio of public good expenditure over tax revenue, and the citizen w's utility function is:

$$v(\tau, \alpha; w) = \left[(1-\tau)w + (1-\alpha)\tau\mu\right] + \max\left\{\frac{\mu}{w}, 1\right\}\sigma(\alpha\tau\mu),$$

where μ is the mean income of this society. In this environment, if the society chooses (τ, α), then its tax revenue is $\tau\mu$ per capita, and its public good expenditure becomes $\alpha\tau\mu$ per capita. Then, $(1-\alpha)\tau\mu$ is the subsidy that every citizen receives through the income redistribution policy. Thus, the choice of (τ, α) implies the choice of redistribution and public good provision in this society. In every citizen's utility function v, the term $(1-\tau)w +$

$(1-\alpha)\,\tau\mu$ represents the citizen w's after-tax income when the policy (τ,α) is implemented; the term $\max\left\{\frac{\mu}{w},1\right\}\sigma\left(\alpha\tau\mu\right)$ represents the citizen w's benefit from the public good provision. Assume that $\lim\limits_{\lambda\to 0}\frac{\partial\sigma(\lambda\mu)}{\partial\lambda\mu}=+\infty$, and for some $\lambda^*\in(0,1)$, $\frac{\partial\sigma(\lambda^*\mu)}{\partial\lambda^*\mu}=1$. Finally, $\mathbb{F}$ is assumed to satisfy **A1** and **A2**, which is characterized by a density function $f\left(w\right)$ over W. Moreover, since $\Omega(t^A,t^B)=\{w\in W\mid v\left(\tau_A,\alpha_A;w\right)-v\left(\tau_B,\alpha_B;w\right)>0\}$, it is easy to see that $\mathbb{F}$ satisfies **A3**. Let $\mathbf{F}$ be a cumulative distribution function on W. It is assumed that $\mathbf{F}\left(\mu\right)>\frac{1}{2}$.

Note that there exists $h^*\in H$ with $w_{h^*}>\mu$ such that $A=\{h\in H\mid w_h<w_{h^*}\}$ and $B=\{h\in H\mid w_h\geq w_{h^*}\}$ with $w_B=\int_{h\in B}w_h\mathrm{d}\mathbb{F}(h)>\mu$, so that $w_A=\int_{h\in A}w_h\mathrm{d}\mathbb{F}(h)<\mu$. It is assumed that $\mathbf{F}\left(w_A\right)=\frac{1}{2}$. Then, by means of the assumptions on $\frac{\partial\sigma(\lambda\mu)}{\partial\lambda\mu}$ and the facts that $w_A<\mu<w_B$, there exist $\alpha^*\in(0,1)$ and $\tau^*\in(0,1)$ such that $\frac{\partial\sigma(\alpha^*\mu)}{\partial\alpha\tau\mu}=\frac{w_A}{\mu}$ and $\frac{\partial\sigma(\tau^*\mu)}{\partial\alpha\tau\mu}=\frac{w_B}{\mu}$. Since $\frac{w_A}{\mu}<\frac{w_B}{\mu}$, $\alpha^*>\tau^*$ holds. We can see that $(1,\alpha^*)$ is the ideal policy for party A, while $(\tau^*,1)$ is the ideal policy for party B.

Given this setting, if β is close enough to $\frac{1}{2}$, then $\left(t^A,t^B\right)=\left(\left(1,\alpha'\right),\left(\tau',1\right)\right)$ with $\frac{w_A}{\mu}<\frac{\partial\sigma(\alpha'\mu)}{\partial\alpha\tau\mu}<1$ and $\frac{w_B}{\mu}>\frac{\partial\sigma(\tau'\mu)}{\partial\alpha\tau\mu}>1$ constitutes a C-**PUNE** with $0<\pi\left(\left(1,\alpha'\right),\left(\tau',1\right)\right)<1$. However, since $\mathbf{F}\left(w_A\right)=\frac{1}{2}$, $\mathbb{F}\left(\Omega(t^A,t^B)\right)>\frac{1}{2}$ holds for the pair of ideal policies $\left(t^A,t^B\right)=\left(\left(1,\alpha^*\right),\left(\tau^*,1\right)\right)$. Thus, if β is close enough to 0, then $\pi\left(\left(1,\alpha^*\right),\left(\tau^*,1\right)\right)=1$, and every **PUNE** $\left(t^A,t^B\right)$ is either $\left(t^A,t^B\right)=\left(\left(1,\alpha^*\right),\left(\tau^*,1\right)\right)$ or $\left(t^A,t^B\right)=\left(\left(1,\alpha^*\right),\left(1,\alpha^*\right)\right)$. ■

Before ending this section, it is worthwhile to point out the following two remarks: First, one may well wonder if the method using Urai-Hayashi fixed point theorem could also be applied to the existence problem of Wittman equilibrium in multi-dimensional political games with uncertainty. For instance, let us consider the problems (1) and (2) with the weights $\alpha^A=\frac{1}{2}=\alpha^B$, and then apply the Urai-Hayashi fixed point theorem as in the proof of Theorem 1. Though this method seems to be useful for the existence problem of Wittman equilibrium, the resulting fixed point is not necessarily a Wittman equilibrium, but definitely a **PUNE**. This is because such a fixed point is a pair of the solutions for the *constrained* Nash bargaining problems *subject to* the upper contour sets of themselves, whereas any Wittman equilibrium should be a pair of the solutions for the *non-constrained* Nash bargaining problems.

Second, though we have focussed on C-**PUNE**, there might well be another interesting refinement of **PUNE**: that is, **PUNE** with endogenous party formations. Though there may be various possible ways to formulate it, one way was proposed by Roemer (2005, p.225), in which it is given by a well-known condition of endogenous party formations [Baron (1993); Caplin and Nalebuff (1997); Gomberg (2004); Gomberg et al. (2004)], in addition to **Definition 1** of this paper. The condition requires that the voting partition resulting from a profile of policy proposals is identical in equilibrium to the membership partition inducing this pair of policy proposals. There are two comments on this specific type of **PUNE**. First, showing even the general existence of M-**PUNE** is quite difficult in the endogenous party models. Note that M-**PUNE** with the endogenous party formations is a specific type of *sorting equilibrium* [Gomberg (2004)], and, as has been discussed by Caplin and Nalebuff (1997), Gomberg (2004), and Gomberg et al. (2004), the general existence of such an equilibrium is faced with difficulty when the dimension of policy space is *even*.[8] Second, in the context of endogenous parties, it seems hard to give a natural interpretation to Militant (and also Reformist) factions. In this context, Militants advocate the average platform, integrated over the space of voters who prefer this party to another, but one may wonder what kind of people in real life uncompromisingly adhere to a platform that often frivolously changes with the decisions of other voters. Of course, Militants may understand that the party is an instrument for representing its members. However, in the above endogenous party models, the left-wing party (or its Militant) may possibly propose a more right-wing policy than the opponent if it happens to collect right-wing citizens as its membership. This paradoxical phenomenon is due to a lack of argument on preserving party members' *loyalty* in determining their platforms: the bureaucracy of the left-wing party would propose a more left-wing policy than the opposing party so as to preserve the loyalty of all left-wing members. Roemer (2005, p.226) proposed an alternative formulation of **PUNE** in endogenous party models with this extra consideration,[9] and the general existence of such an equilibrium can be obtained by modifying the proof method of this paper's Theorem 1, as discussed in Yoshihara (2008a).

[8] Though, as Gomberg, et. al (2004) showed, this does not necessarily imply that the existence of sorting equilibrium is impossible in *any* political game with the *even* dimensionality of policy space.

[9] Roemer (2005) called this *quasi*-**PUNE**.

4 Concluding Remarks

In this paper, we introduced, in general multi-dimensional political games, a refinement of **PUNE**s, C-**PUNE**s, and provided sufficient conditions for the existence of C-**PUNE**s. The sufficient conditions appear natural and plausible, and this implies that there are many reasonable models of multi-dimensional political games in which this refinement exists.

In this paper, we focussed on a specific modeling of party uncertainty, which Roemer (2001: Chapter 2) called the *Error-Distribution Model of Uncertainty.* However, Roemer (2001: Chapter 2) also proposed other types of uncertainty models. We have not considered the existence problems in alternative uncertainty models.

The existence theorems in this paper depend on the assumption of strictly quasi-concave utility functions. However, there are some examples of political games with only weakly quasi-concave utility functions, such as set out by Roemer (1999). The existence of the refined **PUNE**s in political games with only weakly quasi-concave utility functions remains an open question.

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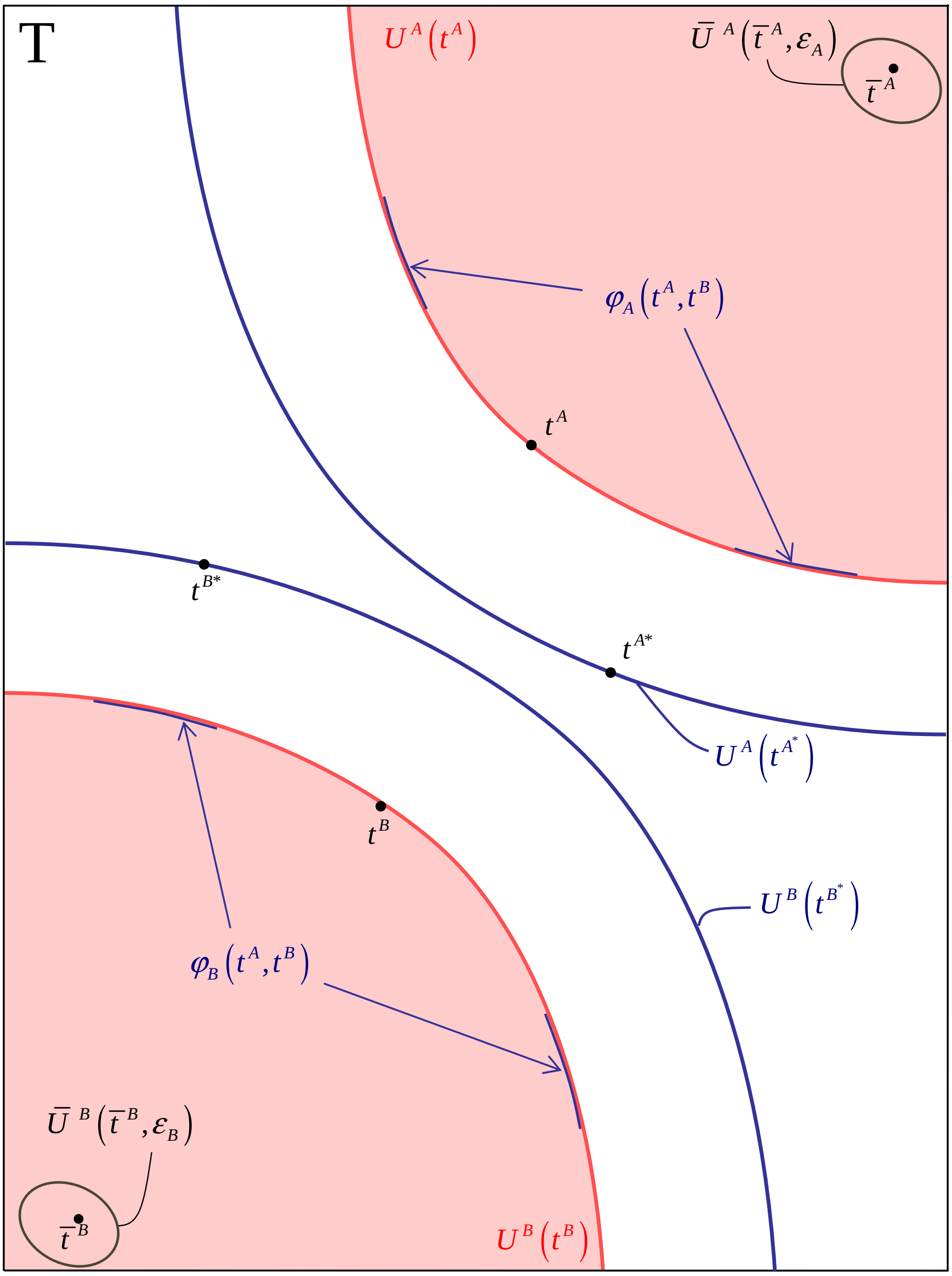

Figure1: $(t^A, t^B) \notin \varphi(t^A, t^B)$

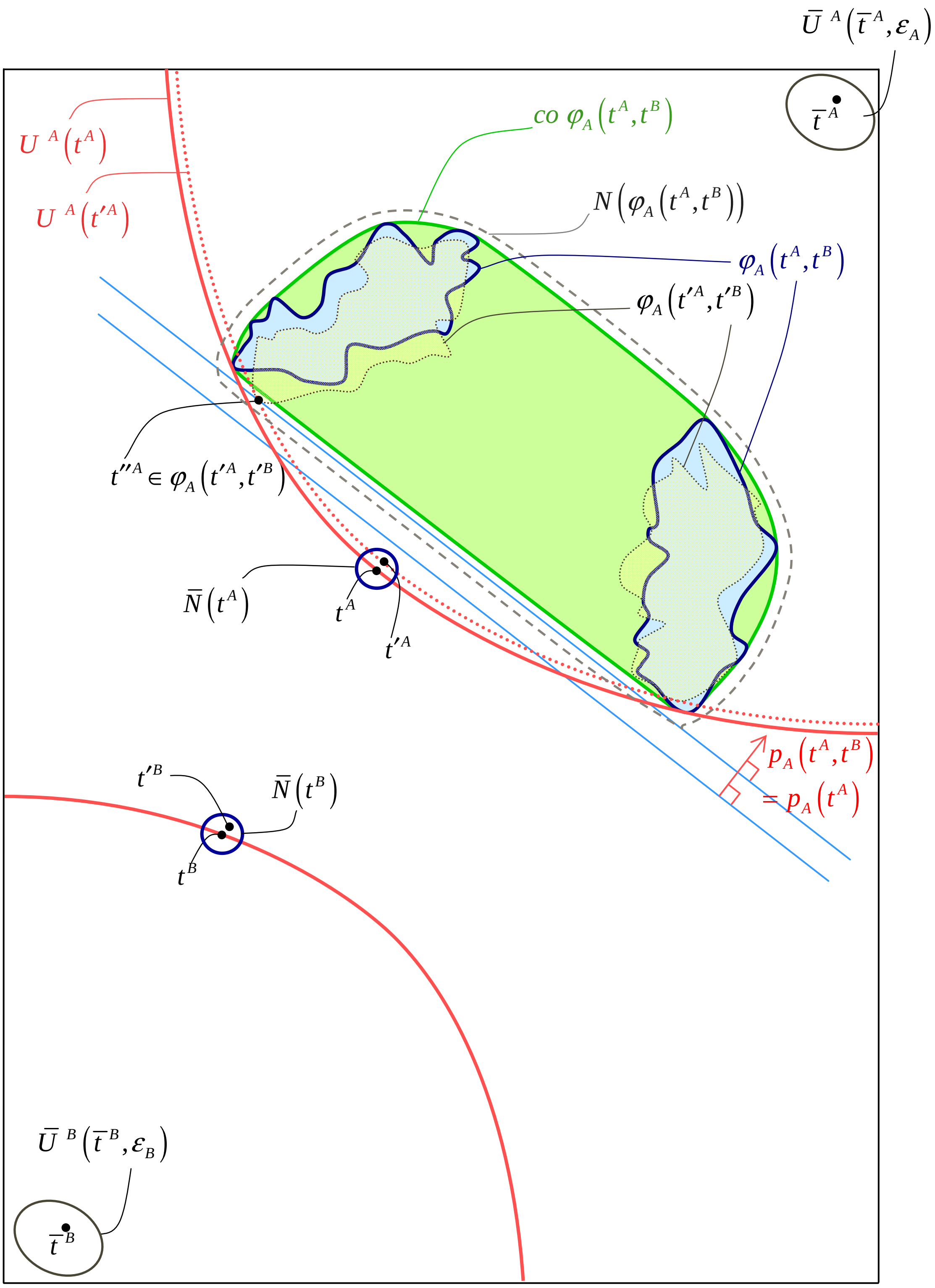

Figure 2: φ satisfies **LDV1**