# Corrigendum to "Class and Exploitation in General Convex Cone Economies" [Journal of Economic Behavior and Organization, 75 (2010) 281-296]

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**Abstract**

In this paper, we correct a flaw in Proposition 1 of Yoshihara (2010). The original proof of Proposition 1 is correct only with the existence of reproducible solutions with zero profit rate. In contrast, this note provides an alternative characterization of the domain of economies under which the existence of reproducible solutions with possibly non-zero maximal profit rates is shown.

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## 1 Introduction

This note corrects a flaw in the proof of Proposition 1 in Yoshihara (2010). In Yoshihara (2010), Proposition 1 argued the necessary and sufficient condition for the domain of aggregate endowments of material productive assets to have a non-trivial RS under such an economy: given a closed convex-cone

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production possibility set $P$ with generic element $(-\alpha_0, -\underline{\alpha}, \overline{\alpha})$, the following set constitutes the domain of aggregate endowments of capital goods,

$$\mathbb{C}^* \equiv \left\{\omega \in \mathbb{R}^m_+ \mid \exists \alpha \in P : \underline{\alpha} \leqq \omega,\, \underline{\alpha} \not< \omega,\, \overline{\alpha} - \alpha_0 b \geqq \underline{\alpha} \;\&\; \overline{\alpha} - \omega - \alpha_0 b \in \partial \widetilde{P}\right\},$$

$$\text{where } \widetilde{P} \equiv \left\{\overline{\alpha} - \underline{\alpha} - \alpha_0 b \in \mathbb{R}^m \mid (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P\right\},$$

under which an RS $(p, 1)$ exists. However, if the domain $\mathbb{C}^*$ is given by this definition, it is true that there exists an RS under any $\omega \in \mathbb{C}^*$, but such an RS is associated only with zero profit rate. However, since in Yoshihara (2010) the main theorems presumed that the economy is under an RS with a positive maximal profit rate, the existence theorem of RSs with zero profit rate *per se* is not sufficient to show the consistency of the economic framework to develop the main analyses. Thus, it is necessary to provide the domain condition of aggregate endowments of capital goods to guarantee the existence of RSs with possibly positive maximal profit rates. In this note, we provide a reformulation of the domain $\mathbb{C}^*$ under which an RS with possibly positive maximal profit rate exists.

# 2 The Basic Model

Following Yoshihara (2010), let $P$ be the production set. $P$ has elements of the form $\alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha})$ where $\alpha_0 \in \mathbb{R}_+$, $\underline{\alpha} \in \mathbb{R}^m_+$, and $\overline{\alpha} \in \mathbb{R}^m_+$. Thus, elements of $P$ are vectors in $\mathbb{R}^{2m+1}$. The first component, $-\alpha_0$, is the direct labor input of process $\alpha$; and the next $m$ components, $-\underline{\alpha}$, are the inputs of goods used in the process; and the last $m$ components, $\overline{\alpha}$, are the outputs of the $m$ goods from the process. The net output vector arising from $\alpha$ is denoted as $\widehat{\alpha} \equiv \overline{\alpha} - \underline{\alpha}$. $P$ is assumed to be a closed convex cone containing the origin in $\mathbb{R}^{2m+1}$. Moreover, it is assumed that:

A 1. $\forall \alpha \in P$ s.t. $\alpha_0 \geq 0$ and $\underline{\alpha} \geqq 0$, $[\overline{\alpha} \geq 0 \Rightarrow \alpha_0 > 0]$;[1] and
A 2. $\forall$ commodity $m$ vector $c \in \mathbb{R}^m_+$, $\exists \alpha \in P$ s.t. $\widehat{\alpha} \geqq c$.
A 3. $\forall \alpha \in P, \forall (-\underline{\alpha}', \overline{\alpha}') \in \mathbb{R}^m_- \times \mathbb{R}^m_+$, $[(-\underline{\alpha}', \overline{\alpha}') \leqq (-\underline{\alpha}, \overline{\alpha}) \Rightarrow (-\alpha_0, -\underline{\alpha}', \overline{\alpha}') \in P]$.

A1 implies that labor is indispensable to produce any non-negative output vector; A2 states that any non-negative commodity vector is producible as a

[1] For all vectors $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, p)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, p)$.

net output; and A3 is a *free disposal* condition, which states that, given any feasible production process $\alpha$, any vector producing (weakly) less net output than $\alpha$ is also feasible using the same amount of labour as $\alpha$ itself.

As a notation, we use, for any set $S \subseteq \mathbb{R}^m$, $\partial S \equiv \{x \in S \mid \nexists x' \in S \text{ s.t. } x' > x\}$. Given a market economy, any price system is denoted by $p \in \mathbb{R}^m_+$, which is a price vector of $m$ commodities. Moreover, a *subsistence vector* of commodities $b \in \mathbb{R}^m_+$ is also necessary in order to supply one unit of labor per day. We assume that the nominal wage rate is normalized to unity when it purchases the subsistence consumption vector only, so that $pb = 1$ holds.

Again, following Yoshihara (2010), denote the set of agents by $N$ with generic element $\nu$. All agents have access to the same technology $P$, but they possess different endowments $\omega^\nu \in \mathbb{R}^m_+$, whose distribution in the economy is given by $(\omega^\nu)_{\nu \in N} \in \mathbb{R}^{Nm}_+$. An agent $\nu \in N$ with $\omega^\nu$ can engage in the following three types of economic activities: she can sell her labor power $\gamma^\nu_0$; she can hire the labor powers of others to operate $\beta^\nu = \left(-\beta^\nu_0, -\underline{\beta}^\nu, \overline{\beta}^\nu\right) \in P$; or she can work for herself to operate $\alpha^\nu = (-\alpha^\nu_0, -\underline{\alpha}^\nu, \overline{\alpha}^\nu) \in P$. Given a price vector $p \in \mathbb{R}^m_+$ and a nominal wage rate $w$, it is assumed that each agent chooses her activities, $\alpha^\nu$, $\beta^\nu$, and $\gamma^\nu_0$, in order to maximize the revenue *subject to the constraints of her capital and labor endowments.*

Thus, given $(p, w)$, each $\nu$ solves the following program $MP^\nu$:

$$\max_{\left(\alpha^\nu;\, \beta^\nu;\, \gamma^\nu_0\right) \in P \times P \times \mathbb{R}_+} \quad [p\,(\overline{\alpha}^\nu - \underline{\alpha}^\nu)] + \left[p\left(\overline{\beta}^\nu - \underline{\beta}^\nu\right) - w\beta^\nu_0\right] + [w\gamma^\nu_0]$$

subject to

$$\begin{aligned} p\underline{\alpha}^\nu + p\underline{\beta}^\nu &\leqq p\omega^\nu \equiv W^\nu, \\ \alpha^\nu_0 + \gamma^\nu_0 &\leqq 1. \end{aligned}$$

Given $(p, w)$, let $\mathcal{A}^\nu(p, w)$ be the set of actions $(\alpha^\nu;\ \beta^\nu;\ \gamma^\nu_0) \in P \times P \times [0, 1]$ which solve $MP^\nu$ at prices $(p, w)$, and let

$$\Pi^\nu(p, w) \equiv \max_{\left(\alpha^\nu;\, \beta^\nu;\, \gamma^\nu_0\right) \in P \times P \times \mathbb{R}_+} [p\,(\overline{\alpha}^\nu - \underline{\alpha}^\nu)] + \left[p\left(\overline{\beta}^\nu - \underline{\beta}^\nu\right) - w\beta^\nu_0\right] + [w\gamma^\nu_0].$$

As in Yoshihara (2010), the equilibrium notion for this model is given as follows:

**Definition 1:** A *reproducible solution* (RS) for the economy specified above is a pair $\left((p, w), (\alpha^\nu;\ \beta^\nu;\ \gamma^\nu_0)_{\nu \in N}\right)$, where $p \in \mathbb{R}^m_+$, $w \geqq pb = 1$, and $(\alpha^\nu;\ \beta^\nu;\ \gamma^\nu_0) \in P \times P \times [0, 1]$, such that:

(a) $\forall \nu \in N$, $(\alpha^\nu;\ \beta^\nu;\ \gamma_0^\nu) \in \mathcal{A}^\nu(p,w)$ (revenue maximization);

(b) $\underline{\alpha} + \underline{\beta} \leqq \omega$ (social feasibility),
where $\underline{\alpha} \equiv \sum_{\nu\in N} \underline{\alpha}^\nu$, $\underline{\beta} \equiv \sum_{\nu\in N} \underline{\beta}^\nu$, and $\omega \equiv \sum_{\nu\in N} \omega^\nu$;

(c) $\beta_0 \leqq \gamma_0$ (labor market equilibrium)
where $\beta_0 \equiv \sum_{\nu\in N} \beta_0^\nu$ and $\gamma_0 \equiv \sum_{\nu\in N} \gamma_0^\nu$; and

(d) $\widehat{\alpha} + \widehat{\beta} \geqq \alpha_0 b + \beta_0 b$ (reproducibility),
where $\widehat{\alpha} \equiv \sum_{\nu\in N}(\overline{\alpha}^\nu - \underline{\alpha}^\nu)$, $\widehat{\beta} \equiv \sum_{\nu\in N}(\overline{\beta}^\nu - \underline{\beta}^\nu)$, and $\alpha_0 \equiv \sum_{\nu\in N} \alpha_0^\nu$.

Given an RS, $\left((p,w), (\alpha^\nu;\ \beta^\nu;\ \gamma_0^\nu)_{\nu\in N}\right)$, let $\alpha^{p,w} \equiv \sum_{\nu\in N} \alpha^\nu + \sum_{\nu\in N} \beta^\nu$, which is the aggregate production activity *actually* accessed in this RS. Thus, the pair $((p,w), \alpha^{p,w})$ is the summary information of this RS. In the following, we sometimes use $((p,w), \alpha^{p,w})$ or only $(p,w)$ for the representation of the RS, $\left((p,w), (\alpha^\nu;\ \beta^\nu;\ \gamma_0^\nu)_{\nu\in N}\right)$.

# 3 Reformulation of Proposition 1

Let $\alpha_0(\omega) \equiv \max\{\alpha_0 \mid \exists \alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P \text{ s.t. } \underline{\alpha} \leqq \omega\}$. Given the production set $P$ and a non-negative number $\pi \geqq 0$, let

$$\widetilde{P}(\pi) \equiv \{\overline{\alpha} - (1+\pi)\underline{\alpha} - \alpha_0 b \in \mathbb{R}^m \mid (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P\}.$$

Then, let

$$\mathbb{C}^{**} \equiv \left\{\omega \in \mathbb{R}_+^m \mid \exists \alpha \in P \ \&\ \exists \pi \geqq 0 : \underline{\alpha} \leqq \omega,\ \underline{\alpha} \not< \omega,\ \widehat{\alpha} \geqq \alpha_0 b \ \&\ \overline{\alpha} - (1+\pi)\omega - \alpha_0 b \in \partial\widetilde{P}(\pi)\right\}$$

**Lemma:** *Under A1∽A3, $\mathbb{C}^{**}$ is a non-empty and closed convex cone.*

**Proof.** It is clear that $\mathbb{C}^{**}$ is a closed convex cone. To see the non-emptyness, let $P(\alpha_0 = 1; b) \equiv \{\alpha \in P \mid \alpha_0 = 1 \ \&\ \widehat{\alpha} \geqq b\}$. If $P(\alpha_0 = 1; b) = \varnothing$, then $\mathbf{0} \in \partial\widetilde{P}(0)$, so that $\omega \equiv \mathbf{0} \in \mathbb{C}^{**}$. Otherwise, then consider $\pi^* \equiv \max_{\alpha\in P(\alpha_0=1;b)} \min_{i=1,\dots,m} \left\{\frac{\widehat{\alpha}_i - b_i}{\underline{\alpha}_i}\right\}$ and $\alpha^* \equiv \arg\max_{\alpha\in P(\alpha_0=1;b)} \min_{i=1,\dots,m} \left\{\frac{\widehat{\alpha}_i - b_i}{\underline{\alpha}_i}\right\}$. Then, by $P(\alpha_0 = 1; b) \neq \varnothing$, $\pi^* \geqq 0$. Moreover, $\overline{\alpha}^* - (1+\pi^*)\underline{\alpha}^* - \alpha_0^* b \in \partial\widetilde{P}(\pi^*)$ holds. To see the last property, suppose not. Then, there exists another $\overline{\alpha}' - (1+\pi^*)\underline{\alpha}' - \alpha_0' b \in \widetilde{P}(\pi^*)$ such that $\overline{\alpha}' - (1+\pi^*)\underline{\alpha}' - \alpha_0' b > \overline{\alpha}^* - (1+\pi^*)\underline{\alpha}^* - \alpha_0^* b$. The last inequality implies that $\min_{i=1,\dots,m}\{\overline{\alpha}_i' - (1+\pi^*)\underline{\alpha}_i' - \alpha_0' b_i\} >$

0 by $\min_{i=1,\ldots,m}\left\{\overline{\alpha}_i^* - (1+\pi^*)\,\underline{\alpha}_i^* - \alpha_0^* b_i\right\} = 0$, which is a contradiction from the definition of $\pi^*$. Thus, $\overline{\alpha}^* - (1+\pi^*)\,\underline{\alpha}^* - \alpha_0^* b \in \partial\widetilde{P}(\pi^*)$ holds, so that $\omega \equiv \underline{\alpha}^* \in \mathbb{C}^{**}$. ■

Then:

**Proposition 1:** *Let* $b \in \mathbb{R}^m_{++}$ *and* $\alpha_0(\omega) \leqq |N|$. *Under A1∽A3, there exists a* reproducible solution (RS), $\left((p,1), (\alpha^\nu;\ \beta^\nu;\ \gamma_0^\nu)_{\nu\in N}\right)$, *for the economy above if and only if* $\omega \in \mathbb{C}^{**}$.

**Proof.** (⇒): Let $\left((p,1), (\alpha^\nu;\ \beta^\nu;\ \gamma_0^\nu)_{\nu\in N}\right)$ be an RS. Given the profile $(\alpha^\nu;\ \beta^\nu;\ \gamma_0^\nu)_{\nu\in N}$, let $\underline{\alpha}^{p,1}$ be the corresponding social production point. At this RS, $p\underline{\alpha}^{p,1} = p\omega$. Thus, $\underline{\alpha}^{p,1} \leqq \omega$ by Definition 1(b), but $\underline{\alpha}^{p,1} \not< \omega$. From Definition 1(d), $\overline{\alpha}^{p,1} - \alpha_0^{p,1} b \geqq \underline{\alpha}^{p,1}$. Also, from Definition 1(a), $\alpha^{p,1} \in \sum_{\nu\in N}\mathcal{A}^\nu(p,1)$, so that $\pi \equiv \frac{p\left(\overline{\alpha}^{p,1}-\omega\right)-\alpha_0^{p,1}}{p\omega}$ is the maximal profit rate at $(p,1)$. Thus, for any other $\alpha' \in P$, $p\overline{\alpha}^{p,1} - (1+\pi)\,p\omega - \alpha_0^{p,1} = 0 \geqq p\overline{\alpha}' - (1+\pi)\,p\underline{\alpha}' - \alpha_0'$ holds. This implies that $\overline{\alpha}^{p,1} - (1+\pi)\,\omega - \alpha_0^{p,1} b \in \partial\widetilde{P}(\pi)$ and $p \in S \equiv \left\{p \in \mathbb{R}^m_+ \mid pb = 1\right\}$ is a supporting price of it. Note that $\left(-\alpha_0^{p,1}, -\omega, \overline{\alpha}^{p,1}\right) \in P$ by $\alpha^{p,1} \in P$, $\underline{\alpha}^{p,1} \leqq \omega$, and A3.

(⇐): Let $\omega \in \mathbb{C}^{**}$. Then, there exists $\alpha^* = (-\alpha_0^*, -\underline{\alpha}^*, \overline{\alpha}^*) \in P$ and $\pi \geqq 0$ such that $\underline{\alpha}^* \leqq \omega$ with $\underline{\alpha}^* \not< \omega$, $\overline{\alpha}^* - \alpha_0^* b \geqq \underline{\alpha}^*$, and $\overline{\alpha}^* - (1+\pi)\,\omega - \alpha_0^* b \in \partial\widetilde{P}(\pi)$. Since $\overline{\alpha}^* - (1+\pi)\,\omega - \alpha_0^* b \in \partial\widetilde{P}(\pi)$, by the supporting hyperplane theorem and A3, there exists $p^* \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ which supports $\overline{\alpha}^* - (1+\pi)\,\omega - \alpha_0^* b$ in $\widetilde{P}(\pi)$. Moreover, since $b \in \mathbb{R}^m_{++}$, $p^*$ can be normalized to $p^* \in S$. Thus, for any $\overline{\alpha} - (1+\pi)\,\underline{\alpha} - \alpha_0 b \in \widetilde{P}(\pi)$, $p^*\overline{\alpha}^* - (1+\pi)\,p^*\omega - \alpha_0^* \geqq p^*\overline{\alpha} - (1+\pi)\,p^*\underline{\alpha} - \alpha_0$ holds. By the cone property of $\widetilde{P}(\pi)$, $p^*\overline{\alpha}^* - (1+\pi)\,p^*\omega - \alpha_0^* = 0$. Moreover, $p^*\overline{\alpha}^* - (1+\pi)\,p^*\underline{\alpha}^* - \alpha_0^* = 0$ by $\underline{\alpha}^* \leqq \omega$ with $\underline{\alpha}^* \not< \omega$. This implies that $\alpha^*$ realizes the maximal profit rate at $(p^*,1)$, and $\pi$ is the associated maximal profit rate.

Let $(\omega^\nu)_{\nu\in N}$ be a distribution of initial endowments such that $\sum_{\nu\in N}\omega^\nu = \omega$. Then, by the cone property of $P$, $\alpha^{*\nu} + \beta^{*\nu} \equiv \frac{p^*\omega^\nu}{p^*\omega}\alpha^*$ and $\gamma^{*\nu} = 1$ constitute an individually optimal solution for each $\nu \in N$. Also, by the property of $\omega$ and $\alpha^*$, Definition 1(b) and 1(d) hold. Finally, since $\alpha_0(\omega) \leqq |N|$, Definition 1(c) holds. Thus, $\left((p^*,1), (\alpha^{*\nu};\ \beta^{*\nu};\ \gamma_0^{*\nu})_{\nu\in N}\right)$ constitutes an RS. ■

Note that the set $\mathbb{C}^*$ introduced in Yoshihara (2010) is a subset of $\mathbb{C}^{**}$, and it can be shown as a corollary of the above Proposition 1 that there exists an RS $(p, 1)$ associated with the maximal profit rate $\pi = 0$ if and only if $\omega \in \mathbb{C}^*$.

# 4 References

Yoshihara, N. (2010): "Class and Exploitation in General Convex Cone Economies," *Journal of Economic Behavior and Organization* **75**, 281-296.