Characterizations of Bargaining Solutions in Production Economies with Unequal Skills*

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Abstract

In production economies with unequal skills, this paper characterizes bargaining solutions by using axioms on allocation rules rather than axioms on classical bargaining solutions. We introduce a new axiom, *Consistency w.r.t. Technological Innovations*, so that the *non-welfaristic* characterizations of bargaining solutions in the production economies are provided. By the characterizations, we can classify the three bargaining solutions (the *Nash*, the *Kalai-Smorodinsky*, and the *Egalitarian solutions*) from the viewpoint of *responsibility* and *compensation* discussed by Dworkin. *Journal of Economic Literature* Classification Numbers: C71, C78, D 63, D 71.

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The running title of this paper: **Bargaining Solutions in Production**

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1 Introduction

In this paper we consider resource allocation problems in production economies with possibly unequal skills, as well as with variable commodities, in which the change in the types of produced commodities is due to the change in production technology. Assuming that the resource allocation is determined via bargaining among individuals, we axiomatically characterize *bargaining solutions* in those economies. However, in contrast to the classical bargaining theory originating with Nash [16], we focus on *allocation rules*, each of which maps each economy to a subset of feasible allocations whose utility values are just the bargaining outcomes, by adopting the axioms which refer explicitly to concrete data on underlying economic environments, rather than just to the geometric data of utility possibility sets. Such an approach is useful to make clear *non-welfaristic properties* of bargaining solutions beyond the *welfaristic* discussions in the Nash-type classical approach. For example, in our setting of production economies, this approach may make it possible to discuss the important issues of whether and/or how the inequality of individuals' production skills, the individuals' developments of "*expensive tastes*" (Dworkin [6, 7]) for which they should be responsible, and the effect of technological innovation respectively should influence the bargaining outcome. Such issues disappear in the classical approach, because of its implicit imposition of the axiom of *Welfarism* (Roemer [20]) which requires solutions to assign the same utility allocation to all the economies giving rise to the same utility possibility sets.

It was Roemer [18, 20] who studied characterizations of bargaining solutions in pure exchange economies with possibly unequal consumption abilities by using the axioms referring to economic information. Through this approach, the non-welfaristic distributive justice was shown to be logically connected with the ethically opposite welfaristic distributive justice. However, Roemer's characterizations rely on a strong axiom, *Consistency of Resource Allocation across Dimension* (**CONRAD**) (Roemer [18, 20]), which is logically implied by the axiom of Welfarism, but the converse relationship also holds on the domain of solutions Roemer considers.

In their recent research, Chen and Maskin [3] characterized the *egalitarian solution* (Kalai [12]) in economic environments with the possibility of individual production without imposing **CONRAD**. Their characterization constitutes a strengthening of Kalai's (Kalai [12]) in the sense that their used axioms are each weaker than their counterpart axioms in Kalai [12]. However, their axioms are not only much stronger than their counterpart axioms in Roemer [19, 20], but also rather inclined to welfarism in the sense that they refer only to each individual's payoff function, which is the composition of each individual's underlying production function and his underlying utility function. So, their axioms have no interest in discriminating the change in payoff functions due to a change in either production functions or utility functions. Moreover, their characterization depends on their rather specific settings of production economies and bargaining situations. In their economic models, production is done individually with each one's using his own production function without using his own labor. Moreover, there is no bargaining regarding distribution of outputs among individuals. So, their result is irrelevant to ethically interesting bargaining problems in cooperative production economies discussed by Moulin and Roemer [15] and Roemer [19].

In contrast to the above works, we model production economies in a more general setting, which can be applied to both the individual production and the cooperative production problems. Particularly, our model is relevant to bargaining problems over the compensation for low skills. By adopting the same axioms as ones in Roemer [19, 20], and introducing a new axiom, *Consistency with respect to Technological Innovation* (**CTI**), instead of adopting **CONRAD**, we provide full characterizations of three classical bargaining solutions in such production economies: the *Nash* (Nash [16]); the *Kalai-Smorodinsk*y (Kalai and Smorodinsky [13]); and the *egalitarian solutions*. **CTI** considers the situation in which a technological innovation occurs, which only involves an invention of a new commodity, so that the new commodity can be produced through the new production technology. However, other characteristics of the economy remain essentially unchanged. Moreover, the appearance of the new commodity does not lead to enhance the potentiality of individuals' welfare. Then, the allocation problem in the new economy should be treated in the coherent way with the allocation problem in the original economy. This is one motivation behind **CTI**. **CTI** also does not imply the axiom of Welfarism even on the domain of solutions which is similar to Roemer's. Thus, all our characterizations provide non-welfaristic foundations of classical bargaining solutions in production economies.

According to the obtained characterizations, we can classify the three solutions from the viewpoint of *responsibility* and *compensation* (Fluerbaey and Maniquet [8, 9]). As in Fluerbaey and Maniquets' settings, here we also assume that each individual is responsible not for his production skill, but for his utility function. We can interpret one axiom as being relevant to responsibility for utility functions, and the other three axioms as being relevant to compensation for low skills. Then, our characterizations show that (1) there is no bargaining solution that meets both the responsibility and the strongest compensation axioms, (2) the egalitarian solution meets the strongest compensation, but no responsibility axiom, and (3) the Nash and the Kalai-Smorodinsky solutions meet both the responsibility and the weaker compensation axioms. From this insight, we may understand that the egalitarian solution is essentially irrelevant to Dworkin's [7] *equality of resources*, while the Nash and the Kalai-Smorodinsky solutions share the same properties with respect to responsibility and compensation.

We should note two points about our characterizations. First, our result on the Nash solution is a strengthening of Roemer's [20] corresponding result, since our characterization is obtained without relying on the stronger version of **CONRAD**, which is essentially equivalent to **Nash IIA** (Nash [16]), while Roemer's relied on it. Second, we show that the characterization of Kalai-Smorodinsky solution in more than two-person problems is different from that in two-person problems even in the context of economic environments. It looks analogous to the arguments on the Kalai-Smorodinsky solution in two- and more than two-person problems under the classical fashion (Kalai and Smorodinsky [13], Roth [21], and Thomson [22]).

For the rest of this paper, section 2 defines a basic model of economies, allocation rules, and classical bargaining solutions. Section 3 introduces the axioms on allocation rules. Section 4 provides the characterizations of the three mentioned bargaining solutions. For the sake of expositional convenience, all the involved proofs are relegated into Appendix.

2 Model

There are (possibly) infinitely many types of commodities and one type of labor input, which is measured in efficiency units and denoted by $x \in \mathbb{R}_+$, to be used to produce commodities, where $\mathbb{R}_+$ denotes the set of non-negative real numbers.[1] The universe of "potential commodities" is denoted by C, and the class of non-empty and finite subsets of C is designated by $\mathcal{M}$, with generic elements, K, L, M,...,. The cardinality of $M \in \mathcal{M}$ is denoted by $\#M$. Given $M \in \mathcal{M}$, let $\mathbb{R}^m_+$, where $m = \#M$, designate the Cartesian product of $\#M$ copies of $\mathbb{R}_+$ indexed by the numbers of M.

[1] As well, $\mathbb{R}_{++}$ denotes the set of positive real numbers.

Given $M \in \mathcal{M}$, one technology that can produce up to M-goods is described by a *production possibility set* $Y \subseteq \mathbb{R}_+ \times \mathbb{R}_+^m$, where it is assumed that:

A.1 $\mathbf{0} \in Y$.

A.2 Y is closed, convex, and comprehensive.

A.3 $\exists t = (x, y) \in Y$ such that $\exists$ a commodity $f \in M$ s.t. $y_f > 0$.

The universal set of such production possibility sets which produce up to M-goods is denoted by $\mathcal{Y}^M$. Let $\mathcal{Y} \equiv \underset{M \in \mathcal{M}}{\cup} \mathcal{Y}^M$. Let $\partial Y \equiv \{(x, y) \in Y \mid \nexists (x', y') \in Y \text{ s.t. } (-x', y') \gg (-x, y)\}$.[2]

The population in the economy is given by the set $N = \{1, \cdots, n\}$, where $2 \leq n < +\infty$. Assume that all individuals have the same upper bound of *labor time* $\overline{x}$, $0 < \overline{x} < +\infty$. Let $\mathcal{U}^M$ be the set of all (real-valued) concave and continuous utility functions defined on $[0, \overline{x}] \times \mathbb{R}_+^m$, such that any $u \in \mathcal{U}^M$ is non-increasing in $[0, \overline{x}]$, non-decreasing in $\mathbb{R}_+^m$, $u(0, \mathbf{0}) = u(\overline{x}, \mathbf{0}) = 0$, and

$$\text{for all } (x, y) \in [0, \overline{x}] \times \mathbb{R}_+^m, \ \lim_{t \to \infty} (1/t) \cdot u(x, ty) = 0. \quad (2.1)$$

Each individual i is also characterized by a *production skill* which is represented by a non-negative real number, $s_i \in \mathbb{R}_+$. The universal set of production skills for all individuals is denoted by $\mathcal{S} \subseteq \mathbb{R}_+$. The production skill $s_i \in \mathcal{S}$ means i's labor input per unit of labor time which is measured in efficiency units. Thus, if his labor time is $x_i \in [0, \overline{x}]$, then it is $s_i x_i \in \mathbb{R}_+$ which implies his labor input into production possibility set measured in efficiency units.

Given $M \in \mathcal{M}$, an *economy with M-goods* is described by a list $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) = (M, (u_i)_{i \in N}, (s_i)_{i \in N}, Y)$, where $M \in \mathcal{M}$, $\mathbf{u} \in \mathcal{U}^{Mn}$, $\mathbf{s} \in \mathcal{S}^n$, $Y \in \mathcal{Y}^M$, and $\mathcal{U}^{Mn}$ and $\mathcal{S}^n$ stand, respectively, for the n-fold Cartesian product of $\mathcal{U}^M$ and that of $\mathcal{S}$. Let $\mathcal{E}^M$ be the class of all such economies with M-goods. Let $\mathcal{E} \equiv \underset{M \in \mathcal{M}}{\cup} \mathcal{E}^M$. Given $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}^M$, a vector $\mathbf{z} = (z_i)_{i \in N} \in ([0, \overline{x}] \times \mathbb{R}_+^m)^n$ is *feasible for* $\mathbf{e} \in \mathcal{E}^M$ if for all $i \in N$, $z_i = (x_i, y_i)$, and $(\sum s_i x_i, \sum y_i) \in Y$.[3] We denote by $Z(\mathbf{e})$ the set of feasible allocations for

[2] For any two vectors $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_p)$, $\mathbf{a} \geq \mathbf{b}$ if and only if $a_i \geq b_i$ $(i = 1, \ldots, p)$, $\mathbf{a} > \mathbf{b}$ if and only if $\mathbf{a} \geq \mathbf{b}$ and *not* $(\mathbf{b} \geq \mathbf{a})$, and $\mathbf{a} \gg \mathbf{b}$ if and only if $a_i > b_i$ $(i = 1, \ldots, p)$.

[3] Since the profiles of labor time and of production skills are respectively $(x_i)_{i \in N}$ and $(s_i)_{i \in N}$, the aggregate amount of labor input in efficiency units is $\sum s_i x_i$, which is transformed into M-commodities through the production possibility set Y.

$\mathbf{e} \in \mathcal{E}^M$. Let $Z(\mathcal{E}) \equiv \underset{\mathbf{e}\in\mathcal{E}}{\cup} Z(\mathbf{e})$. Given $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$ and $\mathbf{z} \in Z(\mathbf{e})$, $\mathbf{z}$ is a *Pareto efficient allocation for* $\mathbf{e}$ if there is no $\mathbf{z}' \in Z(\mathbf{e})$ such that $u_i(z_i') \geq u_i(z_i)$ for all $i \in N$, and $u_j(z_j') > u_j(z_j)$ for some $j \in N$. Given $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$ and $\mathbf{z} \in Z(\mathbf{e})$, $\mathbf{z}$ is a *weakly Pareto efficient allocation for* $\mathbf{e}$ if there is no $\mathbf{z}' \in Z(\mathbf{e})$ such that $u_i(z_i') > u_i(z_i)$ for all $i \in N$. Denote the set of Pareto efficient (resp. weakly Pareto efficient) allocations for $\mathbf{e}$ by $PE(\mathbf{e})$ (resp. $WPE(\mathbf{e})$). Given $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$, the *utility possibility set of* $\mathbf{e} \in \mathcal{E}$ is:

$$S(\mathbf{e}) \equiv \{\overline{\mathbf{u}} = (\overline{u}_i)_{i\in N} \in \mathbb{R}^n_+ \mid \exists\ \mathbf{z} = (z_i)_{i\in N} \in Z(\mathbf{e}), \forall i \in N, \overline{u}_i = u_i(z_i)\}.$$

Note that the utility possibility set $S(\mathbf{e})$ is a compact, comprehensive, convex set in $\mathbb{R}^n_+$ containing the origin.[4] Let $\Sigma \equiv \{S \subseteq \mathbb{R}^n_+ \mid \exists\ \mathbf{e} \in \mathcal{E},\ S = S(\mathbf{e})\}$ be the class of all such utility possibility sets.

Let $d = \mathbf{0} \in \mathbb{R}^n_+$ denote the *disagreement point* in this society. We identify a pair of the utility possibility set, S, and the disagreement point d as a *bargaining game*. Then, a *bargaining solution* is a function $F : \Sigma \times \{d\} \to \mathbb{R}^n_+$ such that for every $S \in \Sigma$, $F(S, d) \in S$. Since $d = \mathbf{0}$ by the assumption of $u_i(0, \mathbf{0}) = 0$ for all $i \in N$,[5] we write only $F(S)$ instead of $F(S, d)$. The universal set of bargaining solutions is denoted by $\mathcal{F}$.

An *allocation rule* is a correspondence $\varphi : \mathcal{E} \twoheadrightarrow Z(\mathcal{E})$ which associates to each $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$, a non-empty subset $\varphi(\mathbf{e})$ of $Z(\mathbf{e})$. The allocation rule φ is assumed to be *essentially a function*; that is, for all $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$, if $\mathbf{z} \in \varphi(\mathbf{e})$ and $\mathbf{z}' \in \varphi(\mathbf{e})$, then $\mathbf{u}(\mathbf{z}) = \mathbf{u}(\mathbf{z}')$, where $\mathbf{u}(\mathbf{z}) = (u_i(z_i))_{i\in N}$ and $\mathbf{u}(\mathbf{z}') = (u_i(z_i'))_{i\in N}$. Moreover, φ is assumed to be a *full correspondence*; that is, for all $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$, if $\mathbf{z} \in \varphi(\mathbf{e})$, $\mathbf{z}' \in Z(\mathbf{e})$, and $\mathbf{u}(\mathbf{z}) = \mathbf{u}(\mathbf{z}')$, then $\mathbf{z}' \in \varphi(\mathbf{e})$. The allocation rule φ *attains* a bargaining solution F if for all $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$, $\mu_\varphi(\mathbf{e}) = F(S(\mathbf{e}))$, where $\mu_\varphi(\mathbf{e}) \equiv \mathbf{u}(\varphi(\mathbf{e}))$. Denote the class of all the allocation rules, each of which attains some bargaining solution, by $\Phi_{\mathcal{F}}$.

[4] Note that even if every utility function is assumed to be strictly monotonic, there is no guarantee that every opportunity set $S(\mathbf{e})$ in this paper is strictly comprehensive; that is, the boundary of the opportunity set may in general include weakly Pareto efficient utility allocations. This is because we consider production economies with possibilities of joint productions. To guarantee strict comprehensiveness of all possible opportunity sets, we should restrict the class of utility functions $\mathcal{U}^M$ into the ones discussed in Diamantaras and Wilkie [5].

[5] By this assumption of the fixed disagreement point, the following analyses of this paper are free from the impossibility theorems in Conley, McLean, and Wilkie [4].

3 Axioms using Economic Information

Let $\mathcal{E}^{M*} \subseteq \mathcal{E}^M$ be the class of economies with M-goods, whose utility possibility sets are strictly comprehensive. Let $\mathcal{E}^* \equiv \underset{M\in\mathcal{M}}{\cup} \mathcal{E}^{M*}$. The following is a sufficient condition for $\mathbf{e}$ to belong to $\mathcal{E}^*$, which is essentially due to Diamantaras and Wilkie [5]:

Lemma 1: *For any* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}^M$, $\mathbf{e} \in \mathcal{E}^{M*}$ *is guaranteed if* $u_i \in \mathcal{U}^M$ $(\forall i \in N)$ *and* $Y \in \mathcal{Y}^M$ *have the following properties:*

(1) $[(x, y) \in Y$ and $x' > x] \Rightarrow [\exists y' > y$ such that $(x', y') \in Y]$;

(2) for all $(x, y), (x', y') \in [0, \overline{x}] \times \mathbb{R}^m_+$ *with* $y \neq \mathbf{0}$ *and* $y' = \mathbf{0}$, $u_i(x, y) > u_i(x', y')$*;*

(3) for all $(x, y), (x', y) \in [0, \overline{x}] \times \mathbb{R}^m_+$ *with* $y \neq \mathbf{0}$, *if* $x < x'$, *then* $u_i(x, y) > u_i(x', y)$

(4) for all $(x, y), (x', y') \in [0, \overline{x}] \times \mathbb{R}^m_+$ *with* $y \neq \mathbf{0}$ *and* $y' \neq \mathbf{0}$, *and all* $\lambda \in (0, 1)$, *if* $u_i(x, y) \geq u_i(x', y')$ *and* $(x, y) \neq (x', y')$, *then*

$$u_i(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y') > u_i(x', y').$$

Proof. To get $\mathbf{e} \in \mathcal{E}^{M*}$, it is sufficient to show that there is no weakly Pareto efficient allocation $\mathbf{z}$ for $\mathbf{e}$, which is weakly dominated by $\mathbf{z}' \in Z(\mathbf{e})$ with $\mathbf{u}(\mathbf{z}') \gg \mathbf{0}$, since $S(\mathbf{e})$ is convex. Suppose that there exists a weakly Pareto efficient allocation $\mathbf{z}$ for $\mathbf{e}$, which is weakly dominated by $\mathbf{z}' \in Z(\mathbf{e})$ with $\mathbf{u}(\mathbf{z}') \gg \mathbf{0}$. Hence, for all $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, and for some $j \in N$, $u_j(z'_j) > u_j(z_j)$. Note that $\mathbf{u}(\mathbf{z}') \gg \mathbf{0}$ implies for all $h \in N$, $y'_h \neq \mathbf{0}$. Suppose $\mathbf{y}' = \mathbf{y}$, that is, for all $h \in N$, $y'_h = y_h$. Then, $0 \leq x'_j < x_j \leq \overline{x}$. So, by (1), some $h' \in N \backslash \{j\}$ must be such that $x'_{h'} > x_{h'}$, so $u_{h'}(z'_{h'}) < u_{h'}(z_{h'})$ by (3), a contradiction. Thus, $\mathbf{y}' \neq \mathbf{y}$. Let $\mathbf{z}'' \equiv \lambda\mathbf{z} + (1 - \lambda)\mathbf{z}'$ for some $\lambda \in (0, 1)$. Since Y is convex, $\mathbf{z}'' \in Z(\mathbf{e})$. If $y_h = \mathbf{0}$, then by (2), $u_h(z''_h) > u_h(z_h)$, since $y''_h \neq \mathbf{0}$ by $y'_h \neq \mathbf{0}$. If $y_h \neq \mathbf{0}$, then by (4), $u_h(z''_h) > u_h(z_h)$. Thus, $\mathbf{z}''$ strictly dominates $\mathbf{z}$, which is a desired contradiction. ■

Since the class of utility functions $\mathcal{U}^M$ has, as its elements, the utility functions having the above properties in **Lemma 1**, it is shown by **Lemma 1** that $\mathcal{E}^*$ is non-empty.

From the next sections, we will provide characterizations of allocation rules φ in $\mathcal{E}^*$ as well as in $\mathcal{E}$. The domain assumptions on φ are:

Axiom $D^{\mathcal{E}}$: *The allocation rule* φ *is a full correspondence which is essentially a function and is defined on the class of economies* $\mathcal{E}$.

Axiom $D^{\mathcal{E}^*}$: *The allocation rule* φ *is a full correspondence which is essentially a function and is defined on the class of economies* $\mathcal{E}^*$.

The following are well-known axioms on allocation rules:

Welfarism (W): *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M', \mathbf{u}', \mathbf{s}', Y') \in \mathcal{E}$, *if* $S(\mathbf{e}) = S(\mathbf{e}')$, *then* $\mu_\varphi(\mathbf{e}) = \mu_\varphi(\mathbf{e}')$.

Pareto Efficiency (PE): *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$ *and* $\mathbf{z} \in \varphi(\mathbf{e})$, $\mathbf{z}$ *is a* Pareto efficient allocation for $\mathbf{e}$.

Weak Pareto Efficiency (WPE): *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$ *and all* $\mathbf{z} \in \varphi(\mathbf{e})$, $\mathbf{z}$ *is a* weak Pareto efficient allocation for $\mathbf{e}$.

Weak Equal Treatment of Equals (WETE): *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$, *if* $u_i = u_j$ *and* $s_i = s_j$ *for all* $i, j \in N$, *then for all* $\mathbf{z} \in \varphi(\mathbf{e})$, $u_i(z_i) = u_j(z_j)$ *for all* $i, j \in N$.

Note that all allocation rules in $\Phi_{\mathcal{F}}$ satisfy **Axiom** $D^{\mathcal{E}}$ and **W**. Moreover, since most meaningful bargaining solutions recommend (weakly) Pareto efficient utility allocations, we should take notice of allocation rules in $\Phi_{\mathcal{F}}$ satisfying **WPE** (or **PE**). If we are interested in bargaining solutions having the symmetric property in the sense that recommending equal utilities for symmetric utility possibility sets, then we should pay attention to allocation rules in $\Phi_{\mathcal{F}}$ satisfying at least **WETE**.

3.1 Axioms on Technological Changes

In this section, we introduce five axioms, each of which stipulates the performance of allocation rules, faced with a particular kind of change in production technology.

Technological Monotonicity (TMON)[6]**:** *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}, Y') \in \mathcal{E}$ *such that* $Y' \supseteq Y$, *and all* $\mathbf{z} \in \varphi(\mathbf{e})$ *and all* $\mathbf{z}' \in \varphi(\mathbf{e}')$, $u_i(z_i) \leq u_i(z_i')$ *for all* $i \in N$.

Given $Y \in \mathcal{Y}^M$ and $f \in M$, let

$$P_f(Y) \equiv \left\{(x, y_f) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \exists y_{-f} \in \mathbb{R}_+^{m-1} : (x, y_f, y_{-f}) \in Y\right\}.$$

[6]This axiom was first introduced by Roemer [19].

Individual Technological Monotonicity (ITMON): *Let* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}, Y') \in \mathcal{E}$ *be such that* $Y' \supseteq Y$, *where there exists a unique commodity* $f \in M$ *such that* $P_f(Y) \subsetneq P_f(Y')$ *and, for any other* $f' \in M \backslash \{f\}$, $P_{f'}(Y) = P_{f'}(Y')$. *Moreover, the commodity* f *is liked only by the agent* $j \in N$. *Then, for all* $\mathbf{z} \in \varphi(\mathbf{e})$ *and all* $\mathbf{z}' \in \varphi(\mathbf{e}')$, $u_j(z_j) \leq u_j(z_j')$.

Given $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}$ and $i \in N$, let

$$m^i(Z(\mathbf{e})) \equiv \left\{ z_i' \in [0, \overline{x}] \times \mathbb{R}_+^m \mid z_i' = \underset{\mathbf{z} \in Z(\mathbf{e}),\ z_i \text{ is } i\text{-th component of } \mathbf{z}}{\arg\max} u_i(z_i) \right\}.$$

Weak Technological Monotonicity (WTMON): *Let* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}, Y') \in \mathcal{E}$ *be such that* $Y' \supseteq Y$, *and* $m^i(Z(\mathbf{e})) \cap m^i(Z(\mathbf{e}')) \neq \varnothing$ *for all* $i \in N$. *Then, for all* $\mathbf{z} \in \varphi(\mathbf{e})$ *and all* $\mathbf{z}' \in \varphi(\mathbf{e}')$, *we have* $u_i(z_i) \leq u_i(z_i')$ *for all* $i \in N$.

Independence of Technological Contraction (ITC)[7]**:** *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}, Y') \in \mathcal{E}$ *such that* $Y' \subseteq Y$, *and all* $\mathbf{z} \in \varphi(\mathbf{e})$, *if* $\mathbf{z} \in Z(\mathbf{e}')$, *then* $\mathbf{z} \in \varphi(\mathbf{e}')$.

Each of these four axioms on allocation rules has its bargaining theory counterpart which is imposed upon utility possibility sets: **Monotonicity** (Kalai [12]) for **TMON**; **Individual Monotonicity** (Kalai and Smorodinsky [13]) for **ITMON** and **WTMON**; and **Nash IIA** (Nash [16]) for **ITC**. It is easy to see that the four axioms are each much weaker than their bargaining theory counterparts.

The next axiom we introduce imposes coherence of allocation rules as we vary the domain of production possibility sets. Given $(x, y) \in [0, \overline{x}] \times \mathbb{R}_+^m$ and $u_i \in \mathcal{U}^M$, let there be $K \subsetneq M$ such that for all $y_K' \equiv (y_f')_{f \in K} \in \mathbb{R}_+^k$, $u_i(x, y_K', y_{M \backslash K}) = u_i(x, y_K, y_{M \backslash K})$, where $y_K \equiv (y_f)_{f \in K}$. Then, we say that *agent* $i \in N$ *is indifferent to each good of* $K \subsetneq M$ *at* (x, y). Given $Y \in \mathcal{Y}^M$ and $K \subsetneq M$, let

$$P_{M \backslash K}(Y) \equiv \left\{ (x, y_{M \backslash K}) \in \mathbb{R}_+ \times \mathbb{R}_+^{m-k} \mid \exists y_K \in \mathbb{R}_+^k : (x, y_K, y_{M \backslash K}) \in Y \right\}.$$

Consistency w.r.t. Technological Innovation (CTI): *Let* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y) \in \mathcal{E}^M$, *and let* $\widehat{\mathbf{z}} = (\widehat{x}_i, \widehat{y}_{Mi})_{i \in N} \in \varphi(\mathbf{e})$ *be weakly Pareto efficient.*[8] *Let* $\mathbf{e}' =$

[7] This axiom was first introduced by Moulin [14] with the name of *IIA*.

[8] When we discuss dimensional changes in consumption spaces, we often denote individual i's consumption vector of M-goods by $y_{Mi} \in \mathbb{R}_+^m$.

$(M \cup L, \mathbf{u}', \mathbf{s}, Y') \in \mathcal{E}^{M\cup L}$, *where* $M \cap L = \varnothing$, *be such that (1)* $P_M(Y') = Y$, (2) *for any* $\mathbf{z} = (x_i, y_{Mi})_{i\in N} \in WPE(\mathbf{e})$, *there exists* $(y_{Li}(\mathbf{z}))_{i\in N} \in \mathbb{R}^{nl}_+$ *such that*

$$u'_i(x_i, y_{Mi}, y_{Li}(\mathbf{z})) = u_i(x_i, y_{Mi}) \ (\forall i \in N) \text{ and } (x_i, y_{Mi}, y_{Li}(\mathbf{z}))_{i\in N} \in WPE(\mathbf{e}'),$$

and (3) every agent $i \in N$ *is indifferent to each good of* L *at* $(\widehat{x}_i, \widehat{y}_{Mi}, \mathbf{0})$. *Then,* $(\widehat{x}_i, \widehat{y}_{Mi}, \mathbf{0})_{i\in N} \in \varphi(\mathbf{e}')$.

A motivation for **CTI** is presented as follows: let, in an economy $\mathbf{e}$ with M-producible commodities, $\widehat{\mathbf{z}} = (\widehat{x}_i, \widehat{y}_{Mi})_{i\in N}$ be a recommendation by the allocation rule φ, and be a (weakly) Pareto efficient allocation. Next, let the economy change from $\mathbf{e} \in \mathcal{E}^M$ to $\mathbf{e}' \in \mathcal{E}^{M\cup L}$, where the economy $\mathbf{e}'$ inherits from $\mathbf{e}$ the characteristics of agents' preferences and production technology on M-commodities in the intimate way that **CTI** postulates. The main difference between $\mathbf{e}$ and $\mathbf{e}'$ comes from the technological change from Y to Y' which makes it possible to consume the new commodities L, although it is a useless innovation in the sense that nobody's opportunity for welfare is enlarged. Then, it may be reasonable that, in the new economy, every agent is guaranteed at least his welfare which is enjoyed by consuming M-commodities in the original economy. It follows from this view that $(\widehat{x}_i, \widehat{y}_{Mi}, \mathbf{0})_{i\in N}$ is a recommendation of φ in $\mathbf{e}'$. In fact, by this new recommendation, nobody loses anything from the environmental change, since, in the new economy, nobody wants to consume L-commodities, and $(\widehat{x}_i, \widehat{y}_{Mi}, \mathbf{0})_{i\in N}$ is (weakly) Pareto efficient.

[Insert Figure 1]

Remark 1: The axiom **W** implies **CTI** under **Axiom** $D^{\mathcal{E}}$ (resp. $D^{\mathcal{E}^*}$), but **CTI** together with **Axiom** $D^{\mathcal{E}}$ (resp. $D^{\mathcal{E}^*}$) does not imply **W**, as shown in **Examples 1**, **2**, and **3** later.

Since all allocation rules in $\Phi_{\mathcal{F}}$ satisfy **W** and **Axiom** $D^{\mathcal{E}}$, the rules in $\Phi_{\mathcal{F}}$ also satisfy **CTI** by Remark 1.

3.2 Axioms on Responsibility and Compensation

In this section, we introduce axioms which are related to the arguments of responsibility and compensation.

3.2.1 Axiom on Responsibility

The first axiom seems to be relevant to *responsibility for individual's utility function*. To define it, let us begin with introducing a few notions: Given $M \in \mathcal{M}$, note that for any utility function $u \in \mathcal{U}^M$, there is a *utility-unit* $b^u \in \mathbb{R}_+$, by which the level of utility assigned by the function u is measured: that is, if $u(z) = b^u$ for some z, it implies that the level of utility $u(z)$ is just "one."[9] Then, for each utility-unit b^u, there is a corresponding set $B(u) \subsetneq [0, \overline{x}] \times \mathbb{R}^m_+$ of *base-consumption for* u such that for all $z \in B(u)$, $u(z) = b^u$.

Now, let us take any two utility functions $u, u' \in \mathcal{U}^M$ for which there is a positive scalar $\lambda > 0$ such that $u' = \lambda \cdot u$. If $\lambda = \frac{b^{u'}}{b^u}$, then u' is just obtained by a *change in utility-units* from b^u to $b^{u'}$, so that u and u' are essentially the same utility representation. In this case, note that $B(u) = B(u')$. In contrast, if $b^u = b^{u'}$, then the change from u to u' can be explained not by the change in utility-units, but rather by a *change in utility intensity*. Note that if the change from u to u' comes from the change in utility intensity, then we have $B(u) \neq B(u')$ and $B(u) \cap B(u') = \varnothing$.[10]

One typical example of the above change in utility intensity is the case of individual development of "*expensive taste*," which was discussed by Dworkin [6]. Consider a case in which an individual develops his expensive taste, so that even if his underlying preference ordering and his risk attitude are invariant, he can no longer enjoy the same level of welfare as he did before developing his expensive taste, without receiving a larger consumption vector than before.[11] This case is simply formulated as a process of a linear transformation of a utility function via a change in utility intensity.

Let us now introduce the first axiom. Given $M \in \mathcal{M}$ and $\mathbf{u} \in \mathcal{U}^{Mn}$, let $\mathbf{b^u} \equiv (b^{u_i})_{i \in N}$. Then, the first axiom is defined as follows:

[9] The author owes the introduction of utility-units in defining the following two axioms to one of the referees in this journal.

[10] We can start from listing $B(u)$ instead of b^u as primitive data. Then, by comparing $B(u)$ with $B(u')$, we can see which type of change occurs when u and u' are correlated by a linear transformation: if $B(u) = B(u')$, it is a change in utility-units, while otherwise, it involves a change in utility intensity.

[11] In this explanation, it is not necessary to assume interpersonal comparability of utilities. The notion of change in utility intensity only presumes intrapersonal comparison of utilities, which seems to be a natural requirement whenever utility functions are cardinally measurable.

Independence of Utility Intensities (IUI): *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}', \mathbf{s}, Y) \in \mathcal{E}$ *with* $\mathbf{b}^{\mathbf{u}} = \mathbf{b}^{\mathbf{u}'}$, *if there exists a vector* $\mathbf{a} = (a_i)_{i \in N} \in \mathbb{R}^n_{++}$ *such that* $u'_i = a_i \cdot u_i$ *for all* $i \in N$, *then* $\varphi(\mathbf{e}) = \varphi(\mathbf{e}')$.

Our motivation for this axiom as one of responsibility is presented as follows: In production economies with differences in production skills, but without differences in consumption abilities among agents, it seems to be that the change in utility intensity of any agent is not a subject for social compensation, but a matter of personal responsiblity. So, the allocation rule should not take into account such an environmental change in determining resource allocations.[12]

Note that there is a similar axiom, *Cardinal Non-comparability*, which was first introduced by Roemer [20] as expressing exactly what Nash intended with his axiom of *Scale Invariance* (Nash [16]). The motivation of Cardinal Non-comparability can be formulated in our model as follows:

Utility-units Invariance (UUI): *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}', \mathbf{s}, Y) \in \mathcal{E}$ *with* $\mathbf{b}^{\mathbf{u}} \neq \mathbf{b}^{\mathbf{u}'}$, *if* $u'_i = \frac{b^{u'_i}}{b^{u_i}} \cdot u_i$ *for all* $i \in N$, *then* $\varphi(\mathbf{e}) = \varphi(\mathbf{e}')$.

Although both **IUI** and **UUI** are respectively implied by Nash's Scale Invariance axiom, their motivations are completely different from each other.

3.2.2 Axioms on Compensation

In contrast with the above axiom, the following axioms are interpreted as ones of compensation, where the motivation behind them should be clear:

Skill Solidarity (SS)[13]**:** *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}', Y) \in \mathcal{E}$, *and all* $\mathbf{z} \in \varphi(\mathbf{e})$ *and all* $\mathbf{z}' \in \varphi(\mathbf{e}')$, *either* $u_i(z_i) \leq u_i(z'_i)$ *for all* $i \in N$, *or* $u_i(z_i) \geq u_i(z'_i)$ *for all* $i \in N$.

Skill Monotonicity (SM)[14]**:** *For all* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}', Y) \in \mathcal{E}$ *such that* $\mathbf{s} \leq \mathbf{s}'$, *and all* $\mathbf{z} \in \varphi(\mathbf{e})$ *and* $\mathbf{z}' \in \varphi(\mathbf{e}')$, $u_i(z_i) \leq u_i(z'_i)$ *for all* $i \in N$.

[12] As discussed above, we may connect the situation that someone's utility intensity decreases with the development of an "expensive taste." Then, **IUI** requires that this person should not be compensated by the allocation rule for his decrease of utility-productivity because of his developed expensive taste.

[13] This axiom was originated by Fleurbaey and Maniquet [9].

[14] This axiom was also originated by Fleurbaey and Maniquet [9].

Independence of Skill Endowments (ISE): *Let* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y), \mathbf{e}' = (M, \mathbf{u}, \mathbf{s}', Y) \in \mathcal{E}$ *be such that* $Z(\mathbf{e}) = Z(\mathbf{e}')$. *Then,* $\varphi(\mathbf{e}) = \varphi(\mathbf{e}')$.

Remark 2: Note that **W** implies **ISE** under **Axiom** $D^{\mathcal{E}}$ (resp. $D^{\mathcal{E}^*}$), but not the converse.

4 Characterization Results

By using the axioms introduced in the previous sections, we will provide characterizations of allocation rules in $\Phi_{\mathcal{F}}$. First, we will provide a basic lemma which is useful to show axiomatic characterizations of any allocation rule in $\Phi_{\mathcal{F}}$.

Lemma 2: *Let* $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathcal{E}$ *be such that* $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^{M*}$ (resp. $\mathcal{E}^{M}$)*,* $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^{L*}$ (resp. $\mathcal{E}^{L}$), *and* $S(\mathbf{e}_1) = S(\mathbf{e}_2)$. *Then, the allocation rule* φ *which satisfies* $D^{\mathcal{E}^*}$ (resp. $D^{\mathcal{E}}$), **PE** (resp. **WPE**), *and* **CTI** *has the following property:* $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_2)$.

Remark 3: The above result of **Lemma 2** depends on imposition of **PE** (resp. **WPE**). The following example shows that only $D^{\mathcal{E}^*}$ and **CTI** without **PE** cannot lead to the result of **Lemma 2**:

Example 1: Let $\#N = 2$. Let $\mathbf{e}_1^{\triangle} = (K, \mathbf{v}^1, \mathbf{s}, Y^{(\#K)}) \in \mathcal{E}^{K*}$ and $\mathbf{e}_2^{\triangle} = (K', \mathbf{v}^2, \mathbf{s}, Y^{(\#K')}) \in \mathcal{E}^{K'*}$ such that $K \cap K' = \varnothing$, $\#K = \#K' = 1$, $Y^{(\#K)} \equiv \mathbb{R}_+ \times [0,1]$, $Y^{(\#K')} \equiv \mathbb{R}_+ \times [0,1]$, and for all $i \in N$, the utility functions $v_i^1 : [0, \overline{x}] \times \mathbb{R}_+^{\#K} \to \mathbb{R}$ and $v_i^2 : [0, \overline{x}] \times \mathbb{R}_+^{\#K'} \to \mathbb{R}$ are defined as follows:

$$\forall (x, y_K) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K},\ v_i^1(x, y_K) = \begin{cases} y_K \text{ if } y_K \in [0,1] \\ 1 \text{ otherwise} \end{cases},$$
$$\forall (x, y_{K'}) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K'},\ v_i^2(x, y_{K'}) = \begin{cases} y_{K'} \text{ if } y_{K'} \in [0,1] \\ 1 \text{ otherwise} \end{cases}.$$

Then, $S(\mathbf{e}_1^{\triangle}) = S(\mathbf{e}_2^{\triangle})$.

Construct $\mathbf{e}^* \equiv \mathbf{e}_1^{\triangle} \wedge \mathbf{e}_2^{\triangle} = (K \cup K', \mathbf{v}^*, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')})$ where for all $i \in N$, $v_i^*(x, y_K, y_{K'}) = \min\{v_i^1(x, y_K), v_i^2(x, y_{K'})\}$, and $Y^{(\#K)} \oplus Y^{(\#K')} \equiv \mathbb{R}_+ \times ([0,1] \times [0,1])$. We can see that $S(\mathbf{e}^*) = S(\mathbf{e}_1^{\triangle})$.

Let φ^{NW_1} be an allocation rule satisfying $D^{\mathcal{E}^*}$ and **CTI**, but not **PE**, such that:

$$\text{for } \mathbf{e}_1^{\triangle},\ \varphi^{NW_1}(\mathbf{e}_1^{\triangle}) \equiv \left\{ \left(\left(x_1, \frac{1}{2} \right), \left(x_2, \frac{1}{2} \right) \right) \in Z(\mathbf{e}_1^{\triangle})\,|x_i \in [0, \overline{x}]\ (\forall i \in N) \right\},$$
$$\text{for } \mathbf{e}_2^{\triangle},\ \varphi^{NW_1}(\mathbf{e}_2^{\triangle}) \equiv \left\{ \left(\left(x_1, \frac{1}{2} \right), \left(x_2, \frac{1}{2} \right) \right) \in Z(\mathbf{e}_2^{\triangle})\,|x_i \in [0, \overline{x}]\ (\forall i \in N) \right\},$$
$$\text{for } \mathbf{e}^*,\ \varphi^{NW_1}(\mathbf{e}^*) \equiv \left\{ \left(\left(x_1, y_1^K, y_1^{K'} \right), \left(x_2, y_2^K, y_2^{K'} \right) \right) \in Z(\mathbf{e}^*) \middle| \min\{y_i^K, y_i^{K'}\} = \frac{1}{4} \right.$$
$$\& \ x_i \in [0, \overline{x}]\ (\forall i \in N)\}.$$

Then, $\mathbf{z} = \left(\left(0, \frac{1}{2}, \frac{1}{4}\right), \left(0, \frac{1}{2}, \frac{1}{4}\right) \right) \in \varphi^{NW_1}(\mathbf{e}^*)$ and $\mathbf{z}' = \left(\left(0, \frac{1}{2}\right), \left(0, \frac{1}{2}\right) \right) \in \varphi^{NW_1}(\mathbf{e}_1^{\triangle})$. Thus, $\mu_{\varphi^{NW_1}}(\mathbf{e}^*) = \left(\frac{1}{4}, \frac{1}{4}\right)$ and $\mu_{\varphi^{NW_1}}(\mathbf{e}_1^{\triangle}) = \left(\frac{1}{2}, \frac{1}{2}\right)$, so that $\mu_{\varphi^{NW_1}}(\mathbf{e}^*) \ll \mu_{\varphi^{NW_1}}(\mathbf{e}_1^{\triangle})$. This implies φ^{NW_1} does not satisfy **W**.

It remains to show that φ^{NW_1} is surely consistent with **CTI**. First, φ^{NW_1} is consistent with **CTI** between $\mathbf{e}_1^{\triangle}$ and $\mathbf{e}^*$, since both agents are not indifferent to K'-good at $\left(x_i, \frac{1}{2}, 0\right)$. The same result holds between $\mathbf{e}_2^{\triangle}$ and $\mathbf{e}^*$. Next, let $\widehat{\mathbf{e}}_1 \equiv (K \cup K', \widehat{\mathbf{v}}^1, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')})$ where for all $i \in N$, $\widehat{v}_i^1(x, y_K, y_{K'}) = v_i^1(x, y_K)$. Then, by **CTI** and $D^{\mathcal{E}^*}$, $\widehat{\mathbf{z}}' = \left(\left(0, \frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right) \right) \in \varphi^{NW_1}(\widehat{\mathbf{e}}_1)$. Since $\widehat{v}_i^1 \geq v_i^*$, we can construct a new economy $\widetilde{\widehat{\mathbf{e}}}_1^{\triangle} \equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^{(\#R)})$ with $R \equiv \{R(1), R(2)\}$ by applying Howe's theorem (Howe [11, Proposition 3]),[15] where there exist $\widehat{w}_i \in \mathcal{U}^{K \cup K' \cup R}$ and $\widehat{y}_{R(1)}, \widehat{y}_{R(2)} \in \mathbb{R}_+$ such that for each $i \in N$, for all $(x, y_K, y_{K'}, y_{R(j)}) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K} \times \mathbb{R}_+^{\#K'} \times \mathbb{R}_+$,

$$\widehat{w}_i(x, y_K, y_{K'}, \widehat{y}_{R(i)}, y_{R(j)}) = \widehat{v}_i^1(x, y_K, y_{K'})$$
$$\widehat{w}_i(x, y_K, y_{K'}, 0, y_{R(j)}) = v_i^*(x, y_K, y_{K'}),$$

and $Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^{(\#R)} \equiv \mathbb{R}_+ \times \left([0,1] \times [0,1] \times [0, \widehat{y}_{R(1)}] \times [0, \widehat{y}_{R(2)}] \right)$. Note that since $S(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}) = S(\widehat{\mathbf{e}}_1) = S(\mathbf{e}_1^{\triangle})$, we see that $\widetilde{\widehat{\mathbf{e}}}_1^{\triangle} \in \mathcal{E}^*$. Since every agent i is indifferent to R-goods at $\widetilde{\widehat{\mathbf{z}}}' = \left(\left(0, \frac{1}{2}, \frac{1}{2}, \mathbf{0}\right), \left(0, \frac{1}{2}, \frac{1}{2}, \mathbf{0}\right) \right)$ under $\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}$, $\widetilde{\widehat{\mathbf{z}}}' \in \varphi^{NW_1}(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle})$ by **CTI**. Thus, $\mathbf{z}'' = \left(\left(0, \frac{1}{2}, \frac{1}{4}, \widehat{y}_{R(1)}, 0\right), \left(0, \frac{1}{2}, \frac{1}{4}, 0, \widehat{y}_{R(2)}\right) \right) \in \varphi^{NW_1}(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle})$ by fullness of φ^{NW_1}. The relationship between $\mathbf{z}$ and $\mathbf{z}''$ is vacuously consistent with **CTI**, since every agent i is not indifferent to R-goods at $\left(\left(0, \frac{1}{2}, \frac{1}{4}, \mathbf{0}\right), \left(0, \frac{1}{2}, \frac{1}{4}, \mathbf{0}\right) \right)$ under $\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}$. ■

[15] The Howe theorem used here is exactly the same version as that used in Roemer [18, 20].

Remark 4: Example 1 shows that $D^{\mathcal{E}^*}$ and **CTI** do not imply **W** unlike the result of Roemer [20], in which $D^{\mathcal{E}^*}$ and **CONRAD** imply **W** in pure exchange economies. Moreover, we can see that even $D^{\mathcal{E}^*}$, **CTI**, and **PE** (resp. **WPE**) together do not imply **W**. It is because **CTI** requires nothing whenever $\mathbf{e}$ and $\mathbf{e}'$ are different in production skills. In fact, **Examples 2** and **3** discussed in section 4.4 give us allocation rules satisfying $D^{\mathcal{E}^*}$, **PE**, and **CTI**, but not **W.**

4.1 Incompatibility between Responsibility and Strong Compensation Axioms of Bargaining Solutions

Our first theorem is of impossibility of bargaining solutions which satisfy both the responsibility and compensation requirements as follows:

Theorem 1: *There is no allocation rule in* $\Phi_{\mathcal{F}}$ *which satisfies* **PE**, **WETE**, **IUI**, *and* **SS**.[16][17]

In the above theorem, the axiom **WETE** is indispensable to keep the implication of the result as incompatibility between responsibility and compensation. Since if **WETE** is deleted, **SS** may have an implication opposite to compensation, as shown in the following possibility result on undesirable bargaining solutions:

Corollary 1: *There is a unique class of allocation rules in* $\Phi_{\mathcal{F}}$ *which satisfy* **PE**, **IUI**, *and* **SS.** *That is the class of* dictatorial rules $\{\varphi^{D_i}\}_{i \in N}$, *each of which attains one of the* dictatorial solutions (Roemer [20]).

These results encourage us to give up either **IUI** or **SS** to get "second best" symmetric bargaining solutions rather than to stick to them leading to the dictatorial rule.

[16] This impossibility result is relevant only to $\Phi_{\mathcal{F}}$. Once we look at rules beyond $\Phi_{\mathcal{F}}$, we can see that the *egalitarian-equivalent rule* (Pazner and Schmeidler [17]) satisfies the above four axioms. I thank F. Maniquet for pointing out this fact. Note that the egalitarian-equivalent rule does not satisfy **CTI**, so it cannot attain any bargaining solution.

[17] We can strengthen this theorem as follows:

Theorem 1*: *There is no allocation rule in* $\Phi_{\mathcal{F}}$ *which satisfies* **PE**, **WETE**, **IUI**, *and* **SM**.

4.2 Characterizations of the Egalitarian Solution

In this section, we will provide characterizations of the *egalitarian solution* (Kalai [12]) in production economies.

Definition 1: *A bargaining solution $E \in \mathcal{F}$ is the* egalitarian *if for any $S \in \Sigma$, $E(S)$ is a (*weak) Pareto efficient outcome on S, *and for any $i, j \in N$, $E_i(S) = E_j(S)$.*

Definition 2: *An allocation rule φ^E is the* egalitarian rule *if it attains the egalitarian solution: for all $\mathbf{e} \in \mathcal{E}$, $\mu_{\varphi^E}(\mathbf{e}) = E(S(\mathbf{e}))$.*

The following three theorems are on characterizations of φ^E in $\mathcal{E}^*$:

Theorem 2: *The allocation rule φ satisfies $D^{\mathcal{E}^*}$*, **PE**, **WETE**, **ISE**, **TMON**, *and* **CTI** *if and only if $\varphi = \varphi^E$.*

Theorem 3: *The allocation rule φ satisfies $D^{\mathcal{E}^*}$*, **PE**, **WETE**, **SS**, *and* **CTI** *if and only if $\varphi = \varphi^E$.*[18]

Theorem 4: *The allocation rule φ satisfies $D^{\mathcal{E}^*}$*, **PE**, **WETE**, **SM**, **TMON**, *and* **CTI** *if and only if $\varphi = \varphi^E$.*

As shown in **Theorem 3**, the egalitarian rule φ^E is obtained by keeping **SS** as well as by deleting **IUI**. In this sense, the egalitarian rule is inclined much more to compensation requirements than to responsibility ones. Roemer [18] discussed that in pure exchange economies with difference in consumption abilities, the egalitarian rule is equivalent to Dworkin's [7] proposal for equality of resources. However, once we mention that Dworkin's motivation for equality of resources was the viewpoint of responsibility and compensation, we may not accept the Roemer view at least in the context of production economies with skill differences, since the egalitarian rule has no property relevant to responsibility as shown in Theorems 2, 3, and 4.

[18] We can strengthen this theorem as follows:

Theorem 3*: *The allocation rule φ satisfies $D^{\mathcal{E}^*}$*, **PE**, **WETE**, **SM**, **ISE**, *and* **CTI** *if and only if $\varphi = \varphi^E$.*

4.3 Characterizations of the Nash and the Kalai-Smorodinsky Solutions

In contrast to the previous subsection, here we will keep **IUI**, and replace **SS** with a weaker compensation axiom, **ISE**. In such a way, we will characterize the *Nash solution* (Nash [16]) as well as the *Kalai-Smorodinsky solution* (Kalai and Smorodinsky [13]).

Definition 3: *A bargaining solution* $Na \in \mathcal{F}$ *is the* Nash solution *if for any* $S \in \Sigma$, $Na(S)$ *is equal to the maximizer in* $\overline{\mathbf{u}} \in S$ *of the "*Nash product*"* $\prod_{i \in N} \overline{u}_i$.

Definition 4: *An allocation rule* φ^{Na} *is the* Nash rule *if it attains the Nash solution: for all* $\mathbf{e} \in \mathcal{E}$, $\mu_{\varphi^{Na}}(\mathbf{e}) = Na(S(\mathbf{e}))$.

The following theorem provides a characterization of φ^{Na} in $\mathcal{E}$:

Theorem 5: *The allocation rule* φ *satisfies* $D^{\mathcal{E}}$, **PE**, **WETE**, **ISE**, **ITC**, **IUI**, *and* **CTI** *if and only if* $\varphi = \varphi^{Na}$.

This theorem constitutes a strengthening of the characterization of the Nash solution in Roemer [20], since Roemer's result depends on **SCONRAD** (Roemer [20]), which is equivalent to **Nash IIA** under $D^{\mathcal{E}}$, while which implies **ITC**, **ISE**, and **CTI** under $D^{\mathcal{E}}$.[19]

Given $S \in \Sigma$ and $i \in N$, let us define $m^i(S) \equiv \max \{\overline{u}_i \in \mathbb{R}_+ \mid \overline{\mathbf{u}} = (\overline{u}_h)_{h \in N} \in S\}$.

Definition 5: *A bargaining solution* $K \in \mathcal{F}$ *is the* Kalai-Smorodinsky solution *if for any* $S \in \Sigma$, $K(S)$ *is a (*weak) Pareto efficient outcome on S, *and there exists a unique value* $\lambda \in (0, 1]$ *such that* $K(S) = \lambda \cdot \mathbf{m}(S)$, $\mathbf{m}(S) \equiv (m^i(S))_{i \in N}$.

Definition 6: *An allocation rule* φ^K *is the* Kalai-Smorodinsky rule *if it attains the Kalai-Smorodinsky solution: for all* $\mathbf{e} \in \mathcal{E}$, $\mu_{\varphi^K}(\mathbf{e}) = K(S(\mathbf{e}))$.

[19] Binmore [2] showed that in two-person exchange economies, the only solution satisfying all the *Nash-like economic axioms* he defined is the Walrasian solution. Since his result depends on a stronger domain restriction and a stronger economic version of **Nash IIA** than **ITC**, we cannot obtain the same relationship between the Nash and the Walrasian solutions in our economic domain.

The following two theorems provide characterizations of φ^K in $\mathcal{E}$ and $\mathcal{E}^*$:

Theorem 6: *Let $\#N = 2$. Then, the allocation rule φ satisfies $D^{\mathcal{E}}$,* **PE**, **WETE**, **ISE**, **ITMON**, **IUI**, *and* **CTI** *if and only if $\varphi = \varphi^K$.*

Theorem 7: *Let $\#N \geq 2$. Then, the allocation rule φ satisfies $D^{\mathcal{E}^*}$,* **PE**, **WETE**, **ISE**, **WTMON**, **IUI**, *and* **CTI** *if and only if $\varphi = \varphi^K$.*

Remark 5: It should be noted that, in contrast to other rules, there is a difference in characterizations of φ^K between two or more persons allocation problems. Since φ^K satisfies **ITMON** even when $\#N > 2$, we can obtain a full characterization of φ^K by the axioms in **Theorem 7** plus **ITMON**, but cannot only by the axioms in **Theorem 6** without the help of **WTMON** when $\#N > 2$. The Figure 2 describes the necessity of **WTMON** for characterizing φ^K in three-person problems:

[Insert Figure 2]

In the Figure 2, there are three economies whose corresponding utility possibility sets are $S(\mathbf{e}_1)$, $S(\mathbf{e}_2)$, and $S(\mathbf{e}_3)$ respectively. Suppose that the economy changes from $\mathbf{e}_1$ to $\mathbf{e}_2$, moreover, to $\mathbf{e}_3$ as the result of technological changes. Note that the performance of φ^K in the change from $\mathbf{e}_1$ to $\mathbf{e}_3$ results in $\mu_{\varphi^K 2}(\mathbf{e}_1) \leq \mu_{\varphi^K 2}(\mathbf{e}_3)$, which both **ITMON** and **WTMON** are necessary to explain: First, the performance of φ^K in the change from $\mathbf{e}_1$ to $\mathbf{e}_2$ which results in $\mu_{\varphi^K 2}(\mathbf{e}_1) \leq \mu_{\varphi^K 2}(\mathbf{e}_2)$, should be explained by **ITMON**. Second, the performance of φ^K in the change from $\mathbf{e}_2$ to $\mathbf{e}_3$ which results in $\mu_{\varphi^K 2}(\mathbf{e}_2) \leq \mu_{\varphi^K 2}(\mathbf{e}_3)$, should be explained not by **ITMON**, but by **WTMON**. Such a change from $\mathbf{e}_1$ to $\mathbf{e}_3$ cannot occur in the two-person problems.

By Theorems 5, 6, and 7, we can see that both the Nash and the Kalai-Smorodinsky rules share the same characteristics regarding responsibility and compensation. A unique element by which both rules are mutually discriminated is the attitude of the rules toward the change in production technology: the Nash rule behaves unconcerned about the contraction of production technology, while the Kalai-Smorodinsky rule behaves sensitive to the weak technological progress in the sense that **ITMON** and **WTMON** presume.

4.4 Independence of Axioms

We can check the independence of axioms for each theorem (**Theorem 2**, **3**, **4**, **5**, **6**, and **7**). First, the independence of $D^{\mathcal{E}^*}$, **PE**, and **CTI** respectively is clear in these theorems. Next, deleting **WETE** in these theorems, we have φ^{D_i}, while the independence of **ISE** in **Theorem 2**, of **SS** in **Theorem 3**, and of **SM** in **Theorem 4** is shown by the following example:

Example 2: Let $\{\mathcal{S}_{sy}^n, \mathcal{S}_{as}^n\}$ be a partition of $\mathcal{S}^n$ such that $\mathcal{S}_{sy}^n$ has all symmetric profiles and $\mathcal{S}_{as}^n$ is its complement, and let $\mathcal{E}^*(\mathcal{S}_i^n)$ be the subset of $\mathcal{E}^*$ such that for each $\mathbf{e} \in \mathcal{E}^*(\mathcal{S}_i^n)$, its profile of production skills belongs to $\mathcal{S}_i^n$ where $i \in \{sy, as\}$. Then, define an allocation rule φ^{NW_2} as follows:

$$\varphi^{NW_2}(\mathbf{e}) \equiv \begin{cases} \varphi^E(\mathbf{e}) & \text{if } \mathbf{e} \in \mathcal{E}^*(\mathcal{S}_{sy}^n) \\ \varphi^{D_i}(\mathbf{e}) & \text{if } \mathbf{e} \in \mathcal{E}^*(\mathcal{S}_{as}^n). \end{cases}$$

Note that φ^{NW_2} satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **TMON**, and **CTI**, but does not satisfy **ISE**, **SS**, or **SM**. ■

The independence of **TMON** in **Theorem 4** is shown by the following:

Example 3: Let $M \in \mathcal{M}$ with $\#M = 1$, and let $Y^{(m)} \equiv \mathbb{R}_+ \times [0, 1]$ and $\mathbf{s}_0 \equiv \mathbf{0} \in \mathbb{R}_+^n$. Then, let $\mathcal{E}^*(Y^{(m)}; \mathbf{s}_0)$ be the class of economies whose production possibility sets and profiles of skills are respectively equal to $Y^{(m)}$ and $\mathbf{s}_0$. Given $\mathbf{e} \in \mathcal{E}^*(Y^{(m)}; \mathbf{s}_0)$ and φ^{Na}, let us say that $\mathbf{e}' \equiv (M', \mathbf{u}', \mathbf{s}_0, Y')$ is a **CTI**-*extension economy of* $\mathbf{e}$ *w.r.t.* φ^{Na} if $P_M(Y') = Y^{(m)}$, and $\mathbf{e}'$ satisfies premises (2) and (3) of **CTI** when the rule is φ^{Na}. Let

$$\begin{aligned} \mathcal{E}^*_{\mathbf{CTI}(\varphi^{Na})}(Y^{(m)}; \mathbf{s}_0) \equiv \; & \{\mathbf{e}' \in \mathcal{E}^* \mid \exists \mathbf{e} \in \mathcal{E}^*(Y^{(m)}; \mathbf{s}_0) : \\ & \mathbf{e}' \text{ is a } \mathbf{CTI}\text{-extension economy of } \mathbf{e} \text{ w.r.t. } \varphi^{Na}\}. \end{aligned}$$

Moreover, for each $\mathbf{e} \in \mathcal{E}^*_{\mathbf{CTI}(\varphi^{Na})}(Y^{(m)}; \mathbf{s}_0)$ and $\mathbf{s} \in \mathcal{S}^n$, we can define an economy $\mathbf{e}'$ which is obtained from $\mathbf{e}$ by replacing $\mathbf{s}_0$ with $\mathbf{s}$. Denote the set of such economies by $\mathcal{E}^*_{\mathbf{CTI}(\varphi^{Na})}(Y^{(m)}; \mathcal{S}^n(\mathbf{s}_0))$. Then, define an allocation rule φ^{NW_3} as follows: For all $\mathbf{e} \in \mathcal{E}^*$,

$$\varphi^{NW_3}(\mathbf{e}) \equiv \begin{cases} \varphi^{Na}(\mathbf{e}) & \text{if } \mathbf{e} \in \mathcal{E}^*_{\mathbf{CTI}(\varphi^{Na})}(Y^{(m)}; \mathcal{S}^n(\mathbf{s}_0)) \\ \varphi^E(\mathbf{e}) & \text{if } \mathbf{e} \in \mathcal{E}^* \backslash \mathcal{E}^*_{\mathbf{CTI}(\varphi^{Na})}(Y^{(m)}; \mathcal{S}^n(\mathbf{s}_0)). \end{cases}$$

Note that φ^{NW_3} satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **SM**, and **CTI**, but does not satisfy **TMOM**. ■

Next, without **IUI** in **Theorems 5** and **6**, φ^E is admissible. Deleting **ISE** in **Theorem 5** (resp. **6**), the following rule is admissible: φ^{NW_4} (resp. φ^{NW_5}) where $\varphi^{NW_4}(\mathbf{e}) \equiv \varphi^{Na}(\mathbf{e})$ (resp. $\varphi^{NW_5}(\mathbf{e}) \equiv \varphi^{K}(\mathbf{e})$) if $\mathbf{e} \in \mathcal{E}^*(\mathcal{S}^n_{sy})$, and $\varphi^{NW_4}(\mathbf{e}) \equiv \varphi^{NW_2}(\mathbf{e})$ (resp. $\varphi^{NW_5}(\mathbf{e}) \equiv \varphi^{NW_2}(\mathbf{e})$) otherwise. Without **ITC** in **Theorem 5**, φ^K is admissible, while without **ITMON** in **Theorem 6**, φ^{Na} is admissible. Also, without **WTMON** in **Theorem 7**, φ^{Na} is admissible.

5 Concluding Remarks

In our characterizations, the assumption of the rather large class of economic environments, where the domain of producible commodities is variable, is crucial. In this point, our analysis entails the same property as that of Roemer [18, 20]. In contrast, Gine and Marhuenda [10], as well as Chen and Maskin [3], considered the economic models with finite dimensional commodity spaces.[20] Under these domain restrictions, they respectively characterized the egalitarian solution by means of Pareto efficiency, weak symmetry, and payoff monotonicity. Weak symmetry is essentially the same as the **WETE** axiom of this paper, while payoff monotonicity can be summarized as follows: if everyone's utility-productivity is improved, then everyone's utility in the bargaining outcome should be improved.

It is also interesting to consider whether or not the egalitarian solution can be characterized by these three axioms even in the context of *cooperative production economies* with a fixed production technology and no skill difference, when the domain of producible commodities is fixed to that of a finite dimension. Unfortunately, in opposition to the works of Chen and Maskin [3] and Gine and Marhuenda [10], that consideration is impossible in this context, as Gine and Marhuenda [10] discussed.

Thus, we may conclude that, in our context of production economies, our assumption of the larger class of economies is not superfluous in deriving a unique bargaining solution. Moreover, I think that such an assumption of the domain is not so inappropriate in bargaining problems under production

[20] In contrast to Chen and Maskin [3]'s specific production economies we discussed before, Gine and Marhuenda [10] focussed on public goods economies with quasi-linear utility functions.

economies. This is because the occurrence of technological innovation which the axiom **CTI** presumes is natural in production economies, and it is important in this context to take into account the problem of how to dispose of the conflict among individuals which such an innovation may entail.

6 Appendix: Proofs of Theorems

Given $Y^{(m)} \in \mathcal{Y}^M$ and $Y^{(l)} \in \mathcal{Y}^L$ with $M \cap L = \varnothing$, let

$$Y^{(m)} \oplus Y^{(l)} \equiv \{(x, y_M, y_L) \in \mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathbb{R}^l_+ \mid (x, y_M) \in Y^{(m)},\ (x, y_L) \in Y^{(l)}\}.$$

Proof of Lemma 2:
1. Given $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^{M*}$ and $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^{L*}$ such that $S(\mathbf{e}_1) = S(\mathbf{e}_2)$, we can find, by Billera and Bixby [1], some pure exchange economy, whose corresponding utility possibility set is equal to $S(\mathbf{e}_1)$, which consists of initial endowments of one unit $k = n(n-1)$ commodities, that is $\mathbf{1} = (1, \cdots, 1) \in [0,1]^k$, and a profile of utility function $\mathbf{v} = (v_i)_{i \in N}$, where for each $i \in N$, $v_i : [0,1]^k \to \mathbb{R}_+$ is continuous, concave, monotonic, and $v_i(\mathbf{0}) = 0$. Based upon this pure exchange economy, we construct a production economy $\mathbf{e}^{\triangle} = (K, \mathbf{v}^{\triangle}, \mathbf{s}, Y^{(k)}) \in \mathcal{E}^{K*}$ with $\#K = k$ and $K \cap M = K \cap L = \varnothing$, which is defined as follows: For each $i \in N$, let $v_i^{\triangle} : [0, \overline{x}] \times \mathbb{R}^k_+ \to \mathbb{R}_+$ be that:

$$\forall (x, y_K) \in [0, \overline{x}] \times \mathbb{R}^k_+,\ v_i^{\triangle}(x, y_K) = \begin{cases} v_i(y_K) & \text{if } y_K \in [0,1]^k \\ v_i((\min\{y_f, 1\})_{f \in K}) & \text{otherwise} \end{cases}.$$

The production possibility set $Y^{(k)}$ is defined as $Y^{(k)} \equiv \mathbb{R}_+ \times [0,1]^k$ with generic element $(x, y_K) \in Y^{(k)}$. Clearly, $v_i^{\triangle} \in \mathcal{U}^K$ for each $i \in N$, and $S(\mathbf{e}^{\triangle}) = S(\mathbf{e}_1) = S(\mathbf{e}_2)$.
2. In the following, we will show that, if φ satisfies $D^{\mathcal{E}^*}$, **PE**, and **CTI**, then $\mu_\varphi(\mathbf{e}^{\triangle}) = \mu_\varphi(\mathbf{e}_1)$. If this claim is shown, then we can show, in the same way, that $\mu_\varphi(\mathbf{e}^{\triangle}) = \mu_\varphi(\mathbf{e}_2)$, so that $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_2)$.
2.1. Let $\mathbf{e}_1 \wedge \mathbf{e}^{\triangle} \equiv (M \cup K, \mathbf{u}^*, \mathbf{s}, Y^{(m)} \oplus Y^{(k)})$ where

$$\forall (x, y_M, y_K) \in [0, \overline{x}] \times \mathbb{R}^m_+ \times \mathbb{R}^k_+,\ u_i^*(x, y_M, y_K) = \min\{u_i^1(x, y_M), v_i^{\triangle}(x, y_K)\}.$$

Then, we will first show that $S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = S(\mathbf{e}_1) = S(\mathbf{e}^{\triangle})$.
(1) $S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) \subseteq S(\mathbf{e}_1) = S(\mathbf{e}^{\triangle})$. Let $\overline{\mathbf{u}} = (\overline{u}_i)_{i \in N} \in S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$. Then, there exists $(x_i, y_{Mi}, y_{Ki})_{i \in N} \in Z(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$ such that for all $i \in N$, $u_i^*(x_i, y_{Mi}, y_{Ki}) =$

$\overline{u}_i$. By definition, $(x_i, y_{Mi})_{i\in N} \in Z(\mathbf{e}_1)$ and $(x_i, y_{Ki})_{i\in N} \in Z(\mathbf{e}^{\triangle})$. Thus, $(u_i^1(x_i, y_{Mi}))_{i\in N} \in S(\mathbf{e}_1)$ and $(v_i^{\triangle}(x_i, y_{Ki}))_{i\in N} \in S(\mathbf{e}^{\triangle})$. By definition, $u_i^*(x_i, y_{Mi}, y_{Ki}) = \min\{u_i^1(x_i, y_{Mi}), v_i^{\triangle}(x_i, y_{Ki})\}$. Thus, by comprehensiveness of $S(\mathbf{e}_1)$ and $S(\mathbf{e}^{\triangle})$, $\overline{\mathbf{u}} \in S(\mathbf{e}_1) \cap S(\mathbf{e}^{\triangle})$.

(2) $S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) \supseteq S(\mathbf{e}_1) = S(\mathbf{e}^{\triangle})$. Let $\overline{\mathbf{u}} = (\overline{u}_i)_{i\in N} \in S(\mathbf{e}_1) \cap S(\mathbf{e}^{\triangle})$. Then, there exist $(x_i, y_{Mi})_{i\in N} \in Z(\mathbf{e}_1)$ and $(x_i, y_{Ki})_{i\in N} \in Z(\mathbf{e}^{\triangle})$ such that $(u_i^1(x_i, y_{Mi}))_{i\in N} = \overline{\mathbf{u}} = (v_i^{\triangle}(x_i, y_{Ki}))_{i\in N}$. Then, $(x_i, y_{Mi}, y_{Ki})_{i\in N} \in Z(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$. Since $u_i^*(x_i, y_{Mi}, y_{Ki}) = \min\{u_i^1(x_i, y_{Mi}), v_i^{\triangle}(x_i, y_{Ki})\} = \overline{u}_i$, we obtain $\overline{\mathbf{u}} \in S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$.

2.2. Now, we will show that $\mu_\varphi(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = \mu_\varphi(\mathbf{e}_1)$. In the same way, we may show that $\mu_\varphi(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = \mu_\varphi(\mathbf{e}^{\triangle})$, so that $\mu_\varphi(\mathbf{e}^{\triangle}) = \mu_\varphi(\mathbf{e}_1)$.

Construct a new economy

$$\widehat{\mathbf{e}}_1 \equiv (M \cup K, \widehat{\mathbf{u}}^1, \mathbf{s}, Y^{(m)} \oplus Y^{(k)})$$

where, for all $i \in N$,

$$\forall (x, y_M, y_K) \in \mathbb{R}_+ \times \mathbb{R}_+^m \times \mathbb{R}_+^k,\ \widehat{u}_i^1(x, y_M, y_K) = u_i^1(x, y_M).$$

Then, by construction, $S(\widehat{\mathbf{e}}_1) = S(\mathbf{e}_1) = S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$.

Note that for any $(x, y_M, y_K) \in [0, \overline{x}] \times \mathbb{R}_+^m \times \mathbb{R}_+^k$,

$$u_i^*(x, y_M, y_K) = \min\{u_i^1(x, y_M), v_i^{\triangle}(x, y_K)\} \leq \widehat{u}_i^1(x, y_M, y_K).$$

So, $Z(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = Z(\widehat{\mathbf{e}}_1)$ and $S(\widehat{\mathbf{e}}_1) = S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$ implies that $WP(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) \subseteq WP(\widehat{\mathbf{e}}_1)$, where $WP(\mathbf{e})$ is the set of (weakly) Pareto efficient allocations for $\mathbf{e}$, and for all $(x, y_M, y_K) \in WP(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) \subseteq WP(\widehat{\mathbf{e}}_1)$, $u_i^*(x, y_M, y_K) = \widehat{u}_i^1(x, y_M, y_K)$.

Moreover, by applying Howe's theorem (Howe [11, Proposition 3]),[21] for each $i \in N$, there exist $w_i \in \mathcal{U}^{M\cup K\cup\{R(i)\}}$ and $\widehat{y}_{R(i)} \in \mathbb{R}_+$such that:

$$\begin{aligned} \forall (x, y_M, y_K) &\in [0, \overline{x}] \times \mathbb{R}_+^m \times \mathbb{R}_+^k, \\ w_i(x, y_M, y_K, \widehat{y}_{R(i)}) &= \widehat{u}_i^1(x, y_M, y_K) \\ \&\ w_i(x, y_M, y_K, 0) &= u_i^*(x, y_M, y_K). \end{aligned}$$

[21] In the following discussion, when we talk about Howe's theorem, it always refers to Howe [11, Proposition 3]. Since we impose the condition (2.1) on admissible utility functions, we can always apply the Howe theorem to any u_i and u_i', whenever they are defined on the same dimensional consumption space with $u_i \geq u_i'$.

Now construct the flat extension of w_i for each $i \in N$:

$$\begin{aligned}\forall(x, y_M, y_K, y_R) &\in [0, \overline{x}] \times \mathbb{R}^m_+ \times \mathbb{R}^k_+ \times \mathbb{R}^n_+,\\ \widehat{w}_i(x, y_M, y_K, y_R) &= w_i(x, y_M, y_K, y_{R(i)}),\\ \text{where } y_R &\equiv (y_{R(j)})_{j\in N}.\end{aligned}$$

Thus, for each $i \in N$, $\widehat{w}_i \in \mathcal{U}^{M\cup K\cup R}$, where $R \equiv \{R(j)\}_{j\in N}$.

Let $Y^{(r)} \equiv \mathbb{R}_+ \times \left(\underset{j\in N}{\times} [0, \widehat{y}_{R(j)}]\right)$. Then, $Y^{(m)} \oplus Y^{(k)} \oplus Y^{(r)} \in \mathcal{Y}^{M\cup K\cup R}$. Thus,

$$\widetilde{\widehat{\mathbf{e}}}_1 \equiv (M \cup K \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(m)} \oplus Y^{(k)} \oplus Y^{(r)}) \in \mathcal{E}^{M\cup K\cup R*}.$$

Note that $S(\widetilde{\widehat{\mathbf{e}}}_1) = S(\widehat{\mathbf{e}}_1) = S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$. So, by construction of $\widetilde{\widehat{\mathbf{e}}}_1$,

$$\begin{aligned}[\forall(y_{Ri})_{i\in N} &\in \mathbb{R}^{nn}_+, (\widehat{w}_i(x_i, y_{Mi}, y_{Ki}, y_{Ri}))_{i\in N} = (\widehat{w}_i(x_i, y_{Mi}, y_{Ki}, \mathbf{0}))_{i\in N} \in \partial S(\widetilde{\widehat{\mathbf{e}}}_1)]\\ &\Leftrightarrow (\widehat{u}^1_i(x_i, y_{Mi}, y_{Ki}))_{i\in N} = (u^*_i(x_i, y_{Mi}, y_{Ki}))_{i\in N} \in \partial S(\widehat{\mathbf{e}}_1) = \partial S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}).\end{aligned}$$

Suppose that $\mu_\varphi(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) \neq \mu_\varphi(\mathbf{e}_1)$ and $(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki})_{i\in N} \in \varphi(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$. By **PE** (resp. **WPE**) of φ, we have $(u^*_i(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}))_{i\in N} = (\widehat{u}^1_i(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}))_{i\in N} \in \partial S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = \partial S(\widehat{\mathbf{e}}_1)$. Thus, under $\widetilde{\widehat{\mathbf{e}}}_1$, every agent is indifferent to each good of R at $(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}, \mathbf{0})_{i\in N}$. By **CTI**, $(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}, \mathbf{0})_{i\in N} \in \varphi(\widetilde{\widehat{\mathbf{e}}}_1)$.

Let $(\widehat{x}'_i, \widehat{y}'_{Mi})_{i\in N} \in \varphi(\mathbf{e}_1)$. By **PE** (resp. **WPE**) of φ, $(u^1_i(\widehat{x}'_i, \widehat{y}'_{Mi}))_{i\in N} \in \partial S(\mathbf{e}_1)$. Then, by **CTI**, $(\widehat{x}'_i, \widehat{y}'_{Mi}, \mathbf{0})_{i\in N} \in \varphi(\widehat{\mathbf{e}}_1)$. By **PE** (resp. **WPE**) of φ, $(\widehat{u}^1_i(\widehat{x}'_i, \widehat{y}'_{Mi}, \mathbf{0}))_{i\in N} \in \partial S(\widehat{\mathbf{e}}_1)$. Since $\partial S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = \partial S(\widehat{\mathbf{e}}_1)$, there exists $(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki})_{i\in N} \in Z(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$ such that $(u^*_i(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}))_{i\in N} = (\widehat{u}^1_i(\widehat{x}'_i, \widehat{y}'_{Mi}, \mathbf{0}))_{i\in N} \in \partial S(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle})$. Then, $(u^*_i(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}))_{i\in N} = (\widehat{u}^1_i(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}))_{i\in N}$. By fullness of φ, $(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki})_{i\in N} \in \varphi(\widehat{\mathbf{e}}_1)$. Since every agent is indifferent to each good of R at $(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}, \mathbf{0})_{i\in N}$ under $\widetilde{\widehat{\mathbf{e}}}_1$, $(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}, \mathbf{0})_{i\in N} \in \varphi(\widetilde{\widehat{\mathbf{e}}}_1)$ by **CTI**. Since $\mu_\varphi(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) \neq \mu_\varphi(\mathbf{e}_1)$, $(\widehat{w}_i(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}, \mathbf{0}))_{i\in N} = (u^1_i(\widehat{x}'_i, \widehat{y}'_{Mi}))_{i\in N}$, and $(\widehat{w}_i(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}, \mathbf{0}))_{i\in N} = (u^*_i(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}))_{i\in N}$, we have $(\widehat{w}_i(\widehat{x}''_i, \widehat{y}''_{Mi}, \widehat{y}''_{Ki}, \mathbf{0}))_{i\in N} \neq (\widehat{w}_i(\widehat{x}_i, \widehat{y}_{Mi}, \widehat{y}_{Ki}, \mathbf{0}))_{i\in N}$, which is a contradiction, because φ is essentially a function. Thus, $\mu_\varphi(\mathbf{e}_1 \wedge \mathbf{e}^{\triangle}) = \mu_\varphi(\mathbf{e}_1)$ holds. ■

Lemma 3: *Let* $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathcal{E}$ *be such that* $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^{M*}$ (resp. $\mathcal{E}^M$), $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^{L*}$ (resp. $\mathcal{E}^L$), *and* $S(\mathbf{e}_1) \supseteq S(\mathbf{e}_2)$. *Then, if the allocation rule* φ *satisfies* $D^{\mathcal{E}^*}$ (resp. $D^{\mathcal{E}}$), **PE** (resp. **WPE**), **TMON**, *and* **CTI**, *then* $\mu_\varphi(\mathbf{e}_1) \geq \mu_\varphi(\mathbf{e}_2)$.

Proof. Given $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^{M*}$ and $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^{L*}$ such that $S(\mathbf{e}_1) \supseteq S(\mathbf{e}_2)$, there exist other economies $\mathbf{e}_1^{\triangle} = (K, \mathbf{v}^1, \mathbf{s}, Y^{(\#K)}) \in \mathcal{E}^{K*}$ and $\mathbf{e}_2^{\triangle} = (K', \mathbf{v}^2, \mathbf{s}, Y^{(\#K')}) \in \mathcal{E}^{K'*}$ such that $K \cap K' = \varnothing$, $S(\mathbf{e}_1^{\triangle}) = S(\mathbf{e}_1)$, and $S(\mathbf{e}_2^{\triangle}) = S(\mathbf{e}_2)$, which are guaranteed in the same way as in step **1.** of the proof of Lemma 2. Note that $Y^{(\#K)} \equiv \mathbb{R}_+ \times [0,1]^{\#K}$ with generic element $(x, y_K) \in Y^{(\#K)}$ and $Y^{(\#K')} \equiv \mathbb{R}_+ \times [0,1]^{\#K'}$ with generic element $(x, y_{K'}) \in Y^{(\#K')}$. Moreover, for each $i \in N$, the utility function $v_i^1 : [0, \overline{x}] \times \mathbb{R}_+^{\#K} \to \mathbb{R}_+$ is defined as:

$$\forall (x, y_K) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K},\ v_i^1(x, y_K) = \begin{cases} v_i(y_K) & \text{if } y_K \in [0,1]^{\#K} \\ v_i((\min\{y_f, 1\})_{f \in K}) & \text{otherwise} \end{cases},$$

and the utility function $v_i^2 : [0, \overline{x}] \times \mathbb{R}_+^{\#K'} \to \mathbb{R}_+$ is defined as:

$$\forall (x, y_{K'}) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K'},\ v_i^2(x, y_{K'}) = \begin{cases} v_i'(y_{K'}) & \text{if } y_{K'} \in [0,1]^{\#K'} \\ v_i'((\min\{y_{f'}, 1\})_{f' \in K'}) & \text{otherwise} \end{cases},$$

where the existence of utility functions $v_i : [0,1]^{\#K} \to \mathbb{R}_+$ and $v_i' : [0,1]^{\#K'} \to \mathbb{R}_+$ are guaranteed by Billera and Bixby [1].

Let us construct a new economy $\mathbf{e}^* \equiv \mathbf{e}_1^{\triangle} \wedge \mathbf{e}_2^{\triangle}$. Let $\widehat{\mathbf{e}}_1 \equiv (K \cup K', \widehat{\mathbf{v}}^1, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')})$ where $\widehat{\mathbf{v}}^1$ is the flat extension of $\mathbf{v}^1$. Note that we can show that $S(\mathbf{e}^*) = S(\mathbf{e}_1^{\triangle}) \cap S(\mathbf{e}_2^{\triangle})$ in the same way as in step **2.1.** of the proof of Lemma 2, so that $S(\mathbf{e}^*) = S(\mathbf{e}_1^{\triangle}) \cap S(\mathbf{e}_2^{\triangle}) = S(\mathbf{e}_2^{\triangle})$. Exactly as in the proof of Lemma 2, construct the Howe's extension economies $\widetilde{\widehat{\mathbf{e}}}_1$ of $\widehat{\mathbf{e}}_1$ and $\widetilde{\mathbf{e}}^*$ of $\mathbf{e}^*$ by:

$$\begin{aligned} \widetilde{\widehat{\mathbf{e}}}_1 &\equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^{(r)}) \\ \widetilde{\mathbf{e}}^* &\equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y_*^{(r)}), \\ \text{where } Y_*^{(r)} &\equiv \mathbb{R}_+ \times \{\mathbf{0}\} \text{ and } \mathbf{0} \in \mathbb{R}_+^n. \end{aligned}$$

By **TMON**, $\mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1) \geq \mu_\varphi(\widetilde{\mathbf{e}}^*)$. It follows that $S(\widetilde{\widehat{\mathbf{e}}}_1) = S(\mathbf{e}_1^{\triangle}) = S(\mathbf{e}_1)$ and $S(\widetilde{\mathbf{e}}^*) = S(\mathbf{e}^*) = S(\mathbf{e}_2)$. Since φ satisfies $D^{\mathcal{E}^*}$ (resp. $D^{\mathcal{E}}$), **PE** (resp. **WPE**), **CTI**, by Lemma 2, $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_1^{\triangle}) = \mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1)$ and $\mu_\varphi(\mathbf{e}^*) = \mu_\varphi(\widetilde{\mathbf{e}}^*) = \mu_\varphi(\mathbf{e}_2)$. Thus, $\mu_\varphi(\mathbf{e}_1) \geq \mu_\varphi(\mathbf{e}_2)$. ■

Lemma 4: *Let $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathcal{E}$ be such that $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^M$ and $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^L$. Moreover, $S(\mathbf{e}_1) \supseteq S(\mathbf{e}_2)$ with $\mu_\varphi(\mathbf{e}_1) \in S(\mathbf{e}_2)$ holds. Then, if the allocation rule φ satisfies $D^{\mathcal{E}}$, **PE**, **ITC**, and **CTI**, then $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_2)$.*

Proof. As in the proof of Lemma 3, let us construct $\mathbf{e}_1^{\triangle} = (K, \mathbf{v}^1, \mathbf{s}, Y^{(\#K)}) \in \mathcal{E}^K$ and $\mathbf{e}_2^{\triangle} = (K', \mathbf{v}^2, \mathbf{s}, Y^{(\#K')}) \in \mathcal{E}^{K'}$ such that $K \cap K' = \varnothing$, $S(\mathbf{e}_1^{\triangle}) = S(\mathbf{e}_1)$, and $S(\mathbf{e}_2^{\triangle}) = S(\mathbf{e}_2)$. Let us also construct $\mathbf{e}^* \equiv \mathbf{e}_1^{\triangle} \wedge \mathbf{e}_2^{\triangle}$ and $\widehat{\mathbf{e}}_1 \equiv (K \cup K', \widehat{\mathbf{v}}^1, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')})$ where $\widehat{\mathbf{v}}^1$ is the flat extension of $\mathbf{v}^1$. Then, as in the proof of Lemma 3, let us construct

$$\widetilde{\widehat{\mathbf{e}}}_1 \equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^{(r)}) \in \mathcal{E}^{K \cup K' \cup R}.$$

Construct, also,

$$\widetilde{\mathbf{e}}^* \equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y_*^{(r)})$$

and assume that each agent $j \in N$ likes the only $R(j)$ in R. By Lemma 2, $\mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1) = \mu_\varphi(\mathbf{e}_1)$ and $\mu_\varphi(\widetilde{\mathbf{e}}^*) = \mu_\varphi(\mathbf{e}_2)$.

Since $S(\widehat{\mathbf{e}}_1) = S(\widetilde{\widehat{\mathbf{e}}}_1)$, there exists $(\widehat{x}_i, \widehat{y}_{Ki}, \widehat{y}_{K'i})_{i \in N} \in \varphi(\widehat{\mathbf{e}}_1)$ such that $(\widehat{v}_i^1(\widehat{x}_i, \widehat{y}_{Ki}, \widehat{y}_{K'i}))_{i \in N} = (\widehat{w}_i(\widehat{x}_i, \widehat{y}_{Ki}, \widehat{y}_{K'i}, (\widehat{y}_R)_i))_{i \in N} = \mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1)$. Since $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\widehat{\mathbf{e}}_1) \in S(\mathbf{e}_2) = S(\widetilde{\mathbf{e}}^*)$, there exists $\zeta = (\widehat{x}_i, \widehat{y}_{Ki}, \widehat{y}_{K'i}, \mathbf{0})_{i \in N} \in Z(\widetilde{\mathbf{e}}^*)$ such that $(\widehat{w}_i(\zeta_i))_{i \in N} = \mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1)$. But, ζ is also a feasible allocation in $\widetilde{\widehat{\mathbf{e}}}_1$. Since φ is a full correspondence, $\zeta \in \varphi(\widetilde{\widehat{\mathbf{e}}}_1)$. By **ITC**, $\zeta \in \varphi(\widetilde{\mathbf{e}}^*)$. Thus, $(\widehat{w}_i(\zeta_i))_{i \in N} = \mu_\varphi(\widetilde{\mathbf{e}}^*) = \mu_\varphi(\mathbf{e}_2)$. This implies $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_2)$. ■

Lemma 5: *Let* $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}$ *be such that* $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^M$ *and* $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^L$ *with* $\mathbf{b}^{\mathbf{u}^1} = \mathbf{b}^{\mathbf{u}^2}$. *Moreover, there exists* $\mathbf{a} = (a_i)_{i \in N} \in \mathbb{R}_+^n$ *such that*

$$\overline{\mathbf{u}} = (\overline{u}_i)_{i \in N} \in S(\mathbf{e}_1) \Leftrightarrow \mathbf{a} \cdot \overline{\mathbf{u}} = (a_i \cdot \overline{u}_i)_{i \in N} \in S(\mathbf{e}_2).$$

Then, if the allocation rule φ *satisfies* $D^{\mathcal{E}}$, **PE**, **IUI**, *and* **CTI**, *then* $\mu_\varphi(\mathbf{e}_2) = \mathbf{a} \cdot \mu_\varphi(\mathbf{e}_1)$.

Proof. Construct $\mathbf{e}^* = (M, \mathbf{u}^*, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^M$ as $u_i^* = a_i \cdot u_i^1$ for all $i \in N$. Then, $S(\mathbf{e}^*) = S(\mathbf{e}_2)$. By Lemma 2, $\mu_\varphi(\mathbf{e}^*) = \mu_\varphi(\mathbf{e}_2)$. Since φ satisfies **IUI**, we obtain $\varphi(\mathbf{e}_1) = \varphi(\mathbf{e}^*)$. Since $\mu_\varphi(\mathbf{e}^*) = \mathbf{a} \cdot \mu_\varphi(\mathbf{e}_1)$, we obtain the desired result. ■

Lemma 6: *Let* $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y^{(m)}) \in \mathcal{E}$ *be such that* $S(\mathbf{e})$ *is a symmetric utility possibility set. Then, if the allocation rule* φ *satisfies* $D^{\mathcal{E}}$, **PE**, **WETE**, **ISE**, *and* **CTI**, *then* $\mu_{\varphi i}(\mathbf{e}) = \mu_{\varphi j}(\mathbf{e})$ *for all* $i, j \in N$.

Proof. Let T be a symmetric utility possibility set arising from the economic environment $\mathbf{e} = (M, \mathbf{u}, \mathbf{s}, Y^{(m)})$. It suffices to show that $T = S(\mathbf{e}^*)$, where $\mathbf{e}^* = (K, \mathbf{u}^*, \mathbf{s}, Y^{(k)}) \in \mathcal{E}$ such that $u_i^* = u_j^*$ for all $i, j \in N$. By Billera and Bixby [1], we can construct a production economy $\mathbf{e}^{\triangle} = (K, \mathbf{v}^{\triangle}, \mathbf{s}, Y^{(k)}) \in \mathcal{E}^K$ with $k = n(n-1)$, $Y^{(k)} \equiv \mathbb{R}_+ \times [0,1]^k$, and $T = S(\mathbf{e}^{\triangle})$ exactly as in step **1.** of the proof of Lemma 2. However, $\mathbf{e}^{\triangle}$ does not necessarily have a profile of symmetric utility functions. Now, by using the data of $\mathbf{v}^{\triangle}$ in $\mathbf{e}^{\triangle}$, we will construct a new economy $\mathbf{e}^*$ with a profile of symmetric utility functions, by which $T = S(\mathbf{e}^*)$ is guaranteed.

Since $S(\mathbf{e}^{\triangle})$ is symmetric, the economy $\mathbf{e}^{\triangle}_{\{1,2\}} \equiv (K, (v_2^{\triangle}, v_1^{\triangle}, \mathbf{v}^{\triangle}_{-\{1,2\}}), \mathbf{s}, Y^{(k)})$ is that $T = S(\mathbf{e}^{\triangle}_{\{1,2\}})$. (Because of the definition of $Y^{(k)}$, we need not take into account the influence of the profile of production skills in determining $S(\mathbf{e}^{\triangle}_{\{1,2\}})$.) Let us define $\mathbf{e}^{12} \equiv \mathbf{e}^{\triangle} \wedge \mathbf{e}^{\triangle}_{\{1,2\}} = (K, (v^{12}, v^{12}, \widehat{\mathbf{v}}^{\triangle}_{-\{1,2\}}), \mathbf{s}, Y_{12})$ where $v^{12} \equiv \min\{v_1^{\triangle}, v_2^{\triangle}\}$, $\widehat{v}_i^{\triangle}$ is the flat extension of $v_i^{\triangle}$ for $i \in N\backslash\{1,2\}$, and $Y_{12} \equiv Y^{(k)} \oplus Y^{(k)}$. Clearly, $T = S(\mathbf{e}^{12})$. Consider economies $\mathbf{e}^{12}_{\{1,3\}} \equiv (K, (\widehat{v}_3^{\triangle}, v^{12}, v^{12}, \widehat{\mathbf{v}}^{\triangle}_{-\{1,2,3\}}), \mathbf{s}, Y_{12})$ and $\mathbf{e}^{12}_{\{2,3\}} \equiv (K, (v^{12}, \widehat{v}_3^{\triangle}, v^{12}, \widehat{\mathbf{v}}^{\triangle}_{-\{1,2,3\}}), \mathbf{s}, Y_{12})$. Since $S(\mathbf{e}^{12})$ is symmetric, so are both $S(\mathbf{e}^{12}_{\{1,3\}})$ and $S(\mathbf{e}^{12}_{\{2,3\}})$. Consider $\mathbf{e}^{12} \wedge \mathbf{e}^{12}_{\{1,3\}}$ and $\mathbf{e}^{12} \wedge \mathbf{e}^{12}_{\{2,3\}}$, then construct

$$\begin{aligned}(\mathbf{e}^{12} \wedge \mathbf{e}^{12}_{\{1,3\}}) \wedge (\mathbf{e}^{12} \wedge \mathbf{e}^{12}_{\{2,3\}}) &= (K, (v^{123}, v^{123}, v^{123}, \widetilde{\widehat{\mathbf{v}}}^{\triangle}_{-\{1,2,3\}}), \mathbf{s}, Y_{123}), \\ \text{where } v^{123} &\equiv \min\{v^{12}, v_3^{\triangle}\},\end{aligned}$$

$\widetilde{\widehat{v}}_i^{\triangle}$ is the flat extension of $v_i^{\triangle}$ for $i \in N\backslash\{1,2,3\}$, and $Y_{123} \equiv (Y_{12} \oplus Y_{12}) \oplus (Y_{12} \oplus Y_{12})$.

Let $\mathbf{e}^{123} \equiv (\mathbf{e}^{12} \wedge \mathbf{e}^{12}_{\{1,3\}}) \wedge (\mathbf{e}^{12} \wedge \mathbf{e}^{12}_{\{2,3\}})$. Clearly, $T = S(\mathbf{e}^{123})$. By repeating such a procedure to n, we obtain $\mathbf{e}^{1\cdots n} \equiv (K, (v^{1\cdots n}, \cdots, v^{1\cdots n}), \mathbf{s}, Y_{1\cdots n})$ such that $T = S(\mathbf{e}^{1\cdots n})$.

Let $\mathbf{e}^* \equiv \mathbf{e}^{1\cdots n}$ with $u_i^* = v^{1\cdots n}$ for all $i \in N$. Then, $T = S(\mathbf{e}^*)$. Let $\mathbf{e}^{**} = (K, \mathbf{u}^*, \mathbf{s}^*, Y_{1\cdots n}) \in \mathcal{E}$ be such that $s_i^* = s_j^*$ for all $i, j \in N$. By definition of $Y_{1\cdots n}$, we can see that $Z(\mathbf{e}^*) = Z(\mathbf{e}^{**})$. Thus, by **ISE**, $\varphi(\mathbf{e}^*) = \varphi(\mathbf{e}^{**})$. By the way, the definition of $\mathbf{e}^{**}$ implies that for all $i, j \in N$, $\mu_{\varphi i}(\mathbf{e}^{**}) = \mu_{\varphi j}(\mathbf{e}^{**})$ by **WETE**. So, we obtain that, for all $i, j \in N$, $\mu_{\varphi i}(\mathbf{e}^*) = \mu_{\varphi j}(\mathbf{e}^*)$. Thus, by Lemma 2, $\mu_{\varphi i}(\mathbf{e}) = \mu_{\varphi j}(\mathbf{e})$ for all $i, j \in N$, which is the desired result. ■

Lemma 7: *Let $\#N = 2$, and $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}$ be such that $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^M$ and $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^L$. Moreover, $S(\mathbf{e}_1) \supseteq S(\mathbf{e}_2)$, and there exists a unique agent $j \in N$ such that for all $i \in N\backslash\{j\}$, $m^i(S(\mathbf{e}_1)) = m^i(S(\mathbf{e}_2))$.*

Then, if the allocation rule φ satisfies $D^{\mathcal{E}}$, **PE**, **ITMON**, *and* **CTI**, *then* $\mu_{\varphi j}(\mathbf{e}_1) \geq \mu_{\varphi j}(\mathbf{e}_2)$.

Proof. 1. Construct $\mathbf{e}_1' = (M', \mathbf{u}^{1\prime}, \mathbf{s}, Y^{(m')}) \in \mathcal{E}^{M'}$ where $\#M' = 1$, $Y^{(m')} \equiv \mathbb{R}_+ \times [0,1]$. Moreover, for each $i \in N$, $u_i^{1\prime} : [0, \overline{x}] \times \mathbb{R}_+ \to \mathbb{R}$ is defined by:

$$\begin{aligned} u_i^{1\prime}(x_i, y_{M'i}) &= p_i^{1\prime}(y_{M'i}) \text{ for all } (x_i, y_{M'i}) \in [0, \overline{x}] \times [0,1], \text{ and} \\ u_i^{1\prime}(x_i, y_{M'i}) &= p_i^{1\prime}(1) \text{ for all } (x_i, y_{M'i}) \in [0, \overline{x}] \times [1, +\infty), \end{aligned}$$

where $\mathbf{p}^{1\prime} : \triangle^{n-1} \to S(\mathbf{e}_1)$ is a homeomorphism such that for each $\mathbf{y}_{M'} = (y_{M'i})_{i \in N} \in \triangle^{n-1}$, there exists a unique $\overline{\mathbf{u}} \in S(\mathbf{e}_1)$ such that $\overline{\mathbf{u}} = (p_i^{1\prime}(y_{M'i}))_{i \in N}$ and $\lambda_{\mathbf{y}_{M'}} \cdot \mathbf{y}_{M'} = \overline{\mathbf{u}}$ for some $\lambda_{\mathbf{y}_{M'}} > 0$. Such $u_i^{1\prime}$ surely belongs to $\mathcal{U}^{M'}$ even in the case that $S(\mathbf{e}_1)$ is not symmetric, since $\#N = 2$. In the same way, we can construct $\mathbf{e}_2' = (L', \mathbf{u}^{2\prime}, \mathbf{s}, Y^{(l')}) \in \mathcal{E}^{L'}$ where $\#L' = 1$, $Y^{(l')} \equiv \mathbb{R}_+ \times [0,1]$, and each $u_i^{2\prime}$ is defined by using a homeomorphism $\mathbf{p}^{2\prime} : \triangle^{n-1} \to S(\mathbf{e}_2)$, which is defined in the same way as $\mathbf{p}^{1\prime}$. Then, $S(\mathbf{e}_1') = S(\mathbf{e}_1)$ and $S(\mathbf{e}_2') = S(\mathbf{e}_2)$. Thus, $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_1')$ and $\mu_\varphi(\mathbf{e}_2) = \mu_\varphi(\mathbf{e}_2')$ by Lemma 2.
2. Construct the flat extension economy of $\mathbf{e}_1'$:

$$\begin{aligned} \widehat{\mathbf{e}}_1' &= \left(M' \cup L', \widehat{\mathbf{u}}^{1\prime}, \mathbf{s}, Y^{(m')} \oplus Y^{(l')}\right) \\ \text{with } \widehat{u}_i^{1\prime}(x, y_{M'}, y_{L'}) &= u_i^{1\prime}(x, y_{M'}), \forall (x, y_{M'}, y_{L'}) \in [0, \overline{x}] \times \mathbb{R}_+^{m'} \times \mathbb{R}_+^{l'}. \end{aligned}$$

Since $S(\mathbf{e}_1') = S(\widehat{\mathbf{e}}_1')$, $\mu_\varphi(\mathbf{e}_1') = \mu_\varphi(\widehat{\mathbf{e}}_1')$ by Lemma 2.
3. Construct the convolution $\mathbf{e}^* = \mathbf{e}_1' \wedge \mathbf{e}_2' = (M' \cup L', \mathbf{u}^*, \mathbf{s}, Y^{(m')} \oplus Y^{(l')})$. In the same way as in step **2.1.** of the proof of Lemma 2, $S(\mathbf{e}^*) = S(\mathbf{e}_1') \cap S(\mathbf{e}_2') = S(\mathbf{e}_2')$, so $\mu_\varphi(\mathbf{e}^*) = \mu_\varphi(\mathbf{e}_2')$ by Lemma 2.
4. Construct $\widehat{\mathbf{e}}_{11}' = (M' \cup L', (\widehat{u}_j^{1\prime}, u_i^*), \mathbf{s}, Y^{(m')} \oplus Y^{(l')})$. Note $S(\widehat{\mathbf{e}}_1') \supseteq S(\widehat{\mathbf{e}}_{11}')$.
5. To show $S(\widehat{\mathbf{e}}_1') \subseteq S(\widehat{\mathbf{e}}_{11}')$: Let $\overline{\mathbf{z}} = \left((\overline{x}_j, \overline{y}_{M'j}), (\overline{x}_i, \overline{y}_{M'i})\right)$ be an allocation in $\mathbf{e}_1'$. Consider in $\widehat{\mathbf{e}}_{11}'$ the allocation $\eta = \left((\overline{x}_j, \overline{y}_{M'j}, 0), (\overline{x}_i, \overline{y}_{M'i}, \overline{y}_{L'i})\right)$ such that for all $i \neq j$, $\overline{y}_{L'i} \geq \overline{y}_{M'i}$,

$$\begin{aligned} \widehat{u}_j^{1\prime}(\overline{x}_j, \overline{y}_{M'j}, 0) &= u_j^{1\prime}(\overline{x}_j, \overline{y}_{M'j}) \text{ and} \\ u_i^*(\overline{x}_i, \overline{y}_{M'i}, \overline{y}_{L'i}) &= \min\left\{u_i^{1\prime}(\overline{x}_i, \overline{y}_{M'i}), u_i^{2\prime}(\overline{x}_i, \overline{y}_{L'i})\right\} \\ &= u_i^{1\prime}(\overline{x}_i, \overline{y}_{M'i}) \leq u_i^{2\prime}(\overline{x}_i, \overline{y}_{L'i}). \end{aligned}$$

The existence of such an allocation as η is checked by setting $\overline{y}_{L'i} = 1$, since $\#N = 2$ and $m^i(S(\mathbf{e}_1)) = m^i(S(\mathbf{e}_2))$. Thus, $\mathbf{u}^{1\prime}(\overline{\mathbf{z}}) \in S(\widehat{\mathbf{e}}_1')$ implies $\mathbf{u}^{1\prime}(\overline{\mathbf{z}}) \in S(\widehat{\mathbf{e}}_{11}')$.

6. From steps **4** and **5**, $S(\widehat{\mathbf{e}}'_1) = S(\widehat{\mathbf{e}}'_{11})$ so $\mu_\varphi(\widehat{\mathbf{e}}'_1) = \mu_\varphi(\widehat{\mathbf{e}}'_{11})$ by Lemma 2.
7. Compare $\widehat{\mathbf{e}}'_{11}$ with $\mathbf{e}^*$. They differ only in the utility function of the agent j and $\widehat{u}_j^{1\prime} \geq u_j^*$. By applying the Howe theorem, there exist $w'_j \in \mathcal{U}^{M' \cup L' \cup \{R(j)\}}$ and $\widehat{y}_{R(j)} \in \mathbb{R}_+$ such that, for all $(x, y_{M'}, y_{L'}) \in [0, \overline{x}] \times \mathbb{R}_+^{m'} \times \mathbb{R}_+^{l'}$,

$$w'_j(x, y_{M'}, y_{L'}, \widehat{y}_{R(j)}) = \widehat{u}_j^{1\prime}(x, y_{M'}, y_{L'}) \text{ and } w'_j(x, y_{M'}, y_{L'}, 0) = u_j^*(x, y_{M'}, y_{L'}).$$

Moreover, for $i \neq j$, define $w_i^*(x, y_{M'}, y_{L'}, y_{R(j)}) = u_i^*(x, y_{M'}, y_{L'})$ for all $(x, y_{M'}, y_{L'}, y_{R(j)}) \in [0, \overline{x}] \times \mathbb{R}_+^{m'} \times \mathbb{R}_+^{l'} \times \mathbb{R}_+$. Then, $w_i^* \in \mathcal{U}^{M' \cup L' \cup \{R(j)\}}$.
8. Construct

$$\begin{aligned} \widetilde{\widehat{\mathbf{e}}}'_{11} &= (M' \cup L' \cup \{R(j)\}, (w'_j, \mathbf{w}^*_{-j}), \mathbf{s}, Y^{(m')} \oplus Y^{(l')} \oplus Y^{R(j)}) \text{ and} \\ \widetilde{\mathbf{e}}^* &= (M' \cup L' \cup \{R(j)\}, (w'_j, \mathbf{w}^*_{-j}), \mathbf{s}, Y^{(m')} \oplus Y^{(l')} \oplus Y_0^{R(j)}) \end{aligned}$$

where $Y^{R(j)} \equiv \mathbb{R}_+ \times [0, \widehat{y}_{R(j)}]$ and $Y_0^{R(j)} \equiv \mathbb{R}_+ \times \{0\}$. Note that $S(\widetilde{\widehat{\mathbf{e}}}'_{11}) = S(\widehat{\mathbf{e}}'_{11})$ and $S(\widetilde{\mathbf{e}}^*) = S(\mathbf{e}^*)$. By Lemma 2, $\mu_\varphi(\widetilde{\widehat{\mathbf{e}}}'_{11}) = \mu_\varphi(\widehat{\mathbf{e}}'_{11})$ and $\mu_\varphi(\widetilde{\mathbf{e}}^*) = \mu_\varphi(\mathbf{e}^*)$.
9. The environment $\widetilde{\widehat{\mathbf{e}}}'_{11}$ and $\widetilde{\mathbf{e}}^*$ differ only in $P_{R(j)}(\cdot)$, which only the agent j likes. Thus, **ITMON** applies, and so $\mu_{\varphi j}(\widetilde{\widehat{\mathbf{e}}}'_{11}) \geq \mu_{\varphi j}(\widetilde{\mathbf{e}}^*)$, which implies $\mu_{\varphi j}(\mathbf{e}_1) \geq \mu_{\varphi j}(\mathbf{e}_2)$ by Lemma 2. ■

Lemma 8: *Let* $\#N \geq 2$, *and* $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}^*$ *be such that* $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}, Y^{(m)}) \in \mathcal{E}^{M*}$ *and* $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}, Y^{(l)}) \in \mathcal{E}^{L*}$. *Moreover,* $S(\mathbf{e}_1) \supseteq S(\mathbf{e}_2)$, *and for all* $i \in N$, $m^i(S(\mathbf{e}_1)) = m^i(S(\mathbf{e}_2))$. *Then, if the allocation rule* φ *satisfies* $D^{\mathcal{E}^*}$, **PE**, **WTMON**, *and* **CTI**, *then* $\mu_\varphi(\mathbf{e}_1) \geq \mu_\varphi(\mathbf{e}_2)$.

Proof. 1. As in the proof of Lemma 3, let us construct $\mathbf{e}_1^\triangle = (K, \mathbf{v}^1, \mathbf{s}, Y^{(\#K)}) \in \mathcal{E}^{K*}$ and $\mathbf{e}_2^\triangle = (K', \mathbf{v}^2, \mathbf{s}, Y^{(\#K')}) \in \mathcal{E}^{K'*}$ such that $K \cap K' = \varnothing$, $S(\mathbf{e}_1^\triangle) = S(\mathbf{e}_1)$, and $S(\mathbf{e}_2^\triangle) = S(\mathbf{e}_2)$. Construct the convolution $\mathbf{e}^* = \mathbf{e}_1^\triangle \wedge \mathbf{e}_2^\triangle = (K \cup K', \mathbf{v}^*, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')}) \in \mathcal{E}^{K \cup K'*}$. In the same way as in step **2.1.** of the proof of Lemma 2, $S(\mathbf{e}^*) = S(\mathbf{e}_1^\triangle) \cap S(\mathbf{e}_2^\triangle) = S(\mathbf{e}_2)$, so $\mu_\varphi(\mathbf{e}^*) = \mu_\varphi(\mathbf{e}_2)$ by Lemma 2.
2. Construct the flat extension economy of $\mathbf{e}_1^\triangle$:

$$\begin{aligned} \widehat{\mathbf{e}}_1^\triangle &= \left(K \cup K', \widehat{\mathbf{v}}^1, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')}\right) \\ \text{with } \widehat{v}_i^1(x, y_K, y_{K'}) &= v_i^1(x, y_K), \forall (x, y_K, y_{K'}) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K} \times \mathbb{R}_+^{\#K'}. \end{aligned}$$

Since $S(\mathbf{e}_1) = S(\widehat{\mathbf{e}}_1^\triangle)$, $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\widehat{\mathbf{e}}_1^\triangle)$ by Lemma 2. Compare $\mathbf{e}^*$ with $\widehat{\mathbf{e}}_1^\triangle$. By definition, $\widehat{v}_i^1 \geq v_i^*$ for all $i \in N$. Since $m^i(S(\mathbf{e}_1)) = m^i(S(\mathbf{e}_2))$ for all $i \in$

N and $S(\mathbf{e}_1) = S(\widehat{\mathbf{e}}_1^{\triangle}) \supseteq S(\mathbf{e}^*) = S(\mathbf{e}_2)$, we obtain $m^i(S(\widehat{\mathbf{e}}_1^{\triangle})) = m^i(S(\mathbf{e}^*))$ for all $i \in N$.
3. By applying the Howe theorem, for each $i \in N$, there exist $w_i \in \mathcal{U}^{K \cup K' \cup \{R(i)\}}$ and $\widehat{y}_{R(i)} \in \mathbb{R}_+$ such that, for all $(x, y_K, y_{K'}) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K} \times \mathbb{R}_+^{\#K'}$,

$$w_i(x, y_K, y_{K'}, \widehat{y}_{R(i)}) = \widehat{v}_i^1(x, y_K, y_{K'}) \text{ and } w_i(x, y_K, y_{K'}, 0) = v_i^*(x, y_K, y_{K'}).$$

Construct

$$\begin{aligned} \widetilde{\widehat{\mathbf{e}}}_1^{\triangle} &= \left(K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^{(r)}\right) \text{ and} \\ \widetilde{\mathbf{e}}^* &= \left(K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y_*^{(r)}\right) \end{aligned}$$

where $R \equiv \{R(i)\}_{i \in N}$, $Y^{(r)}$ and $Y_*^{(r)}$ are defined exactly as in the proof of Lemma 3, and, for each $i \in N$,

$$\widehat{w}_i(x, y_K, y_{K'}, y_R) = w_i(x, y_K, y_{K'}, (y_R)_i) \text{ for all } (x, y_K, y_{K'}, y_R) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K} \times \mathbb{R}_+^{\#K'} \times \mathbb{R}_+^n.$$

Then, $S(\widehat{\mathbf{e}}_1^{\triangle}) = S(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}) \supseteq S(\widetilde{\mathbf{e}}^*) = S(\mathbf{e}^*)$.
4. Since $m^i(S(\widehat{\mathbf{e}}_1^{\triangle})) = m^i(S(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}))$ and $m^i(S(\widetilde{\mathbf{e}}^*)) = m^i(S(\mathbf{e}^*))$ for all $i \in N$, we have $m^i(S(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle})) = m^i(S(\widetilde{\mathbf{e}}^*))$ for all $i \in N$. This implies that $m^i(Z(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle})) \cap m^i(Z(\widetilde{\mathbf{e}}^*)) \neq \varnothing$ for all $i \in N$, since $Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^{(r)} \supseteq Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y_*^{(r)}$. Then, by **WTMON**, $\mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}) \geq \mu_\varphi(\widetilde{\mathbf{e}}^*)$. Thus, by Lemma 2, $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\widehat{\mathbf{e}}_1^{\triangle}) = \mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1^{\triangle}) \geq \mu_\varphi(\widetilde{\mathbf{e}}^*) = \mu_\varphi(\mathbf{e}^*) = \mu_\varphi(\mathbf{e}_2)$. ■

Proof of Theorem 1: Note that any allocation rule which attains a bargaining solution should satisfy $D^{\mathcal{E}^*}$ and **CTI**. By Theorem 3, we see that φ^E is a unique allocation rule which satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **SS**, and **CTI**. However, φ^E does not satisfy **IUI**, so the theorem is proved. ■

Given $\mathbf{s} \in \mathcal{S}^n$, let $\mathcal{E}(\mathbf{s}) \subsetneq \mathcal{E}$ (resp. $\mathcal{E}^*(\mathbf{s}) \subsetneq \mathcal{E}^*$) be the class of economies with the profile of production skills $\mathbf{s}$ fixed.

Proof of Theorem 2: (1) It is easy to see that φ^E satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **ISE**, **TMON**, and **CTI**.
(2) Suppose that the allocation rule φ satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **ISE**, **TMON**, and **CTI**. Then, for any $\mathbf{s} \in \mathcal{S}^n$, μ_φ satisfies, on $\mathcal{E}^*(\mathbf{s})$, all three

axioms which together characterize the egalitarian solution E (Kalai [12]), which is followed by Lemmas 3 and 6. Thus, for any $\mathbf{s} \in \mathcal{S}^n$, μ_φ on $\mathcal{E}^*(\mathbf{s})$ is always the outcome of the egalitarian solution E. This implies that φ attains E, so that $\varphi = \varphi^E$. ■

To prove Theorem 3, we need the following two lemmas:

Lemma 9: *If φ satisfies $D^{\mathcal{E}}$,* **WPE**, **ISE**, *and* **CTI**, *then φ satisfies* **W**.

Proof. Let $\mathbf{e}_1 = (M, \mathbf{u}^1, \mathbf{s}^1, Y^{(m)}) \in \mathcal{E}^M$ and $\mathbf{e}_2 = (L, \mathbf{u}^2, \mathbf{s}^2, Y^{(l)}) \in \mathcal{E}^L$ be such that $S(\mathbf{e}_1) = S(\mathbf{e}_2)$. Let φ satisfy $D^{\mathcal{E}}$, **WPE**, **ISE**, and **CTI**. We will show that $\mu_\varphi(\mathbf{e}_1) = \mu_\varphi(\mathbf{e}_2)$.

Let us define $\mathbf{e}_1^{\triangle} = (K, \mathbf{v}^{\triangle}, \mathbf{s}^1, Y^{(k)}) \in \mathcal{E}^K$ and $\mathbf{e}_2^{\triangle} = (K, \mathbf{v}^{\triangle}, \mathbf{s}^2, Y^{(k)}) \in \mathcal{E}^K$ as defining $\mathbf{e}^{\triangle}$ in step **1.** of the proof of Lemma 2. Then, $S(\mathbf{e}_1^{\triangle}) = S(\mathbf{e}_1) = S(\mathbf{e}_2) = S(\mathbf{e}_2^{\triangle})$. By Lemma 2, $\mu_\varphi(\mathbf{e}_1^{\triangle}) = \mu_\varphi(\mathbf{e}_1)$ and $\mu_\varphi(\mathbf{e}_2^{\triangle}) = \mu_\varphi(\mathbf{e}_2)$. Thus, it suffices to show $\mu_\varphi(\mathbf{e}_1^{\triangle}) = \mu_\varphi(\mathbf{e}_2^{\triangle})$. Note that, by construction of $Y^{(k)}$, $Z(\mathbf{e}_1^{\triangle}) = Z(\mathbf{e}_2^{\triangle})$. Thus, by **ISE**, $\varphi(\mathbf{e}_1^{\triangle}) = \varphi(\mathbf{e}_2^{\triangle})$, so that $\mu_\varphi(\mathbf{e}_1^{\triangle}) = \mu_\varphi(\mathbf{e}_2^{\triangle})$. ■

Lemma 10: *If φ satisfies $D^{\mathcal{E}^*}$,* **PE**, **SS**, *and* **CTI**, *then φ satisfies* **TMON**.[22]

Proof. Let $\mathbf{e}_1 = (M, \mathbf{u}, \mathbf{s}, Y_1^{(m)}) \in \mathcal{E}^{M*}$ and $\mathbf{e}_2 = (M, \mathbf{u}, \mathbf{s}, Y_2^{(m)}) \in \mathcal{E}^{M*}$ be such that $Y_1^{(m)} \supseteq Y_2^{(m)}$. Let φ satisfy $D^{\mathcal{E}^*}$, **PE**, **SS**, and **CTI**. We will show that $\mu_\varphi(\mathbf{e}_1) \geq \mu_\varphi(\mathbf{e}_2)$.

Note that $S(\mathbf{e}_1) \supseteq S(\mathbf{e}_2)$. Let $\mathbf{e}_1^{\triangle} = (K, \mathbf{v}^1, \mathbf{s}, Y^{(\#K)}) \in \mathcal{E}^{K*}$ be such that $S(\mathbf{e}_1^{\triangle}) = S(\mathbf{e}_1)$, which is defined as in step **1.** of the proof of Lemma 2. In the same way, let $\mathbf{e}_2^{\triangle} = (K', \mathbf{v}^2, \mathbf{s}, Y^{(\#K')}) \in \mathcal{E}^{K*}$ be such that $S(\mathbf{e}_2^{\triangle}) = S(\mathbf{e}_2)$, where $K \cap K' = \varnothing$. Let $\mathbf{e}^* = \mathbf{e}_1^{\triangle} \wedge \mathbf{e}_2^{\triangle} = (K \cup K', \mathbf{v}^*, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')})$. Let $\widehat{\mathbf{e}}_1 \equiv (K \cup K', \widehat{\mathbf{v}}^1, \mathbf{s}, Y^{(\#K)} \oplus Y^{(\#K')})$ where $\widehat{\mathbf{v}}^1$ is the flat extension of $\mathbf{v}^1$. Note that $S(\mathbf{e}^*) = S(\mathbf{e}_2^{\triangle})$. Thus, by Lemma 2, $\mu_\varphi(\mathbf{e}^*) = \mu_\varphi(\mathbf{e}_2^{\triangle})$ and $\mu_\varphi(\widehat{\mathbf{e}}_1) = \mu_\varphi(\mathbf{e}_1^{\triangle})$.

By Howe's theorem, for each $i \in N$, there exist $w_i \in \mathcal{U}^{K \cup K' \cup \{R(i)\}}$ and $\widehat{y}_{R(i)} \in \mathbb{R}_+$ such that:

$$\forall (x, y_K, y_{K'}) \in [0, \overline{x}] \times \mathbb{R}_+^{\#K} \times \mathbb{R}_+^{\#K'},$$

[22] We can strengthen this lemma as follows:

Lemma 10*: *If φ satisfies $D^{\mathcal{E}}$,* **WPE**, **SM**, **ISE**, *and* **CTI**, *then φ satisfies* **TMON**.

In fact, the proof of Lemma 10 can directly apply to the proof of Lemma 10*. By replacing Lemma 10 with Lemma 10*, we can derive the proofs of Theorems 1* and 3*.

$$\begin{aligned} w_i(x, y_K, y_{K'}, \widehat{y}_{R(i)}) &= \widehat{v}_i^1(x, y_K, y_{K'}) \\ \& \; w_i(x, y_K, y_{K'}, 0) &= v_i^*(x, y_K, y_{K'}). \end{aligned}$$

Now construct the flat extension of w_i for each $i \in N$:

$$\begin{aligned} \forall (x, y_K, y_{K'}, y_R) &\in [0, \overline{x}] \times \mathbb{R}_+^{\#K} \times \mathbb{R}_+^{\#K'} \times \mathbb{R}_+^n, \\ \widehat{w}_i(x, y_K, y_{K'}, y_R) &= w_i(x, y_K, y_{K'}, y_{R(i)}), \\ \text{where } y_R &= (y_{R(j)})_{j \in N}. \end{aligned}$$

Let $\mathbf{s}^1 = (s_i^1)_{i \in N}$ be a new profile of production skills such that for all $i \in N$, $s_i^1 = 1$. Also, let $\mathbf{s}^* = (s_i^*)_{i \in N}$ be a new profile of production skills such that for all $i \in N$, $s_i^* = 0$. Let, for each $R(i)$,

$$Y^{R(i)} \equiv \left\{ (x, y_{R(i)}) \in \mathbb{R}_+^2 \,\middle|\, y_{R(i)} \leq \min \left\{ \frac{\widehat{y}_{R(i)}}{n\overline{x}} x, \widehat{y}_{R(i)} \right\} \right\}.$$

Now, define

$$\begin{aligned} \widetilde{\widehat{\mathbf{e}}}_1 &\equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}^1, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^R) \\ \widetilde{\mathbf{e}}^* &\equiv (K \cup K' \cup R, \widehat{\mathbf{w}}, \mathbf{s}^*, Y^{(\#K)} \oplus Y^{(\#K')} \oplus Y^R) \\ \text{where } Y^R &\equiv Y^{R(1)} \oplus \cdots \oplus Y^{R(i)} \oplus \cdots \oplus Y^{R(n)}. \end{aligned}$$

By this definition, $S(\widetilde{\widehat{\mathbf{e}}}_1) = S(\widehat{\mathbf{e}}_1)$ and $S(\widetilde{\mathbf{e}}^*) = S(\mathbf{e}^*)$. By Lemma 9, $\mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1) = \mu_\varphi(\widehat{\mathbf{e}}_1)$ and $\mu_\varphi(\widetilde{\mathbf{e}}^*) = \mu_\varphi(\mathbf{e}^*)$, since **SS** together with $D^{\mathcal{E}^*}$ and **PE** imply **ISE**. Since φ satisfies **PE** and **SS**, we have $\mu_\varphi(\widetilde{\widehat{\mathbf{e}}}_1) \geq \mu_\varphi(\widetilde{\mathbf{e}}^*)$. This implies $\mu_\varphi(\mathbf{e}_1^\triangle) \geq \mu_\varphi(\mathbf{e}_2^\triangle)$, so that, by Lemma 2, $\mu_\varphi(\mathbf{e}_1) \geq \mu_\varphi(\mathbf{e}_2)$. ■

Proof of Theorem 3: (1) It is easy to see that φ^E satisfies **SS**.
(2) If φ satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **SS**, and **CTI**, then φ satisfies **TMON** and **ISE** by Lemma 10. Thus, by Theorem 2, $\varphi = \varphi^E$. ■

Proof of Theorem 4: (1) It is easy to see that φ^E satisfies **SM**.
(2) By Fleurbaey and Maniquet [9], **PE**, **SM**, and **TMON** together imply **SS**. Thus, by Theorem 2, if φ satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **SM**, **TMON**, and **CTI**, then $\varphi = \varphi^E$. ■

Let $\{\mathcal{U}^n(\mathbf{b}_\lambda)\}_{\lambda \in \Lambda}$ be a partition of $\mathcal{U}^n$ such that for every $\lambda \in \Lambda$, every n-tuple utility functions in $\mathcal{U}^n(\mathbf{b}_\lambda)$ has the same profile of utility-units $\mathbf{b}_\lambda$.

Given $\mathbf{s} \in \mathcal{S}^n$, let $\mathcal{E}(\mathbf{s}; \mathbf{b}_\lambda) \subsetneq \mathcal{E}(\mathbf{s})$ be the class of economies with the profiles of production skills $\mathbf{s}$ and of utility-units $\mathbf{b}_\lambda$ fixed.

Proof of Theorem 5: (1) It is easy to see that φ^{Na} satisfies $D^{\mathcal{E}}$, **PE**, **WETE**, **ISE**, **ITC**, **IUI**, and **CTI**.
(2) Suppose that the allocation rule φ satisfies $D^{\mathcal{E}}$, **PE**, **WETE**, **ISE**, **ITC**, **IUI**, and **CTI**. Then, for any $\mathbf{s} \in \mathcal{S}^n$ and any $\mathbf{b}_\lambda$, μ_φ satisfies, on $\mathcal{E}(\mathbf{s}; \mathbf{b}_\lambda)$, all four axioms which together characterize the Nash solution Na (Nash [16]), which is followed by Lemmas 4, 5, and 6. Thus, for any $\mathbf{s} \in \mathcal{S}^n$ and any $\mathbf{b}_\lambda$, μ_φ on $\mathcal{E}(\mathbf{s}; \mathbf{b}_\lambda)$ is always the outcome of the Nash solution Na. This implies that φ attains Na, so that $\varphi = \varphi^{Na}$. ■

Proof of Theorem 6: (1) It is easy to see that φ^K satisfies $D^{\mathcal{E}}$, **PE**, **WETE**, **ISE**, **ITMON**, **IUI**, and **CTI**.
(2) Suppose that the allocation rule φ satisfies $D^{\mathcal{E}}$, **PE**, **WETE**, **ISE**, **ITMON**, **IUI**, and **CTI**. Then, for any $\mathbf{s} \in \mathcal{S}^n$ and any $\mathbf{b}_\lambda$, μ_φ satisfies, on $\mathcal{E}(\mathbf{s}; \mathbf{b}_\lambda)$, all four axioms which together characterize the Kalai-Smorodinsky solution K when $\#N = 2$ (Kalai and Smorodinsky [13]), which is followed by Lemmas 5, 6, and 7. Thus, for any $\mathbf{s} \in \mathcal{S}^n$ and any $\mathbf{b}_\lambda$, μ_φ on $\mathcal{E}(\mathbf{s}; \mathbf{b}_\lambda)$ is always the outcome of the Kalai-Smorodinsky solution K. This implies that φ attains K, so that $\varphi = \varphi^K$. ■

Proof of Theorem 7: (1) It is easy to see that φ^K satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **ISE**, **WTMON**, **IUI**, and **CTI**.
(2) Suppose that the allocation rule φ satisfies $D^{\mathcal{E}^*}$, **PE**, **WETE**, **ISE**, **WTMON**, **IUI**, and **CTI**. Then, for any $\mathbf{s} \in \mathcal{S}^n$ and any $\mathbf{b}_\lambda$, μ_φ satisfies, on $\mathcal{E}^*(\mathbf{s}; \mathbf{b}_\lambda)$, all four axioms which together characterize the Kalai-Smorodinsky solution K when $\#N \geq 2$ (Thomson [22, Proposition 1, Remark 2]), which is followed by Lemmas 5, 6, and 8. Thus, for any $\mathbf{s} \in \mathcal{S}^n$ and any $\mathbf{b}_\lambda$, μ_φ on $\mathcal{E}^*(\mathbf{s}; \mathbf{b}_\lambda)$ is always the outcome of the Kalai-Smorodinsky solution K. This implies that φ attains K, so that $\varphi = \varphi^K$. ■

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