On the Ricardian Invariable Measure of Value: A General Possibility of the Standard Commodity*

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Abstract

The purpose of this paper is to examine the critical arguments made by Burmeister, Samuelson, and others with respect to Sraffa (1960). In his arguments about the standard commodity, Sraffa assumed that a change in income distribution has no effect on the output level and choice of techniques. However, critics argue that the interdependence among changes in income distribution, output level, and choice of techniques should be considered in the arguments on the invariable measure of value and the linearity of income distribution. Given this debate, this paper defines a generalisation of the standard commodity by considering general economies with non-increasing returns to scale, in which such interdependence is a universal feature. Moreover, it is shown that the generalised standard commodity can serve as an invariable measure of value in those general economies. Finally, this paper characterises the necessary and sufficient condition under which the linear functional relation of income distribution is obtained in those economies.

Keywords: Ricardo's invariable measure of value, Sraffa's standard commodity, Linear relation of income distribution

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1 Introduction

As is well known, in his later years, Ricardo searched intensively for an invariable measure of value.[1] His struggle to find such a measure is shown in, among other publications, his *Principles* and his papers entitled 'Absolute Value and Exchangeable Value', which were written in the last few weeks of his life (Ricardo, 1951D, pp. 361–412).

An invariable measure of value can be defined as a measure that is invariable with respect to changes in both income distribution and technique (Ricardo, 1951A, chap. 1). The advantage of an invariable measure of value is that, when relative prices change, we can use it to distinguish between variations that belong to the commodity itself and those occasioned by a variation in the medium by which values or prices are expressed (Ricardo, 1951A, p. 48). For Ricardo, the pursuit of the invariable measure of value was directly related to completing the embodied labour theory of value.[2] However, the importance of the invariable measure of value was not adequately understood by his contemporaries, such as Malthus.[3]

Although it is true that the embodied labour theory of value cannot hold generally when the rate of profit is positive, this does not mean that the invariable measure of value is no longer significant. Ricardo wished to construct an invariable measure of value as a foundation on which to measure variables such as national income or

national wealth, as well as to compare those variables intertemporally. No one can deny the importance of the invariable measure of value, even today.

In the 20th century, Sraffa (1960) revived the concern about the invariable measure of value, which had fallen into oblivion since the so-called Marginal Revolution. Unlike Ricardo, he divided the problem of identifying an invariable measure of value into two parts: (i) searching for a measure of value that is invariable with respect to changes in technique, leaving aside the possibility of changing income distribution; and (ii) searching for a measure of value that is invariable with respect to the change in income distribution, leaving aside the possibility of changing technique. Sraffa concentrated on the latter of the two by constructing a special, composite commodity termed the *standard commodity*.[4] He also demonstrated an interesting relationship with respect to income distribution if the standard commodity is adopted as the numéraire: the linear relationship of income distribution. Although many economists have paid great attention to Sraffa's results, it seems there is no consensus on their evaluation of these results. Some economists appreciate him, while others do not unconditionally admit the significance of the standard commodity and the linearity of income distribution. In particular, those who are critical of Sraffa regard the assumption of a fixed technique without constant returns to scale as being too restrictive, and thus, downgrade the relevance of Sraffa's findings. Among others,

Burmeister (1968, 1975, 1977, 1980, 1984), Samuelson (2000, 2008), and Samuelson and Etula (2006) have claimed that Sraffa's analyses are irrelevant without the assumption of constant returns to scale.

We think that the views of Burmeister, Samuelson, and others are worth examining, because they point out problems from the viewpoint of Ricardo (1951A) and modern economic theory that Sraffa had not addressed. In his arguments about the standard commodity, Sraffa (1960) assumed that a change in income distribution has no effect on the output level and choice of techniques, which he used as an analytical device to construct a model. However, it is plausible that a change in income distribution is related to changes in output level or choice of techniques in a real economy. In fact, almost all modern economic theories, including Ricardo (1951A), admit an interdependence among changes in income distribution, output level, and choice of techniques, although the theories differ on the logical consequences of the interdependence. Even those who favour Sraffa cannot deny this interdependence. Curiously enough, there is little literature on whether an invariable measure of value and linearity of income distribution can be obtained in a model in which the above-mentioned interdependence is allowed.[5] Therefore, we attempt to examine the critical arguments of Burmeister, Samuelson, and others with respect to Sraffa (1960). That is, assuming quite general economies with non-increasing returns to scale, where the

above-mentioned interdependence is a universal feature, we define a generalisation of the standard commodity and show that it still serves as an invariable measure of value in such general economies. In addition, we specify the conditions under which the linear relationship of income distribution is preserved in such general economies.

This paper is organised as follows: In Section 2, we present a brief review of the concept of Ricardo's invariable measure of value and Sraffa's standard commodity. We then briefly review the history of the debates on Sraffa's standard commodity and the linear relationship of income distribution. In Section 3, we apply the standard commodity to a more general production economy than Sraffa's (1960), and discuss the main results in terms of the invariable measure of value and the linear relationship of income distribution. Lastly, Section 4 concludes the paper.

2 The Invariable Measure of Value and Debates concerning Sraffa (1960)

In this section, we briefly review the concept of Ricardo's invariable measure of value and Sraffa's standard commodity. We also review the debates concerning the significance of the standard commodity and linearity of income distribution.

2.1 Ricardo's invariable measure of value

Ricardo asserted that the conditions necessary to make a measure of value perfect are that it should itself have a value, and that the value itself should be invariable (Ricardo, 1951D, p. 361). Concerning the first condition, he clearly asserted that the labour content embodied in such a commodity represents the exchange value of the commodity. The second condition, the invariance of the value of such a commodity, perplexed him throughout his life.

Why is it difficult to obtain an invariable measure of value? First, the technique to produce the measure must remain unchanged. In other words, a commodity eligible to become an invariable measure of value is one 'which now and at all times required precisely the same quantity of labour to produce it'. However, Ricardo realised that 'of such a commodity, we have no knowledge, and consequently are unable to fix on any standard of value' (Ricardo, 1951A, p. 17, n. 3). In fact, Ricardo regarded money (i.e. gold) as the invariable measure of value, but that it is only 'as near as approximation to a standard measure of value as can be theoretically conceived' (Ricardo, 1951A, p. 45). The justification for this view of money is based on his recognition that the techniques used to produce gold are subject to fewer variations (Ricardo, 1951A, p. 87) and his supposition that money can be regarded as the so-called 'average commodity'(Ricardo, 1951A, pp. 45–6).[6]

With respect to the second condition, even though the technique to produce gold is unchanged, it cannot be an invariable measure of value. This is because all industries have different proportions of capital and labour, different proportions of circulating and fixed capital, different degrees of durability of fixed capital, and require different periods to bring the commodity to market. In this situation, a change in the level of wage rates causes changes in relative prices. Therefore, as already mentioned, we cannot precisely know the price changes, because the price of gold itself (the standard of value) is subject to relative variation. The perfect invariable measure of value does not exist in reality. However, according to Ricardo (1951A, p. 45), the effect of a change in income distribution on relative prices is smaller than the effect of a change in technique. Therefore, Ricardo considered the deviation of value from the embodied quantity of labour to be sufficiently slight (Ricardo, 1951B, p. 66), and was reluctantly content to say that money can be regarded as an invariable measure of value, at least as a first approximation.

2.2 Sraffa's standard commodity and income distribution

Sraffa (1960) revived the concern about the invariable measure of value. As already mentioned, Ricardo had defined the conditions that the invariable measure of value should satisfy: the invariance of the measure of value with respect to changes in

both income distribution and technique. Ricardo was perplexed by the conditions, because he attempted to solve the two simultaneously. In contrast, Sraffa concentrated on finding a measure of value that is invariable with respect to a change in income distribution, while ignoring changes in techniques. Furthermore, it was Sraffa's breakthrough idea to find a special, composite commodity, termed the *standard commodity*, which plays the role of the invariable measure of value. In contrast, Ricardo had attempted to find a single commodity to play this role.

Let us briefly review the concept of the standard commodity in a single product system. The price system is defined as follows:

$$\mathbf{p} = (1 + \pi)\,\mathbf{p}\boldsymbol{A} + w\boldsymbol{L}, \tag{1}$$

where $\mathbf{p}$, $\boldsymbol{L}$, and $\boldsymbol{A}$ denote the price vector, the labour coefficient vector, and the physical input coefficient matrix, respectively. For the sake of simplicity, $\boldsymbol{A}$ is assumed to be an indecomposable and productive matrix. In addition, π and w denote the rate of profit and the wage rate, respectively. In order to escape from the impasse that Ricardo faced, Sraffa attempted to find an (imaginary) industry with a value-ratio of the net product to means of production such that the increase in profit is exactly offset by the decrease in wages when the wage rate is reduced. This value-ratio is the solution to the system that Sraffa (1960, p. 20) called the *standard*

system:

$$(1+\Pi^*)\,\boldsymbol{A}\mathbf{q}^* = \mathbf{q}^*, \tag{2}$$

$$\boldsymbol{L}\mathbf{q}^* = 1. \tag{3}$$

Here, Π^* is the value-ratio, now termed the *standard ratio*, and is related to the Frobenius root, λ_A, as $\lambda_A = \frac{1}{1+\Pi^*}$. The standard ratio is equal to the maximum rate of profit. Then, $\mathbf{q}^*$ is the corresponding eigenvector, which denotes the output level of the industry that has the standard ratio. This output level is now termed the standard commodity. Since we assume the productiveness and indecomposability of $\boldsymbol{A}$, the above system of equations has the solution of $\Pi^* > 0$ and $\mathbf{q}^* > 0$, from the Perron–Frobenius theorem (Pasinetti, 1977, pp. 95–7). From equation (2), we obtain such a relationship as follows:

$$\frac{\mathbf{p}\,[\boldsymbol{I}-\boldsymbol{A}]\,\mathbf{q}^*}{\mathbf{p}\boldsymbol{A}\mathbf{q}^*} = \Pi^*, \tag{4}$$

where $\boldsymbol{I}$ denotes the identity matrix. Relationship (4) means that the ratio of the net product to the means of production, measured by the standard commodity, is always constant, irrespective of price variations. Therefore, Π^* is a real ratio that is

independent of prices. Sraffa defined the standard net product and chose it as the numéraire, as follows:

$$\mathbf{p}\left[\boldsymbol{I}-\boldsymbol{A}\right]\mathbf{q}^{*}=1. \tag{5}$$

Although the price of *any* numéraire is invariant by definition, the standard commodity is special in that the cause of a price change as a result of a change in income distribution is absent in the industry producing it. This can only be ensured when the numéraire is the standard commodity. Therefore, the standard commodity is eligible to become the invariable measure of value under the assumption of a fixed technique.[7] Note that the standard commodity does not need to be actually produced. It is a 'purely auxiliary construction' (Sraffa, 1960, p. 31).

In Sraffa's model, nothing except income distribution ever changes; the technique in use, output level, and proportion of means of production to labour are all fixed. Therefore, according to Sraffa (1960, p. v), no assumption on returns to scale is required. Under these assumptions, he exclusively analysed the change in relative prices caused by a change in income distribution. Based on (4) and (5), there is no need for a variation in the price of $\mathbf{q}^{*}$ to restore the surplus or deficit in the industry that produces that commodity when the wage rate is reduced. Therefore, the variation in relative prices caused by a change in income distribution is solely

attributed to the variation in the prices of the measured commodities based on the invariance property of the numéraire defined by the standard commodity.

Furthermore, the adoption of the standard commodity as the numéraire shows us the useful relation of income distribution. From (1) and (5), we obtain:

$$\pi = \Pi^* (1 - w) . \tag{6}$$

Here, w denotes the wage rate or the wage share in terms of the standard commodity, whereas π is the actual rate of profit. The distributional relation is expressed by the straight line. The important implication of function (6) is that the rate of profit can be obtained without knowing prices, once we know the wage in terms of the standard commodity. In other words, the standard commodity enables us to treat income distribution independently of prices. As Pasinetti (2006, p. 154) pointed out, the relevance of function (6) does not lie in its linearity, but in the fact that it is independent of prices.

We conclude that Sraffa partially resolved the problem that Ricardo could not. However, he did not consider the problem of the measure of value being invariable with respect to a change in technique.

2.3 After Sraffa (1960)

There have been many reactions to Sraffa (1960) since its publication. The debates have focused not only on the invariable measure of value, but also on the usefulness of the standard commodity and function (6). Some arguments appreciate Sraffa's achievements, especially his contribution in constructing the standard commodity as the invariable measure of value (e.g. Roncaglia, 2009). Others are critical of Sraffa. First, some economists have argued that the standard commodity does not play the role of the Ricardian invariable measure of value, pointing out the flaw in Sraffa's analysis. Flaschel (1986) is a typical example. The second critical argument was that the standard commodity and function (6) are so restrictive as to be impractical for relevant analyses. Note that these arguments were mainly raised by neoclassical economists, who were interested in variations in output and proportions of means of production.

Let us examine Flaschel (1986) first. According to him, there is a specific and complete solution to the problem of determining the conditions for the invariable measure of value, but Sraffa's standard commodity does not fulfil those conditions. It seems to us that his definition of invariance is different to those of Ricardo and Sraffa. According to his definition, given $\mathbf{e} - \boldsymbol{A}\mathbf{e}$ as the numéraire, where $\mathbf{e}$ is a vector, all the elements of which are units, an arbitrary composite commodity $\mathbf{b}$ has

the invariance property if and only if $\mathbf{pb} = 1$ holds for any non-negative and non-zero $\mathbf{p}$, with $\mathbf{p}\left[\boldsymbol{I} - \boldsymbol{A}\right]\mathbf{e} \equiv 1$ (Flaschel, 1986, pp. 597–8).[8] Certainly, Sraffa (1960, p.11) adopted $\left[\boldsymbol{I} - \boldsymbol{A}\right]\mathbf{e}$ as the numéraire, but the numéraire adopted in this context is irrelevant to the issue of the invariable measure of value, and his arguments on the standard commodity have nothing to do with this numéraire. Therefore, Flaschel's critique of the standard commodity seems pointless.[9]

With regard to the second criticism of Sraffa, a typical example is that of Burmeister (1968). The conclusions he derived are summarised as follows:

1) The economic significance of the standard commodity is dubious.

2) The linearity of the distributional relation does not hold if wages are paid at the beginning of the production period rather than at the end.

3) Without the assumption of constant returns to scale and a fixed coefficients matrix, Sraffa's analysis is meaningless if the quantity produced by an arbitrary industry changes.

Burmeister later repeated similar conclusions (Burmeister, 1975, 1977, 1980, 1984). However, he obviously misunderstood some aspects of Sraffa (1960).

Burmeister's first conclusion is a serious misunderstanding. He regarded the standard commodity as the *actual* consumption basket by which the real wage rate, w, in function (6) is measured.[10] Therefore, he argued that the standard commodity has

no economic significance; 'Sraffa's weights used to construct his basket of goods are seen to be determined completely from the technology without regard for consumption preferences' (Burmeister, 1984, p. 509). However, the adoption of the standard commodity as the numéraire does not imply that people must actually consume each commodity in the same proportion as that given by the standard commodity. Moreover, it does not imply that each commodity is actually produced in the same proportion as that given by the standard commodity (see Kurz and Salvadori, 1987, pp. 876–7).

Burmeister's second conclusion is correct. However, though the linearity no longer holds in this case, as Pasinetti (1977, p. 131) showed, the distributional relation is independent of prices.

The third conclusion is controversial. Samuelson (2000) and Samuelson and Etula (2006) also argued that constant returns to scale is an indispensable assumption in order to retain the significance of Sraffa's analysis. Against these arguments, some proponents of Sraffa argued that the assumption on returns to scale is unnecessary in Sraffa's analysis. The characteristic of the analysis is that it is based on the classical surplus approach. In the approach, the analysis of the distribution of physical surplus comes first. Eatwell (1977) emphasised the difference in the analytical basis between classical and neoclassical economics. In classical economics, the size and

composition of the output, technique in use, and real wage are the *data*, on the basis of which the distribution of the surplus, price formulation, and quantities of input and labour employed are obtained. In contrast, in neoclassical economics, the preferences of individuals, initial endowment of commodities and/or factors of production, distribution of the initial endowments among individuals, and technology are the *data*, and all variables are determined by the interaction between supply and demand. The latter is based on the marginal method, and thus the assumption on returns to scale is necessary in neoclassical economics. Eatwell (1977) thus argued that the assumption of constant returns to scale is irrelevant in Sraffa's analysis, because it is based on the classical surplus approach.

However, Burmeister and Samuelson considered what happens to the model when the output level changes. Unless constant returns to scale are assumed, the technique generally changes as the output level changes. Since each coefficient matrix has the specific standard ratio, the standard ratio also changes when the technique in use changes. Therefore, function (6) no longer gives us any useful information on income distribution when a change in the output level causes a change in technique in economies without constant returns to scale. Burmeister (1977, pp. 69–70) thus replied to Eatwell: 'I conclude that constant returns to scale is irrelevant for Sraffa's analysis only if one is content to pose irrelevant questions.'

Although it is true that Burmeister's interpretation of Sraffa included the misunderstanding, it is also true that he raised important questions that Sraffa had not addressed. The first was whether an invariable measure of value exists in economies in which income distribution and technical choice are both available. The second was that, if the measure does exist in such an economy, what kind of relationship between the invariable measure of value and income distribution holds. We believe it is worthwhile examining these questions. From the viewpoint of modern economic theories as well as Ricardo (1951A), these are natural questions, because nearly all economic theories allow for interdependence among changes in income distribution, output level, and techniques. In fact, Sraffa (1925, 1926) himself considered the relationship between returns to scale and choice of techniques, although his consideration was related to the critique of Marshallian partial equilibrium analysis.

3 The Standard Commodity and Income Distribution under Non-increasing Returns to Scale

In this section, we investigate the conditions for the invariable measure of value and the linearity of income distribution under a non-increasing returns to scale production technology.

3.1 Generalisation of the standard commodity

In order to analyse the invariable measure of value and income distribution in economies in which changes in technique are allowed, we introduce the production possibility set, P, with non-increasing returns to scale. Here, P is the set of available production processes. A production process is defined as $\alpha \equiv (-\alpha_l, -\underline{\alpha}, \overline{\alpha})$, where α_l is the non-negative *effective* labour input of the process, $\underline{\alpha}$ is the non-negative vector of the inputs of the produced goods used in the process, and $\overline{\alpha}$ is the non-negative vector of the outputs of the n goods. There are small, mild restrictions on the properties of P: (i) not activating any available production processes is possible; (ii) to produce any non-negative vector of the n goods as a net output, there is at least one production process available in P; (iii) to produce any non-negative and non-zero vector of commodities, the inputs of labour and at least one type of capital good are indispensable; and (iv) if there are two production processes available in P, any proportion, say $t \in (0, 1)$, of one of the processes and the remaining proportion, $1 - t$, of the other process can be jointly activated. Production set P satisfying these restrictions is so general that various types of technologies, such as Leontief production models, with or without technical choices, joint production models, and even neoclassical differentiable production functions are subject to analysis here.[11]

Let us assume that one *economy* is represented by a production set, P. We can

then define the standard commodity in economy P as follows.

Definition 1: For any economy, P, a *standard commodity* is a positive vector, $y > \mathbf{0}$, such that there exists a vector $\alpha = (-1, -x, x + y)$ on P satisfying the following properties; (i) $\frac{y_i}{x_i} = \frac{y_j}{x_j}$, for any $i, j = 1, \ldots, n$; and (ii) there is no other $\alpha' = (-1, -x', x' + y')$ with $\frac{y'_j}{x'_i} = \frac{y'_j}{x'_j}$, for any $i, j = 1, \ldots, n$, $\frac{y'_i}{x'_i} > \frac{y_i}{x_i}$, for any $i = 1, \ldots, n$, and $y' > y$.

The standard commodity defined here is a generalisation of Sraffa's definition. Firstly, Definition 1 implies that the standard commodity is defined as the net product, y, that can be produced by process $\alpha = (-1, -x, x + y)$ with labour input $\alpha_l = 1$, physical inputs $\underline{\alpha} = x$, and gross outputs $\overline{\alpha} = x + y$. Moreover, condition (i) of Definition 1 implies that under this process, the ratio of net product to means of production is uniform, $\frac{y_i}{x_i} = \frac{y_j}{x_j}$, for any $i, j = 1, \ldots, n$. Secondly, condition (ii) of Definition 1 is a generalisation of the maximality condition of the uniform ratio of net product to means of production. The ratio corresponds to the standard ratio Π^* of equation (2) in Section 2.2. Therefore, Definition 1 is an extension of the Sraffian definition of the standard commodity characterised by equations (2) and (3) to a more general economy, P. In addition, production process $\alpha = (-1, -x, x + y)$ is regarded as the *standard system* in economy P.

This definition is well defined in that the standard commodity given by Definition

1 exists.[12]

3.2 The invariable measure of value and the linear relation of income distribution

This section examines whether the standard commodity defined in Definition 1 can function as an invariable measure of value in economy P.

Consider a price system, (p, w), which is a non-negative and non-zero vector with $p \neq \mathbf{0}$. Let there be the maximal rate of profit, $\pi \geqq 0$, and a production process, $\alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha})$, on P associated with (p, w), such that

$$\begin{aligned} p\overline{\alpha} &= (1+\pi)\, p\underline{\alpha} + w\alpha_l \text{ and} \\ p\overline{\alpha}' &\leqq (1+\pi)\, p\underline{\alpha}' + w\alpha_l' \end{aligned}$$

hold for any $\alpha' = (-\alpha_l', -\underline{\alpha}', \overline{\alpha}')$ on P. Then, let us call such a price system an *equilibrium price*.[13] Consider a situation in which an equilibrium price changes from (p, w) to (p', w'). Moreover, let π (resp. π') be the maximal rate of profit associated with the price system, (p, w) (resp. (p', w')). Then, let $\triangle p \equiv p' - p$, $\triangle w \equiv w' - w$, and $\triangle \pi \equiv \pi' - \pi$. Then, the following definitions are a generalisation of the invariable

measure of value, based on Baldone (2006):

Definition 2: Given an economy, P, let (p, w) and (p', w') be two different equilibrium prices, and π and π' be their respective maximal profit rates. Then, a commodity bundle, $y > \mathbf{0}$, serves as an *invariable measure of value with respect to a change from* (p, w) *to* (p', w') if and only if there exist a non-negative and non-zero vector, x, and a positive number, $k > 0$, such that the process $(-k, -x, x + y)$ is feasible within economy P, $(-k, -x, x + y) \in P$, and $\triangle py = 0$ holds whenever this price change involves a *redistribution between profit and wage,* i.e. $\triangle \pi px + \triangle wk = 0$.

Definition 3: Given an economy, P, a commodity bundle, $y > \mathbf{0}$ serves as an *invariable measure of value* if and only if for any different vectors of the equilibrium prices (p, w) and (p', w'), it serves as the invariable measure of value with respect to the change from (p, w) to (p', w').

That is, a commodity bundle serves as an invariable measure of value if and only if for any change in the price system involving a redistribution of profit and wage, the price of this commodity bundle is invariable. The definitions faithfully follow that of Sraffa reviewed in Section 2. More precisely, let us consider the counterfactual situation of a change in income distribution from (π, w) to (π', w'), while keeping the commodity price vector p constant, such that the increase (*resp.* decrease) in

profit is equal to the decrease (*resp.* increase) in wage in the production process $(-k, -x, x+y)$ of the targeted commodity bundle, y. Such a change may be derived from a purely political conflict on the income distribution between capital and labour, or it may involve a change in technique. However, whatever the cause, it results in a change in commodity prices from p to p'. Then, the commodity bundle y can serve as an invariable measure of value with respect to the change from (p, w) to (p', w') whenever $py = p'y$. Furthermore, if the commodity bundle satisfies such an invariable property for any change in price systems, with its corresponding redistribution between wage and profit, it can serve as an invariable measure of value.

It is worth emphasising that in the above definitions, the invariable property must hold regardless of the cause of such a price change. For instance, even if the price change associated with the corresponding redistribution is generated *by a technical change* so that the selected production process is changed in equilibrium,[14] the value of the commodity bundle is required to be invariable.

Theorem 1 provides the necessary and sufficient condition for the standard commodity to serve as an invariable measure of value.

Theorem 1: *For any economy, P, the standard commodity, y^*, associated with the standard system, $\alpha^* = (-1, -x^*, x^* + y^*)$,* serves as an invariable measure of value *if and only if for any equilibrium prices, (p, w) and (p', w'), such that $\triangle \pi p x^* + \triangle w =$*

0, *there exist non-negative numbers,* δ *and* δ'*, such that* $py^* = \pi px^* + w - \delta px^*$*,* $p'y^* = \pi' p'x^* + w' - \delta' p'x^*$*, and* $\delta = \delta'$ *hold.*

Proof: See the Appendix.

In Theorem 1, δ (resp. δ') represents the shortfall of profit rate from the maximal level, when generated by operating the standard system, α^*, at the equilibrium price, (p, w) (resp. (p', w')). The standard commodity can serve as an invariable measure of value with respect to a change from (p, w) to (p', w') if and only if the shortfall of the profit rate generated by operating the standard system is invariable with respect to such a change in prices whenever it involves an income redistribution, $\triangle \pi px^* + \triangle w = 0$. It then follows that the standard commodity can serve as an invariable measure of value if and only if the shortfall in profit generated by operating the standard system is invariable with respect to any change in equilibrium prices involving an income redistribution.

Theorem 1 can be used to investigate, for each given economy, whether the standard commodity can serve as an invariable measure of value. Such an investigation is particularly relevant in a general economy, P, in which a change in prices could be associated interdependently with a change in technique and/or a change in produced outputs. For the standard system, α^* is not necessarily a profit-rate maximiser in such an economy. Therefore, a positive shortfall, $\delta > 0$, of profit rate from the max-

imal level is available, unlike in the case of the single-product system discussed in Section 2. Even in the case of $\delta > 0$, as Theorem 1 suggests, the standard commodity can serve as an invariable measure of value whenever the amount of the shortfall is invariable.

The following corollary gives us a typical situation in which the standard commodity serves as the invariable measure of value.

Corollary 1: *For any economy, P, let us take any equilibrium prices, (p, w) and (p', w'), at each of which the standard system $\alpha^* = (-1, -x^*, x^* + y^*)$ is a profit rate maximiser. Then, the standard commodity, y^*, serves as an invariable measure of value with respect to a change from (p, w)* to *(p', w').*

Proof: See the Appendix.

Note that when the production set is represented by a simple Leontief technology, the standard system, α^*, is a profit rate maximiser for any equilibrium price vector. In an economy with a simple Leontief technology, an equilibrium price system is associated with an equal rate of profit available within every industry. This implies that any efficient production process, including the standard system, is a profit rate maximiser. Thus, as Corollary 1 shows, the standard commodity, y^*, can be an invariable measure of value with respect to any change in equilibrium prices.

Let us now assume that the standard commodity, y^*, is selected as the numéraire. Then, by definition, any non-negative and non-zero price vector, p, is normalised as $py^* = 1$. Given such a situation, we need to examine whether and under what condition the linear distributional relationship between profit and wage is preserved in economy P. The following theorem is our second main result.

Theorem 2: *Given an economy, P, the linear functional relation of income distribution $\pi' = \Pi(1 - w')$holds for any equilibrium price vector (p', w') associated with the maximal profit rate, π', if and only if $p'y^* = \pi' p' x^* + w'$ holds for any equilibrium price vector (p', w') associated with π', where $\alpha^* = (-1, -x^*, x^* + y^*)$ is the standard system and Π is the standard ratio.*

Proof: See the Appendix.

The crucial point for the above analysis is whether the standard system $\alpha^* = (-1, -x^*, x^* + y^*)$ is a profit rate maximiser at all equilibrium prices available in P. This property is trivially satisfied in single-product systems, such as Leontief production economies and as in Sraffa (1960), as discussed above. In contrast, economy P allows for the possibility of joint production as well as for technical choices, under which the standard system may not be a profit rate maximiser at some equilibrium prices. In such a case, Theorem 2 suggests that the linearity of income distribution

no longer holds.

Theorem 2 *per se* suggests that the linear functional relation of income distribution does not hold in most of the general economies other than Leontief production economies. However, it is quite interesting that, as in the following theorem, the standard commodity serves as an invariable measure of value in general economies.

Theorem 3: *For any economy, P, the standard commodity, y^*,serves as an invariable measure of value.*

Proof: See the Appendix.

In more detail, firstly, according to Corollary 1, the standard commodity y^* serves as an invariable measure of value with respect to a price change from (p, w) to (p', w') whenever the standard system α^* is a profit rate maximiser at both equilibrium prices. Secondly, it follows from a simple analysis with the application of Theorem 2 that any change of equilibrium prices from (p, w) to (p', w') involves the income redistribution, $\triangle \pi p x^* + \triangle w = 0$, in terms of Definition 2 if and only if the corresponding profit rate shortfalls are identical, $\delta = \delta'$. Therefore, by Definition 2 and Theorem 1, the standard commodity y^* serves as an invariable measure of value even in such a general case of a price change.

As discussed above, Definition 1 in this study is a faithful generalisation of Sraffa's

(1960) own definition of the standard commodity formulated within the domain of single-product systems. The standard commodity given in Definition 1 also satisfies both the watershed and recurrence conditions (Schefold, 1986, 1989). Moreover, Definitions 2 and 3 are faithful to Sraffa's (1960) own analysis and are generalisations of Baldone (2006). Therefore, Theorem 3 implies a general possibility theorem for the standard commodity as an invariable measure of value in a large class of general economies.

4 Concluding Remarks

This paper identified the necessary and sufficient condition for the standard commodity à la Sraffa (1960) to serve as an invariable measure of value under a general domain of economies with non-increasing returns to scale technology, and allowing for the interdependence among changes in income distribution, output levels, and technical choices. Based on the identified condition, the standard commodity is shown to serve as an invariable measure of value in such general economies. Therefore, this paper significantly generalises the result of Baldone (2006), who did not consider a change in price systems involving a change in technique.

Another main contribution of this paper is to identify the necessary and sufficient condition for the linear relationship of income distribution to hold when the standard

commodity is adopted as the numéraire in such a general domain of economies. According to the identified condition, the linearity is obtained if and only if the standard commodity is a profit rate maximiser whatever the equilibrium price system is. Most economies within such a domain, other than economies with single product systems, would not satisfy this condition. Ricardo (1952A, p. 194) conjectured that 'the great questions of Rent, Wages, and Profits must be explained by the proportions in which the whole produce is divided between landlords, capitalists, and labourers, and which are not essentially connected with the doctrine of value', which suggests that the rate of profit can be obtained without knowing the structure of prices. Our second result demonstrates that Ricardo's conjecture is not generally valid.

This sharp contrast between the performance of the two basic functions of the standard commodity is quite interesting, and cannot appear in the standard single-product system. However, it is a distinctive feature in more general economies.

Appendix: Mathematical Formulation of Production Possibility Set and Proofs of Theorems

In the Appendix, we rigorously formulate our model presented in Section 3.

Let $\mathbb{R}_+$ be the set of all non-negative real numbers, and $\mathbb{R}_{++}$ be the set of all

positive numbers. Let $\mathbb{R}^n_+$ (resp. $\mathbb{R}^n_{++}$) be the n-fold Cartesian product of $\mathbb{R}_+$ (resp. $\mathbb{R}_{++}$). For any $x, y \in \mathbb{R}^n_+$, we write $x \geqq y$ to mean $[x_i \geqq y_i$ for all $i = 1, \ldots, n]$, $x \geq y$ to mean $[x_i \geqq y_i$ for all $i = 1, \ldots, n$ and $x \neq y]$, and $x > y$ to mean $[x_i > y_i$ for all $i = 1, \ldots, n]$.

Let there be n reproducible commodities. Let $\mathbf{0}$ denote the null vector. Production technology is represented by a production set, P, which has elements of the form $\alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha})$, where $\alpha_l \in \mathbb{R}_+$ is the *effective* labour input of the process; $\underline{\alpha} \in \mathbb{R}^n_+$ are the inputs of the produced goods used in the process; and $\overline{\alpha} \in \mathbb{R}^n_+$ are the outputs of the n goods. Thus, elements of P are vectors in $\mathbb{R}^{2n+1}$. The following assumptions are imposed on production set P.

Assumption 0 (A0). P is closed and convex in $\mathbb{R}^{2n+1}$ and $\mathbf{0} \in P$.

Assumption 1 (A1). For all $\alpha \in P$, if $\overline{\alpha} \geq \mathbf{0}$, then $\alpha_l > 0$ and $\underline{\alpha} \geq \mathbf{0}$.

Assumption 2 (A2). For all $c \in \mathbb{R}^n_+$, there is an $\alpha \in P$ such that $\widehat{\alpha} \equiv \overline{\alpha} - \underline{\alpha} \geqq c$.

For each production possibility set, P, let us denote $\partial P \equiv \{\alpha \in P \mid \nexists \alpha' \in P : \alpha' > \alpha\}$, which is the boundary of production set P.

The model of the production sets with **A0~A2** covers a broad class of production technologies. For instance, it contains the class of von Neumann production models as a subclass. It also contains a convex combination of multiple Leontief production

models, which is an example of an economy with the possibility of technical choices and without joint production.

Under **A0~A2**, Theorem 1 presented in Section 3 can be rigorously expressed and proven, as follows.

Theorem 1: *For any economy* P satisfying ***A0~A2****, the standard commodity* y^* *associated with the standard system* $\alpha^* = (-1, -x^*, x^* + y^*) \in \partial P$ serves as an invariable measure of value *if and only if for any equilibrium prices* (p, w) *and* (p', w') *such that* $\triangle\pi px^* + \triangle w = 0$*, there exist non-negative numbers* $\delta, \delta' \geqq 0$ *such that* $py^* = \pi px^* + w - \delta px^*$*,* $p'y^* = \pi' p'x^* + w' - \delta' p'x^*$*, and* $\delta = \delta'$ *hold.*

Proof: Let us take any equilibrium prices (p, w) and (p', w') such that $p \neq \mathbf{0} \neq p'$, and $\triangle\pi px^* + \triangle w = 0$. By Definition 1, $y^* = \Pi x^*$ holds for some $\Pi > 0$. Since $\alpha_l^* = 1$, $py^* \leqq \pi px^* + w$ and $p'y^* \leqq \pi' p'x^* + w'$ generally hold. Therefore, there are non-negative numbers $\delta, \delta' \geqq 0$ such that $py^* = \pi px^* + w - \delta px^*$ and $p'y^* = \pi' p'x^* + w' - \delta' p'x^*$ hold. Then,

$$\begin{aligned} \triangle py^* &= (\pi + \triangle\pi)\triangle px^* + (\triangle\pi px^* + \triangle w) - (\delta' p'x^* - \delta px^*) \\ &= (\pi + \triangle\pi)\triangle px^* + (\triangle\pi px^* + \triangle w) - (\delta + \triangle\delta)\triangle px^* - \triangle\delta px^*, \quad \text{(a.1)} \end{aligned}$$

where $\triangle\delta \equiv \delta' - \delta$. Since $y^* = \Pi x^*$, equation (a.1) can be reduced to

$$\triangle py^* = \frac{1}{\Pi}\left((\pi + \triangle\pi) - (\delta + \triangle\delta)\right)\triangle py^* + (\triangle\pi px^* + \triangle w) - \triangle\delta px^*. \quad \text{(a.2)}$$

Thus, unless $\frac{1}{\Pi}(\pi' - \delta') = 1$, we have

$$\triangle py^* = \left[1 - \frac{1}{\Pi}(\pi' - \delta')\right]^{-1}\left((\triangle\pi px^* + \triangle w) - \triangle\delta px^*\right). \quad \text{(a.3)}$$

Suppose that y^* serves as an invariable measure of value with respect to a change from (p, w) to (p', w'). Then, by Definition 2, $\triangle\pi px^* + \triangle w = 0$ implies $\triangle py^* = 0$. Then, by equation (a.2), $\triangle\delta px^* = 0$ must hold. Since $x^* > \mathbf{0}$ by Definition 2 and $y^* = \Pi x^*$, $px^* > 0$ holds, so that $\triangle\delta = 0$ must hold. Therefore, $\delta = \delta'$ holds.

Conversely, let there be $\delta, \delta' \in \mathbb{R}_+$ such that $py^* = \pi px^* + w - \delta px^*$, $p'y^* = \pi' p'x^* + w' - \delta' p'x^*$, and $\delta = \delta'$ hold. Since we consider a redistribution of wages and profit, $\triangle\pi \neq 0 \neq \triangle w$ holds. Therefore, $(\pi' - \delta') \neq (\pi - \delta)$ holds. Then, at least one of $(\pi' - \delta')$ and $(\pi - \delta)$ is not equal to Π. Thus, without loss of generality, let $(\pi' - \delta') \neq \Pi$. Then, equation (a.3) implies that $\triangle py^* = 0$ follows from $\triangle\pi px^* + \triangle w = 0$. Thus, by Definition 2, y^* serves as an invariable measure of value with respect to a change from (p, w) to (p', w'). Since (p, w) to (p', w') are any equilibrium prices, y^* serves as an invariable measure of value by Definition 3. ■

By letting $P(p,w) \equiv \left\{ \alpha \in P \mid \alpha = \arg\max_{\alpha'} \frac{p\overline{\alpha}' - p\underline{\alpha}' - w\alpha_l'}{p\underline{\alpha}'} \right\}$, Corollary 1 in Section 3 can be rigorously expressed and proven, as follows.

Corollary 1. *Under any economy,* P, satisfying ***A0˜A2****, take any equilibrium prices* (p,w) *and* (p',w')*, such that* $\alpha^* \in P(p,w) \cap P(p',w')$ *holds. Then, the standard commodity* y^* *serves as an invariable measure of value with respect to a change from* (p,w) *to* (p',w')*.*

Proof. Note that $\alpha^* = (-1, -x^*, x^* + y^*) \in P(p,w) \cap P(p',w')$ implies that $py^* = \pi px^* + w - \delta px^*$ and $p'y^* = \pi' p'x^* + w' - \delta' p'x^*$ hold for $\delta = 0 = \delta'$. Then, by Theorem 1, the desired result immediately follows. ■

Let us define the set of price vectors measured by the standard commodity as $\Delta^{y^*} \equiv \left\{ (p,w) \in \mathbb{R}_+^{n+1} \middle| \, py^* = 1 \right\}$. Then, Theorem 2 can be rigorously expressed and proven under **A0˜A2**, as follows.

Theorem 2: *Given* P *satisfying* ***A0˜A2****, the linear functional relation of income distribution,* $\pi' = \Pi(1 - w')$*, holds for any equilibrium price vector* $(p',w') \in \Delta^{y^*}$ *associated with the maximal profit rate* π' *if and only if* $p'y^* = \pi' p'x^* + w'$ *holds for any equilibrium price vector* (p',w') *associated with* π'*.*

Proof. By definition of an equilibrium price $(p',w') \in \Delta^{y^*}$ associated with the maximal profit rate π', it is generally true that $p'y^* = \pi' p'x^* + w' - \delta' p'x^*$ for some

$\delta' \geqq 0$. Then, since $p'y^* = 1$ and $y^* = \Pi x^*$ holds for some $\Pi > 0$, it follows that $\Pi p'x^* = 1$, and so $1 = \frac{1}{\Pi}\pi' + w' - \frac{1}{\Pi}\delta'$. The last equation implies that $\pi' = \Pi(1 - w') + \delta'$. Thus, we have a functional relation $\pi' = \Pi(1 - w') + \delta(p', w')$ for any equilibrium price $(p', w') \in \Delta^{y^*}$, where $\delta(p', w')$ represents a non-negative real-value function $\delta : \Delta^{y^*} \to \mathbb{R}_+$ such that for any $(p', w') \in \Delta^{y^*}$, $\delta(p', w') = 0$ holds if and only if $\alpha^* \in P(p', w')$. As a result of this functional relation, for any equilibrium price $(p', w') \in \Delta^{y^*}$, $\pi' = \Pi(1 - w')$ holds if and only if $\delta(p', w') = 0$. Therefore, $\pi' = \Pi(1 - w')$ holds for any equilibrium price $(p', w') \in \Delta^{y^*}$ if and only if $\alpha^* \in P(p', w')$ for any equilibrium price $(p', w') \in \Delta^{y^*}$. Since $\alpha^* \in P(p', w')$ implies $p'y^* = \pi' p'x^* + w'$, we obtain the desired result. ■

Theorem 3: *For any economy,P, satisfying **A0~A2**, the standard commodity y^* serves as an invariable measure of value.*

Proof. From the proof of Theorem 2, we have a functional relation of wages and profit rates, such that $\frac{1}{\Pi}\pi + w = 1 + \frac{1}{\Pi}\delta(p, w)$ for any equilibrium price $(p, w) \in \Delta^{y^*}$, where $\delta(p, w) = 0$ holds if and only if $\alpha^* \in P(p, w)$. Consider a change of wages and profit rate from (π, w) to (π', w'):

$$\begin{aligned} \triangle\pi &= \pi' - \pi = \Pi(1 - w') - \Pi(1 - w) + (\delta(p', w') - \delta(p, w)) \\ &= -\Pi\triangle w + \triangle\delta. \end{aligned}$$

Thus, a change of wages and profit rate is represented by $\frac{1}{\Pi}\triangle\pi+\triangle w=\frac{1}{\Pi}\triangle\delta$, which is rewritten as $\triangle\pi px^*+\triangle w=\triangle\delta px^*$. Therefore, since $px^*>0$ by $x^*>\mathbf{0}$, the change of wage and profit constitutes a change in income distribution $\triangle\pi px^*+\triangle w=0$ if and only if $\triangle\delta=0$.

Let us take any equilibrium prices, $(p,w),(p',w')\in\mathbb{R}^{n+1}_{+}$, with $p\neq\mathbf{0}\neq p'$ associated with the maximal profit rates $\pi\geqq 0$ and $\pi'\geqq 0$, respectively. If $\alpha^*\in P(p,w)\cap P(p',w')$, then y^* serves as an invariable measure of value with respect to a change from (p,w) to (p',w'), as shown by Corollary 1.

Suppose that $\alpha^*\in P(p,w)\setminus P(p',w')$. Then, since $py^*=\pi px^*+w$ and $p'y^*=\pi'p'x^*+w'-\delta'p'x^*$ with $\delta'>0$, $\triangle\delta\neq 0$ holds. Therefore, $\triangle\pi px^*+\triangle w\neq 0$ holds. Thus, by Definition 2, y^* trivially serves as an invariable measure of value with respect to a change from (p,w) to (p',w'). Note that the same argument can be applied to the case $\alpha^*\in P(p',w')\setminus P(p,w)$.

Suppose that $\alpha^*\notin P(p,w)\cup P(p',w')$. Then, $py^*=\pi px^*+w-\delta px^*$ and $p'y^*=\pi'p'x^*+w'-\delta'p'x^*$ for some $\delta,\delta'>0$. If $\delta\neq\delta'$, then $\triangle\pi px^*+\triangle w\neq 0$ holds from $\triangle\delta\neq 0$. Thus, by Definition 2, y^* trivially serves as an invariable measure of value with respect to a change from (p,w) to (p',w'). If $\delta=\delta'$, then $\triangle\pi px^*+\triangle w=0$ holds. Then, by Definition 2 and Theorem 1, y^* serves as an invariable measure of value with respect to a change from (p,w) to (p',w').

In summary, for any equilibrium prices $(p, w), (p', w') \in \mathbb{R}^{n+1}_+$, with $p \neq \mathbf{0} \neq p'$, y^* serves as an invariable measure of value with respect to a change from (p, w) to (p', w'). Thus, by Definition 3, y^* serves as an invariable measure of value. ■

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Notes

[1]Ricardo's concern about an invariable measure of value had already appeared in his contributions to the 'bullionist' controversy. Here, he pointed out the need for an invariable measure of value that would enable an intertemporal comparison of values. He argued that no such measure existed, although money could be regarded as an invariable measure of value, at least as a first approximation (Ricardo, 1951C, p. 65). However, his arguments at this stage were not based on the theory of value. See Kurz and Salvadori (1993) concerning the conceptual transition of Ricardo's invariable measure of value.

[2]Ricardo (1952C, p. 358) said, 'As soon as we are in possession of the knowledge of the circumstances which determine the value of commodities, we are enabled to say what is necessary to give us an invariable measure of value.' See also Sraffa (1951) for details on the transition of Ricardo's theory of value.

[3]See Porta (1992) concerning the debates between Ricardo and Malthus.

[4]Pasinetti's (1981, 1993) *dynamic standard commodity* is an example of a construction that focuses on the first of the two parts.

[5]One of the exceptions is Yagi (2012). Following Pasinetti (1981, 1993), he constructed a model to compare two different economic systems (called Period 1 economy and Period 2 economy) intertemporally. Moreover, he investigated the invariable

measure of value and the linearity of income distribution.

[6]The 'average commodity' is one in which the proportion between labour and the value of the means of production is equal to the social average. However, the 'average commodity' cannot play the role of an invariable measure of value except in such trivial cases that there is only one basic commodity (Sraffa, 1960, p. 8) in an economic system. See Roncaglia (2009, p. 85).

[7]Schefold (1986, 1989) emphasised that the watershed condition and the recurrence condition must hold for the standard commodity to serve as an invariable measure of value. The two conditions imply that the industry producing the standard commodity, as well as all industries that produce the means of production necessary to produce the standard commodity, must adopt the 'watershed' proportion of means of production to labour (Sraffa, 1960, p. 16), so as to make the industry neither earns a surplus nor incurs a deficit, by which there is no need to change its price when income distribution changes. The industry producing q^* obviously satisfies both conditions. Note that, the watershed condition alone is sufficient for the existence of the standard commodity, insofar as the proof is based on the Perron–Frobenius theorem.

[8]In Flaschel (1986, p. 597), this is explicitly written as 'the problem of invariance cannot be described unless a measure of value has already been assumed. This fact is implicitly taken into account by Sraffa ([1960], Ch. 3) in his assumption

$\mathbf{p}(\mathbf{e} - \boldsymbol{A}\mathbf{e}) \equiv 1. \cdots$ the search for (conditions for) a "measure of value" relative to an already given measure of value! But what can be expected from the solution of such a problem?'

[9]See also Baldone (2006) and Bellino (2004) concerning Flaschel (1986).

[10]Samuelson (2008) came to the same incorrect conclusion. Moreover, Samuelson (1990) mistakenly related the standard commodity to the amelioration of the fault of the labour theory of value.

[11]A rigorous formulation of production sets is given in the Appendix.

[12]The existence of the standard commodity is shown in Theorem A1 of the addendum. The unique existence of the standard commodity is not necessarily guaranteed, though the standard ratio uniquely exists. If production set P is more suitably specified, the unique existence of the standard commodity can be shown by applying the non-linear Frobenius theorem (Fujimoto, 1979, 1980).

[13]Note that this concept is consistent with the 'price' in Sraffa (1960). Although it is true that Sraffa avoided using the term 'equilibrium', according to Roncaglia (2009, pp. 121–2), the equality of the rate of profit in Sraffa's system implies that the mobility of capital between sectors, in the search of maximum profitability, would ultimately bring out a tendency of the rates of profit to converge towards this benchmark position. Moreover, the uniform rate of profit in Sraffa's system does not require

the equality of demand and supply, in contrast to the concept of 'equilibrium' used by 'marginalists'. He also asserted that it is only in this sense that one can speak of an 'equilibrium' price within the Sraffa's system. Our concept of 'equilibrium price' also only requires the achievement of the maximum rate of profit in all activated 'sectors' under a production process α.

[14]Such a situation is not relevant when we assume a single product system, as in Baldone (2006).

1 Addendum

For each production possibility set, P, let us denote $SP \equiv \{\alpha \in P \mid \nexists \alpha' \in P : \alpha' \geq \alpha\}$, which is the efficiency frontier of the production set P. Moreover, given $k > 0$, let $P(\alpha_l = k) \equiv \{\alpha \in P \mid \alpha_l = k\}$ and

$$\partial P(\alpha_l = k) \equiv \{\alpha \in P(\alpha_l = k) \mid \nexists \alpha' \in P(\alpha_l = k) : (-\underline{\alpha}', \overline{\alpha}') > (-\underline{\alpha}, \overline{\alpha})\}.$$

1.1 Examples of production models satisfying A0~A2

The following two examples are typical types of production models in the class of the production sets satisfying **A0~A2** presented in the Appendix.

Example 1: Given a von Neumann technology $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times m$ non-negative matrices and $\boldsymbol{L}$ is a $1 \times m$ positive vector. Suppose that for each sector $j = 1, \ldots, m$, there exists at least one commodity $i = 1, \ldots, n$ such that $a_{ij} > 0$. we can define a production set $P_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{L})}$ as

$$P_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{L})} \equiv \left\{\alpha \in \mathbb{R}_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+ \mid \exists x \in \mathbb{R}^m_+ : \alpha \leqq (-\boldsymbol{L}x, -\boldsymbol{A}x, \boldsymbol{B}x)\right\}.$$

Note that for each $\alpha \in SP_{(A,\boldsymbol{B},\boldsymbol{L})}$, there exists $x \in \mathbb{R}^m_+$ such that $\alpha = (-\boldsymbol{L}x, -\boldsymbol{A}x, \boldsymbol{B}x)$. The set $P_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{L})}$ satisfies all of **A0~A2**. As a special case of the von Neumann tech-

nology, we can consider the case that $m = n$ and $\boldsymbol{B} = \boldsymbol{I}$, which implies a Leontief technology $(\boldsymbol{A}, \boldsymbol{I}, \boldsymbol{L})$. Then, we can define $P_{(\boldsymbol{A},\boldsymbol{L})} \equiv P_{(\boldsymbol{A},\boldsymbol{I},\boldsymbol{L})}$ as in the definition of $P_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{L})}$. ■

Example 2: Let us consider a class of Leontief technology $\left\{\left(\boldsymbol{A}^k, \boldsymbol{L}^k\right)\right\}_{k=1,\ldots,m}$, where for each $k = 1, \ldots, m$, $\boldsymbol{A}^k$ is an $n \times n$, non-negative, productive, and indecomposable matrix and $\boldsymbol{L}^k$ is a $1 \times n$ positive vector, such that for any $k, k' = 1, \ldots, m$, and for any non-negative $n \times 1$ vectors x^k and $x^{k'}$, $\boldsymbol{A}^k x^k = \boldsymbol{A}^{k'} x^{k'}$ implies $x^k = x^{k'}$ and $\boldsymbol{L}^k x^k = \boldsymbol{L}^{k'} x^{k'}$. Given this, we can define a production set $P_{\left(\boldsymbol{A}^k,\boldsymbol{L}^k\right)_{k=1,\ldots,m}}$ as

$$
\begin{aligned}
P_{\left(\boldsymbol{A}^k,\boldsymbol{L}^k\right)_{k=1,\ldots,m}} \equiv \; & \left\{\alpha \in \mathbb{R}_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+ \mid \exists S \equiv \left\{k^1, \ldots, k^S\right\} \subseteq \{1, \ldots, m\}, \right. \\
& \left. \exists \left\{x^{k^s}\right\}_{k^s \in S} \subseteq \mathbb{R}^n_+ : \alpha \leqq \left(-\sum_{k^s \in S} \boldsymbol{L}^{k^s} x^{k^s}, -\sum_{k^s \in S} \boldsymbol{A}^{k^s} x^{k^s}, \sum_{k^s \in S} x^{k^s} \right) \right\}.
\end{aligned}
$$

By the supposition of $\left\{\left(\boldsymbol{A}^k, \boldsymbol{L}^k\right)\right\}_{k=1,\ldots,m}$, the production set $P_{\left(\boldsymbol{A}^k,\boldsymbol{L}^k\right)_{k=1,\ldots,m}}$ satisfies **A0˜A2**. ■

1.2 The existence of the standard commodity

To provide a general existence of the standard commodity, let us introduce the following additional assumptions on the production set:

Assumption 3 (A3). For all $\alpha \in P$, and for all $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \in \mathbb{R}_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+$,

if $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \leqq \alpha$, then $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \in P$.

Assumption 4 (A4). There exists $r \in \mathbb{R}_{++}$, with $r \leqq 1$, such that for all $\alpha \in P$, and for any $k > 0$, $(-k\alpha_l, -k\underline{\alpha}, k^r\overline{\alpha}) \in P$.

The model of production sets with **A0~A4** still covers a broad class of production technologies. Indeed, it still contains the class of von Neumann production models and the class of Leontief production models with the possibility of technical choices, as in **Example 1** and **Example 2**.

Given the above setup of the model and the definition of the standard commodity presented in Section 3, the general existence of the standard commodity is proven as follows.

Theorem A1: *Under* ***A0~A4****, there uniquely exists the standard commodity* $y^* \in \mathbb{R}^n_{++}$ *associated with* $\alpha^{***} \in \partial P(\alpha_l = 1)$ *and* $\widehat{\alpha}^{***} = y^*$.

Proof: Given $P(\alpha_l = 1)$ which is convex, let $P_{\alpha_l=1}$ be the minimal closed convex cone containing $P(\alpha_l = 1)$. By definition, $P_{\alpha_l=1}$ is a closed convex cone with $P_{\alpha_l=1}(\alpha_l = 1) = P(\alpha_l = 1)$. If $r = 1$, $P_{\alpha_l=1} = P$. Given $P_{\alpha_l=1}$, let $\overline{P}_{\alpha_l=1} \equiv \left\{\alpha \in P_{\alpha_l=1} \mid \sum_{i=1,\ldots,n} \overline{\alpha}_i = 1\right\}$. Let $F : P_{\alpha_l=1} \to \mathbb{R}_+$ be such that for each $\alpha \in P_{\alpha_l=1}$,

$F(\alpha) = \min_{i=1,\ldots,n} \frac{\overline{\alpha}_i}{\underline{\alpha}_i}$ where

$$\frac{\overline{\alpha}_i}{\underline{\alpha}_i} \equiv \begin{cases} 0 & \text{if } \overline{\alpha}_i = 0 \\ +\infty & \text{if } \underline{\alpha}_i = 0 \text{ and } \overline{\alpha}_i > 0. \end{cases}$$

This mapping is continuous and well-defined by **A1**. Note that, by **A2** and **A4**, there exists $\alpha' \in \partial P(\alpha_l = 1)$ such that $\widehat{\alpha}' > \mathbf{0}$. Hence, for $\frac{\alpha'}{\sum_{i=1,\ldots,n} \overline{\alpha}'_i} \in \partial \overline{P}_{\alpha_l=1}$, $F\left(\frac{\alpha'}{\sum_{i=1,\ldots,n} \overline{\alpha}'_i}\right) > 0$. This implies $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) > 0$. Suppose that $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) = +\infty$. Then, there exists a sequence $\left\{\alpha^k\right\} \subseteq \overline{P}_{\alpha_l=1}$ such that $\alpha^k \to \alpha^*$ with $\lim_{k \to +\infty} F\left(\alpha^k\right) = F(\alpha^*) = \sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha)$. By definition of F, $F(\alpha^*) = +\infty$ implies that $\alpha^* = (-l, \mathbf{0}, \overline{\alpha}^*)$ for some $l \geqq 0$ and some $\overline{\alpha}^* > \mathbf{0}$. Since $\overline{P}_{\alpha_l=1}$ is closed, $\alpha^* \in \overline{P}_{\alpha_l=1}$. By construction, $\overline{P}_{\alpha_l=1}$ satisfies **A1**, which is a contradiction of $\alpha^* \in \overline{P}_{\alpha_l=1}$. Thus, $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) < +\infty$. Then, $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) = \max_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha)$. Let $\alpha^* \in \arg\max_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha)$. Then, by the cone property, $\frac{\alpha^*}{\alpha^*_l} \in P(\alpha_l = 1)$ and $\frac{\alpha^*}{\alpha^*_l} \in \arg\max_{\alpha \in P(\alpha_l=1)} F(\alpha)$. Hence, without loss of generality, let $\alpha^* \in \arg\max_{\alpha \in P(\alpha_l=1)} F(\alpha)$. Then, $\alpha^* \in \partial P(\alpha_l = 1)$. Since there exists $\alpha' \in \partial P(\alpha_l = 1)$ such that $F(\alpha') > 0$, $\max_{\alpha \in P(\alpha_l=1)} F(\alpha) > 0$ holds, which implies that $\overline{\alpha}^* > \mathbf{0}$.

Define $V \equiv \{\overline{\alpha} - F(\alpha^*)\,\underline{\alpha} \mid (-1, -\underline{\alpha}, \overline{\alpha}) \in P(\alpha_l = 1)\}$. Then, V is a closed convex set with $V \cap \mathbb{R}^n_{++} = \varnothing$. Therefore, there exists $p^* \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $p^*\left[\overline{\alpha} - F(\alpha^*)\,\underline{\alpha}\right] \leqq 0$ for all $\alpha \in P(\alpha_l = 1)$ and $p^* z > 0$ for all $z \in \mathbb{R}^n_{++}$. This

implies that if there exists $i \in \{1,\ldots,n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} > F(\alpha^*)$, then $p_i^* = 0$. By $p^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, there exists $i \in \{1,\ldots,n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} = F(\alpha^*)$ and $p_i^* > 0$. Thus, $p^*\left[\overline{\alpha}^* - F(\alpha^*)\,\underline{\alpha}^*\right] = 0$. Hence, p^* is a supporting vector of $\alpha^* \in \partial P(\alpha_l = 1)$. Let $\alpha^{**} \in P(\alpha_l = 1)$ be such that for each $i \in \{1,\ldots,n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} > F(\alpha^*)$, $(\underline{\alpha}_i^{**}, \overline{\alpha}_i^{**}) \in \mathbb{R}_{++}^2$ with $\frac{\overline{\alpha}_i^{**}}{\underline{\alpha}_i^{**}} \equiv F(\alpha^*)$. (Note that such a construction is possible by **A3**.) Furthermore, for each $i \in \{1,\ldots,n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} = F(\alpha^*)$, $(\underline{\alpha}_i^{**}, \overline{\alpha}_i^{**}) \equiv (\underline{\alpha}_i^*, \overline{\alpha}_i^*)$. Then, by construction, $p^*\left[\overline{\alpha}^{**} - F(\alpha^*)\,\underline{\alpha}^{**}\right] = 0$, which implies that $\alpha^{**} \in \partial P(\alpha_l = 1)$. Note that $\underline{\alpha}^{**} > \mathbf{0}$ and $\overline{\alpha}^{**} = F(\alpha^*)\,\underline{\alpha}^{**}$.

Denote the set of such production processes as α^{**} by $P(F)$. Then, for any $\alpha^{**} \in P(F)$, $\alpha^{**} \in \partial P(\alpha_l = 1)$ and $F(\alpha^{**}) \geqq F(\alpha')$ hold for all $\alpha' \in P(\alpha_l = 1)$. Since $P(F)$ is compact, there exists $\alpha^{***} \in P(F)$ such that for any $\alpha^{**} \in P(F)$, $\overline{\alpha}^{***} - \underline{\alpha}^{***} \geqq \overline{\alpha}^{**} - \underline{\alpha}^{**}$.

Let $y^* \equiv \overline{\alpha}^{***} - \underline{\alpha}^{***}$. Remember that there exists $\alpha' \in \partial P(\alpha_l = 1)$ such that $\widehat{\alpha}' > \mathbf{0}$ and $F(\alpha') > 0$, which implies $F(\alpha^*) \geqq F(\alpha') > 1$. Therefore, $y^* > \mathbf{0}$. Then, there exists a positive number $\Pi > 0$ such that $\Pi x^* = y^*$ for $x^* \equiv \underline{\alpha}^{***} > \mathbf{0}$. By Definition 1, $y^* > \mathbf{0}$ is a standard commodity of the economy P. Note that $1 + \Pi = \max_{\alpha \in P(\alpha_l = 1)} F(\alpha) = F(\alpha^{**})$ and $y^* \geqq \overline{\alpha}^{**} - \underline{\alpha}^{**}$ for any $\alpha^{**} \in P(F)$. This guarantees the uniqueness of the standard ratio Π for the economy P. ■