2017年度上級マクロ経済学講義ノート: 連続時間OLGモデル

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平成29年6月11日

1 Introduction

In this lecture note, we will discuss (1) a tractable model of overlapping generations with a lot of different generations, and (2) solution methods for dynamic optimization problems in continuos time. As we discussed in previous lectures, OLG models are very powerful and useful tools to describe various problems such as the burden of national debts, inter-generational transfer such as pensions and taxes, as well as the theoretical consideration of the role of money. However, it is also true that the standard OLG models with only two generations, old and young, are by far too simple to be applied to the real world problems. If we try to deal with more than two generations, suddenly we need either heavy numerical computations or unreasonably strong assumptions. In many cases, numerical computations of general OLG models include many parameters, which makes it difficult to interpret the results, although such extensions are quite necessary to investigate the real world problems. In 1985, Oliver Blanchard wrote a very influential paper[1]. In the paper, based on works by Yaari (1965)[2], he introduced a notion of "perpetual youth", which enables us to extend the analysis by Diamond (1965) to the economy with higher frequencies, not 30 years length, but say, annual or quarters. Of course, we need a lot of assumptions. Some of them are not realistic. Therefore, the model is not for statistical, but for theoretical analyses. Because of its tractability, a lot of researchers use and extend Blanchard's perpetual youth model since the model is by far the only tractable OLG model with business cycle frequency.

[1]Blanchard, Olivier j. (1985): "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, 93(2), 223–247.

[2]Yaari, Menahem E., (1965), Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, Review of Economic Studies, 32, issue 2, p. 137-150.

To solve Blanchard's perpetual youth model, we need to use some mathematical tricks, (1) continuous time and (2) dynamic optimization technique developed by L.S. Pontryagin, a mathematical genius in Russia. The optimal control theory by Pontryagin has become one of the standard techniques in optimization. In various academic and nonacademic worlds, the optimal control theory has been utilized. Economics is no exception. Pontryagin's control theory is used in various economic fields such as optimal taxation, principal-agent analyses, labor economics as well as macroeconomics. Although it is very technical, we need to master it. In this lecture note, we will briefly review the basics of continuous time and some useful characteristics of integrals. We also discuss some results of the optimal control theory by Pontryagin.

2 Continuos time

2.1 Discount Rate

Blanchard (1985) took advantages of various tricks coming from continuos time setting and integration. In general, continuous time is mathematically more complicated than discrete time models. However, as long as the system is small, such as two dimensions, continuos time models turn out to be much more tractable than discrete time model. We can see the usefulness of continuous time model in compound interest calculation.

Let's consider a value of 1 yen 10 years later. Suppose the annual interest rate is fixed at r. If the interest is paid every year, the discounted value of 1 yen 10 years later will be

$$\text{Discounted Value of 1 yen 10 years later} = \frac{1}{(1+r)^{10}}. \tag{1}$$

Next, let's suppose the interest are paid every month. Of course, the interests will be smaller than the annual rate, that is, monthly interest rate will be $r/12$. In such a case, the discounted value will be

$$\text{Discounted Value of 1 yen 10 years later} = \frac{1}{(1+\frac{r}{12})^{12\times 10}}. \tag{2}$$

Let's go further. Assume the time is continuous, that is, interests are paid continuously. This can be done by introducing a parameter, n, and move it to infinity, that is,

$$\text{Discounted Value of 1 yen 10 years later} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{r}{n}\right)^{n\times 10}}. \tag{3}$$

Although the above formula is easy to interpret, it is not easy to use it because we need take a limit. Let's change the formula a bit, with time notation, t, $\frac{1}{\left(1+\frac{r}{n}\right)^{nt}} = \left(1+\frac{r}{n}\right)^{-nt}$.

$$\frac{1}{\left(1+\frac{r}{n}\right)^{nt}} = \left(1+\frac{r}{n}\right)^{-nt}.$$

Define a parameter m as,

$$m = \frac{n}{r}.$$

Then, we get [3]:

$$\left(1+\frac{r}{n}\right)^{-nt} = \left(1+\frac{1}{m}\right)^{m(-rt)} \tag{4}$$

It is easy to show that

$$\lim_{n\to\infty} \frac{1}{\left(1+\frac{r}{n}\right)^{nt}} = \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{m(-rt)}. \tag{5}$$

By the way, let's recall the definition of the base of the natural logarithm, e,

$$e \equiv \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{m}. \tag{6}$$

Therefore, we an eliminate the limit by writing,

$$\lim_{n\to\infty} \frac{1}{\left(1+\frac{r}{n}\right)^{nt}} = \left(\lim_{m\to\infty} \left(1+\frac{1}{m}\right)\right)^{-rt} \tag{7}$$

$$= e^{-rt}. \tag{8}$$

If the interest rate varies over time, that is, if interest rate is a function of time, $r(t)$, the interest rate between time 0 and t should be the sum of the interests over the time, that is, the integral of $r(t)$,

$$\int_0^t r(s)\, ds. \tag{9}$$

Therefore, the discount rate becomes,

$$\exp\left[-\int_0^t r(s)\, ds\right]. \tag{10}$$

In sum, the future discount value of 1 yen 10 years later in continuos time is

$$\text{Discounted Value of 1 yen 10 years later} = \exp\left[-\int_0^{10} r(s)\, ds\right]. \tag{11}$$

[3] Assume r is not zero.

If the interest rate is constant,

$$\text{Discounted Value of 1 yen 10 years later} = e^{-10r}. \tag{12}$$

This is very simple, isn't it? The biggest advantage of this formula is that the function e^x satisfies the following relationship,

$$\frac{de^{rx}}{dx} = re^{rx}, \tag{13}$$

$$\int_0^t e^{rx}dx = \frac{1}{r}\left[e^{rt} - 1\right]. \tag{14}$$

Later in this lecture note, this very nature of e will help you quite a lot.

2.2 Reviews of Integration

The fundamental theorem of integration:

$$\frac{d\int_t^x (f(z))}{dx} = f(x). \tag{15}$$

Integration by parts:

$$\int_t^x \left(\frac{dg(z)}{dz} f(z)\right) dz = [g(z) f(z)]_t^x - \int_t^x \left(g(z)\frac{df(z)}{dz}\right) dz. \tag{16}$$

Differentiation of integrals:

$$F(a,b,c) = \int_a^b (f(x,c))\,dx. \tag{17}$$

$$\frac{\partial F}{\partial c} = \int_a^b \left(\frac{\partial f(x,c)}{\partial c}\right) dx,$$

$$\frac{\partial F}{\partial a} = -f(a,c),$$

$$\frac{\partial F}{\partial b} = f(b,c).$$

$$F(a,b,c) = \int_{a(c)}^{b(c)} (f(x,c))\,dx. \tag{18}$$

$$\frac{\partial F}{\partial c} = \int_{a(c)}^{b(c)} \left(\frac{\partial f(x,c)}{\partial c}\right) dx + f(b(c),c)\frac{\partial b}{\partial c} - f(a(c),c)\frac{\partial a}{\partial c}.$$

2.3 Perpetual Youth

The main objective of this lecture note is to introduce an overlapping generations model with many different generations in a tractable manner. We need a lot of strong assumptions to do this. One of them is an assumption of "perpetual youth". Assume that people face the same probability of death, p, regardless of their ages. That is, an old guy has exactly the same expected life length as a very young guy because they face the same probability of death. This assumption, the independence of the probability of dying from age, is called "perpetual youth".

Let's think about the time until death in an economy with perpetual youth. Remember the definitions of the exponential and the Poisson distributions. If the arrival rate of some event is constant, the number of arrivals given time follows the Poisson distribution, while the expected time for the next arrival follows the exponential distribution. Therefore, the time until death in this economy is a random variable that follows the exponential distribution with the following density function,

$$f_x(t) = p \exp[-pt]. \tag{19}$$

The expected value is

$$\begin{aligned} E[x] &= \int_0^\infty t f_x(t)\, dt && (20) \\ &= \int_0^\infty tp \exp[-pt]\, dt && (21) \\ &= [-t \exp[-pt]]_0^\infty + \int_0^\infty \exp[-pt]\, dt && (22) \\ &= 0 + \left[-\frac{1}{p} \exp[-pt] \right]_0^\infty && (23) \\ &= \frac{1}{p}. && (24) \end{aligned}$$

That is, the expected life expectancy is $1/p$ in this economy for all the survivors.

The fraction of p of the total population always die. To keep the size of the total population constant, we need to assume that at time t, population increases by the rate p. Suppose the size of cohort that are born at time s is h at their birth. Then, at time t, the total number of the cohort becomes $h \exp[-p(t-s)]$. The number of the total population can be obtained by summing these numbers over all the cohorts, that is,

$$\int_{-\infty}^{t} h \exp\left[-p\left(t-s\right)\right] ds = h\frac{1}{p}\left[\exp\left[-p\left(t-s\right)\right]\right]_{-\infty}^{t} \tag{25}$$

$$= h\frac{1}{p} \tag{26}$$

If the size of each cohort at birth is $p\,(=h)$, the total population become unity. Because the nominal size is not important, just a normalization, hereafter, we assume that the size of each cohort is always p, so that the total population is unity all the time.

2.4 Insurance (Yaari (1965))

In the previous subsection, we showed that number of total population is constant over time. This means that although each individual faces risks of dying or surviving, the whole economy does not face such risks. Because the random variable of living status is not correlated across individuals, we can use the law of large number to eliminate aggregate risks. In such an economy, insurance can improve welfare of individuals.

Suppose each individual does not have bequest motives, that is, they do not care about future generations. Because of the assumption of perpetual youth, each individual has an incentive to hold some amount of savings to fiance their future consumption. If an individual happens to die, he/she will leave all his/her savings as an accidental bequest. If they are allowed to have debts, unintended accidental debts (negative bequest) will be left. If a no Ponzi-game (NPG) condition is imposed, the amount of accidental bequest should be always non-negative. Suppose there is an insurance company that pays a fixed amount of annuity as an exchange for receiving their savings in the event of death. If the market of insurance is competitive (free entry and exit), people will make a contract that states that people give all the nonhuman wealth (financial or physical), v_t, to the insurance company in the event of death, while the insurance company will pay them pv_t as long as they survive. Because the number of total population is always unity with the identical probability of dying, the total revenue of the insurance company at time t is pv_t, while the total amount of payments of the insurance company is also pv_t. That is, the profit of the insurance company is always zero.

2.5 Utility Maximization

Let's denote $c(s,t)$, $y(s,t)$, $v(s,t)$,and $h(s,t)$ as consumption, labor income, non-human wealth, and human wealth at time t, respectively of an individual born at time s. For a while, let's concentrate on a particular generation so that we can drop the generation index, s.

The utility function at time t is assumed to be,

$$E_t\left[\int_t^{\infty} u(c(z))\exp\left[-\rho(z-t)\right]dz\right]. \tag{27}$$

$\rho(>0)$ is the subjective discount rate. The expectation operator exists because of the risks of dying. Given the density function, we can get rid of the operator. For simplicity, let's assume the instantaneous utility function is logarithmic.[4] That is,

$$\begin{aligned}
&E_t\left[\int_t^{\infty} u(c(z))\exp\left[-\rho(z-t)\right]dz\right] && (28)\\
&= \int_t^{\infty} u(c(z))\exp\left[-\rho(z-t)\right]\exp\left[-p(z-t)\right]dz && (29)\\
&= \int_t^{\infty} \ln(c(z))\exp\left[-\rho(z-t)\right]\exp\left[-p(z-t)\right]dz && (30)\\
&= \int_t^{\infty} \ln(c(z))\exp\left[-(\rho+p)(z-t)\right]dz. && (31)
\end{aligned}$$

The budget constraint for the individual is,

$$\frac{dv(z)}{dz} = \left[r(z)+p\right]v(z) + y(z) - c(z). \tag{32}$$

The above budget constraint should be easy to interpret. $r(z)$ is the return of nonhuman wealth at time z. To prohibit accumulating debts and leaving them in the even of death, we need to impose a NPG condition,

$$\lim_{z\to\infty}\exp\left[-\int_t^{z}(r(m)+p)\,dm\right]v(z) = 0, \tag{33}$$

that is, the growth rate of debts should be smaller than $r(m)+p$ on average.

To keep the notation simple, let's define the discount factor between time t and z as

$$R(t,z) \equiv \exp\left[-\int_t^{z}(r(m)+p)\,dm\right]. \tag{34}$$

Therefore, the NPG condition becomes,

[4] See the original paper by Blanchard (1985) for more general cases such as CRRA utility function.

$$\lim_{z\to\infty} R(t,z)\, v(z) = 0 \tag{35}$$

Next, multiply both sides of the budget constructing by $R(t,z)$ and integrate it from t to infinity,

$$\int_t^{\infty} \left(R(t,z)\frac{dv(z)}{dz} - R(t,z)\left[r(z)+p\right] v(z) \right) dz = \int_t^{\infty} R(t,z)\left(y(z)-c(z)\right) dz. \tag{36}$$

This might seem intimidating. This integration will appear very frequently in macroeconomics. So, you should get used to it. First, let's differentiate $v(z)\, R(t,z)$ as follows,

$$\begin{aligned} \frac{d\left(v(z)\, R(t,z)\right)}{dz} &= R(t,z)\frac{dv(z)}{dz} + v(z)\frac{dR(t,z)}{dz} \\ &= R(t,z)\frac{dv(z)}{dz} - R(t,z)\, v(z)\left(r(z)+p\right) \end{aligned} \tag{37}$$

Next, integrate $\frac{d(v(z)R(t,z))}{dz}$ from t to infinity,

$$\int_t^{\infty} \frac{d\left(v(z)\, R(t,z)\right)}{dz} dz = \int_t^{\infty} \left(R(t,z)\frac{dv(z)}{dz} - R(t,z)\, v(z)\left(r(z)+p\right) \right) dz \tag{38}$$

The L.H.S. becomes

$$\int_t^{\infty} \frac{d\left(v(z)\, R(t,z)\right)}{dz} dz = \lim_{z\to\infty} v(z)\, R(t,z) - v(t)\, R(t,t) \tag{39}$$

From the NPG condition and $R(t,t) = 1$, we get,

$$\int_t^{\infty} \frac{d\left(v(z)\, R(t,z)\right)}{dz} dz = -v(t) \tag{40}$$

Therefore,

$$-v(t) = \int_t^{\infty} \left(R(t,z)\frac{dv(z)}{dz} - v(z)\, R(t,z)\left(r(z)+p\right) \right) dz \tag{41}$$

Return to the budget constraint,

$$\int_t^{\infty} \left(R(t,z)\frac{dv(z)}{dz} - v(z)\, R(t,z)\left[r(z)+p\right] \right) dz = \int_t^{\infty} R(t,z)\left(y(z)-c(z)\right) dz. \tag{42}$$

We know,

$$\int_t^{\infty} \left(R(t,z)\frac{dv(z)}{dz} - v(z)\, R(t,z)\left[r(z)+p\right] \right) dz = -v(t)\,. \tag{43}$$

Therefore, the intertemporal budget constraint, with the NPG condition, becomes,

$$-v(t) = \int_t^\infty R(t,z)(y(z) - c(z))\,dz. \tag{44}$$

Let's denote the human wealth as follows,

$$h(t) = \int_t^\infty R(t,z)\,y(z)\,dz \tag{45}$$

Then, the budget constraint becomes very simple,

$$\int_t^\infty R(t,z)(c(z))\,dz = v(t) + h(t). \tag{46}$$

The interpretation of the above equation should be straightforward. The discounted sum of future consumption must be equal to the current nonhuman wealth (financial and physical assets) and the discounted sum of future labor income (human wealth).

The maximization problem of the consumer can be formulated as,

$$\max \int_t^\infty \ln(c(z)) \exp[-(\rho + p)(z - t)]\,dz \tag{47}$$

$$\text{s.t.} \quad \int_t^\infty R(t,z)(c(z))\,dz = v(t) + h(t). \tag{48}$$

2.6 Hamiltonian and the maximum principle

Using Lagrangian multiplier, we can write the maximization problem as,

$$L = \int_t^\infty \ln(c(z)) \exp[-(\rho + p)(z - t)]\,dz + \lambda \left[v(t) + h(t) - \int_t^\infty R(t,z)(c(z))\,dz \right] \tag{49}$$

Because $v(t)$ and $h(t)$ are exogenous, the only variables consumer can choose are $c(z)$ for $z \geq t$. The first order condition is

$$\frac{\exp[-(\rho + p)(z - t)]}{c(z)} = \lambda R(t,z) \tag{50}$$

Taking the logarithms,

$$-(\rho + p)(z - t) - \ln c(z) = \ln \lambda - \int_t^z (r(m) + p)\,dm. \tag{51}$$

Taking the derivative with respect to z, we can get,

$$-(\rho + p) - \frac{dc(z)}{dz}\frac{1}{c(z)} = -(r(z) + p) \tag{52}$$
$$\frac{dc(z)}{dz}\frac{1}{c(z)} = (r(z) + p) - (\rho + p)$$
$$= r(z) - \rho.$$

That is,

$$\frac{dc(z)}{dz} = c(z)(r(z) - \rho). \tag{53}$$

This is nothing but the Euler equation, with which you are supposed to be familiar. Additionally,we need the NPG condition as one of the conditions.

$$\lim_{z\to\infty} R(t,z)\, v(z) = 0.$$

Without this condition, the maximization problem becomes trivial and meaningless.

The above procedure of solving the maximization problem requires integration of the budget constraint over time, which is sometimes time consuming. Pontryagin's control theory allows us to solve the problem more easily. Let's state the optimization as follows,

$$\max \int_t^\infty \ln(c(z)) \exp[-(\rho + p)(z - t)]\, dz \tag{54}$$

$$\text{s.t. } \frac{dv(z)}{dz} = [r(z) + p]\, v(z) + y(z) - c(z) \text{ for all } z \tag{55}$$

$$\lim_{z\to\infty} R(t,z)\, v(z) = 0 \tag{56}$$

Obviously, $v(z)$ is not exogenous in this problem. We call this variable as a state variable. Additional complexity comes from the existence of differential operator, $dv(z)/dz$, in the constraint. Following Pontryagin's maximum principle, first we define the following Hamiltonian,

$$H_z = \ln(c(z)) \exp[-(\rho + p)(z - t)] + \lambda_z ([r(z) + p]\, v(z) + y(z) - c(z)) \tag{57}$$

Because the discount factor is added in the first term, the above Hamiltonian is called "present value Hamiltonian" function. Notice that λ_z is not a parameter as in the Lagrangian function, but a function of time, which might vary over time. We call λ_z as the costate variable, implying close relationship with the state variable, $v(z)$. The first order conditions are as follows,

$$\frac{dH_z}{dc\left(z\right)} = 0, \tag{58}$$

$$\frac{d\lambda_z}{dz} = -\frac{dH_z}{dv\left(z\right)},$$

$$\lim v\left(z\right)\lambda_z = 0$$

The last condition is called as "Transversality condition". The second condition is called as the "Hamiltonian dynamics". Let's calculate the above conditions in our optimization problem,

$$\frac{dH_z}{dc\left(z\right)} = \exp\left[-\left(\rho+p\right)\left(z-t\right)\right]\frac{1}{c\left(z\right)} - \lambda_z \tag{59}$$

$$= 0, \tag{60}$$

$$-\frac{dH_z}{dv\left(z\right)} = -\lambda_z\left[r\left(z\right)+p\right] = \frac{d\lambda_z}{dz}. \tag{61}$$

The meanings of these equations are not clear. Let's eliminate λ_z.

$$\lambda_z = \exp\left[-\left(\rho+p\right)\left(z-t\right)\right]\frac{1}{c\left(z\right)} \tag{62}$$

$$\frac{d\lambda_z}{dz} = \frac{-1}{c\left(z\right)^2}\frac{dc\left(z\right)}{dz}\exp\left[-\left(\rho+p\right)\left(z-t\right)\right] - \left(\rho+p\right)\exp\left[-\left(\rho+p\right)\left(z-t\right)\right]\frac{1}{c\left(z\right)}$$

$$\frac{d\lambda_z}{dz}\frac{1}{\lambda_z} = \frac{-1}{c\left(z\right)}\frac{dc\left(z\right)}{dz} - \left(\rho+p\right)$$

$$-\left[r\left(z\right)+p\right] = \frac{d\lambda_z}{dz}\frac{1}{\lambda_z} \tag{63}$$

Then, we get

$$\frac{-1}{c\left(z\right)}\frac{dc\left(z\right)}{dz} - \left(\rho+p\right) = -\left[r\left(z\right)+p\right] \tag{64}$$

$$\frac{1}{c\left(z\right)}\frac{dc\left(z\right)}{dz} = r\left(z\right) - \rho$$

$$\frac{dc\left(z\right)}{dz} = c\left(z\right)\left(r\left(z\right)-\rho\right).$$

Obviously, the derived condition is identical to the Euler equation derived from using the Lagrangian,(53).

In general, when solving the following maximization problem,

$$\max\int_0^\infty u\left(x_t,k_t\right)e^{-\rho t}dt \tag{65}$$

$$s.t.\frac{dk_t}{dt} = f\left(x_t,k_t;z_t\right)$$

We can use the Hamiltonian,

$$H_t = u(x_t, k_t) e^{-\rho t} + \lambda_t f(x_t, k_t, z_t) \tag{66}$$

with the following first order conditions and the transversality condition,

$$\begin{aligned} \frac{dH_t}{dx_t} &= 0, \\ \frac{d\lambda_t}{dt} &= -\frac{dH_t}{dk_t}, \\ \lim_{t\to\infty} \lambda_t k_t &= 0. \end{aligned} \tag{67}$$

Some researchers and textbooks use the following "current value Hamiltonian",

$$H_t = u(x_t, k_t) + \lambda_t f(x_t, k_t, z_t). \tag{68}$$

The corresponding first order conditions and the transversality condition are,

$$\begin{aligned} \frac{dH_t}{dx_t} &= 0, \\ \frac{d\lambda_t}{dt} - \rho\lambda_t &= -\frac{dH_t}{dk_t}, \\ \lim_{t\to\infty} \lambda_t e^{-\rho t} k_t &= 0. \end{aligned} \tag{69}$$

The biggest advantage of using the current value Hamiltonian is that at the steady sate (if any), λ_t becomes constant, while the costate variable in the present value Hamiltonian is not generally constant even at the steady state. The choice of these two Hamiltonians is completely matter of taste. As long as the Hamiltonian and the conditions for optimality are consistent with each other, you may use what you feel comfortable.[5]

The formal derivations of the maximum principle requires a lot of math and is very difficult. If you would like to know more about the technique including the second order condition and the mathematical justification for the transversality condition, a classic textbook by Arrow and Kurz (1970)[6] is still worth reading because it contains a lot of useful discussions. For more comprehensive treatments, see Kamien and Schwartz (1991)[7].

[5] I personally feel uneasy when using the current value Hamiltonian because the Hamiltonian dynamics include a term, $-\rho\lambda_t$, without any appearance of ρ in the current value Hamiltonian.

[6] *Public Investment, The Rate of Return, and Optimal Fiscal Policy.* BY KENNETH J. ARROW AND MORDECAI Kurz. Baltimore: The Johns Hopkins University Press, 1970.

[7] Kamien, M. I. and Schwartz, N. L. (1991), *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management* , Vol. 31 of Advanced Textbooks in Economics , second edn, Elsevier B.V., Amsterdam.

2.7 Poincaré-Bendixson theorem

In discrete time model, even if the dynamic system contains just one variable, say,

$$x_{t+1} = f(x_t),$$

the system can create very complicated transition dynamics. A very well-known example is,

$$x_{t+1} = ax_t(1 - x_t).$$

If $1 + \sqrt{6} < a < 4$, staring from any point in $(0, 1)$, x_t never converges to any steady state, nor exhibits any periodic motions. Such dynamic behavior is called chaos. In continuous time model, such chaotic behavior appears only when there are three or more variables. That means, as long as there are only two endogenous variables, such that,

$$\frac{dx}{dt} = f(x, y),$$
$$\frac{dy}{dt} = g(x.y).$$

the dynamic system is either 1) converging to a steady state, 2) exploding to infinity, or 3) converging to a periodic cycle (limit cycle). This result comes from a famous Poincaré-Bendixson theorem. Most textbooks on advanced differential equation contain the proof of this theorem. If you are interested in nonlinear dynamic system, Perko (2001) is a nice starting point.[8] Because of this theorem, many macroeconomic models with continuous time contain up to two endogenous variables. Blanchard (1985)'s perpetual youth model is no exception.

2.8 Aggregation

So far, we have derived the optimal consumption rule for each individual. Because the model contains a lot of heterogeneous individuals, the behavior of the aggregate consumption might be different from those of individual level consumption. In this section, we will move from "micro" to "macro" by aggregating the micro units all over the economy. First, take a look at the first order condition and the intertemporal budget constraint for an individual,

[8] Perko, Lawrence (2001) *Differential Equations and Dynamical Systems*, Springer-Verlag New York.

$$\frac{dc\left(z\right)}{dz}=c\left(z\right)\left(r\left(z\right)-\rho\right), \tag{70}$$

$$\int_{t}^{\infty}R\left(t,z\right)\left(c\left(z\right)\right)dz=v\left(t\right)+h\left(t\right). \tag{71}$$

It is easy to derive the integrals for the rest of their lives,

$$\begin{aligned}\int_{t}^{\infty}R\left(t,z\right)\frac{dc\left(z\right)}{dz}dz &= \left[R\left(t,z\right)c\left(z\right)\right]_{t}^{\infty}+\int_{t}^{\infty}c\left(z\right)R\left(t,z\right)\left(r\left(z\right)+p\right)dz\\ &= c\left(t\right)+\int_{t}^{\infty}c\left(z\right)R\left(t,z\right)\left(r\left(z\right)+p\left(z\right)\right)dz\\ &= \int_{t}^{\infty}R\left(t,z\right)c\left(z\right)\left(r\left(z\right)-\rho\right)dz.\end{aligned}$$

Therefore, individual consumption level can be written as,

$$\begin{aligned}c\left(t\right) &= \int_{t}^{\infty}c\left(z\right)R\left(t,z\right)\left(r\left(z\right)+p\right)dz-\int_{t}^{\infty}R\left(t,z\right)c\left(z\right)\left(r\left(z\right)-\rho\right)dz \qquad (72)\\ &= \int_{t}^{\infty}c\left(z\right)R\left(t,z\right)\left(p+\rho\right)dz\\ &= \left(p+\rho\right)\left(v\left(t\right)+h\left(t\right)\right).\end{aligned}$$

This implies that the individual level consumption is a linear function of the total individual wealth!

Define the aggregate consumption as follows,

$$C\left(t\right)\equiv\int_{-\infty}^{t}c\left(s,t\right)p\exp\left[-p\left(t-s\right)\right]ds. \tag{73}$$

Remember that the size of generation s at time t is $p\exp\left[-p\left(t-s\right)\right]$. The aggregate consumption is the weighted sum of individual consumption. The weights are their population sizes. Similarly, define the aggregate nonhuman and human wealth as,

$$W\left(t\right)\equiv\int_{-\infty}^{t}v\left(s,t\right)p\exp\left[-p\left(t-s\right)\right]ds. \tag{74}$$

$$H\left(t\right)\equiv\int_{-\infty}^{t}h\left(s,t\right)p\exp\left[-p\left(t-s\right)\right]ds. \tag{75}$$

Because of the linearity, it is easy to show,

$$C\left(t\right)=\left(p+\rho\right)\left(W\left(t\right)+H\left(t\right)\right). \tag{76}$$

To derive the aggregate dynamics, we need to specify how the labor income is determined. Simply assume that all the individuals receive the same labor income, that is,

$$y(s,t) = aY(t), a > 0$$

That is, the income level of generation s at time t is does not depend on their ages, $(t-s)$, but depends on the aggregate level of income,$Y(t)$. The feedback from the aggregate to micro, $Y(t)$ to $y(s,t)$, creates some complexity. We need to impose a parameter constraint. The definition of $Y(t)$ is,

$$Y(t) \equiv \int_{-\infty}^{t} y(s,t)\, p \exp[-p(t-s)]\, ds. \tag{77}$$

By eliminating $y(s,t)$,we can get

$$\begin{aligned} Y(t) &= \int_{-\infty}^{t} aY(t)\, p \exp[-p(t-s)]\, ds \qquad (78) \\ &= apY(t) \int_{-\infty}^{t} \exp[-p(t-s)]\, ds \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &= ap \int_{-\infty}^{t} \exp[-p(t-s)]\, ds \qquad (79) \\ &= a \end{aligned}$$

That is, a must be unity. Because the size of population is unity, it is obvious that a must be equal to one.

The human wealth becomes,

$$h(s,t) = \int_{t}^{\infty} R(t,z)\, y(s,z)\, dz \tag{80}$$

$$= \int_{t}^{\infty} R(t,z)\, Y(z)\, dz \tag{81}$$

This implies that the human wealth of generation s does not depend on the generation, but depend only on the current and future aggregate income.

The aggregate human wealth is very simple,

$$H(t) = h(s,t) = \int_{t}^{\infty} R(t,z)\, Y(z)\, dz. \tag{82}$$

Because the number of total population is constant at unity, and because all the surviving individuals have identical human wealth, the aggregate human wealth is just the same as the expected value of the human wealth of one individual. The time derivative of $H(t)$ is,

$$\frac{dH(t)}{dt} = -Y(z) + (r(t) + p) H(t). \tag{83}$$

By integrating the above differential equation, we can obtain (82) if we impose

$$\lim_{z \to \infty} R(t, z) H(z) = 0. \tag{84}$$

This constraint is similar to no Ponzi-game-condition for nonhuman wealth. The meaning is that the growth rate of human wealth should not be greater than the average interest rate and the probability of dying. Since human wealth cannot take negative value, and the wage is usually exogenous for households, this constraint is not a part of the conditions for the optimality. If the growth rate of human wealth is very large, the present discount value of human wealth becomes infinite, which enables people to consume infinite amount of goods. So, we need this assumption to keep our model meaningful.

The aggregate nonhuman wealth is defined as,

$$W(t) \equiv \int_{-\infty}^{t} v(s, t) p \exp[-p(t-s)] ds. \tag{85}$$

The time derivative is,

$$\frac{dW(t)}{dt} = pv(t, t) - pW(t) + \int_{-\infty}^{t} \frac{dv(s, t)}{dt} p \exp[-p(t-s)] ds. \tag{86}$$

Because we assume no bequest is left in the event of death, an individual does not have financial or physical wealth at birth, that is, we assume $v(t, t) = 0$. Using the budget constraint (55),

$$\begin{aligned} \int_{-\infty}^{t} \frac{dv(s, t)}{dt} p \exp[-p(t-s)] ds &= \int_{-\infty}^{t} ([r(t) + p] v(s, t) + y(s, t) - c(s, t)) p \exp[-p(t-s)] ds \qquad (87) \\ &= [r(t) + p] W(t) + Y(t) - C(t) \end{aligned}$$

That is,

$$\frac{dW(t)}{dt} = [r(t) + p] W(t) + Y(t) - C(t) - pW(t) \tag{88}$$

$$= r(t) W(t) + Y(t) - C(t) \tag{89}$$

At individual level, the return on nonhuman wealth is $[r(t) + p]$,however, at aggregate level, the return is $r(t)$.

So far, we have obtained the following dynamic system at aggregate level,

$$C(t) = (p+\rho)(W(t) + H(t)) \tag{90}$$
$$\frac{dH(t)}{dt} = -Y(z) + (r(t) + p) H(t)$$
$$\frac{dW(t)}{dt} = r(t) W(t) + Y(t) - C(t)$$
$$\lim_{z\to\infty} R(t,z) H(z) = 0.$$

To make the system smaller, take a time derivative of the aggregate consumption,

$$\begin{aligned}
\frac{dC(t)}{dt} &= (p+\rho)\left(\frac{dW(t)}{dt} + \frac{dH(t)}{dt}\right) \qquad (91)\\
&= (p+\rho)(-Y(z) + (r(t)+p) H(t) + r(t) W(t) + Y(t) - C(t))\\
&= (p+\rho)((r(t)+p) H(t) + r(t) W(t) - C(t))\\
&= (p+\rho)(r(t)+p) H(t) + (p+\rho)(r(t) W(t) - C(t))\\
&= (r(t)+p) C(t) - (r(t)+p)(p+\rho) W(t) + (p+\rho)(r(t) W(t) - C(t))\\
&= (r(t)-\rho) C(t) - (p+\rho) pW(t).
\end{aligned}$$

Therefore, the dynamic system becomes,

$$\frac{dC(t)}{dt} = (r(t)-\rho) C(t) - (p+\rho) pW(t) \tag{92}$$
$$\frac{dW(t)}{dt} = r(t) W(t) + Y(t) - C(t).$$

In the above system, we still have four endogenous variables, $C(t), W(t), Y(t)$, and $r(t)$. Take a close look at the dynamics of consumption. The change in consumption depends on the level of nonhuman wealth. In the standard optimal growth model, such term does not exist. If we set $p = 0$, then, the term of nonhuman wealth disappears, which leads us to the well-known consumption Euler equation in the standard growth model. To close the model, we need to specify the determination of interest rates.

2.9 Production

Following the standard growth theory, let's assume that there is a competitive market for goods with the neoclassical production function with capital, K, and labor, L. Assuming fixed labor inputs at unity, we define the aggregate production function as,

$$F(K) \equiv F(K,1) - \delta K, \delta > 0 \tag{93}$$

while $F(K, L)$ is the standard neoclassical production function. δ is the rate of depreciation for capital. Assuming competitive factor and asset markets, we can obtain,

$$\begin{aligned} r(t) &= F'(K(t)) \\ Y(t) &= F(K) - r(t) K(t) \\ W(t) &= K(t). \end{aligned} \tag{94}$$

To keep the notation simple, let's drop time subscript from now on. Then, the dynamic system becomes,

$$\begin{aligned} \frac{dC}{dt} &= (F'(K) - \rho) C - (p + \rho) pK \\ \frac{dK}{dt} &= F(K) - C. \end{aligned} \tag{95}$$

It is simply amazing that after all the works of a dynamic general equilibrium with many heterogeneous agents, we have reached to such a simple two-dimensional dynamic system! Let's draw the phase diagram. $dK(t)/dt = 0$ locus is easy to draw. Since $F(0) = 0$, $dK(t)/dt = 0$ locus must include the origin. The curve has a single peak, at which $F'(K) = 0$. Remember that at the golden rule, the capital return must be equal to the rate of population growth. In our model, the population is constant. Therefore, zero-return corresponds to the golden rule level. Drawing $dC/dt = 0$ locus is a bit more complicated. On the locus, we get

$$(F'(K) - \rho) C - (p + \rho) pK = 0. \tag{96}$$

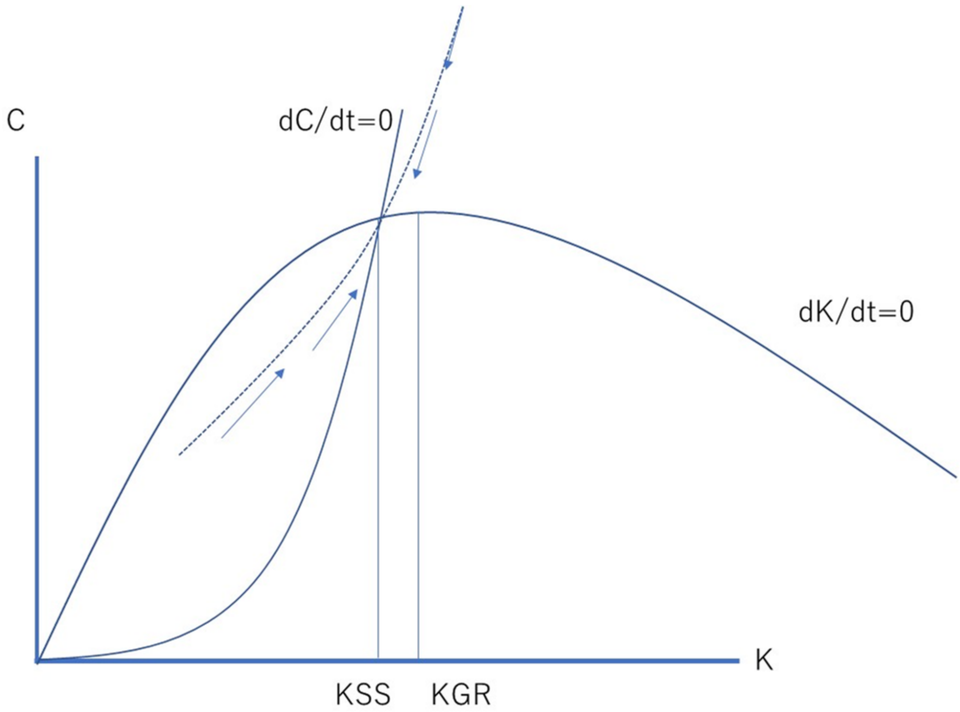
From the implicit function formula,

$$\left.\frac{dC}{dK}\right|_{dC/dt=0} = \frac{-CF''(K) + (p + \rho) p}{(F'(K) - \rho)}. \tag{97}$$

As long as $F'(K) - \rho > 0$, dC/dK is positive, implying the slope is always positive. As $F'(K) \to \rho$, the slope becomes steeper. If $F'(K) - \rho < 0$, there is no steady state with positive consumption level. Under reasonable assumptions, we can show that $dC/dt = 0$ locus passes the origin. Thus, we can draw the phase diagram.

The steady state is saddle, implying that the economy has a unique path to converge to the steady state. The origin is also a steady state. What can we say about the steady state level of capital, K^*? It is obvious that $F'(K^*) > \rho$. Next, suppose that

$$F'(K^*) = \rho + p(1 + \varepsilon), \varepsilon > 0. \tag{98}$$



Then, from $dC/dt = 0$,

$$\begin{aligned}&\left(\rho + p\left(1+\varepsilon\right) - \rho\right) C^* - \left(p+\rho\right) pK^* \\ &= \left(p\left(1+\varepsilon\right)\right) C^* - \left(p+\rho\right) pK^* = 0\end{aligned} \tag{99}$$

$$\left(1+\varepsilon\right) C^* = \left(p+\rho\right) K^*. \tag{100}$$

From $dK/dt = 0$,

$$\begin{aligned}\left(1+\varepsilon\right) F\left(K^*\right) &= \left(p+\rho\right) K^* \\ &= \left(p + F'\left(K^*\right) - p\left(1+\varepsilon\right)\right) K^* \\ &= \left(F'\left(K^*\right) - p\varepsilon\right) K^* \\ &= K^* F'\left(K^*\right) - p\varepsilon K^*.\end{aligned} \tag{101}$$

That is,

$$K^* F'\left(K^*\right) = \left(1+\varepsilon\right) F\left(K^*\right) + p\varepsilon K^* > F\left(K^*\right). \tag{102}$$

However, because production function is assumed to be concave, we must have,

$$\frac{F(K^*)}{K^*} \geq F'(K^*) \tag{103}$$
$$F(K^*) \geq K^* F'(K^*)$$

This is a contradiction. Therefore, at the steady state, we have,

$$F'(K^*) < \rho + p. \tag{104}$$

In sum, we obtain the following inequalities,

$$\rho < F'(K^*) < \rho + p. \tag{105}$$

These inequalities restrict the possible values of the steady state values of capital. Note that that the non-trivial steady state is always dynamically efficient. Note that the golden rule is $\frac{\partial F(K^*,1)}{\partial K} - \delta = 0$ that is, $F'(K^*) = 0$. $F'(K^*) > \rho$ implies $\frac{\partial F(K^*,1)}{\partial K} - \delta > \rho$. That is, $\frac{\partial F(K^*,1)}{\partial K} > \delta$. This result is not robust, unfortunately. Blanchard (1985) showed that under different specification of income distribution across different age groups, specifically, if the income is decreasing with age, it is possible to derive dynamically inefficient steady state. See his original paper for detail.

3 Fiscal Policy

Suppose the government levies lump sum tax, T, to finance her expenditure, G. The government expenditure itself does not affect consumption or saving directly. Denote the public debt as B with interest rate, $r(t)$. The budget constraint for the government is

$$\frac{dB(z)}{dz} = r(z)B(z) + G(z) - T(z). \tag{106}$$

Note that the interest rate is $r(t)$, not $r(t) + p$.

The NPG condition is

$$\lim_{z\to\infty} B(z)\exp\left[-\int_t^\infty r(m)\,dm\right] = 0. \tag{107}$$

Integrating (106) with the NPG condition of the L.H.S.,

$$\int_t^\infty \frac{dB(z)}{dz}\exp\left[-\int_t^z r(m)\,dm\right]dz \tag{108}$$
$$= \left[B(z)\int_t^z r(m)\,dm\right]_t^\infty + \int_t^\infty r(z)B(z)\exp\left[-\int_t^z r(m)\,dm\right]dz$$
$$= \lim_{z\to\infty} B(z)\int_t^z r(m)\,dm - B(t) + \int_t^\infty r(z)B(z)\exp\left[-\int_t^z r(m)\,dm\right]dz$$
$$= -B(t) + \int_t^\infty r(z)B(z)\exp\left[-\int_t^z r(m)\,dm\right]dz.$$

Similarly, the R.H.S. becomes,

$$\int_t^\infty \left[(r(z) B(z) + G(z) - T(z)) \exp\left[-\int_t^z r(m)\, dm \right] \right] dz \tag{109}$$
$$= \int_t^\infty \left[r(z) B(z) \exp\left[-\int_t^z r(m)\, dm \right] \right] dz + \int_t^\infty [G(z) - T(z)] \exp\left[-\int_t^z r(m)\, dm \right] dz.$$

Therefore,

$$-B(t) = \int_t^\infty [G(z) - T(z)] \exp\left[-\int_t^z r(m)\, dm \right] dz. \tag{110}$$

The meaning of the above equation should be clear. The current level of debt should be equal to the discounted sum of future primary surpluses.

Let's introduce B, G, and T in our dynamic system.

$$\begin{aligned} C(t) &= (p + \rho)(W(t) + H(t)) \\ V(t) &= K(t) + B(t) \\ H(t) &= \int_t^\infty R(t, z)(Y(z) - T(z))\, dz \\ \frac{dV(t)}{dt} &= r(t) V(t) + Y(t) - C(t) - T(t). \end{aligned} \tag{111}$$

The system of differential equations is,

$$\frac{dC}{dt} = (F'(K) - \rho) C - (p + \rho) p (B + K) \tag{112}$$
$$\frac{dK}{dt} = F(K) - C - G$$
$$\frac{dB}{dt} = F'(K) B + G - T. \tag{113}$$

At the steady state, we have,

$$\begin{aligned} (F'(K^*) - \rho) C^* &= (p + \rho) p (B^* + K^*) \\ F(K^*) - C^* - G &= 0 \\ F'(K^*) B^* + G - T &= 0 \end{aligned} \tag{114}$$

Obviously,

$$C^* = F(K^*) - G \tag{115}$$

Using the implicit function theorem,

$$\begin{aligned} \frac{dK^*}{dB} &= \frac{(p + \rho) p}{F''(K^*) C^* - (p + \rho) p + (F'(K^*) - \rho)(F'(K^*))} \\ &= \frac{(p + \rho) p}{F''(K^*) C^* - (p + \rho) p + (r^* - \rho) r^*}. \end{aligned} \tag{116}$$

At the steady state,

$$\begin{aligned}
(r^* - \rho) C^* &= (p + \rho) p (B^* + K^*) \\
(p + \rho) p &= \frac{(r^* - \rho) C^*}{p (B^* + K^*)}
\end{aligned} \tag{117}$$

We also have,

$$r^* (B^* + K^*) + Y^* - C^* - T = 0. \tag{118}$$

Therefore,

$$\begin{aligned}
r^* &= \frac{T + C^* - Y^*}{(B^* + K^*)}, \\
(r^* - \rho) r^* &= (r^* - \rho) \frac{T + C^* - Y^*}{(B^* + K^*)}.
\end{aligned} \tag{119}$$

Combining them,

$$\begin{aligned}
-(p + \rho) p + (r^* - \rho) r^* &= \frac{-(r^* - \rho) C^*}{p (B^* + K^*)} + (r^* - \rho) \frac{T + C^* - Y^*}{(B^* + K^*)} \\
&= (r^* - \rho) \frac{T - Y^*}{(B^* + K^*)}.
\end{aligned} \tag{120}$$

Therefore,

$$\frac{dK^*}{dB} = \frac{(p + \rho) p}{F'' (K^*) C^* + (r^* - \rho) \frac{T - Y^*}{(B^* + K^*)}}. \tag{121}$$

Since we know at the steady state, $r^* > \rho$, if $T - Y < 0$, that is, if the lump sum tax is smaller than wage as is supposed to be, the denominator is always negative. This means, an increase in governmental bond decreases the steady state level of capital! So, the Ricardian equivalence does not hold in this economy in the long run.

This model has been extended to various directions by many researchers. See Boucekkine et al. (2002)[9], Heijra and Romp (2009)[10], and Garleanu and Panageas (2015)[11] for recent progresses.

[9] Boucekkine, R., D. de la Croix, and O. Licandro (2002), "Vintage human Capital, demographic trends and endogenous growth," *Journal of Economic Theory* 104, 340-375.

[10] Heijdra, Ben J. & Romp, Ward E., 2009. "Human capital formation and macroeconomic performance in an ageing small open economy," *Journal of Economic Dynamics and Control*, Elsevier, vol. 33(3), pages 725-744, March.

[11] Nicolae Garleanu & Stavros Panageas, 2015."Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing," *Journal of Political Economy*, University of Chicago Press, vol. 123(3), pages 670-685.