20010年度上級マクロ経済学講義ノート(3):最適成長

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1 Objectives

The growth model by Solow assumes constant saving rate. The assumption might be appropriate for research on long run economic growth. But, in short run, it is apparently not a good assumption because if capital level is very small, the marginal return of capital becomes high, which leads to high interest rate. If interest rate is high, typical consumers will increase their savings. So, the constant saving rate is inappropriate for an economy that is not at the steady state, or at the balanced growth path.

In this lecture note, two players are introduced to the model, household and firm. All the households have the identical utility function and maximize it over infinite horizon. Firms are price takers and have the same linear homogenous technology. Interaction between firms and households via market will be discussed.

In this lecture, everything is assumed to be certain. That is, both firms and households know what is going to happen 100 years later. Of course, this is very unrealistic assumption. But it might be OK as long as we are interested in the long run behavior, not business cycle. Stochastic elements will be introduced in RBC part.

2 Households

A large number of identical households,H. Each household is growing its size at rate n,

H: the number of households

L : the total population of the economy

C: the consumption of each member of the household

u: the instantaneous utility function.

The household's utility function

$$U = \int_{t=0}^{\infty} e^{-\rho t} u\left(C_t\right) \frac{L_t}{H} dt \tag{1}$$

ρ: a constant subjective discount rate.

$$u(C_t) = \frac{C_t^{1-\theta} - 1}{1-\theta} \text{ if } \theta \neq 1 \tag{2}$$

$$= \log(C_t) \text{ if } \theta = 1 \tag{3}$$

where $\theta > 0$. θ is the measure of concavity of the instantaneous utility function.[1]

Caution: the utility LEVEL becomes negative if $\theta > 1$, but there is no problem in it. Consider why?

3 Specifications of Utility Function

Utility function of the power form is called CRRA, Constant Relative Risk Aversion. The Arrow-Pratt of coefficient of relative risk aversion is

$$R = -\frac{U''C}{U}. \tag{4}$$

Apparently, if the utility function is CRRA,

$$R = \theta, \tag{5}$$

which is constant. Since there is no uncertainty in the model here, the attitude toward risk is not relevant. But the parameter θ DOES have a significant meaning!

3.1 Elasticity of Intertemporal Substitution

Consider discrete time case. There are only two periods. The consumer receives exogenous income y_t at time t. The price of asset that can be used as saving is q at time 1.

So, the utility maximization problem can written as

$$\max U(c_1) + \beta U(c_2)$$

$$c_1 + qa_1 = y_1$$

$$c_2 = y_2 + a_1$$

$$c_1, c_2 \geq 0$$

or

$$c_1 + \frac{c_2}{(1+r)} = y_1 + \frac{y_2}{(1+r)}, 1 + r = \frac{1}{q}$$

[1] By assuming the above utility function, we have imposed several strong characteristics on the model. (1) Additive time separability, (2) Constant discount rate, (3) Consumption Smoothing, and (4) relative risk aversion has one-to-one correspondence with the degree of interpomral substitution. There have been a number of papers that examine the validity of above characteristics. Most of them point out that data does not match to the model. However, we use this CRRA type of utility function because still CRRA is regarded as a benchmark that is simple, tractable, and easy to interpret.

The first order condition is

$$U'(c_1) = \beta(1+r)U'(c_2)$$

This is nothing but the Euler Equation.

Now consider the role of interest rate, r.

(1) An increase in r will lower the price of consumption at the second period. This will lower the consumption at time 1 and increase the future consumption: The substitution effects.

(2) An increase in r will lower the price of consumption at the second period. This will make the household richer, then positive income effects will emerge: The income effects.

(3) An increase in r will lower the discounted value of the income at the second period: The human capital effects.

So, there are three competing effects. Which one will dominate?

If the instantaneous utility function is CRRA,

$$U = \frac{c^{1-\theta} - 1}{1-\theta}$$

The elasticity of intertemporal substitution is

$$is(c_2, c_1) = -\frac{\left[\frac{d\frac{c_2}{c_1}}{\frac{c_2}{c_1}}\right]}{\left[\frac{d\frac{1}{1+r}}{\frac{1}{1+r}}\right]} = -\left[\frac{d\frac{c_2}{c_1}}{d\frac{1}{1+r}}\right]\left[\frac{\frac{1}{1+r}}{\frac{c_2}{c_1}}\right]$$

The first order condition is

$$\frac{\beta U'(c_2)}{U'(c_1)} = \frac{1}{1+r}$$

When it is CRRA

$$\frac{c_2}{c_1} = \left(\frac{1}{\beta(1+r)}\right)^{-\frac{1}{\theta}}$$

By differentiating it,

$$\frac{d\frac{c_2}{c_1}}{d\frac{1}{1+r}} = -\frac{1}{\beta\theta}\left(\frac{1}{\beta(1+r)}\right)^{-\frac{1}{\theta}-1}$$

Then, it is easy to obtain

$$is(c_2, c_1) = \frac{1}{\beta\theta}\left(\frac{1}{\beta(1+r)}\right)^{-\frac{1}{\theta}-1} * \left(\frac{1}{\beta(1+r)}\right)^{\frac{1}{\theta}} = \frac{1}{\theta}$$

So, CRRA utility has the constant elasticity of the intertemporal substitution, which turns out to be just the inverse of the risk aversion rate! This implies that if consumers have CRRA type utility function, he/she should be risk averse if his/her elasticity of intertemporal substitution is small. Note that the two concept, the risk aversion and the elasticity between the two periods, are completely different things. A person can be very prudent and has a large elasticity. CRRA does not allow us to consider such households.

It is possible to generalize CRRA so that the risk aversion and the intertemporal substitution can be separated. Consider the following utility function/

$$u_t = \left[c_t^{1-\theta} + \beta u_{t+1}^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{6}$$

Then, it is easy to show that this is nothing but a CES utility function with

$$u_t = \left[\sum_{j=0}^{\infty} \beta^j c_{t+j}^{1-\theta}\right]^{\frac{1}{1-\theta}}. \tag{7}$$

Now, let's introduce uncertainty. Usually, when introducing uncertainty, we take the expectation of the utility, which gives us the expected utility function. But this time, consider

$$u_t = \left[c_t^{1-\theta} + \beta\left(E_t\left(u_{t+1}^{1-\gamma}\right)\right)^{\frac{1-\theta}{1-\gamma}}\right]^{\frac{1}{1-\theta}}. \tag{8}$$

Notice that we do not take the expected utility for the future, but take the non-linear transformation of the expected utility. This class of utility function was first considered by Koopmance long time ago, and formalized by Kreps and Porteus (1978). Since this uses the non-linear transformation of the future expected utility, this is sometimes called non expected utility function. Now, we have two independent parameters, γ and θ, which represent the risk aversion and the intertemporal substitution, respectively.

If we assume $\theta = \gamma$, we will get CRRA utility function.

This class of utility function have been used mainly in finance literature. Epstein and Zin (1989) applied this type of utility function to empiricial analysis. Hamori (1996)(in Japanese) is a useful guide to Consumption CAPM and this class of utility function. To solve the intertemporal utility maximization above, we need to use Dynamic Programming technique. As far as economic growth is concerned, day to day uncertainty is not a serious problem. We will not discuss this type of utility function further. But it is worth noting that non expected utility function is one of the fields both micro and macro people are excited and pursuing.

As long as the utility function is CRRA, the first order condition

$$\frac{c_2}{c_1} = \left(\frac{1}{\beta(1+r)}\right)^{-\frac{1}{\theta}}$$

and

$$c_1 + \frac{c_2}{(1+r)} = y_1 + \frac{y_2}{(1+r)}$$

gives us

$$c_2 = \left(\frac{1}{\beta(1+r)}\right)^{-\frac{1}{\theta}} c_1$$

$$c_1 \left(1 + \left(\frac{1}{\beta}\right)^{-\frac{1}{\theta}} \left(\frac{1}{1+r}\right)^{1-\frac{1}{\theta}}\right) = y_1 + \frac{y_2}{(1+r)}$$

$$c_1 = \left(1 + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}-1}\right)^{-1} \left(y_1 + \frac{y_2}{(1+r)}\right)$$

Therefore

$$\begin{aligned}
\frac{\partial c_1}{\partial (1+r)} &= -\frac{y_2}{(1+r)^2} \left(1 + \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\theta}-1}\right)^{-1} \\
&\quad - \left(y_1 + \frac{y_2}{(1+r)}\right) \left(1 + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\sigma}-1}\right)^{-2} \left(\beta^{\frac{1}{\theta}} \left(\frac{1}{\sigma} - 1\right) (1+r)^{\frac{1}{\theta}-2}\right) \\
&= -\frac{y_2}{(1+r)^2} \left(1 + \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\theta}-1}\right)^{-1} - \left(1 + \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}-1}\right)^{-1} c_1 \left(\beta^{\frac{1}{\theta}} \left(\frac{1}{\sigma} - 1\right) (1+r)^{\frac{1}{\theta}-2}\right) \\
&= -\left(1 + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}-1}\right)^{-1} \left[\frac{y_2}{(1+r)^2} + c_1 \left(\beta^{\frac{1}{\theta}} \left(\frac{1}{\sigma} - 1\right) (1+r)^{\frac{1}{\theta}-2}\right)\right]
\end{aligned}$$

The first term of $\frac{y_2}{(1+r)^2} + c_1 \left(\beta^{\frac{1}{\theta}} \left(\frac{1}{\sigma} - 1\right) (1+r)^{\frac{1}{\theta}-2}\right)$ is the effect through human capital, the rest terms are the sum of income effects and substitution effects.

If $\theta = 1$ that is the utility function if logarithms, the income and substitution effects cancel out each other. The only remaining effect is the human capital effects. This implies if there is no income at the second period, a change in interest rate does not affect the consumption at the first period.

If $0 < \theta < 1$, that is, when the elasticity of substitution if large, the second term become positive. The substitution effects dominate the income effects. Under this circumstance, the consumption at the first period will decrease. On the other hand, if $\theta > 1$, that is, when the elasticity of substitution is small, the second term becomes negative. The income effects dominate. The total effects depend on the relative size of the human capital effect and the second effects.

3.2 Discount Rate and Time Inconsistency

So far, we have assumed that the time discount rate is constant over time. Let's relax this assumption for a while. Consider

$$U = \int_{s}^{\infty} u(c_t) D[t, t-s, x(t)] dt \tag{9}$$

So, the discount rate depends both on the calendar time, t and the time interval, t-s. The other factor, $x(t)$ such as consumption, income, can affect the discount rate as well. But let's forget about it.

Now, consider that the consumer plans at time τ_1 and τ_2 $(\tau_1 < \tau_2)$ about the future consumptions at time t_1 and t_2 $(t_1 < t_2)$ Let's assume $\tau_1 < \tau_2 < t_1 < t_2$

When the consumer plans at time τ_1, the marginal rate of substitution between t_1 and t_2 becomes

$$\frac{u'(c_{t_1}) D[t_1, t_1 - \tau_1]}{u'(c_{t_2}) D[t_2, t_2 - \tau_1]}. \tag{10}$$

Later, at time τ_2, the consumer rethinks the decision he/she made before. Then the marginal substitution will be changed to

$$\frac{u'(c_{t_1}) D[t_1, t_1 - \tau_2]}{u'(c_{t_2}) D[t_2, t_2 - \tau_2]}. \tag{11}$$

As long as $D[t_1, t_1 - \tau_1] = D[t_2, t_2 - \tau_2]$, the two marginal substitution rates are the same. But, in general, they are different from each other. This means the optimal plan made before is no longer optimal later. Note that uncertainty does not play any role in this discussion. The plan made at time τ_1 is *time inconsistent*. As far as the future is discounted by exponential function with constant rate, time inconsistency does not occur. Time inconsistency can be a serious problem. In macroeconomic literature, time inconsistency has been mainly discussed in policy related issues. See, for example, Barro and Gordon (1983) and Kydland and Prescott (1977)[2]. But some people are very serious about the time inconsistency coming from non-exponential discounting. One example is hyperbolic discounting. Many research on human and animals often find that the discount rate depends on the time and discount rate is greater for near future. Of course, consumption decision with hyperbolic discount is time inconsistent. See Laibson (1996) for more details. Like non-expected utility function, hyperbolic discount is one of the hot fields, particularly among people in experimental economics and behavior finance. Whether hyperbolic discount is a serious issue in macroeconomics is still under debate.

4 Firms

A lot of competitive identical firms that have the same technology as in the Solow's model,

$$Y = F(K, AL) \tag{12}$$

A grows at constant rate, g.

[2]Chapter 10 in Romer's textbook has some explanation of Time inconsistency (or subgame perfection).

If g is significantly greater than ρ, the economy might grow faster than the discount rate, which gives us positive infinite value of utility function. To avoid this case, let's assume

$$\rho - n - (1-\theta)\, g > 0 \tag{13}$$

Firms are facing competitive goods and factor markets. The profit maximization gives us

$$r_t = f'(k_t) \tag{14}$$

$$w_t = A_t \left[f(k_t) - k_t f'(k_t) \right] \tag{15}$$

where r_t : the real interest rate, w_t : the wage.

5 Households' Budget Constraint

Each household can lend and borrow at the rate, r_t at time t. Each member of the household has K_0/H amounts of asset at time 0.

Intertemporal budget constraint is

$$\int_{t=0}^{\infty} e^{-R_t} C_t \frac{L_t}{H} dt \le \frac{K_0}{H} + \int_{t=0}^{\infty} e^{-R_t t} W_t \frac{L_t}{H} dt \tag{16}$$

where

$$R_t = \int_{s=0}^{t} r_s ds. \tag{17}$$

Arranging a bit,

$$\frac{K_0}{H} + \int_{t=0}^{\infty} e^{-R_t} \left[W_t - C_t \right] \frac{L_t}{H} dt \ge 0. \tag{18}$$

Notice

$$\frac{d \int_{s=0}^{t} r_s ds}{dt} = r_t. \tag{19}$$

Write

$$\frac{dK_t}{dt} = L_t \left[W_t - C_t \right] + r_t K_t \tag{20}$$

That means, incomes of each household come from two sources: labor incomes and interest revenues. Income minus consumption becomes savings. Calculate

$$\frac{d\left(K_t e^{-Rt}\right)}{dt} = \dot{K}_t e^{-Rt} - r_t K_t e^{-Rt} \tag{21}$$

Taking the integral from 0 to s,

$$\begin{aligned} \int_{t=0}^{s} \frac{d\left(K_t e^{-Rt}\right)}{dt} dt &= K_s e^{-R_s} - K_0 \\ &= \int_{t=0}^{s} e^{-Rt} \left(L_t \left[W_t - C_t \right] \right) dt \end{aligned}$$

Therefore,

$$\frac{K_s}{H} = \int_{t=0}^{s} e^{R_s - R_t} \left(\frac{L_t}{H} \left[W_t - C_t \right] \right) dt + \frac{e^{R_s} K_0}{H}. \tag{22}$$

Notice that $W_t - C_t$ can be negative. If we allow $\frac{K_s}{H} < 0$, the household can borrow infinite amount of money to finance any level of consumption. We have to eliminate this situation to get meaningful consumption plan. The following condition does this job,

$$\lim_{s \to \infty} e^{-R_s} \frac{K_s}{H} \geq 0 : \text{ The No Ponzi Condition.} \tag{23}$$

6 Households' Maximization Problem

Define

$$c_t = \frac{C_t}{A_t} \tag{24}$$

Then,

$$\begin{aligned} \frac{(A_t c_t)^{1-\theta}}{1-\theta} &= A_t^{1-\theta} \frac{(c_t)^{1-\theta}}{1-\theta} \\ &= A_0^{(1-\theta)} e^{-(1-\theta)gt} \frac{c_t^{1-\theta}}{1-\theta} \end{aligned}$$

The Lifetime utility becomes

$$\begin{aligned} U &= \int_{t=0}^{\infty} e^{-\rho t} u(C_t) \frac{L_t}{H} dt \\ &= \int_{t=0}^{\infty} e^{-\rho t} u(C_t) \frac{L_0}{H} e^{gt} dt \\ &= \int_{t=0}^{\infty} e^{-\rho t} A_0^{(1-\theta)} e^{-(1-\theta)gt} \frac{c_t^{1-\theta}}{1-\theta} \frac{L_0}{H} e^{gt} dt \\ &= \frac{L_0 A_0^{(1-\theta)}}{H} \int_{t=0}^{\infty} e^{-(\rho - n - (1-\theta)g)t} \frac{c_t^{1-\theta}}{1-\theta} dt \end{aligned}$$

Remember that we have assumed

$$\rho - n - (1-\theta) g > 0$$

Define

$$\begin{aligned} \beta &= \rho - n - (1-\theta) g \\ B &= \frac{L_0 A_0^{(1-\theta)}}{H}, \end{aligned}$$

Then,

$$U = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}dt. \tag{25}$$

The budget constraint becomes

$$\int_{t=0}^{\infty} e^{-R_t}C_t\frac{L_t}{H}dt \leq \frac{K_0}{H} + \int_{t=0}^{\infty} e^{-R_t t}W_t\frac{L_t}{H}dt$$

$$\int_{t=0}^{\infty} e^{-R_t}A_t c_t\frac{L_t}{H}dt \leq \frac{k_0 L_0 A_0}{H} + \int_{t=0}^{\infty} e^{-R_t t}w_t\frac{A_t L_t}{H}dt$$

$$\int_{t=0}^{\infty} e^{-R_t}A_0 c_t\frac{L_0}{H}e^{(g+n)t}dt \leq \frac{k_0 L_0 A_0}{H} + \int_{t=0}^{\infty} e^{-R_t t}w_t e^{(g+n)t}\frac{A_0 L_0}{H}dt$$

$$\int_{t=0}^{\infty} e^{-R_t}c_t e^{(g+n)t}dt \leq k_0 + \int_{t=0}^{\infty} e^{-R_t t}w_t e^{(g+n)t}dt. \tag{26}$$

The no-Ponzi-game condition becomes

$$\lim_{s\to\infty} e^{-R_s}e^{(g+n)t}k_s \geq 0. \tag{27}$$

7 Maximization

Max

$$U = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}dt. \tag{28}$$

s.t.

$$\int_{t=0}^{\infty} e^{-R_t}c_t e^{(g+n)t}dt \leq k_0 + \int_{t=0}^{\infty} e^{-R_t t}w_t e^{(g+n)t}dt, \tag{29}$$

and

$$\lim_{s\to\infty} e^{-R_s}e^{(g+n)t}k_s \geq 0. \tag{30}$$

7.1 Using Lagrangian:

$$L = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}dt+\lambda\left(k_0 + \int_{t=0}^{\infty} e^{-R_t t}w_t e^{(g+n)t}dt - \int_{t=0}^{\infty} e^{-R_t}c_t e^{(g+n)t}dt\right).$$

The F.O.C. for c_t :

$$Be^{-\beta t}c_t^{-\theta} = \lambda e^{-R_t}e^{(g+n)t}.$$

Taking the natural logarithms,

$$\begin{aligned}\ln B - \beta t - \theta \ln c_t &= \ln\lambda - R_t + (g+n)\,t \\ &= \ln\lambda - \int_{s=0}^{t} r_s ds + (g+n)\,t.\end{aligned}$$

Taking the time derivatives of both sides,

$$-\beta - \theta \frac{\dot{c_t}}{c_t} = (g+n) - r_{t.}$$

That is,

$$\frac{\dot{c_t}}{c_t} = \frac{r_{t.} - \theta g - \rho}{\theta} \tag{31}$$

7.2 Using Hamiltonian:

Rewrite

$$\frac{dK_t}{dt} = L_t \left[W_t - C_t \right] + r_t K_t \tag{32}$$

as

$$\frac{dA_t L_t k_t}{dt} = \frac{d\left(A_0 L_0 e^{(g+n)t} k_t\right)}{dt} = A_0 L_0 e^{(g+n)t} \left(w_t - c_t\right) + A_0 L_0 e^{(g+n)t} r_t k_t,$$

therefore,

$$\dot{k_t} = (w_t - c_t) + r_t k_t - (g+n)\, k_t. \tag{33}$$

Set up Hamiltonian as

$$H_t = \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left((w_t - c_t) + r_t k_t - (g+n)\, k_t \right). \tag{34}$$

The F.O.C. in terms of c_t

$$c_t^{-\theta} = \lambda_t.$$

The Hamiltonian Dynamics:

$$\begin{aligned} \dot{\lambda_t} - \beta \lambda_t &= -H_k \\ &= \lambda_t \left(g + n - r_t \right). \end{aligned} \tag{35}$$

BTW,

$$\dot{\lambda_t} = -\theta c_t^{-\theta} \frac{\dot{c_t}}{c_t}.$$

Therefore,

$$\dot{\lambda_t} - \beta \lambda_t = -\theta c_t^{-\theta} \frac{\dot{c_t}}{c_t} - \beta c_t^{-\theta} = -c_t^{-\theta} \left(\theta \frac{\dot{c_t}}{c_t} + \beta \right) = -\lambda_t \left(\theta \frac{\dot{c_t}}{c_t} + \beta \right). \tag{36}$$

Combining the above and (35), we can get

$$\theta \frac{\dot{c_t}}{c_t} = -\left(g + n - r_t + \beta \right),$$

$$\frac{\dot{c}_t}{c_t} = \frac{(r_t - \rho - \theta g)}{\theta} \tag{37}$$

This is the same as (31).

This equation is called "*the consumption Euler equation*". The interpretation of the Euler equation is easy. If the real interest rate is high, it is beneficial for the household to consume not now, but future (save more). If θ is a large number, the change in consumption is small, that is, consumption does not change a lot. This is because of the concavity of the utility function. This phenomena is called "*the consumption smoothing*". The smoothing does not occur when θ is close to zero. In such a case, the instantaneous utility function becomes linear. A person with linear utility function does not have any incentives to smooth his/her consumption over time since two consumption levels at different time are perfect substitutes.

8 The Equilibrium

In an equilibrium, wages and interest rates are determined to meet demand and supply in factor markets. Therefore, we obtain

$$\frac{\dot{c}_t}{c_t} = \frac{(f'(k_t) - \rho - \theta g)}{\theta}, \tag{38}$$

$$\begin{aligned} \dot{k}_t &= (w_t - c_t) + r_t k_t - (g+n) k_t && (39) \\ &= (f(k_t) - f'(k_t) k_t - c_t) + f'(k_t) k_t - (g+n) k_t && (40) \end{aligned}$$

$$\dot{k}_t = f(k_t) - c_t - (g+n) k_t. \tag{41}$$

Two equations, (38) and (41) gives us a two dimensional ordinary differential equation system. (41) is a resource constraint, or IS-equation, which says the output is either consumed or invested (with adjustment of economic growth).

9 The Ramsey Problem

Consider the central planner's problem.

Max

$$\int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt$$

s.t.

$$\dot{k}_t = f(k_t) - c_t - (g+n) k_t. \tag{42}$$

Set the Hamiltonian,

$$H_t = \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left(f\left(k_t\right) - c_t - \left(g+n\right) k_t \right).$$

The F.O.C. for consumption is

$$c_t^{-\theta} = \lambda_t.$$

The Hamiltonian Dynamics:

$$\begin{aligned} \dot{\lambda}_t - \beta\lambda_t &= -H_k \qquad (43) \\ &= \lambda_t \left(g + n - f'\left(k_t\right) \right). \end{aligned}$$

The two equations give us

$$\frac{\dot{c}_t}{c_t} = \frac{\left(f'\left(k_t\right) - \rho - \theta g\right)}{\theta}.$$

Therefore, the market equilibrium is equivalent to the solution of the central planner's problem. This is not surprising because the market equilibrium is Pareto efficient from the first welfare theorem. In macroeconomics, it is often redundant to set up a model with markets since as long as markets are efficient, from the second welfare theorem, we can support any efficient allocation as a market equilibrium. Therefore, in many occasions, we use the central planner's problem to describe market economies.

10 The Modified Golden Rule

At the steady state, we get

$$f'\left(k^*\right) - \rho - \theta g = 0 \qquad (44)$$

Recall that the golden rule says

$$f'\left(k^{GR}\right) - n = 0$$

Since we assume

$$\rho - n - \left(1-\theta\right) g > 0,$$

we can obtain

$$f'\left(k^*\right) > n.$$

Because of concavity of the production function, the above equation implies

$$k^{GR} > k^*.$$

That is, the steady state level of capital is smaller than the golden rule level. The formula (44), is called "*the modified golden rule*". The (original) golden rule is the level at which the steady state level of consumption is maximized. This is not efficient in the Ramsey problem with positive discount rate since the household prefers current consumption more than future consumption.

11 The Linearized System

We can approximate the system by a linear system at the steady state.

$$\dot{c}_t = \frac{c_t \left(f'(k_t) - \rho - \theta g\right)}{\theta}, \tag{45}$$

$$\dot{k}_t = f(k_t) - c_t - (g+n) k_t \tag{46}$$

Linearize the system at the steady state defined by

$$k^* : f'(k^*) - \rho - \theta g = 0,$$

$$c^* : f(k^*) - c^* - (g+n) k^* = 0.$$

Then, we can get

$$\begin{pmatrix} \dot{c}_t \\ \dot{k}_t \end{pmatrix} = \begin{pmatrix} 0 & \frac{c^* f''(k^*)}{\theta} \\ -1 & f'(k^*) - (g+n) \end{pmatrix} \begin{pmatrix} c_t - c^* \\ k_t - k^* \end{pmatrix}. \tag{47}$$

Recall

$$\beta = \rho - n - (1-\theta) g.$$

Therefore

$$f'(k^*) - (g+n) = \rho + \theta g - g - n = \beta$$

The characteristic function is

$$\lambda^2 - \beta\lambda + \frac{c^* f''(k^*)}{\theta} = 0. \tag{48}$$

The eigenvalues are

$$\lambda = \frac{\beta \pm \sqrt{\beta^2 - \frac{4c^* f''(k^*)}{\theta}}}{2}.$$

Because of the concavity of the production function, λ is always a real number. It is also clear that one of the eigenvalues is positive and the other is negative. Therefore, the steady state is saddle.

Romer (2006) utilizes a very special nature of this economy to obtain the saddle path. That is clever and worth learning. But here, we will use more general technique to obtain the saddle path.

Define the Jacobian of the system as

$$A = \begin{pmatrix} 0 & a \\ -1 & \beta \end{pmatrix}$$

where $a = \frac{c^* f''(k^*)}{\theta}$.

Eigenvectors of A can be obtained by solving the

$$\begin{pmatrix} 0 & a \\ -1 & \beta \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ x \end{pmatrix},$$

which gives us

$$P = \begin{pmatrix} 1 & 1 \\ \frac{\lambda_1}{a} & \frac{\lambda_2}{a} \end{pmatrix}$$

where the ith column is the eigenvector for λ_i. Without loss of generality, assume $\lambda_1 < \lambda_2$.

Then, by Jordan decomposition,

$$A = P \times \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \times P^{-1}. \tag{49}$$

Define

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix}. \tag{50}$$

Then,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Therefore,

$$z_{2t} = z_{20} e^{\lambda_2 t}.$$

Since $\lambda_2 > 0$, we have to have $z_{20} = 0$ to assure finite values for c and k. BTW,

$$P^{-1} = \frac{1}{\frac{\lambda_2}{a} - \frac{\lambda_1}{a}} \begin{pmatrix} \frac{\lambda_2}{a} & -1 \\ -\frac{\lambda_1}{a} & 1 \end{pmatrix}.$$

Therefore,

$$z_{20} = \frac{1}{\frac{\lambda_2}{a} - \frac{\lambda_1}{a}} \begin{pmatrix} -\frac{\lambda_1}{a} & 1 \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} = 0$$

That is,

$$\frac{\lambda_1}{a} (c - c^*) + (k - k^*) = 0.$$

Or,

$$c_t - c^* = \frac{c^* f''(k^*)}{\theta \lambda_1} (k_t - k^*). \tag{51}$$

This is the saddle path in this economy. The saddle path gives us one-to-one correspondence between k and c. Remember that the capital level is given at time t. The consumption can jump at time t. The saddle path tells us to which point the consumption must jump to be optimal. That is, the saddle path gives us the optimal level of consumption given the capital level. Sometimes, it is called as *the Policy Function*, since it tells us the optimal consumption plan for the household.

This approach is straightforward, but a bit time consuming to calculate eigenvectors and its inverse matrix. C.Sims suggests using the left eigenvector defined by

$$xA = \lambda x$$

where x is a $1 \times n$ vector. Left eigenvalues are the same as the normal eigenvalues. This approach reduces calculations significantly. Try it if you are interested in.

12 Several Remarks

Discount rate: Why does the household discount the future? The assumption of the positive discount rate implies that each household thinks the current generation more important than future ones. But how can we justify? Actually, in the original paper, Ramsey assumed zero discount rate. This creates some mathematical difficulties since the utility level diverges. There are a number of papers that assume zero discount rate. But, of course, we have to redefine the maximization problem to accommodate infinite level of the utility. In public finance literature, this problem is known as the social discount rate problem. There are lots of debates on the ethical aspects of the assumption. However, concerning the mathematical difficulty, the complexity mainly comes from the assumption of the infinite horizon that is also difficult to defend.

Infinite horizon: Why does the household have infinite horizon perspective? Everybody knows eventually the solar system will decay. The assumption is not only difficult to defend from economic point of view, but also raises several very serious mathematical difficulties. For example, taking the integral from zero to infinity raises a problem of convergence and existence of the integral. So, why do we have to assume infinite horizon? The only answer I can come up with is that we do not have a certain terminal point, either. If we know that the comet Halley will destroy the earth for sure in the next month, we have a certain terminal point, the end of the human being. But this is not the case, fortunately. The economic implication of having the definite terminal point is huge because everything will become worthless at that time. No incentives will remain to save and invest. People will consume everything at the end of the universe. We should not include such a drastic aspect in our model!

No heterogeneity among consumers: The aim of this model is to describe "macro economic" movements of economic variables. We use linear approximation to solve the system. As long as we use linearization, the distribution of assets or consumption does not have significant effects unless the economy has some imperfection. If this is not the case, for example, if the economy has incomplete capital market, or there are public goods determined by some political process such as voting, distribution becomes a very important matter in the economy.

No uncertainty: We will introduce uncertainty to this model in RBC section.

13 Further Readings

[1] Mas-Colell, Whinston, and Green, (1995) Microeconomics

Chapter 20 of this book describes more mathematical arguments of this class of model.

[2] Solow, (1999), Neoclassical Growth Theory, in *Handbook of Macroeconomics, IA*This chapter gives us a very brief introduction to various growth theories by the Novel winner. Very easy to understand.