Global Liquidity Trap*

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Abstract

We consider the optimal coordination of monetary policy where two countries are simultaneously caught by liquidity trap. We analyze the properties of optimal commitment and discretionary policy using a standard two-country New Open Economy Macroeconomics (NOEM) model. Our main findings are as follows: (1) in contrast to previous studies in NOEM, under liquidity trap, efficiency cannot be achieved by inward-looking policy even when the producer currency pricing is assumed; (2) there, the optimal monetary policy under commitment and discretion reflects international interdependence. The optimal policy responses under global liquidity trap are substantially complicated compared to those under closed economy reflecting international interdependence; (3) yet, similarly to previous studies on liquidity trap in closed economy, the optimality of history dependent policy under commitment is confirmed even under global liquidity trap. By keeping nominal interest rates at very low level even after the adverse shock to the economy disappears, the optimal allocations and prices can be attained.

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1 Introduction

The economic downturn following a financial turmoil since year 2007 has resulted in the monetary policy with virtually zero lower bounds of nominal interest rates in a number of countries, including Japan, United Kingdom, and United States, simultaneously. Figure 1 shows nominal interest rates from year 2000 to 2009 in several advanced countries. All nominal interest rates exhibit drastic decreases from their levels in 2007. Moreover, the room for further monetary easing is considerably limited, as they are already set to nearly zero. Virtually, the Bank of Japan (BOJ), Bank of England (BOE), and the Federal Reserve Board (FRB) have already cut their policy rates to almost zero:

"*Lowering of the Bank's target for the uncollateralized overnight call rate by 20 basis points; it will be encouraged to remain at around 0.1 percent.*" (December 12, 2008, Statements on Monetary Policy, BOJ),

"*The Bank of England's Monetary Policy Committee today voted to reduce the official Bank Rate paid on commercial bank reserve by 0.5 percentage point to 0.5%, and ...*" (5 March 2009, News Release, BOE),

"*The Committee will maintain the target range for the federal funds rate at 0 to 1/4 percent and anticipate that economic conditions are likely to warrant exceptionally low levels of the federal funds rate for an extended period.*" (March 18, 2009, Press Release, FRB).

In addition, the European Central Bank (ECB) and the Bank of Canada (BOC) are also very close to *de facto* zero interest rate policy:

"*Bank of Canada lowers overnight rate target by 1/4 percentage point to 1/4 per cent and, conditional on the inflation outlook, commits to hold current policy rate until the end of the second quarter of 2010.*" (April 21, 2009, Press Release, BOC).

Liquidity trap is no longer an exceptional experience of Japan, but has become an international concern that can be solved by international monetary cooperation.

Existing studies do not, however, provide appropriate theoretical framework as to how monetary policies should be conducted under global liquidity trap of now. To date, zero bound problem on nominal interest rates has been mostly considered in a closed economy model, e.g. Eggertsson and Woodford (2003a, b), Jung, Teranishi, and Watanabe (2005) and Adam and Billi (2006, 2007). How monetary policy cooperations should be designed when multiple countries simultaneously face the zero bound of nominal interest rates has not yet been scrutinized in the literature.

We investigate the optimal coordination of monetary policy where two countries are simultaneously caught by liquidity trap. We analyze the properties of optimal commitment and discretionary policy using a standard two-country New Open Economy Macroeconomics (NOEM) model. Our main findings are as follows: (1) in contrast to previous studies in NOEM, under liquidity trap, efficiency cannot be achieved by inward-looking policy even when the producer currency pricing is assumed; (2) there, the optimal monetary policy under commitment and discretion reflects international interdependence. The optimal policy responses under global liquidity trap are substantially complicated compared to those under closed economy reflecting international interdependence; (3) yet, similarly to previous studies on liquidity trap in closed economy, the optimality of history dependent policy under commitment is confirmed even under global liquidity trap. By keeping nominal interest rates at very low level even after the adverse shock to the economy disappears, the Ramsey optimal allocations and prices can be achieved. At the same time, although higher social welfare can be achieved under commitment, making credible international commitments to future policies is considered a difficult task in open economies. Since imperfect credibility seems the most appropriate assumption in the open economies, central banks must understand the paths under the optimal monetary policy cooperation with discretion as well as commitment, since they are quite different under global liquidity trap.

The structure of the paper is as follows. Section 2 discusses the related literatures on liquidity trap. Section 3 derives the two-country NOEM model and the world loss function that central banks under cooperation should aim at minimizing. In section 4, we first show how international dependence affects the optimal monetary policy under global liquidity trap. Then, simulation results under optimal monetary policy are presented. We discuss the characteristics of the optimal monetary policy cooperation under both commitment and discretion. In Section 5, we conduct sensitivity analysis under stochastic environment. Section 6 concludes.

2 Related Literatures on Liquidity Trap

Reflecting the experience of the zero interest rate policy in Japan, many studies, such as Reifschneider and Williams (2000), Eggertsson and Woodford (2003a, b), Jung, Teranishi, and Watanabe (2005), Sugo and Teranishi (2005), Kato and Nishiyama (2005), Adam and Billi (2006, 2007), and Nakov (2008), have outlined the characteristics of desirable monetary policy under the zero lower bound on the nominal interest rate in closed economy.

Reifschneider and Williams (2000) investigate how the stabilization policy should be conducted in a low interest rate environment using the FRBUS model. Central bank should set out a zero interest rate policy preemptively in a wake of adverse shock. On the other hand, it is recommended that once the economy is caught by the zero lower bound, zero interest rate policy should be maintained for a while even after adverse shock disappears. They stress the importance of such a history dependent monetary policy. Although their analysis is very powerful and reasonable and the optimality of such a history dependent policy is later justified by following theoretical studies, they did not conduct any rigid welfare analysis.

Eggertsson and Woodford (2003a, b) and Jung, Teranishi, and Watanabe (2005) derive optimal targeting rules in a purely forward-looking economy under the standard New Keynesian model consisting of forward-looking IS and Phillips curves. According to these studies, the zero interest rate policy should be continued even after improvements in the economic situation resume. Thanks to committing to such a history dependent policy, central banks can stimulate the economy caught by liquidity trap through higher inflation expectations and resulting low real interest rates. Kato and Nishiyama (2005), Adam and Billi (2006, 2007) and Nakov (2008) extend above analyses to stochastic environment.[1] Again, the importance of history dependence is confirmed as the key feature of optimal monetary policy under liquidity trap. In addition, they also show that uncertainty makes the monetary policy more history-dependent.

Coenen and Wieland (2003), Svensson (2001), and Nakajima (2008) study the properties of the desirable zero interest rate policy in open economy. Coenen and Wieland (2003) and Svensson (2001) ascertain the merits of commitment policy for higher inflation expectation. In addition, they report that such the effectiveness of history dependent monetary policy is further intensified by the depreciation of nominal exchange rates. Using a similar model to Svensson (2001), however, Nakajima (2008) finds that nominal exchange rates should appreciate for a country conducting the zero interest rate policy under the optimal commitment policy because of the uncovered interest rate parity condition. These studies, however, only consider situations where nominal interest rates hit the zero lower bound in a single country. Therefore, they are not very appropriate for the analysis on the current economic situations, namely global liquidity trap.

3 The Model

Our two-country New Open Economy Macroeconomics model is a conventional one, as those used in Clarida, Galí, and Gertler (2002) and Benigno and Benigno (2003). The economy consists of domestic country H, and foreign country F. Labor is not mobile and it is used to produce a continuum of differentiated goods on the unit intervals $[0, 1]$ in both countries. Consumption indices in both countries, C and C^* are made of bundles of differentiated domestically produced goods and foreign produced goods, C_H and C_F, respectively, and private-agents in both countries consume such consumption indices. We denote the weight for the bundle of domestically produced goods by n, and that of foreign

[1] Eggertsson and Woodford (2003a, b) analysed the optimal monetary policy under stochastic environment. Yet, they assumed a Markov process with absorbing state. Therefore, once the economy comes back to normal steady state, there is no deviation from it.

produced goods by $1-n$. Actually, n (and $1-n$) represents the relative size of home (foreign) country. The domestic (foreign) central bank manipulates nominal interest rates i_H (i_F) so as to affect the economic decisions of private agents.

In what follows, we make use of the history notation. Letting $s_t \in S$ be a set of all possible states of the economy that can occur in period t, we denote the history from $t=0$ up until period t by

$$s^t = (s_0, s_1, ..., s_t),$$

where the probability of each history s^t is given by $\mu\left(s^t\right)$.

3.1 Households

A representative household in the domestic country H has the following preference:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) \left\{u\left[C\left(s^t\right)\right] - v\left[h\left(s^t\right)\right]\right\},$$

where $0 < \beta < 1$. $C\left(s^t\right)$ and $h\left(s^t\right)$ denote consumption and the labor supply in history s^t, respectively. The domestic household budget constraint is given by

$$W\left(s^t\right) h\left(s^t\right) + \Pi\left(s^t\right) + B\left(s^{t-1}\right) \geq \sum_{s_{t+1}} Q\left(s_{t+1}, s^t\right) B\left(s_{t+1}, s^t\right) + P\left(s^t\right) C\left(s^t\right) + T\left(s^t\right).$$

where $W\left(s^t\right)$, $\Pi\left(s^t\right)$ and $T\left(s^t\right)$ denote the wage rate, lump sum profits and taxes, in domestic currency units. The object $B\left(s_{t+1}, s^t\right)$ is an Arrow security, that delivers a unit of domestic currency in period $t+1$ if state s_{t+1} is realized, conditional on history, s^t. $Q\left(s_{t+1}, s^t\right)$ and $P\left(s^t\right)$ are the price of Arrow security, and that of consumption index for domestic household, respectively.

The lifetime utility of a representative household in the foreign country F, is given by

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) \left\{u\left[C^*\left(s^t\right)\right] - v\left[h^*\left(s^t\right)\right]\right\},$$

where superscript * denotes foreign variables. The budget constraint for foreign household is given by

$$W^*\left(s^t\right) h^*\left(s^t\right) + \Pi^*\left(s^t\right) + \frac{B^*\left(s^{t-1}\right)}{\mathcal{E}\left(s^t\right)} \geq \sum_{s_{t+1}} \frac{Q\left(s_{t+1}, s^t\right) B^*\left(s_{t+1}, s^t\right)}{\mathcal{E}\left(s^t\right)} + P^*\left(s^t\right) C^*\left(s^t\right) + T^*\left(s^t\right).$$

$\mathcal{E}\left(s^t\right)$ is the nominal exchange rate, that is defined as units of domestic currency per unit of foreign currency.

The households in the two countries maximize utilities subject to their own budget constraints, taking prices, wages, and exchange rates as given.

3.2 Firms

A homogeneous domestically produced good $Y\left(s^t\right)$ is produced by a representative competitive firm using the following technology:

$$Y\left(s^t\right) = \left[\int_0^1 Y\left(s^t, i\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{1}$$

where $\varepsilon > 1$. Homogeneous good producing firms take input prices $Y\left(s^t, i\right)$ and their output prices $P_H\left(s^t\right)$ as given and beyond their control.

The i^{th}, $i \in (0,1)$, intermediate good is produced by a monopolist using the following technology:

$$Y\left(s^t, i\right) = Z\left(s^t\right) h\left(s^t, i\right),$$

where $Z\left(s^t\right)$ is the technology common across i, and the only stochastic disturbance in our economy. The marginal cost of production for the i^{th} monopolist is given by

$$MC\left(s^t\right) = \left[1 - \tau\left(s^t\right)\right] \frac{W\left(s^t\right)}{Z\left(s^t\right) P_H\left(s^t\right)}, \tag{2}$$

where $\tau\left(s^t\right)$ denotes a tax subsidy associated with the supply of labor, financed by a lump sum tax on households. The i^{th} monopolist maximizes profits subject to its demand curve derived from equation (1), and the Calvo (1983) price frictions. In particular, the monopolist may optimize its price, $P_H\left(s^t, i\right)$, with probability $1-\theta$ and with probability θ it sets its price as follows:

$$P_H\left(s^t, i\right) = P_H\left(s^{t-1}, i\right).$$

The domestic consumption good index is produced by a competitive, representative firm that has the following production function:

$$C\left(s^t\right) = \left[\frac{C_H\left(s^t\right)}{n}\right]^n \left[\frac{C_F\left(s^t\right)}{1-n}\right]^{1-n}, \tag{3}$$

where $0 \leq n \leq 1$ is the relative size of the domestically produced goods, to the foreign produced goods. $C_H\left(s^t\right)$ and $C_F\left(s^t\right)$ denote the domestically produced goods consumed in domestic country, and foreign produced goods that is imported to the domestic country, respectively. The corresponding price for domestic consumption goods index is given by

$$P\left(s^t\right) = P_H\left(s^t\right)^n P_F\left(s^t\right)^{1-n},$$

where $P_F\left(s^t\right)$ is the price of foreign produced goods imported to the domestic country.

The same production technologies are used in foreign country. We denote the homogeneous foreign produced good, the price of foreign produced goods in foreign country, domestically produced goods imported to foreign country, foreign produced goods consumed in foreign country, and the price of foreign consumption goods index respectively by $Y^*\left(s^t\right)$, $P_H^*\left(s^t\right)$, $P_F^*\left(s^t\right)$, $C_H^*\left(s^t\right)$, $C_F^*\left(s^t\right)$ and $P^*\left(s^t\right)$.

3.3 Market Clearing Conditions

Market clearing conditions for domestic labor market, domestic homogenous goods market, and financial market are given below. The same conditions hold for foreign markets as well.

Clearing in the domestic labor market requires

$$h\left(s^t\right) = \int_0^1 h\left(s^t, i\right) \mathrm{d}i.$$

Clearing in domestic homogeneous goods market requires

$$nY\left(s^t\right) = nC_H\left(s^t\right) + (1-n)\, C_H^*\left(s^t\right). \tag{4}$$

Clearing in financial market requires

$$B\left(s_{t+1}, s^t\right) + B^*\left(s_{t+1}, s^t\right) = 0.$$

The output of the homogeneous domestic good is related to aggregate employment in domestic country. Using a framework given in Yun (2005), we can derive:

$$Y\left(s^t\right) = \Delta\left(s^t\right) Z\left(s^t\right) h\left(s^t\right), \tag{5}$$

where the relative price distortion term for domestic goods $\Delta\left(s^t\right)$ is defined as

$$\Delta\left(s^t\right) \equiv \left\{\int_0^1 \left[\frac{P_H\left(s^t, i\right)}{P_H\left(s^t\right)}\right]^{-\varepsilon} \mathrm{d}i\right\}^{-1}.$$

Under Calvo price frictions, the dynamics of distortion is given by

$$\Delta\left(s^t\right) = \frac{1}{(1-\theta)\left\{\frac{1-\theta\left[1+\pi_H\left(s^t\right)\right]^{\varepsilon-1}}{1-\theta}\right\}^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\left[1+\pi\left(s^t\right)\right]^{\varepsilon}}{\Delta\left(s^{t-1}\right)}},$$

where

$$\pi_H\left(s^t\right) = \frac{P_H\left(s^t\right)}{P_H\left(s^{t-1}\right)} - 1.$$

3.4 Equilibrium Condition for Financial Assets

From the first order necessary conditions with respect to holdings of Arrow securities,

$$\frac{u'\left[C^*\left(s^{t+1}\right)\right] q\left(s^t\right)}{u'\left[C^*\left(s^t\right)\right] q\left(s^{t+1}\right)} = \frac{u'\left[C\left(s^{t+1}\right)\right]}{u'\left[C\left(s^t\right)\right]},$$

where the real exchange rate $q\left(s^t\right)$ is defined by

$$q\left(s^t\right) = \frac{\mathcal{E}\left(s^t\right) P^*\left(s^t\right)}{P\left(s^t\right)}.$$

Under the assumption of symmetric preferences, the real exchange rate is always unity. As a result, the equilibrium relative consumption also equals unity with suitable initial wealth conditions under unit elasticity of substitution between domestic and foreign goods,[2] namely

$$C\left(s^t\right) = C^*\left(s^t\right). \tag{6}$$

This is the arbitrage condition for aggregate consumption indices.

3.5 Preference and Parameter

We assume that $U(\cdot)$, $U^*(\cdot)$, $V(\cdot)$ and $V^*(\cdot)$ are isoelastic functions as

$$U(X) = U^*(X) = \frac{X^{1-\sigma}}{1-\sigma},$$

[2] For the formal proof on this point, see proposition in Nakajima (2008).

and

$$V(X) = V^*(X) = \frac{X^{1+\omega}}{1+\omega},$$

where σ is the inverse of the intertemporal elasticity of substitution in consumption and ω is the Frisch elasticity of labor supply. In the following analysis, we assume $\sigma^{-1} = 0.2$, 1, and 1.5. For other parameters, we follow Woodford (2003) as in Table 1. They are on quarterly bases. Since we assume symmetric two countries, structural parameters take the same values in both countries.

3.6 The Linearized System of Equations

Using the equilibrium conditions above, we can derive the log-linearlized system of equations, which is used for simulations below[3] The aggregate supply conditions are given by the New Keynesian Phillips curves:

$$\pi_H\left(s^t\right) = \gamma_H x_H\left(s^t\right) + \gamma_{H,F}(1-n)\,x_F\left(s^t\right) + \beta\sum_{s^{t+1}}\mu\left(s^{t+1}\right)\pi_H\left(s^{t+1}\right), \tag{7}$$

for the domestic country, and

$$\pi_F^*\left(s^t\right) = \gamma_{H,F} n x_H\left(s^t\right) + \gamma_F x_F\left(s^t\right) + \beta\sum_{s^{t+1}}\mu\left(s^{t+1}\right)\pi_F^*\left(s^{t+1}\right), \tag{8}$$

for the foreign country, where

$$\gamma_H \equiv \frac{(1-\theta)(1-\beta\theta)\left[1+\omega+(\sigma-1)n\right]}{\theta(1+\omega\varepsilon)},$$

$$\gamma_F \equiv \frac{(1-\theta)(1-\beta\theta)\left[1+\omega+(\sigma-1)(1-n)\right]}{\theta(1+\omega\varepsilon)},$$

and

$$\gamma_{H,F} \equiv \frac{(1-\theta)(1-\beta\theta)(\sigma-1)}{\theta(1+\omega\varepsilon)}.$$

The aggregate demand conditions are given by the dynamic IS curves:

$$\begin{aligned} i_H\left(s^t\right) &= \sum_{s^{t+1}}\mu\left(s^{t+1}\right)\left\{\begin{array}{c}\left[1+(\sigma-1)n\right]x_H\left(s^{t+1}\right) \\ +(\sigma-1)(1-n)\,x_F\left(s^{t+1}\right)+\pi_H\left(s^{t+1}\right)\end{array}\right\} \\ &\quad -\left[1+(\sigma-1)n\right]x_H\left(s^t\right)-(\sigma-1)(1-n)\,x_F\left(s^t\right)+r_H^n\left(s^t\right), \end{aligned} \tag{9}$$

[3] For the details of derivations, see, for example, Clarida, Galí, and Gertler (2002), Benigno and Benigno (2003), and Nakajima (2008).

for the domestic country, and

$$i_F\left(s^t\right) = \sum_{s^{t+1}} \mu\left(s^{t+1}\right) \left\{ \begin{array}{c} \left[1+(\sigma-1)(1-n)\right] x_F\left(s^{t+1}\right) \\ +(\sigma-1) n x_H\left(s^{t+1}\right) + \pi_F^*\left(s^{t+1}\right) \end{array} \right\} \\ -\left[1+(\sigma-1)(1-n)\right] x_F\left(s^t\right) - (\sigma-1) n x_H\left(s^t\right) + r_F^n\left(s^t\right), \tag{10}$$

for the foreign country. The output gaps $x_H\left(s^t\right)$ and $x_F\left(s^t\right)$ are defined by the log deviation of outputs from their flexible price levels $Y_n\left(s^t\right)$ and $Y_n^*\left(s^t\right)$:

$$x_H\left(s^t\right) = \log\left[Y\left(s^t\right)\right] - \log\left[Y_n\left(s^t\right)\right],$$

and

$$x_F\left(s^t\right) = \log\left[Y^*\left(s^t\right)\right] - \log\left[Y_n^*\left(s^t\right)\right].$$

The natural rate of interest $r_H^n\left(s^t\right)$ and $r_F^n\left(s^t\right)$ are determined by $Z\left(s^t\right)$ and $Z^*\left(s^t\right)$.

There exist zero lower bounds of nominal interest rates:

$$i_H\left(s^t\right) \geq 0, \tag{11}$$

and

$$i_F\left(s^t\right) \geq 0. \tag{12}$$

The linearized system of equations consists of four equations: (7), (9), (8), and (10), two non-negativity constraints on the nominal interest rate: (11) and (12), and two optimal monetary policies derived by minimizing the world loss function, for six endogenous variables: $x_H\left(s^t\right)$, $x_F\left(s^t\right)$, $\pi_H\left(s^t\right)$, $\pi_F^*\left(s^t\right)$, $i_H\left(s^t\right)$ and $i_F\left(s^t\right)$.

3.7 Welfare Criteria

Following Woodford (2003), we approximate consumers' welfare in the second order.[4] Central banks cooperate to minimize the world loss L, derived by the second order approximation of the world welfare:

$$n \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) \left\{u\left[C\left(s^t\right)\right] - v\left[h\left(s^t\right)\right]\right\} \\ +(1-n) \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) \left\{u\left[C^*\left(s^t\right)\right] - v\left[h^*\left(s^t\right)\right]\right\} \simeq -\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) L\left(s^t\right). \tag{13}$$

[4] For the derivation of the welfare loss function under policy cooperation, see Clarida, Galí, and Gertler (2002) and Benigno and Benigno (2003).

where

$$
\begin{aligned}
L\left(s^{t}\right) &= \frac{\gamma_H n}{\varepsilon} x_H\left(s^{t}\right)^2 + \frac{2\gamma_{H,F} n\left(1-n\right)}{\varepsilon} x_H\left(s^{t}\right) x_F\left(s^{t}\right) + \frac{\gamma_F\left(1-n\right)}{\varepsilon} x_F\left(s^{t}\right)^2 \\
&\quad + n\pi_H\left(s^{t}\right)^2 + \left(1-n\right)\pi_F^{*}\left(s^{t}\right)^2 .
\end{aligned}
$$

4 Optimal Monetary Policy Cooperation

This section describes the optimal policy cooperation, in response to the adverse shocks that make two countries hit by the zero lower bounds simultaneously. We first discuss the role of the intertemporal elasticity of substitution on international interdependence via model structure and monetary policy. Then, we show optimal responses under global liquidity trap via simulation.

4.1 The Role of σ for International Interdependence

The intertemporal elasticity of substitution, namely σ^{-1}, is the key parameter for equilibrium dynamics in our model. In addition to the standard roles in the intertemporal as well as intratemporal mechanisms in Euler equations, in the two country model with the zero lower bounds, σ plays important roles in determining the degree of interdependence through the model structure and through the optimal targeting rule.

4.1.1 Interdependence through Model Structure

In a two country model, equilibrium allocations and prices in one country are affected by those in the other country. Especially, marginal cost of domestic production, and intertemporal substitution of domestic consumption index, are tied to the foreign variables through the terms of trade defined by $\mathcal{E}\left(s^{t}\right) P_F^{*}\left(s^{t}\right)/P_H\left(s^{t}\right)$, which becomes $P_F\left(s^{t}\right)/P_H\left(s^{t}\right)$ under producer currency pricing. How much the two countries interdepend on each other is determined by the parameter σ. When σ takes unity, our model becomes as if it were a model with two closed economies although there exist trades of goods and financial assets.[5]

[5] As shown by Benigno and Benigno (2003), when σ equals to the intertemporal elasticity of substitution between domestically produced goods and foreign produced goods, stabilization problem for two countries

Time paths of nominal interest rates become quite different depending on σ.[6]

Let us check on this point by deriving expression for real marginal cost with two equations. The first equations is the optimality condition for leisure-consumption decision:

$$\begin{aligned} v'\left[h\left(s^t\right)\right] &= u'\left[C\left(s^t\right)\right]\frac{W\left(s^t\right)}{P\left(s^t\right)} \\ &= C\left(s^t\right)^{-\sigma}\frac{MC\left(s^t\right)Z\left(s^t\right)}{1-\tau\left(s^t\right)}\left[\frac{P_H\left(s^t\right)}{P_F\left(s^t\right)}\right]^{1-n}. \end{aligned} \tag{14}$$

We use equation (2) and the definition of the consumer price index derived as the Hicksian demand function from equation (3). At the same time, domestic consumption is related to the domestic output through the terms of trade. By using equations (4), (5) and (6) together with the Marshallian demand functions derived from equation (3), we can derive the second equation:

$$C\left(s^t\right) = \left[\frac{P_H\left(s^t\right)}{P_F\left(s^t\right)}\right]^{1-n}\Delta\left(s^t\right)Z\left(s^t\right)h\left(s^t\right). \tag{15}$$

On the effect of the terms of trade on the domestic activities, equation (14) shows that given domestic consumption and leisure, an improvement of the terms of trade lowers marginal cost. On the other hand, equation (15) indicates that given domestic output, an improvement of the terms of trade raises domestic consumption. In equation (14), higher consumption increases the marginal rate of substitution between consumption and leisure. Consequently, marginal cost becomes higher.[7] The two effects cancel out when $\sigma = 1$, and marginal costs of two countries are not affected from each other. For $\sigma > (<)\, 1$, an improvement in the terms of trade decreases (increases) marginal cost. This is related to the argument by Tille (2001) that depending on the elasticity of substitution, monetary policy becomes 'begger-thy-neighbor' or 'begger-thyself.' Under the liquidity trap, increase

is reduced to that for two separate countries. In the current paper, we follow the proposition in Nakajima (2008) and assume the intertemporal elasticity is unity, to avoid unrealistic assumption about initial wealth. Here, $\sigma = 1$ becomes the only relevant condition for the claim of Benigno and Benigno (2003) holds. With $\sigma = 1$, there exists no spillover effects between countries, and therefore no gain from policy cooperation.

[6] For details on this point, see Tille (2001).

[7] Clarida, Galí, and Gertler (2002) term the first effect as the terms of trade effect and the latter as the risk sharing effects.

in the marginal cost can lessen the severeness of the zero lower bound to achieve higher inflation expectation. Monetary policy in cooperation aims at increase the marginal cost and therefore output in the country still caught by the lower bound though the direction of policy rate change depends on σ.

4.1.2 Interdependence through Monetary Policy

Optimal monetary policy determines the sequence of nominal interest rates $\left\{i_H\left(s^t\right), i_F\left(s^t\right)\right\}_{t=1}^{\overline{S}}$ by minimizing the social loss (13) subject to equations (7), (9), (8), (10), (11) and (12). Under commitment, central banks commit to the future paths of the nominal interest rates considering the effects of their action on the private agent's expectations. On the other hand, under discretion, central banks take expected variables as given.

If the zero bound is not the binding constraint, namely central banks optimize without subjecting to equations (11), and (12), the equilibrium conditions for monetary policy, that is the targeting rules, under commitment, are given by

$$\pi_H\left(s^t\right) + \frac{1}{\varepsilon}\left[x_H\left(s^t\right) - x_H\left(s^{t-1}\right)\right] = 0, \tag{16}$$

and

$$\pi_F^*\left(s^t\right) + \frac{1}{\varepsilon}\left[x_F\left(s^t\right) - x_F\left(s^{t-1}\right)\right] = 0. \tag{17}$$

Similarly, under discretion, we can obtain

$$\pi_H\left(s^t\right) + \frac{1}{\varepsilon}x_H\left(s^t\right) = 0, \tag{18}$$

and

$$\pi_F^*\left(s^t\right) + \frac{1}{\varepsilon}x_F\left(s^t\right) = 0. \tag{19}$$

Under the producer currency pricing as assumed in this paper,[8] monetary policy becomes inward looking. Similarly to the result pointed out in Nakajima (2008), the targeting rules in our two-country model are collapsed to those obtained for the closed economy, when

[8] As shown by Devereux and Engel (2003), under the local currency pricing, the optimal monetary policy aims at stabilizing nominal exchange rates and therefore takes foreign variables into consideration. To examine the optimal monetary policy under both liquidity trap and the local currency pricing is left for our future research.

zero bound constraints are not binding. Here, in response to the two adverse shocks in the two countries, the world welfare is maximized if each central bank follows a feedback rule responding to variables in her country. Note that this inward-lookingness of optimal monetary policy does not hinge on the size of σ.

This is not the case when nominal interest rates can be bounded by zero. Even only with technology shocks and the producer currency pricing, foreign variables must be included in the domestic targeting rule. The degree of influences from foreign variables are determined again by σ. Denoting the Lagrangian multipliers associated with inequalities (11) and (12) by $\phi_H\left(s^t\right)$ and $\phi_F\left(s^t\right)$, first order condition under commitment policy policy yields the following targeting rules:[9]

$$\begin{aligned} & \gamma_H n_H \varepsilon^{-1} x_H(s^t) + \gamma_{HF} n_H n_F \varepsilon^{-1} x_F(s^t) - \gamma_H n_H (1-L)^{-1}\left[\beta^{-1}\phi_H(s^{t-1}) - \pi_H(s^t)\right] \\ & -n_H n_F (1-L)^{-1}\left(\beta^{-1}\phi_F(s^{t-1}) - \pi_F^*(s^t)\right) + n_H\left[1+(\sigma-1)n_H\right]\phi_H(s^t) \\ & -\beta^{-1} n_H\left[1+(\sigma-1)n_H\right]\phi_H(s^{t-1}) + n_H n_F(\sigma-1)\phi_F(s^t) - \beta^{-1} n_H n_F(\sigma-1)\phi_F(s^{t-1}) \\ = \; & 0, \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \gamma_F n_F \varepsilon^{-1} x_F(s^t) + \gamma_{HF} n_H n_F \varepsilon^{-1} x_H(s^t) - \gamma_F n_F (1-L)^{-1}\left[\beta^{-1}\phi_F(s^{t-1}) - \pi_F^*(s^t)\right] \\ & -n_H n_F (1-L)^{-1}\left(\beta^{-1}\phi_H(s^{t-1}) - \pi_H(s^t)\right) + n_F\left[1+(\sigma-1)n_F\right]\phi_F(s^t) \\ & -\beta^{-1} n_F\left[1+(\sigma-1)n_F\right]\phi_F(s^{t-1}) + n_H n_F(\sigma-1)\phi_H(s^t) - \beta^{-1} n_H n_F(\sigma-1)\phi_H(s^{t-1}) \\ = \; & 0, \end{aligned} \tag{21}$$

where L is a lag operator. Similarly, under discretionary policy, the targeting rules take the forms of

$$\begin{aligned} & \varepsilon^{-1} x_H\left(s^t\right)\gamma_H + \varepsilon^{-1} x_F\left(s^t\right)\gamma_{HF} n_F + \pi_H\left(s^t\right)\gamma_H \\ & +\pi_F^*\left(s^t\right) n_F + \phi_H\left(s^t\right)\left[1+(\sigma-1)n_H\right] + \phi_F\left(s^t\right) n_F(\sigma-1) \\ = \; & 0, \end{aligned} \tag{22}$$

[9] See details in Appendix.

and

$$
\begin{aligned}
& \varepsilon^{-1} x_F \left(s^t\right) \gamma_F + \varepsilon^{-1} x_H \left(s^t\right) \gamma_{HF} n_H \\
& + \pi_F^* \left(s^t\right) \gamma_F + \pi_H \left(s^t\right) n_H + \phi_F \left(s^t\right) \left[1 + (\sigma - 1) n_F\right] + \phi_H \left(s^t\right) n_H (\sigma - 1) \\
= \; & 0. \qquad (23)
\end{aligned}
$$

Only when both $\phi_H\left(s^t\right)$ and $\phi_F\left(s^t\right)$ are zeros, these couples of two rules given by equations (16) and (17) or equations (18) and (19) can be transformed to those given by equations (20) and (21) or equations (22) and (23). Therefore, unless both $\phi_H\left(s^t\right)$ and $\phi_F\left(s^t\right)$ are zero, namely the zero bounds are not binding constraints, the targeting rules contain variables in the counterpart country and the degree of interdependence through policy is affected by σ. With the possibility that the economy caught by the zero lower bound, the dynamic IS curves become constraints for optimal monetary policy. As a result, unless $\sigma = 1$, the optimality of inward looking policy is not supported. In a liquidity trap, policy makers cannot achieve economic efficiency by 'keeping one's house in order.'[10]

4.2 Simulation

In simulations below, economy is at the steady state level at $t = 0$. Then, unexpected declines of natural rates of interest occur in both countries at period $t = 1$. The natural rates of interest $r_H^n\left(s^t\right)$ and $r_F^n\left(s^t\right)$ are decreased from the steady state value $\overline{r} = \frac{1-\beta}{\beta}$ to a negative value $\underline{r} = -0.02/4$,[11] and remains at that level for a while. These adverse shocks are temporary, and $r_H^n\left(s^t\right)$ and $r_F^n\left(s^t\right)$ revert back to normal levels at period T_H and T_F respectively.[12] Following Jung, Teranishi, and Watanabe (2005), we assume that both private-agents and central banks completely foresee the sequence of natural interest rates $\left\{r_H^n\left(s^t\right), r_F^n\left(s^t\right)\right\}_{t=1}^{\overline{S}}$ at period $t = 1$. Central banks under cooperation set their nominal interest rates to minimize the welfare loss associated with these adverse shocks,

[10] Due to this, we guess that the gains from cooperation should be higher under liquidity trap. To test this conjecture is left for our future research.

[11] For the discussion on the negative natural rate of interest, see Adam and Billi (2006).

[12] Following Eggertson and Woodford (2003a, b), we assume that natural rates of interest return to normal within finite periods $\overline{S}$.

and private-agents supply labor and consume, given the set of nominal interest rates and equilibrium prices.

We show the time paths of nominal interest rates, inflation rates, output gaps in the two countries, and the terms of trade. Interest rates and Inflation rates are in annual basis, and output gaps are in quarterly basis. The details of numerical algorithm employed in this paper is shown in Appendix.

4.2.1 Commitment

We first show the paths under commitment. We examine two cases. In the first simulation, adverse shocks in the two countries die out simultaneously. Therefore, $T_H = T_F = 10$. In the second simulation, a shock to natural rate of interest lasts longer in the domestic than in the foreign country. Hence, $T_H = 15$ and $T_F = 10$. Thus, the terms of trade is invariant in the former, while it improves in the latter reflecting the difference in technological progresses between two countries.

Figure 2 display the response of economy to the symmetric adverse shocks. First, both countries are caught by liquidity trap. They terminate zero interest rate policy at one or two quarters after the natural rate of interest recovers to $\overline{r}$. As emphasized in former studies such as Eggertson and Woodford (2003a, b) and Jung, Teranishi, and Watanabe (2005), central banks should aim at rising inflation expectations by committing to low nominal interest rates even for the periods after adverse shocks disappear. As the terms of trade is invariant in this simulation thanks to the symmetry in shocks, differences in time paths of nominal interest rates by σ simply reflect the effect of σ stemming solely from intertemporal as well as intratemporal mechanisms in Euler equations. Hence, they have nothing to do with interdependence.

Figure 3 illustrates the time paths under asymmetric shocks, where natural rate of interest in foreign country returns to its steady state level earlier than in domestic country. Improvements in the terms of trade for the domestic country imply that interdependence between countries matters unless $\sigma = 1$. Reflecting the differences in shocks, the foreign nominal interest rate is raised first. Again, similarly to the symmetric simulations, history dependence is observed. The dates when terminating the zero interest rate policy are later than T_H and T_F respectively. Zero nominal interest rate is maintained even after adverse shocks have gone. Right after foreign adverse shock dies out at $t = 15$, nominal interest rates in the foreign country becomes lower (higher) than the steady state natural rate of interest rate when $\sigma^{-1} = .2$ (1.5). This reflects the interdependence through model structure via equations (14) and (15) and monetary policy as discussed above.

4.2.2 Discretion

Next, we compute responses under discretionary policy. Similarly to the exercises above, we examine cases with both symmetric and asymmetric shocks. Since there is no endogenous state variables in the standard NOEM model considered in this paper, nominal interest rates are set to zero only when adverse shocks are hitting the economy.

Figure 4 shows the time paths of variables in response to the symmetric shocks. The nominal interest rate is set to zero while the adverse shocks hit the economy and returns to $\overline{r}$ immediately after shock disappears. Although movements of nominal interest rates are unaffected by the size of σ, those of inflation output gaps vary with σ. This simply implies that through intertemporal as well as intratemporal mechanisms in Euler equations, policy effects differs with different σ. Naturally, the recession becomes more severe under discretion than under commitment.

Figure 5 shows the time paths of variables in response to asymmetric shocks. While domestic nominal interest rate shows the same path as under symmetric shock and therefore is determined independently from σ, foreign nominal interest is set higher (lower) than $\overline{r}$ for $\sigma^{-1} = .2$ (1.5). Whether σ is larger or smaller than unity determines the sign of domestic monetary policy on foreign marginal cost. To ease the severity under high real interest rate due to the zero lower bound on nominal interest rates, a country which escapes from liquidity trap earlier should adjust its policy rule depending on the size of σ.

4.3 Discussion: Commitment vs Discretion

As shown in Table 2 which demonstrates the relative size of welfare loss under commitment over that under discretionary policy when $T_H = 15$ and $T_H = 10$, since higher social welfare can be achieved by committing such a history dependent policy, the central banks under cooperation should conduct the commitment monetary policy, implying the lower interest rates for the future under global liquidity trap. This is a clear and strong implication for the real monetary policies.

In real economy, however, making credible international coordination is considered a very difficult task. Different central bank has different aim and situation. Actually, as in the introduction, the FRB and BOC intend to commit the future monetary policy by the words of "*for an extended period*" and "*until the end of the second quarter of 2010*", respectively, but BOJ and BOE do not declare the future monetary policy in the zero interest rate policies. Moreover, since agents across the globe must completely understand the statements made by foreign central banks regardless of whether they are written in their own language or not, imperfect credibility seems the most appropriate assumption in the global economy.

The central banks must understand the paths under the optimal monetary policy cooperation with discretion as well as commitment, since they are quite different in global liquidity trap.

5 Sensitivity Analysis: Stochastic Simulations

In this section, we examine the robustness of the findings under the stochastic environment. Here, we follow Eggertsson and Woodford (2003a, b). The timing when each adverse shock dies out is now uncertain. Once the natural rates of interest in the two countries become negative at $\underline{r}$ at $t = 1$, each natural rate of interest remains $\underline{r}$ with constant probability p, and returns to its steady state value $\overline{r}$ with constant probability $1 - p$. If the latter is realized at the subsequent period, the natural rate of interest in that country takes $\overline{r}$ after all. If not, private agents continues to face the uncertainty of the same probability as before.

In contrast to the environments where sequence of natural rates of interest are known at period $t = 1$, private-agents and central banks in the stochastic environment forecast the upcoming states of period $t + 1$, conditional on the information set available at period t. Suppose that both adverse shocks are present at period t, there are four possible states of an economy at period $t + 1$, depending on the realization of natural rates of interest. In what follows, we set $p = .25$, following Eggertsson and Woodford (2003a, b). To keep the computation from being complicated, we further assume that both T_H and T_F are smaller than an integer $\overline{S}$.

The time paths under the commitment policy are given in Figure 6. In stochastic environment, both T_F and T_H are randomly chosen integers that fall between 2 and $\overline{S}$. There are $\left(\overline{S} - 1\right)^2$ possible states of an economy, that differ from themselves, by the realization of T_F and T_H. Among the states, we choose a state where $T_F = 10$ and $T_H = 15$.

5.1 Commitment

Figure 6 illustrates that even under stochastic shocks, the optimal commitment policy is characterized by history dependence. That is, central bank keeps each nominal interest rate low enough for several quarters even after its own shock dies out. We can again conclude that the optimal monetary policy under global liquidity trap is very much characterized by the history dependence and the interdependence through σ.

5.2 Discretion

Figure 7 shows the time paths of variables under discretionary policy. The property of optimal discretionary policy does not alter very much even under stochastic simulation. Optimal discretionary monetary policy is characterized by the policy interdependence through σ. The policy interest rate in a country getting out of the zero lower bound earlier becomes higher or lower than the steady state level of the natural rate of interest rate to increase the marginal cost in the counterpart country depending on σ.

6 Concluding Remarks

In this paper, using a standard two-country New Open Economy Macroeconomics model, we show responses for the natural rate shocks under optimal monetary policy cooperation with both commitment and discretion, when the two countries simultaneously fall into the liquidity trap. Under optimal commitment policy, which naturally attains the highest social welfare, central bank keeps its nominal interest rate at very low level, for several quarters even after the adverse shock dies out. Therefore the optimality of history dependent policy, the effectiveness of which in closed economy is stressed by such previous studies as Reifschneider and Williams (2000), Eggertsson and Woodford (2003a, b), Jung, Teranishi, and Watanabe (2005), Sugo and Teranishi (2005), Kato and Nishiyama (2005), Adam and Billi (2006, 2007), and Nakov (2008), is still maintained under global liquidity trap. At the same time, however, the optimal paths of nominal interest rates become substantially complicated due to the degree of interdependence between two countries characterized by the intertemporal elasticity of substitution. For example, in contrast to previous studies in NOEM, under liquidity trap, efficiency cannot be achieved by inward-looking policy even when the producer currency pricing is assumed.

Although higher social welfare can be achieved under commitment, making credible commitment to future policy is considered a difficult task in open economies. Central banks need to try best efforts in informing the contents of commitments to the citizens not only in their home country but also in foreign countries. At the same time, agents across the globe must completely understand the statements made by foreign central banks regardless of whether they are written in their own language or not. Thus, as imperfect credibility seems the most appropriate assumption in the open economies, central banks must understand the paths under the optimal monetary policy cooperation with discretion as well as commitment, since they are quite different under global liquidity trap.

In the current paper, we restrict our attention to an economy with two identical countries, where only a nature of adverse shock that hits the country is different. Optimal monetary policy cooperation surely change, as this assumption is relaxed. For example, we can calibrate the relative weight of two countries n, to the Japan-U.S., or Canada-U.S. relationships. Furthermore, we should check how different the optimal paths of nominal interest rate between under cooperation and non-cooperation. These extensions are left for future research.

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Table 1: Parameter values

Parameters	Values	Explanation
β	0.99	Subjective discount factor
σ^{-1}	0.2, 1, 1.5	Intertemporal elasticity of substitution
κ	0.024	Elasticity of inflation with respect to output
θ	0.66	Probability of price change
ρ	1	Elasticity of substitution between domestic and foreign goods
ε	7.88	Elasticity of substitution among differentiated goods
ω	0.47	Frish elasticity
n	0.5	Country size

Table 2: Welfare comparison

	$\sigma^{-1} = 0.2$	$\sigma^{-1} = 1$	$\sigma^{-1} = 1.5$
$T_F = 10$ and $T_H = 15$	0.0092	0.0074	0.0067

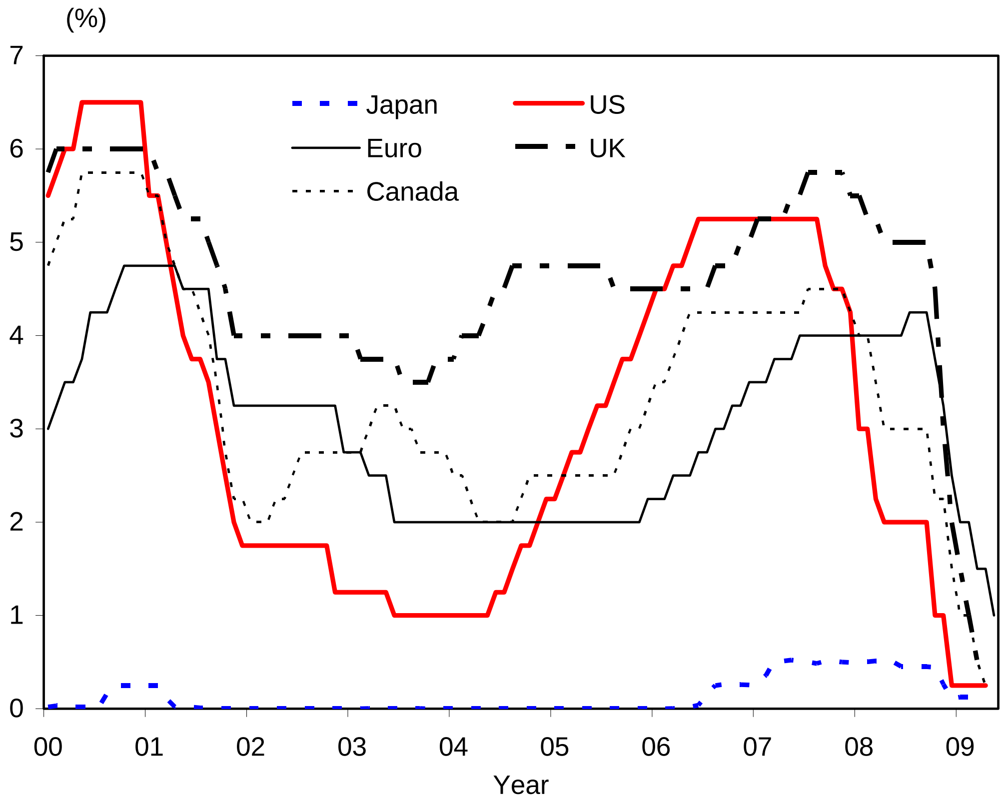

Figure 1: Policy rates for developed countries.

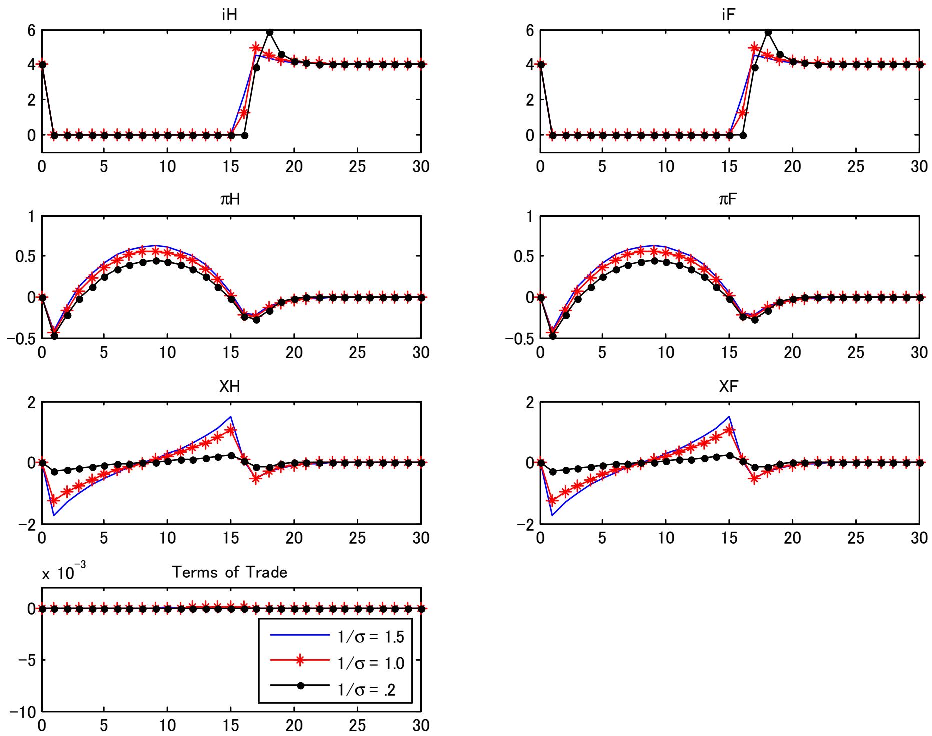

Figure 2: Case for $T_H = T_F = 15$ under the commitment policy with deterministic shock.

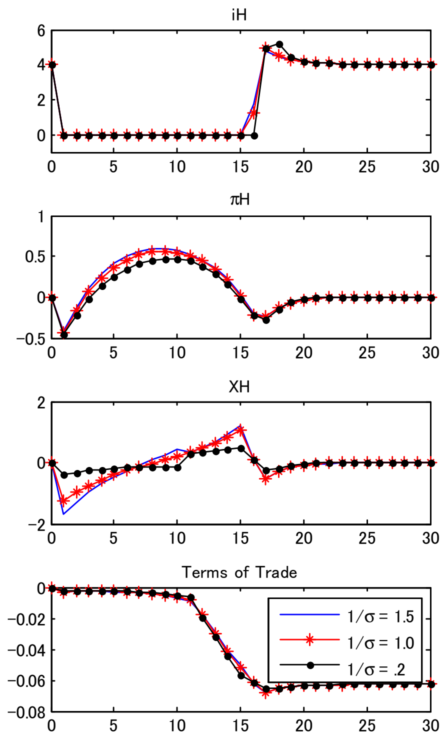

Figure 3: Case for $T_F = 10$ and $T_H = 15$ under the commitment policy with deterministic shock.

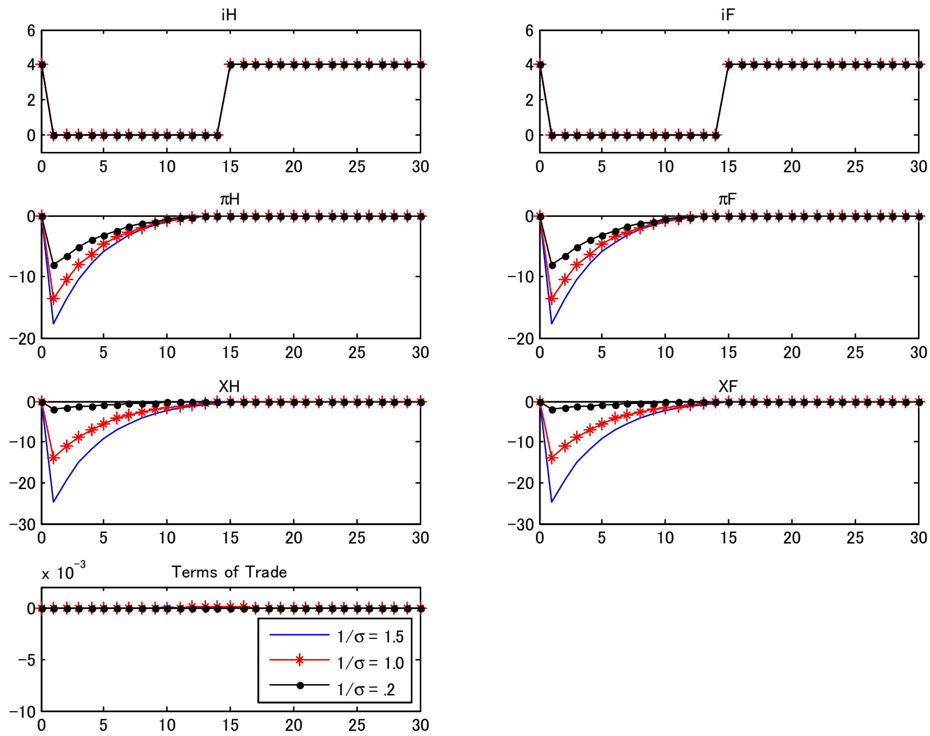

Figure 4: Case for $T_H = T_F = 15$ under the discretionary policy with deterministic shock.

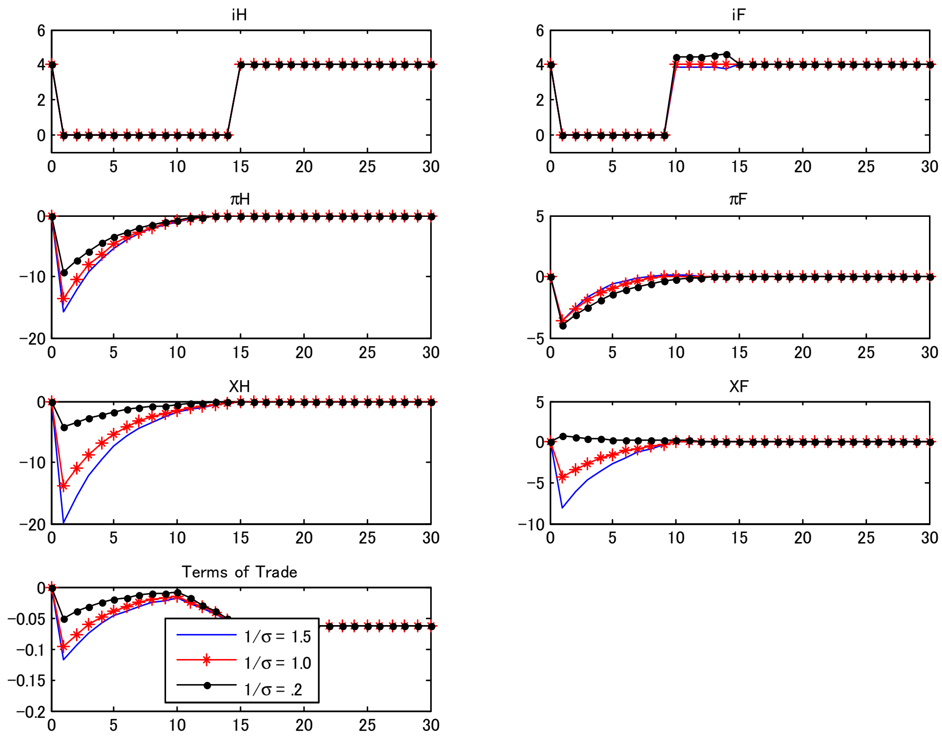

Figure 5: Case for $T_H = 15$ and $T_F = 10$ under the discretionary policy with deterministic shock.

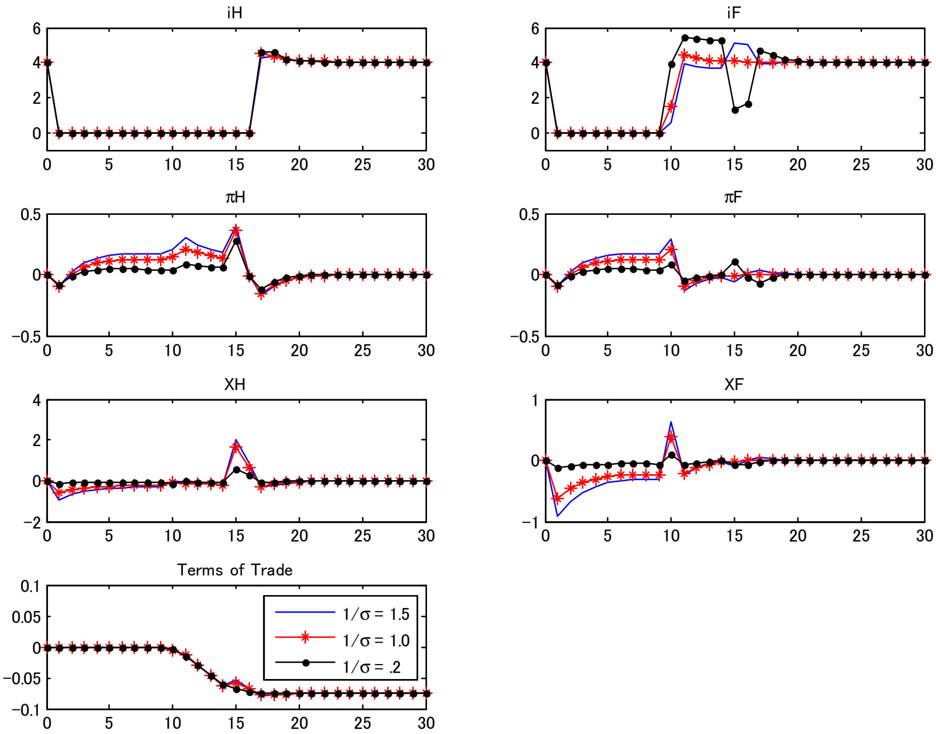

Figure 6: Case for $T_H = 15$ and $T_F = 10$ under the commitment policy with stochastic shock.

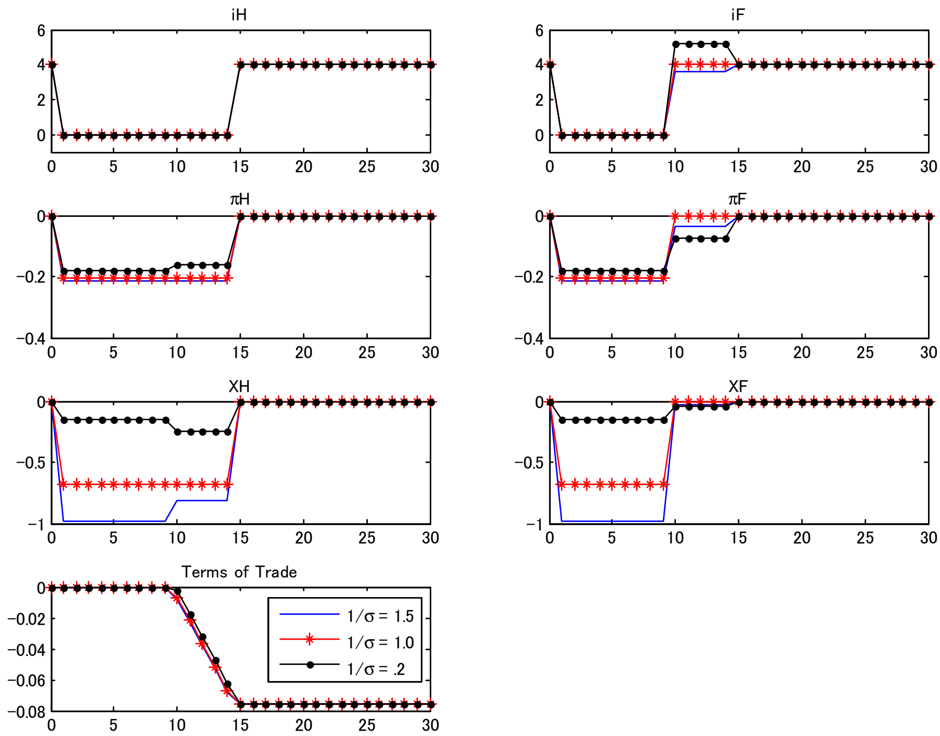

Figure 7: Case for $T_H = 15$ and $T_F = 10$ under the discretionary policy with stochastic shock.

A Numerical Algorithm for Simulation

A.1 Solution Method for Commitment: Deterministic Case

Our solution method for deterministic simulation is closely related to those employed in Eggertson and Woodford (2003) and Nakajima (2008). We assume that the natural rates of interest for the two countries r_H^n and r_F^n become unexpectedly negative for period $t = 1$, and revert back to the steady state in T_F and T_F, respectively. Without loss of generality, we consider that both T_H and T_F are smaller than a integer S.

Optimal commitment solution is given as the sequence of nominal interest rates $\{i_t, i_t^*\}_{t=0}^{S}$ that solves (13), subject to four equations given by equations (7), (8), (9), and (10), with two non-negativity constraints on the nominal interest rate by equations (11) and (12).

Let $k_H(T_H, T_F)$ and $k_F(T_F, T_F)$ be integers that fall in the range between $-\max\{T_H, T_F\}+1$ and S. Suppose $T_H + k_H(T_H, T_F)$ be a period when i_t stops taking zero, and $T_F + k_F(T_F, T_F)$ be the period when i_t^* starts taking positive value, we choose $k_H(T_H, T_F)$ and $k_F(T_H, T_F)$ that are consistent with the equilibrium conditions under the optimal commitment for given T_H and T_F. We first presume a set $\{k_H,\ k_F\}$ and calculate the sequence $\{\pi_{H,t},\ \pi_{F,t},\ x_{H,t},\ \ x_{F,t},\ i_{H,t},\ i_{F,t}^*,\ \psi_{H,t},\ \psi_{F,t},\ \phi_{H,t},\ \phi_{F,t}\}_{t=0}^{S}$ using the FOCs of the commitment solution. $\psi_{H,t}$, $\psi_{F,t}$, $\phi_{H,t}$, $\phi_{F,t}$ are Lagrangian multipliers associated with equations (7) and (8) and inequalities (11) and (12), respectively, in the optimization problem. We then check if the time path are consistent from each other, and try another sets for $\{k_H,\ k_F\}$, if otherwise holds.

Structure of the equilibrium conditions at time t is tied to where the economy locates over the time path. That is, the system of the economy at period t depends on whether non-negativity constraints on the nominal interest rates are biting for the two countries. We provide below the equilibrium conditions for an economy where both of the nominal interest rates are positive (phase (i)), for an economy where the nominal interest rate in one of the countries is set to zero (phase (ii)), and for an economy where the nominal interest rate in both countries are set to zero (phase (iii)). Description is limited to the states where natural rate of foreign recovers the first and that of home recovers the last,

$T_F < T_H$, both k_H and k_F are positive, and k_H is sufficiently small, $T_F + k_F, < T_H$. The equilibrium conditions that hold in other states are easily obtained from the extensions.

When the economy is in phase (i), zero bound cease to bite for both countries, and natural rates of the interest, and the nominal interest rates are positive. It implies that t is greater than or equal to $T_H + k_H$. We have

$$\left[\begin{array}{l} N\pi_t - \beta N\pi_{t+1} - \Lambda x_{t+1} = 0 \\ \Lambda x_t - \theta\Lambda\psi_t + A\left(\phi_t - \beta^{-1}\phi_{t-1}\right) = 0 \\ \pi_t + \psi_t - \psi_{t-1} - \beta^{-1}\phi_{t-1} = 0 \\ \phi_t = \phi_{t+1} = 0 \end{array}\right], \tag{24}$$

where $\pi_t \equiv (\pi_{H,t}, \pi_{F,t})'$, $x_t \equiv (x_{H,t}, x_{F,t})'$, $\psi_t \equiv \left(\psi_{H,t}, \psi_{F,t}\right)'$, $\phi_t \equiv \left(\phi_{H,t}, \phi_{F,t}\right)'$ and matrix N, Λ and A are defined by

$$N \equiv \left[\begin{array}{ll} n_H & 0 \\ 0 & n_F \end{array}\right],$$

$$\Lambda \equiv \frac{1}{\theta}\left[\begin{array}{ll} \gamma_H n_H & \gamma_{HF} n_H n_F \\ \gamma_{HF} n_H n_F & \gamma_F n_F \end{array}\right],$$

and

$$A \equiv \left[\begin{array}{ll} n_H\left[1 + (\sigma - 1)\, n_H\right] & (\sigma - 1)\, n_F n_H \\ (\sigma - 1)\, n_F n_H & n_F\left[1 + (\sigma - 1)\, n_F\right] \end{array}\right].$$

Under our premise described above, phase (ii) holds for the sub-periods where $T_F + k_F, T_F \leq t < T_H + k_H$. During phase (ii), nominal interest rate of foreign country is positive while that of home country stays negative. Phase (iii) lasts while $T_F + k_F > t$ holds. In phase (iii), both countries are subject to the zero bound. Equilibrium conditions for phase (ii) and phase (iii) are given by (24) with the last two equations replaced by the following equations respectively. That is

$$\left[\begin{array}{l} \left[\begin{array}{ll} 1 & 0 \end{array}\right]\left[A x_t - A x_{t+1} - N\pi_{t+1} - r_t^n\right] = 0 \\ \phi_{F,t+1} = \phi_{F,t} = 0 \end{array}\right], \tag{25}$$

for phase (ii) and

$$A x_t - A x_{t+1} - N\pi_{t+1} - r_t^n = 0, \tag{26}$$

for phase (iii), where $r_t^n \equiv \left(r_{H,t}^n, r_{F,t}^n\right)'$.

A.2 Solution Method for Discretion: Deterministic Case

Under discretionary policy, the two central banks solve the welfare function, taking the expected values for output gap and inflation as given. Equilibrium conditions are altered since future values are no longer relevant for the optimal time path of the nominal interest rates.

Optimal discretionary solution is given as the sequence of nominal interest rates $\{i_t, i_t^*\}_{t=0}^{S}$ that minimizes (13) subject to four equations given by equations (7), (8), (9), (10), (11), and (12). Forward variables $x_{t+1,H}$, $x_{t+1,F}$, $\pi_{t+1,H}$ and $\pi_{t+1,F}$ are taken given for the central banks.

Assuming the same conditions as to the ordering of T_H, $T_H + k_H$ and T_F, the sign for k_H, k_F, the system of phase (i) is obtained as

$$\begin{bmatrix} N\pi_t - \beta N\pi_{t+1} - \Lambda x_{t+1} = 0 \\ \Lambda x_t - \theta\Lambda\psi_t + A\phi_t = 0 \\ \pi_t + \psi_t = 0 \\ \phi_t = \phi_{t+1} = 0 \end{bmatrix}. \tag{27}$$

Notice that lag terms no longer appear other than the first two equations of the system. For the phase (ii) and phase (iii), the equilibrium condition are the same as (27) except that the two equations in the last two rows are replaced by (25) and (26).

A.3 Solution Method for Stochastic Cases

In the stochastic economy, the private-agents and the two central banks are uncertain as to the timing when the natural interest rates recover to positive. Similarly to Eggertsson and Woodford (2003), we assume that the natural rates of interest for the two countries r_H^n and r_F^n become unexpectedly negative for period $t = 0$, and revert back to the steady state value with probability q_H and q_F in every period. During the periods for $t \geq \max\{T_H, T_F\}$, agents in the economy face no uncertainty as both two natural rates have already returned to their normal values. For periods where $t < \min\{T_H, T_F\}$, agents expect the reverting of natural interest rate for home country with probability q_H,the reverting of natural interest rate for foreign country with probability q_F, and the reverting of both natural interest rates with probability $q_F q_H$. For the rest of the time path, uncertainty associated with either one of the natural rates of interest is present.

Optimal commitment solution is given as the sequence of nominal interest rates $\{i_t, i_t^*\}_{t=0}^S$ that solves (13), subject to these six equations (7), (8), (9), (10), (11), and (12). All of the future variables appearing in the equations are replaced with expected values conditional on the information set at period t. Optimal discretionary solution is given as those that solves (13), subject to the same six equations, taking these expected values as given. We describe below the set of equilibrium conditions for optimal commitment solutions in stochastic environment.

For phase (i), there is no uncertainties associated with the natural rates of interest in the economy. The equilibrium conditions is then given by (24). Supposing that the realization of the stochastic process of r_H^n and r_F^n are such that $T_F < T_H$, equilibrium condition for phase (ii) is written as

$$\begin{bmatrix} N\pi_t - \beta N \mathrm{E}_t \pi_{t+1} - \Lambda \mathrm{E}_t x_{t+1} = 0 \\ \Lambda x_t - \theta \Lambda \psi_t + A\left(\phi_t - \beta^{-1}\phi_{t-1}\right) = 0 \\ \pi_t + \psi_t - \psi_{t-1} - \beta^{-1}\phi_{t-1} = 0 \\ \begin{bmatrix} 1 & 0 \end{bmatrix} [Ax_t - Ax_{t+1} - N\pi_{t+1} - r_t^n] = 0 \\ \phi_{F,t+1} = \phi_{F,t} = 0 \end{bmatrix},$$

where E_t is the expectation operator conditional on the information set available at period t. For $t < T_F$, agents foresee the three possible outcomes as to the states in period $t+1$. For $T_H \leq t < T_F$, agents foresee the two possible outcomes as to the states in period $t+1$. For phase (ii), equilibrium condition is given by

$$\begin{bmatrix} N\pi_t - \beta N \mathrm{E}_t \pi_{t+1} - \Lambda \mathrm{E}_t x_{t+1} = 0 \\ \Lambda x_t - \theta \Lambda \psi_t + A\left(\phi_t - \beta^{-1}\phi_{t-1}\right) = 0 \\ \pi_t + \psi_t - \psi_{t-1} - \beta^{-1}\phi_{t-1} = 0 \\ Ax_t - A\mathrm{E}_t x_{t+1} - N\mathrm{E}_t \pi_{t+1} - r_t = 0 \end{bmatrix}.$$

The same arguments hold to the expected value for future variables.