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A Class of Acceptable and Practical Social Welfare Functions with Variable Populations: A Stepwise Rank-Dependent Utilitarianism and Its Application

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A Class of Acceptable and Practical Social Welfare Functions with Variable Populations: A Stepwise Rank-Dependent Utilitarianism and Its Application

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Abstract

This study proposes a new class of social welfare orderings, *stepwise rank-dependent utilitarianisms*, which generalize rank-dependent utilitarianisms in the setting of social choice with variable population size. It is shown that the stepwise rank-dependent utilitarianism is equivalent to a social welfare ordering that satisfies the desirable axioms: strong Pareto, anonymity, Pigou-Dalton transfer equity, continuity, rank-separability, cardinal full comparability, and consistency for population replication. This social welfare ordering is a kind of rank-dependent utilitarianisms designed to have the same weight for each proportion of the population, with the obvious advantage that allows weights to be freely chosen for assessing well-being inequality. As a practical application of the stepwise rank-dependent utilitarianism,

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we propose a *k-quantile mean comparison method*, which evaluates social welfare by comparing each quantile's average income in an approximate lexicographic manner. The method has a methodological advantage in that it makes flexible consideration on distributive justice possible compared with the traditional GDP per capita. Furthermore, we show a representation theorem of a social welfare function that generalizes the stepwise rank-dependent utilitarianism for the problem of optimal population size. In addition, the paper reexamines and reformulates several impossibility theorems with a variable population and shows that they are not so serious in the context of cardinal full comparability. Combining the previous results with our findings, theoretical correspondence between interpersonal comparability of individual well-being and acceptable social welfare orderings would be clarified. If cardinal partial comparability were admissible, stepwise rank-dependent Kolm-Pollack or Atkinson-Blackorby-Donaldson social welfare functions should be recommended. If ordinal full comparability were admissible, a stepwise leximin should be recommended. The stepwise rank-dependent utilitarianism should be used if cardinal full comparability were admissible.

Key Words: Social Welfare Ordering, Stepwise Rank-Dependent Utilitarianism, k-Quantile Mean Comparison Method

JEL Code: D63, D71, H43, I31, I32, J18, Q56

1. Introduction

Policymakers must often compare total well-being and its distribution between groups with different demographics, such as nations, local areas, and specific groups in order to implement reasonable policies for public health, economic growth, and preparation for natural disasters. Also, they must evaluate various policies that affect future demographic composition, such as educational policies and effective preventions for global climate change, taking into account of sustainable environment, biodiversity, and future generations' well-being. If we need to compare social welfare between different population sizes, how should we aggregate individual interests and assess social states with variable populations? If a certain degree of interpersonal comparability of individual well-being were admissible, what kind of aggregation method should be appropriate to use?

At the starting point of welfare economics for assessing social situations with different population sizes, previous studies tried to avoid Arrow's impossibility theorem (Arrow 1951; 1963) and explore theoretical extensions of social choice theory by admitting interpersonal comparability of individual well-being. In their seminal study, Blackorby and Donaldson (1984) proposed a new class of social welfare functions, *critical-level generalized utilitarianism*. The critical-level generalized utilitarianism evaluates social states with different population sizes by comparing their total values obtained by subtracting the given critical level from individual's utility levels. Parfit (1976) pointed out the problem of the *repugnant conclusion* with a simple utilitarianism in the sense that a utilitarian social welfare ordering prefers a situation with many individuals having low utility to that with a few individuals having high utility. In Blackorby and Donaldson's critical-level utilitarianism, since a value obtained by subtracting a critical-level from an extremely low utility level could be negative, a situation with many individuals having low utility would be socially worse than that with relatively few individuals having high utility. Hence, it can avoid Parfit's repugnant conclusion.

The celebrated findings of critical-level generalized utilitarianism by Blackorby and Donaldson

(1984) galvanized leading theorists into studies on various axiomatic characterizations and explorations of variants of social welfare orderings with variable populations[1], e.g. critical-level leximin, number-dampened critical-level generalized utilitarianism, rank-discounted critical-level generalized utilitarianism, and rank-additive social welfare ordering (Ng 1986; 1989; Ebert 1988b; Blackorby, Bossert and Donaldson 1995; 1996; 1997; 2001; 2005; Asheim and Zuber 2014; Pivato 2020). Almost all social welfare orderings proposed by these studies are variants of *separable* generalized utilitarianisms that satisfy strong independence conditions defined on utility profiles. Furthermore, Blackorby et al. (1999) proved that there is no efficient and equitable social welfare ordering that satisfies strong independence under the assumption of cardinal full comparability of individual well-being.[2] When the independence condition is slightly weakened, an average utilitarianism could only survive. Of course, the average utilitarianism can satisfy strong Pareto principle, but it is not equitable at all.

Does this result imply that there exists no efficient and equitable social welfare function under cardinal full comparability? The answer to this question would be quite the opposite if the society could agree to drop the independence condition or separability on utility profiles. In social choice theory with variable populations, there is a class of versatile, efficient, and equitable social welfare orderings that can make a consistent judgment for any changes in population size. The basic idea of these desirable social welfare orderings is derived from the generalized Gini index (or rank-dependent utilitarianism) that was analyzed and proposed by Weymark (1981). The generalized Gini index is defined by weighted sum of

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[1] See Blackorby et al. (2002; 2005) for excellent summaries on social choice theory with variable population. Blackorby et al. (2005) is a must-read textbook written by the pioneers themselves in this field.

[2] This impossibility theorem is directly proved from the classical result in Dechamps and Gevers (1978). They show that under the assumption of cardinal full comparability, social welfare orderings that satisfy strong Pareto, anonymity, separability are weak utilitarianism or leximin. In the framework of social choice with variable population, social welfare orderings that satisfy the axioms of independence, strong Pareto, and anonymity are weak utilitarianism or leximin with the same population. If a positive affine transformation is applied for both rules in comparing utility profiles with different population, it becomes impossible to make a consistent evaluation.

utilities where weights are given in order of relative rank of individual utility profiles. Therefore, this social welfare ordering can satisfy the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and continuity.[3] Obviously, this rule cannot satisfy separability, and may not provide a consistent ordering for population replication. As a result, the class of rank-dependent utilitarianisms have been excluded in the literature by the setting where the independence or separability are required.

There are, however, two exceptional applications of rank-dependent utilitarianisms in social choice theory with variable populations. The first one is the *rank-discounted critical-level generalized utilitarianism* proposed by Asheim and Zuber (2014). Given a *rank-discounted rate* $\beta \in (0, 1)$, this social welfare ordering has a similar form of critical-level generalized utilitarianisms, except that each value is rank-discounted by the power of the fixed discounted rate.[4] Obviously, if the critical level is zero, then this social welfare ordering is a kind of rank-dependent utilitarianisms whose weights are given by the power of the rank-discounted rate. Also, due to the very similar structure to standard rank-dependent utilitarianisms, even if the critical level is not zero, this ordering can satisfy the desirable axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and continuity. However, it violates the consistency condition for any changes of population replication[5]. Furthermore, since the weights are based on the *fixed*

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[3] See Weymark (1981) for theoretical properties of the generalized Gini index. Ebert (1988a) shows that this rule is characterized by the axiom of strong Pareto, anonymity, rank-separability, and continuity under the assumption of cardinal full comparability. Furthermore, when continuity is not required, the generalized leximin can be characterized by the above axioms (Sakamoto 2020).

[4] The definition of rank-discounted critical-level generalized utilitarianism is as follows:
For all utility profiles u_N and v_M, u_N is at least as good as v_M if and only if

$$\sum_{i=1}^{n} \beta^{[i]} [g(u_{[i]}) - g(c)] \geqq \sum_{i=1}^{m} \beta^{[i]} [g(v_{[i]}) - g(c)],$$

where β is a rank discounted rate in (0, 1), g is a concave function, $u_{[i]}$ is an individual well-being with the i-th lowest utility value in the profile u_N, c is a critical value, $|N|= n$ and $|M|= m$.

[5] The fact that this social welfare ordering violates the consistency of population replication can be easily shown in the following example. Suppose that $u_N = (2, 4)$, $v_N = (1, 6)$, the discount rate $\beta = 0.5$, the critical level $c = 0$, and the function g is given by an identity mapping. Then, by using the rank discounted critical level generalized utilitarianism, it holds that 1/2×2 + 1/4×4 = 2 = 1/2×1 + 1/4×6. Hence, u_N and v_N are indifferent. On the other hand, in the evaluation of $(u_N, u_N) = (2, 4, 2, 4)$ and $(v_N, v_N) = (1, 6, 1, 6)$, it holds that 1/2×2 + 1/4×2 + 1/8×4 +

discounted rate, it has a disadvantage that the degree of freedom in selecting weights is low. The second one is the single-parameter Gini social welfare ordering proposed by Donaldson and Weymark (1980). This social welfare ordering is a generalization of social welfare functions which are an equality scale based on Gini coefficients multiplied by average incomes.[6] This ordering is useful because it can satisfy not only the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, continuity, and cardinal full comparability, but also the consistency condition for any population replications. However, even this social welfare ordering has almost no degree of freedom in selecting weights for evaluation of inequality because the weight vector is almost fixed, except for single parameters.[7]

The purpose of this study is to avoid the unfreedom problem of weight selection in the above two rank-dependent utilitarianisms, and to find a new class of versatile, efficient, and equitable social welfare orderings that can satisfy the condition of population consistency under cardinal full comparability. In fact, if the society requires rank-separability instead of separability, the class of efficient and equitable social welfare orderings extends from average utilitarianisms to rank-dependent utilitarianisms. In addition, even if the society requires a class of rank-dependent utilitarianisms to satisfy population consistency, there are a new class of social welfare orderings, *stepwise rank-dependent utilitarianisms*, which have complete

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1/16×4 = 18/8> 15/8 = 1/2×1 + 1/4×1 + 1/8×6 + 1/16×6. Hence, (u_N, u_N) is strictly better than (v_N, v_N). Therefore, the ordering fails to provide a consistent judgment for replication changes.

[6] The single-parameter Gini social welfare ordering is defined as follows:
For all utility profiles u_N and v_M, u_N is at least as good as v_M if and only if

$$\frac{1}{n^\delta}\sum_{i=1}^{n}[\ i^\delta - \ (i-1)^\delta]\,\tilde{u}_i \geqq \ \frac{1}{m^\delta}\sum_{i=1}^{m}[\ i^\delta - \ (i-1)^\delta]\,\tilde{v}_i,$$

where δ is a parameter that is greater than 1, and a tilde of u means a utility profile sorted in descending order. By definition, if δ is 2, this ordering simply judges the profiles following average incomes × (1 - Gini coefficients). Obviously, this social welfare ordering satisfies the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and Dalton's *population principle* (the axiom is called *replication equivalence* in this paper).

[7] The sensitivity of inequality aversion in this ordering is given by the single parameter δ. Although it may be an advantage in terms of saving social decision costs to determine a single parameter, if we would like to evaluate inequalities following various normative aspects, it could be disadvantageous in the sense that a single parameter cannot take various consideration on inequalities.

freedom to choose weights for evaluations on inequalities. This rule divides each profile with *n* individuals into *n*-quantiles from the bottom utility to the top utility, assigns a weight deduced by its share of a fixed weight function for each quantile, and calculate the weighted sum. To satisfy a consistency condition for population replication, this social welfare ordering maps any population onto the closed interval [0, 1] and computes the weighted sum defined by the product of step functions of utility profiles and integral parts of a fixed weight function on [0, 1]. Intuitively speaking, a stepwise rank-dependent utilitarianism judges the profiles by comparing the weighted sums of utility quantiles[8]. By definition, this social welfare ordering is a kind of rank-dependent utilitarianisms, and is not only compatible with the axioms of efficiency, equity, and population consistency, but it also has an obvious advantage that the weights can be freely selected. One of the practical applications of the stepwise rank-dependent utilitarianism is a *k-quantile mean comparison method*, where income or consumption levels are divided into groups of *k* quantiles and the average income of each group are compared in an approximate lexicographic manner. This comparison method makes it possible to compare well-being distributions in an intuitive and easy-to-understand manner, and is a practical method that considers both efficiency and equity more than the traditional comparison of GDP per capita. Furthermore, this paper shows a generalized representation theorem of social welfare orderings that includes a class of stepwise rank-dependent utilitarianism[9] so that policymakers can consider the problem of optimal population size. In addition, this study reexamines some impossibility theorems shown in the literature. If these results were reformulated and interpreted following the context of cardinal full comparability of individual well-being, all impossibility theorems would not be so serious in essence.[10]

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[8] This social welfare ordering divides the utility profile into *n* quantiles by $1/n$ %. Then, a specific weight is given to each quantile, and social welfare is judged by its weighted sum. According to this method, the weight of top 1% in the 100-individual economy is equal to the weight of top 1% in 10,000-individual economy.

[9] This result is immediately derived from the important representation theorem by Blackorby et al. (2001).

[10] The impossibility theorems considered in this paper are related to the problem of the *repugnant conclusion* (Parfit 1976; 1984) and the *sadistic conclusion* (Arrhenius 2000). In the repugnant conclusion, the situation with many individuals having low utility is strictly better than that with a few individuals having high utility. In the

The main contributions of this paper can be summarized as follows. First, this study succeeds in finding a new class of efficient, equitable, and consistent social welfare orderings, *stepwise rank-dependent utilitarianisms*, which are equivalent to social welfare orderings satisfying the desirable axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, continuity, and population consistency under cardinal full comparability. Second, a general functional form of the stepwise rank-dependent utilitarianism is obtained for considering the problem of optimal population size. Third, the paper reformulates the impossibility theorems in the literature under cardinal full comparability and shows that these impossibility results are not so serious. Fourth, the paper proposes a *k*-quantile mean comparison method as a practical application of the stepwise rank-dependent utilitarianism and use actual data to illustrate how the results differ from GDP per capita method. Fifth, by combining the results of previous studies and this paper, theoretical correspondence between acceptable social welfare orderings and interpersonal comparability of individual well-being is clarified in the problem of social choice with variable populations.

The structure of this paper is as follows. Section 2 explains the notations, definitions, and axioms in this paper. Section 3 axiomatically characterizes a stepwise rank-dependent utilitarianism and shows a representation theorem of a generalized stepwise rank-dependent utilitarianisms for considering the problem of optimal population size. Section 4 reformulates some impossibility theorems in social choice theory with variable populations and shows that a generalized form of stepwise rank-dependent utilitarianisms can cope with these difficulties. Section 5 compares social welfare among eight developed countries by the *k*-quantile mean comparison method and shows its differences from the GDP per capita method. The last section offers a summary and discusses the remaining issues.

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sadistic conclusion, adding a small population with negative utilities is strictly better than adding a large population with positive utilities. If the stepwise rank-dependent utilitarianism is used for social evaluation, it is easy to show that these conclusions are not so serious in the context of cardinal full comparability. Furthermore, this study provides some solutions to the tyranny of aggregation and non-aggregation shown by Fleurbaey and Tungodden (2010).

2. Notations and Definitions

This section explains notations, definitions, and axioms in this paper. Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{++}$, and $\mathbb{R}_{--}$ be the sets of natural numbers, real numbers, positive real numbers, and negative real numbers, respectively. The sets $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$ included by $\mathbb{N}$ are typical elements with different population size. The set of all possible well-being vectors is denoted by $U = \cup_{N\subseteq\mathbb{N}} \mathbb{R}^N$. For all $u_N \in U$, let $u_{[N]} = (u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ be a non-decreasing rearrangement of the well-being vector u_N, that is, $u_{[1]} \leqq u_{[2]} \leqq \ldots \leqq u_{[n]}$. The set of ranks is denoted by $[N] = \{[1], [2], \ldots, [n]\}$. For an arbitrary set X, a binary relation defined on X is an *ordering* if and only if it satisfies completeness and transitivity.[11] Let a *social welfare ordering* $\succcurlyeq$ be defined on U. For all $u_N, v_M \in U$, $u_N \succcurlyeq v_M$ means that u_N is at least as socially good as v_M. Asymmetric and symmetric parts of $\succcurlyeq$ are given by $\succ$ and ~, respectively.

Note that all social welfare orderings are social *ordering* functions because they always generate an ordering defined on the set of well-being profiles.[12] Each individual well-being is assumed to be cardinal full comparable as follows:

Cardinal Full Comparability: $\forall u_N, v_N \in U$, $\forall a \in \mathbb{R}$, $\forall b \in \mathbb{R}_{++}$, $u_N \succcurlyeq v_N \Leftrightarrow (a+bu_i)_{i\in N} \succcurlyeq (a+bv_i)_{i\in N}$.

Next, let us define a series of traditional axioms in social choice theory. First, as an axiom of efficiency, the paper requires strong Pareto principle.

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[11] Completeness requires that for all x, y in X, $x \succcurlyeq y$ or $y \succcurlyeq x$. Transitivity requires that for all x, y, z in X, ($x \succcurlyeq y$ & $y \succcurlyeq z$) implies $x \succcurlyeq z$.

[12] This paper implicitly assumes that a social ordering function satisfies *an independence condition* and *Pareto indifference*. The theoretical relationships between welfarism, neutrality, independence, and Pareto principle in the setting of variable populations are examined in Blackorby et al. (1999, Theorems 1-3).

Strong Pareto: $\forall u_N, v_N \in U$, if $u_N \geqq v_N$, then $u_N \succcurlyeq v_N$. Moreover, if $u_N > v_N$, then $u_N \succ v_N$.[13]

Throughout the paper, all social welfare orderings must treat each individual well-being equally, and this requirement is represented by the following anonymity axiom.

Anonymity: $\forall$ bijections π on N, $\forall u_N \in U$, $u_N \sim u_{\pi(N)}$.

This paper considers two types of continuity of social welfare orderings. The first continuity axiom demands that both the upper contour set and the lower contour set of social welfare ordering should be closed with *the same* population.

Continuity: $\forall u_N \in U$, both $\{v_N \in U \mid v_N \succcurlyeq u_N\}$ and $\{v_N \in U \mid u_N \succcurlyeq v_N\}$ are closed.

The second continuity is an extended version of the above continuity condition. This continuity axiom demands that both the upper contour set and the lower contour set of social welfare ordering should be closed in the setting of *variable* populations.

Extended Continuity: $\forall u_N \in U$, both $\{v_M \in U \mid v_M \succcurlyeq u_N\}$ and $\{v_M \in U \mid u_N \succcurlyeq v_M\}$ are closed.

Separability requires social welfare orderings to ignore well-being information about indifferent individuals between two profiles. This axiom plays a central role in the famous joint characterization theorem (Dechamps and Gevers 1978), where Paretian and anonymous social welfare orderings must be weak utilitarian, leximin, or leximax rules with the same population. In the setting of variable populations, it is shown that an efficient, separable, and population consistent social welfare ordering must be an

[13] For all $u_N, v_N \in U$, [$u_N \geqq v_N$ iff $u_i \geqq v_i$ for all i] and [$u_N > v_N$ iff $u_i \geqq v_i$ for all i and $u_j > v_j$ for some j].

average utilitarianism under cardinal full comparability.

Separability: $\forall u_N, v_N, u'_N, v'_N \in U$, if $\exists M \subseteq N$, $(\forall i \in M, u_i = u'_i$ & $v_i = v'_i)$ and $(\forall j \in N \setminus M, u_j = v_j$ & $u'_j = v'_j)$, then $u_N \succcurlyeq v_N \Leftrightarrow u'_N \succcurlyeq v'_N$.

The weaker version of separability is called the following rank-separability.

Rank-Separability: $\forall u_{[N]}, v_{[N]}, u'_{[N]}, v'_{[N]} \in U$, if $\exists [M] \subseteq [N]$, $(\forall i \in [M], u_{[i]} = u'_{[i]}$ & $v_{[i]} = v'_{[i]})$ and $(\forall j \in [N] \setminus [M], u_{[j]} = v_{[j]}$ & $u'_{[j]} = v'_{[j]})$, then $u_{[N]} \succcurlyeq v_{[N]} \Leftrightarrow u'_{[N]} \succcurlyeq v'_{[N]}$.

This axiom requires social welfare orderings to ignore well-being information about the same well-being in the same ranks between two profiles. Obviously, separability implies rank-separability under the assumption of anonymity. The next section shows that simply imposing rank-separability instead of separability yields a versatile class of distribution-sensitive social welfare orderings.

Let us introduce an axiom of equity. Pigou-Dalton transfer equity states the following: Given that the well-being of other persons is fixed, and there is a well-being gap between two individuals, the same amount of transfer that improves the gap will not at least reduce social welfare.

Pigou-Dalton Transfer Equity: $\forall u_N, v_N \in U$, $\forall \varepsilon \in \mathbb{R}_{++}$, if $\exists i, j \in N$, $v_i - \varepsilon = u_i \geqq u_j = v_j + \varepsilon$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succcurlyeq v_N$.

Finally, consider two consistency conditions of population replication. The first consistency of population replication, *replication equivalence* requires that social welfare does not change if a well-being distribution remains the same no matter how many times a utility profile is replicated. This requirement is the same as the principle of population proposed by Dalton (1920). To define replication equivalence, let

$k{*}u_N$ denote a k-replica of well-being profile u_N (i.e., $k{*}u_N = (\underbrace{u_N, \dots, u_N}_{k\ times})$). Then, the axiom is defined as follows.

Replication Equivalence: $\forall k \in \mathbb{N}, \forall u_N \in U, u_N \sim k{*}u_N$.

Since replication equivalence considers only well-being distributions, population growth or decline cannot affect social welfare. Although it may be an appropriate axiom for inequality measurements, it would be too strong for social welfare measurements. The next replication invariance requires that social welfare judgments on any two profiles remain the same with the replicated population. By definition, replication equivalence implies replication invariance.

Replication Invariance: $\forall k \in \mathbb{N}, \forall u_N, v_N \in U, u_N \succcurlyeq v_N \Leftrightarrow k{*}u_N \succcurlyeq k{*}v_N$.

3. Stepwise Rank-Dependent Utilitarianism and Its Variations

This section proposes and characterizes a class of rank-dependent utilitarianisms with flexible weight functions, a *stepwise rank-dependent utilitarianism*, that satisfies the axioms of efficiency, equity, and replication consistency defined in the previous section. To define this social welfare ordering, let us define a step function on utility profiles as follows.

Definition: A function $u_{[N]} : [0, 1] \rightarrow \mathbb{R}$ is called a rank-dependent step function on u_N if and only if for all u_N, for all t in [0, 1], $u_{[N]}(t) = u_{[i]}$ whenever t in $[\,[i-1]/n, [i]/n\,)$.

Next, let a weight function w be defined on the closed interval [0, 1], where $w(t) \geqq w(t') \geqq 0$ for all t, t' with $t < t'$ and $\int_0^1 w(t)dt = 1$. Using these definitions, a stepwise rank-dependent utilitarian social welfare function is simply defined as an integral value of a product of this weight function and the rank-dependent step function on utility profiles on the closed interval [0, 1]. By definition, this social welfare function could be interpreted as a generalization of rank-dependent utilitarianisms with the same population size.[14]

[14] Note that a stepwise rank-dependent utilitarian social welfare function is well-defined for all *finite* natural numbers. In the setting of infinite populations, the rank-dependent step function cannot be defined and there may be no computable algorithm for admissible social welfare orderings (Basu and Mitra 2003; Zame 2007; Lauwers 2010). As an alternative definition of this ordering, by calculating each weight of a person with $[i]$-th well-being in n-population by $\int_{[i-1]/n}^{[i]/n} w(t)dt$, the ordering can be represented by a family of standard rank-dependent utilitarianisms with the weight vectors defined in the above manner for each population. However, this alternative definition is not useful for expressing a generalized form of stepwise rank-dependent utilitarianisms discussed later, and is not considered in this paper.

Definition: A social welfare ordering $\succcurlyeq^{\mathrm{SRDU}}$ is a stepwise rank-dependent utilitarianism if and only if $\forall$ $u_N, v_M \in U$, $u_N \succcurlyeq^{\mathrm{SRDU}} v_M \Leftrightarrow \int_0^1 w(t)u_{[N]}(t)dt \geqq \int_0^1 w(t)v_{[M]}(t)dt$.

Note that the stepwise rank-dependent utilitarianism includes the single parameter Gini social welfare ordering proposed by Donaldson and Weymark (1980). In fact, if the weight function is defined as $w(t) = (1+\delta)(1-t)^{\delta}$ for all t in [0, 1], then a stepwise rank-dependent utilitarianism with this weight function is equivalent to the single-parameter Gini index. A definition of a stepwise rank-dependent utilitarianism is so simple that the weights for any population size is easy to compute.[15] Furthermore, each weight function is not necessarily required to be continuous and only needs to be a measurable function. In this sense, the degree of freedom in selecting weight vectors is very high.

Since a stepwise rank-dependent utilitarianism is a generalization of standard rank-dependent utilitarianisms with the same population framework, it obviously satisfies the axioms of strong Pareto, anonymity, continuity, Pigou-Dalton transfer equity, rank-separability and cardinal full comparability, which are used in characterizing a class of standard rank-dependent utilitarianisms with the same population (Ebert 1988a; Sakamoto 2020). Also, it is trivial that the stepwise rank-dependent utilitarianism satisfies replication equivalence by its definition. Moreover, whenever a social welfare ordering satisfies these axioms, it can be represented by a stepwise rank-dependent utilitarianism with a certain weight function.[16]

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[15] In Donaldson and Weymark (1980), Ebert (1988b), and Bossert (1990), they consider theoretical properties of a series of weight vectors of standard rank-dependent utilitarianisms satisfying replication equivalence. However, the functional form of weights is given by the differences of specific function defined on ratios of population and is difficult to compute for comparing various situations.

[16] If each weight vector is defined for a family of rank-dependent utilitarianisms to satisfy replication equivalence, then a social welfare ordering satisfying the above axioms is simply represented in the form of a stepwise rank-dependent utilitarianism. Since more complicated expressions are possible for representing the social welfare ordering, note that Theorem 1 is not an axiomatic characterization. However, it seems to be essentially the same as the characterization result.

Theorem 1: *A stepwise rank-dependent utilitarianism satisfies the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, continuity, cardinal full comparability, and replication equivalence. Moreover, if a social welfare ordering* $\succcurlyeq$ *satisfies the above axioms, then it is represented by a stepwise rank-dependent utilitarianism with a certain weight function, that is,*

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \Leftrightarrow \int_0^1 w(t)u_{[N]}(t)dt \geqq \int_0^1 w(t)v_{[M]}(t)dt.$$

[Proof] It is easy to prove that a stepwise rank-dependent utilitarianism satisfies the above axioms. We only prove that a social welfare ordering satisfying the axioms can be represented by a stepwise rank-dependent utilitarianism through the following four claims. Let $\succcurlyeq$ satisfy the above axioms.

[Claim 1] $\forall n \in \mathbb{N}$, $\exists$ an weight vector $(w^n{}_{[i]}) \in (0, 1)^N$ with $w^n{}_{[1]} \geqq \dots \geqq w^n{}_{[n]} \geqq 0$ & $\sum_{[i]\in[N]} w^n{}_{[i]}=1$, $\forall u_N, v_N \in U, u_N \succcurlyeq v_N \Leftrightarrow \sum_{i=1}^n w_{[i]}^n u_{[i]} \geqq \sum_{i=1}^n w_{[i]}^n v_{[i]}$.

By Ebert (1988a)'s characterization theorem, a social welfare ordering with the same population is a rank-dependent utilitarianism if and only if it satisfies the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, and cardinal full comparability. Hence, Claim 1 holds true. ■

[Claim 2] $\forall n \in \mathbb{N}$, $\exists$ an weight function w^n with $w_{[i]}^n = \int_{[i-1]/n}^{[i]/n} w^n(t)dt$ for all $[i]$, $\forall u_N, v_N \in U$, $u_N \succcurlyeq^{\mathrm{SRDU}} v_N \Leftrightarrow \sum_{i=1}^n w_{[i]}^n u_{[i]} \geqq \sum_{i=1}^n w_{[i]}^n v_{[i]}$.

Given $n \in \mathbb{N}$, consider an weight function w^n such that $\forall [i] \in [N]$, $\forall t \in [[i\text{-}1]/n, [i]/n)$, $w^n(t) = nw_{[i]}^n$. With this weight function, it obviously holds that $w_{[i]}^n = \int_{[i-1]/n}^{[i]/n} w^n(t)dt$ and $u_N \succcurlyeq^{\mathrm{SRDU}} v_N \Leftrightarrow \sum_{i=1}^n w_{[i]}^n u_{[i]} \geqq \sum_{i=1}^n w_{[i]}^n v_{[i]}$. ■

[Claim 3] $\forall u_N, v_M \in U, u_N \succcurlyeq v_M \Leftrightarrow \frac{lcm(n,m)}{n} * u_N \succcurlyeq \frac{lcm(n,m)}{m} * v_M$, where $lcm(n, m)$ is the least common multiple of n and m.

By replication equivalence, $\forall u_N, v_M \in U$, $\frac{lcm(n,m)}{n} * u_N \sim u_N \succcurlyeq v_M \sim \frac{lcm(n,m)}{m} * v_M$. Transitivity implies that $u_N \succcurlyeq v_M \Leftrightarrow \frac{lcm(n,m)}{n} * u_N \succcurlyeq \frac{lcm(n,m)}{m} * v_M$. ■

[Claim 4] $\forall n, m \in \mathbb{N}$, $\forall [i] \in [N]$, $\forall [j] \in [M]$,

$\int_{[i-1]/n}^{[i]/n} w^{lcm(n,m)}(t)dt = \int_{[i-1]/n}^{[i]/n} w^n(t)dt$ and $\int_{[j-1]/m}^{[j]/m} w^{lcm(n,m)}(t)dt = \int_{[j-1]/m}^{[j]/m} w^m(t)dt$.

By Claims 1-3, it immediately follows that $\forall u_N, v_M \in U$, $u_N \succcurlyeq v_M \Leftrightarrow \int_0^1 w^{lcm(n,m)}(t)u_{[N]}(t)dt \geqq \int_0^1 w^{lcm(n,m)}(t)v_{[M]}(t)dt$. Then, if the equations in Claim 4 do not hold, Claim 3 cannot hold. ■

Claim 4 guarantees that any weights induced by some weight functions can be calculated by finer weight functions. Then, for all rational number t in [0, 1], let the $w(t) = \lim_{n\to\infty} w^n(t)$ If the weight function $w^n(t)$ does not converge for some rational numbers, there are at most finite discontinuous points in the set of rational numbers belonging to [0, 1], since the weight of $[i]/n$ must be Riemann-integrable for each natural number n and $[i]$. For all discontinuous points, we assume that the value of w is equal to the maximal value of limits.[17] This weight function is easily extended to the set of real numbers in [0, 1] because there at most discontinuous points in [0, 1]. Thus, by using w defined as the above, the following equation holds.

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \Leftrightarrow \int_0^1 w(t)u_{[N]}(t)dt \geqq \int_0^1 w(t)v_{[M]}(t)dt. \blacksquare$$

Theorem 1 shows that a stepwise rank-dependent utilitarianism is an efficient, equitable,

[17] Ebert (1988b) states that any weights are representable by some continuous function (Theorem 8, p. 155). However, weight functions can be discontinuous in some cases. Consider $w(t) = 3/2$ if t in [0, 1/2) and $w(t) = 1/2$ otherwise (i.e. this weight function is discontinuous at 1/2). Then, a stepwise rank-dependent utilitarianism with this weight function obviously satisfies all axioms in Theorem 1.

consistent, and versatile social welfare ordering in the sense that it satisfies the axioms of strong Pareto, Pigou-Dalton transfer equity, and replication equivalence, and has a high degree of freedom in selecting weight functions. An interesting application of stepwise rank-dependent utilitarianisms is a *k-quantile mean comparison method*. This method divides population into k quantiles and compares each average well-being level for each quantile in an *approximate* lexicographic manner. For example, suppose that the weight of j-th quantile (j = 1, …, k-1) is given by $w_{j\text{-th}} = 1 - \Sigma_{h=1, \ldots, j-1} w_{h\text{-th}} - 1/1000^{j}$ and $w_{k\text{-th}}$ must be $1/1000^{k-1}$. If $k = 3$, then $w_{1st} = 0.999$, $w_{2nd} = 0.000999$, and $w_{3rd} = 0.000001$. By giving a huge priority to populations with lower well-being, the k-quantile mean comparison method approximately ignores the levels of well-being in the upper quantiles and is almost the same as a generalized leximin (Sakamoto 2020). This method is clearly more distributive-sensitive than the conventional GDP per capita method, and provides an intuitive and simple method for comparing income distributions.

It seems that, however, replication equivalence would be too strong as a condition of consistency for population replication. In fact, a stepwise rank-dependent utilitarianism states that a one-person economy with a high well-being should be better than a 10,000-person economy where all persons have a slightly lower well-being.[18] To avoid such a problem, positive responsiveness of social welfare to population increment must be considered. Fortunately, this difficulty can be solved by requiring extended continuity (a stronger version of continuity) and replication invariance (a weaker version of replication equivalence) due to the celebrated theorem in Blackorby et al. (2001).

Theorem 2: *If a social welfare ordering satisfies the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, extended continuity, cardinal full comparability, and replication invariance, then it is represented by a generalized form of stepwise rank-dependent utilitarianism as follows:*

[18] This problem is called the *reverse repugnant conclusion* and is analyzed by Blackorby et al. (1998).

$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \Leftrightarrow W(n, V(u_N)) \geqq W(m, V(v_M)),$

where W: $\mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ *is continuous and increasing in its second argument and V is a function that represents a stepwise rank-dependent utilitarianism.*

[Proof] We prove Theorem 2 by using the following three claims. Let $\succcurlyeq$ satisfy the above axioms.

[Claim 1] $\exists$ a function W: $\mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\exists$ a function V^n: $\mathbb{R}^n \rightarrow \mathbb{R}$ for all n such that $\forall u_N, v_M \in U$, $u_N \succcurlyeq v_M \Leftrightarrow W(n, V^n(u_N)) \geqq W(m, V^m(v_M))$, where W is continuous and increasing in its second argument.
By the famous representation theorem in Blackorby et al. (2001), if a social welfare ordering satisfies extended continuity and weak Pareto,[19] then Claim 1 holds true. Since strong Pareto implies weak Pareto, Claim 1 obviously holds. ■

[Claim 2] $\forall n \in \mathbb{N}$, $\exists$ an weight vector $(w^n_{[i]}) \in (0, 1)^N$ with $w^n_{[1]} \geqq \ldots \geqq w^n_{[n]} \geqq 0$ & $\sum_{[i] \in [N]} w^n_{[i]} = 1$, $\forall u_N, v_N \in U, u_N \succcurlyeq v_N \Leftrightarrow \sum_{i=1}^{n} w^n_{[i]} u_{[i]} \geqq \sum_{i=1}^{n} w^n_{[i]} v_{[i]}$.
Claim 2 is trivial since it is directly shown by Claim 1 in Theorem 1. ■

[Claim 3] $\forall k \in \mathbb{N}, \forall u_N \in U, V^n(u_N) = V^{kn}(k{*}u_N)$.
Given u_N, suppose that $V^n(u_N) = \bar{u}$. By replication invariance and $u_N \sim n{*}(\bar{u})$, it holds that for all k, $k{*}u_N \sim kn{*}(\bar{u})$. Hence, $V^n(u_N) = V^{kn}(k{*}u_N)$. ■

By Claim 3, if an ordering is represented by a function V, then it satisfies replication equivalence. Hence, this function is equivalent to a stepwise rank-dependent utilitarianism with a certain weight function by the proof of Theorem 1. ■

[19] Weak Pareto is defined as follows: $\forall u_N, v_N \in U$, if $\forall i \in N, u_i > v_i$, then $u_N \succ v_N$.

If separability is required for a social welfare ordering instead of rank-separability, the functional form of Theorems 1 and 2 must be an *average utilitarianism* rather than a stepwise rank-dependent utilitarianism by the famous joint characterization theorem in Dechamps and Gevers (1978). The requirement of separability has virtually the same effect as some independence conditions on utility profiles. Therefore, as previous results show, only average utilitarianism can survive under a requirement of separability.[20] In this sense, when cardinal full comparability is admissible, both separability and independence should be abandoned in order to take a consideration on well-being distributions.

Note that Theorem 2 is a simple representation theorem based on a stepwise rank-dependent utilitarianism and says nothing on specific functional forms and positive responsiveness to population increment. In fact, a class of social welfare orderings represented in Theorem 2 includes a *reverse number-dampened* social welfare function; the form is given by a function $f(n)$ multiplied by a stepwise rank-dependent utilitarianism, where $f(n)$ is a strictly decreasing function with respect to a natural number. If the society would like some properties on positive responsiveness to population increment, the additional axiom should be needed. Moreover, if seeking some type of additive social welfare functions such as a number-dampened stepwise rank-dependent utilitarianism, additivity axioms would be required. This paper dose not provide a definitive conclusion on the question of which specific function forms should be used in the class of efficient and equitable social welfare functions that generalize stepwise rank-dependent utilitarianisms. In order to solve this ambivalent and difficult problem, the society must consider the issues of ethical values of social inequality, degrees of freedom for selecting parameters, and population growth. It may be possible to solve this difficulty by experimental, empirical, and ethical analyses.

In addition, following the celebrated results in Blackorby and Donaldson (1982) and Ebert (1988b), theoretical relationships between interpersonal comparability of well-being and stepwise forms

[20] Blackorby et al. (2005) provide an axiomatic characterization of average utilitarianism in which they require not separability but an independence condition. In general, some types of independence conditions imply the axiom of separability. In fact, the same-number independence (Blackorby et al. 2005, Ch. 5) is equivalent to separability.

of acceptable social welfare functions could be obtained.[21] If translation-scale full comparability were admissible, a stepwise rank-dependent Kolm-Pollack function should be used. If ratio-scale full comparability were admissible, a stepwise rank-dependent Atkinson-Blackorby-Donaldson function should be used.[22] However, since these stepwise function forms might become slightly complicated, both the rank-dependent Kolm-Pollack and Atkison-Blackorby-Donaldson social welfare functions may well be defined in the easy form of normal functions restricting weight vectors to satisfy replication equivalence.[23] Using a similar proof to Theorems 1, the above two social welfare functions can be characterized by the same axioms, except for cardinal full comparability, changing it to the corresponding invariance condition. Moreover, as well as the discussion of Theorem 2, there are the two generalized forms of these stepwise rank-dependent social welfare functions that satisfy both population invariance and positive responsiveness to population increment.

Finally, if ordinal full comparability were admissible,[24] one candidate for acceptable social welfare orderings would be the following stepwise leximin.

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[21] If the society requires separability instead of rank-separability, then the class of acceptable social welfare orderings is no longer in the stepwise forms, but average forms of Kolm-Pollack and Atkinson-Blackorby-Donaldson social welfare functions. If continuity of social welfare ordering is not required, acceptable social welfare ordering with the same population size must be a generalized leximin rule (Sakamoto 2020).

[22] Translation-scale full comparability and ratio-scale full comparability are defined as follows.

Translation-Scale Full Comparability: $\forall u_N, v_N \in U$, $\forall a \in \mathbb{R}$, $u_N \succcurlyeq v_N \Leftrightarrow (a+u_i)_{i\in N} \succcurlyeq (a+v_i)_{i\in N}$.

Ratio-Scale Full Comparability: $\forall u_N, v_N \in U$, $\forall b \in \mathbb{R}_{++}$, $u_N \succcurlyeq v_N \Leftrightarrow (bu_i)_{i\in N} \succcurlyeq (bv_i)_{i\in N}$.

[23] The easier formulations of these social welfare orderings are as follows. As well as the alternative definition of stepwise rank-dependent utilitarianism, by calculating each weight of a person with $[i]$-th well-being in n-population by $\int_{[i-1]/n}^{[i]/n} w(t)dt$, the ordering can be represented by a family of standard rank-dependent Kolm-Pollack or Atkinson-Blackorby-Donaldson social welfare functions with the weight vectors defined in the above manner.

[24] Ordinal full comparability is defined as follows.

Ordinal Full Comparability: $\forall u_N, v_N \in U$, $\forall$ an increasing function φ, $u_N \succcurlyeq v_N \Leftrightarrow (\varphi(u_i))_{i\in N} \succcurlyeq (\varphi(v_i))_{i\in N}$.

Definition: A social welfare ordering $\succcurlyeq^{SL}$ is a *stepwise leximin* if and only if $\forall u_N, v_M \in U, u_N \succcurlyeq^{SL} v_M \Leftrightarrow$ $[\forall t$ in $[0, 1], u_{[N]}(t) = v_{[M]}(t)]$ or $[\exists t'$ in $[0, 1], \forall t < t', u_{[N]}(t) = v_{[M]}(t)$ and $u_{[N]}(t') > v_{[M]}(t')]$.

Obviously, the stepwise leximin is efficient, equitable, and consistent because it satisfies the axioms of strong Pareto, Pigou-Dalton transfer equity, and replication equivalence. With a slight modification of the famous Hammond's theorem (Hammond 1976), it is shown that this social welfare ordering is characterized by the axioms of strong Pareto, anonymity, Hammond equity, separability, ordinal full comparability, and replication equivalence. If the society would like the stepwise leximin to have positive responsiveness to population increment and replication invariance, then the following number-dampened stepwise leximin could be a candidate for acceptable social welfare orderings under the assumption of ordinal full comparability.[25]

Definition: A social welfare ordering $\succcurlyeq^{NSL}$ is a *number-dampened stepwise leximin* if and only if $\exists$ an increasing function $f: \mathbb{N} \rightarrow \mathbb{R}_{++}, \forall u_N, v_M \in U, u_N \succcurlyeq^{NSL} v_M \Leftrightarrow f(n) \cdot u_N \succcurlyeq^{SL} f(m) \cdot v_M$.

Hence, theoretical relationships between interpersonal comparability of well-being and acceptable social welfare functions could be clarified. The following table summarizes these relationships.

[25] Note that the stepwise leximin and the number-dampened stepwise leximin can be defined in the discrete population manner. Consider the following lexicographic ordering:

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_M \Leftrightarrow \frac{lcm(n,m)}{n} * u_N \succcurlyeq^{L} \frac{lcm(n,m)}{m} * v_M,$$

where $\succcurlyeq^{L}$ is a lexicographic ordering with the same population. The above ordering is obviously equivalent to the stepwise leximin. The number-dampened stepwise leximin can be defined in the similar way.

Table 1. Theoretical Relationships of Interpersonal Comparability and Admissible SWOs with Variable Population

Invariance	Admissible Value Functions with Variable Population	Generalized Forms of Admissible Value Functions with Variable Population
Ordinal Full Comparability (OFC)	Stepwise Leximin (=SP∧A∧SEP∧HE∧RE∧OFC)	Number-Dampened Stepwise Leximin
Translation-Scale Full Comparability (TFC)	Average KP (=SP∧A∧SEP∧C∧ RE∧TFC), Stepwise Rank-Dependent KP (=SP∧A∧R-SEP∧C∧PD∧RE∧TFC)	Number-Dampened Forms of Average KP, Stepwise Rank-Dependent KP, etc.
Ratio-Scale Full Comparability (RFC)	Average ABD (=SP∧A∧SEP∧C∧RE∧RFC), Stepwise Rank-Dependent ABD (=SP∧A∧R-SEP∧C∧PD∧RE∧RFC)	Number-Dampened Forms of Average ABD, Stepwise Rank-Dependent ABD, etc.
Cardinal Full Comparability (CFC)	Average U (=SP∧A∧SEP∧C∧RE∧CFC), Stepwise Rank-Dependent U (=SP∧A∧R-SEP∧C∧PD∧RE∧CFC)	Number-Dampened Forms of Average U, Stepwise Rank-Dependent U, etc.

Each abbreviation is defined as follows: SP: Strong Pareto; A: Anonymity; SEP: Separability; HE: Hammond Equity; RE: Replication Equivalence, R-SEP: Rank-Separability; C: Continuity; PD: Pigou-Dalton Transfer Equity; KP: Kolm-Pollack Social Welfare Function; ABD: Atkinson-Blackorby-Donaldson Social Welfare Function; U: Utilitarianism.

4. Reexamination of Impossibility Theorems[26]

This section investigates the causes and mechanisms of some impossibility theorems in social choice with variable populations and shows that these impossibilities are not so serious under the assumption of cardinal full comparability.

First, let us consider the commonly known as *repugnant conclusion* problem by Parfit (1976; 1982; 1984). A simple utilitarianism prefers the situation where 20,000 individuals have a utility level of 1 (total utility 20,000) to the situation where 1,000 individuals have a utility level of 10 (total utility 10,000). That is, utilitarian judgments often lead to the repugnant conclusion in the sense that a situation where a relatively small population has a very high utility is socially worse than a situation where a huge population has a very low utility. Previous studies demonstrate that combining positive responsiveness to population increment with Pigou-Dalton transfer equity implies the repugnant conclusion (Ng 1989; Carlson 1998; Broome 2004; Blackorby et al. 2005). However, this repugnant conclusion is caused by the specific requirement on positive responsiveness to population increment.[27] The impossibility result can be easily avoided if the positive responsiveness to population increment is replaced by the following axiom.

Positive Responsiveness to Population Growth: $\forall u_N \in U,\ \exists \bar{u} \in \mathbb{R}, (u_N,\ \bar{u}) \succcurlyeq u_N.$

This axiom only requires that there should be a threshold for positive responsiveness of social welfare when population increment occurs. Under cardinal full comparability, it is impossible to set a constant value as a threshold of well-being for additional population increment because such a threshold

[26] Marcus Pivato pointed out our misunderstanding on the concept of the sadistic conclusion in the former draft. We sincerely appreciate his kindness and thoughtful suggestions.

[27] Previous studies often set a constant value as a threshold of additional individual's utility for positive responsiveness to population increment (Ng 1989; Blackorby and Donaldson 1984; Blackorby et al. 1995; 1997; 1998; 2002; 2005; Carlson 1998; Arrhenius 2000; Asheim and Zuber 2014; Greaves 2017; Pivato 2020).

is easily varied for any positive affine transformations. Hence, the requirement for positive responsiveness to population increment should refrain from justifying a certain value as a threshold.[28]

Next, let's formulate the repugnant conclusion in the context of social choice with variable populations. The repugnant conclusion says that for some n, m with $m > n$, a situation where more than m individuals have very low utilities is always socially better than a situation where n individuals have high utilities. Following Blackorby et al. (2005), let avoidance of the repugnant conclusion be defined as follows.[29]

Avoidance of the Repugnant Conclusion: $\exists\, \bar{u}, \bar{v} \in \mathbb{R}_{++}$ with $\bar{u} > \bar{v}, \exists\, n \in \mathbb{N}, \forall\, m > n,\ n{*}(\bar{u}) \succcurlyeq m{*}(\bar{v})$.

It is easily shown that the number-dampened stepwise rank-dependent utilitarianism satisfies Pigou-Dalton transfer equality and positive responsiveness to population growth while avoiding the repugnant conclusion if a function $f(n)$ is bounded and monotonically increasing.[30]

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[28] There is another reason why we do not set such a threshold *exogenously*. The concept of the utility threshold of *lives worth living* invokes serious ethical issues such as the *slippery slope argument* and comprehensive discussions on *basic human rights*. Instead, the authors interpret that the addition of an individual who has a significantly lower utility than the other people damages the dignity of society, undermines social justice, and decreases social welfare. A social welfare function measures *relative* desirability of society. It seems that whether an individual is worth living should not be simply judged by social welfare or his/her utility level.

[29] See Blackorby et al. (2005), Asheim and Zuber (2014), Greaves (2017), and Pivato (2020) on various properties of the repugnant conclusion and the sadistic conclusion. Note that even critical-level generalized utilitarianisms cannot avoid a modified version of the repugnant conclusion. For simplicity, suppose that a social welfare ordering takes a form of a simple critical-level utilitarianism. Consider the case where n individuals have $c + \varepsilon$ well-being where c is a given critical level and ε is a sufficiently small positive number. This social welfare is given by $n\varepsilon$. Obviously, it is easy to construct examples where relatively small populations enjoy sufficiently good lives but its social welfare is lower than $n\varepsilon$.

[30] For example, let a function $f(n) = 1-\beta^n$ for $0<\beta<1$. Then, the number-dampened stepwise rank-dependent utilitarianism can avoid both the repugnant conclusion.

Proposition 3: *Suppose that a function f is bounded and monotonically increasing. Then, the number-dampened stepwise rank-dependent utilitarianism satisfies Pigou-Dalton transfer equality and positive responsiveness to population growth and avoids the repugnant conclusion.*

[Proof] Since a function f is bounded and monotonically increasing, there exists a real number γ such that $\gamma > f(n) \geqq f(1)$ for all n. For all $\bar{u} \in \mathbb{R}$, choose $\bar{v}$ with $f(1)\bar{u} > \gamma\bar{v}$. Then, for all n, m with $m > n$, the following holds true:

$$f(n)\int_0^1 w(t)u_{[N]}(t)dt = f(n)\bar{u} \geqq f(1)\bar{u} > \gamma\bar{v} > f(m)\bar{v} = f(m)\int_0^1 w(t)v_{[M]}(t)dt.$$

Therefore, the number-dampened stepwise rank-dependent utilitarianism satisfies avoidance of the repugnant conclusion.[31] In addition, this ordering obviously satisfies positive responsiveness to population growth because adding $\bar{v}$ with $\bar{v} > u_{[n]}$ to u_N always increases social welfare in the stepwise rank-dependent utilitarianism. ■

Next, let us consider the sadistic conclusion problem pointed out by Arrhenius (2000). The sadistic conclusion says that adding a few individuals with negative utilities to a utility profile is socially better than adding many individuals with positive utilities to it. Arrhenius (2000) shows that a social welfare ordering satisfying a specific combination of axioms leads to the repugnant conclusion or the sadistic conclusion. Formally, Blackorby et al. (2005) defines avoidance of the sadistic conclusion as follows:

Avoidance of the Sadistic Conclusion: $\forall u_N \in U, \forall v_M \in \cup_{N\subseteq\mathbb{N}} \mathbb{R}^N_{++}, \forall s_L \in \cup_{N\subseteq\mathbb{N}} \mathbb{R}^N_{--}, (u_N, v_M) \succcurlyeq (u_N, s_L)$.

[31] This proof shows that the number-dampened stepwise rank-dependent utilitarianism satisfies a *stronger* version of avoidance of the repugnant conclusion. That is, this social welfare function satisfies the following condition: $\forall \bar{u} \in \mathbb{R}, \exists \bar{v} \in \mathbb{R}$ with $\bar{u} > \bar{v}$, $\forall n, m \in \mathbb{N}$ with $m > n$, $n*(\bar{u}) \succcurlyeq m*(\bar{v})$.

This requirement seems to be plausible at first glance, but it includes the irritating aporia in the definition. Consider the situation (-100, 100∗1) where 100 individuals with utility 1 are added to one individual with utility -100 and the situation (-100, -1) where one individual with utility -1 are added to one individual with utility -100. Suppose that the former is socially better than the latter because the former's average utility is higher than that of the latter. This judgment is consistent to not only avoidance of the sadistic conclusion but our common sense.

Next, consider two utility profiles: (100∗100, 100∗1) and (100∗100, -1). These profiles are the same as above except for replacing the original position (one individual with utility -100) by (100∗100). To avoid the sadistic conclusion, (100∗100, 100∗1) must be socially better than (100∗100, -1). This judgment seems to be, however, somehow weird. Comparing the situation where there are gaps of 100 and 1 in half of the society and the situation where there is only one gap of 100 and -1 in the society, it seems reasonably valid that the latter is socially better than the former. Furthermore, in order to satisfy both cardinal full comparability and avoidance of the sadistic conclusion, (100∗101, 100∗2) must be socially better than (100∗101, 0). If social inequality in the former were taken seriously, avoidance of the sadistic conclusion would not have a high ethical priority. That is why the sadistic conclusion seems to be awkward.

The stepwise rank-dependent utilitarianism can judge that the latter is preferable to the former in both cases.[32] Under cardinal full comparability, neither positive nor negative utility values have any implication, so the original version of sadistic conclusion itself has no ethical significance in aggregating social welfare.[33] Hence, to make the problem of sadistic conclusion somewhat significant, a new version

[32] If the society would like the former to be better the latter in both cases, then a class of the number-dampened stepwise rank-dependent utilitarianisms could be useful. Depending on the shape of a function $f(n)$, it is easy to create a situation in which the former is socially better than the latter.

[33] In order to avoid such a situation, the sadistic conclusion usually does not consider any invariance axioms. However, requiring no invariance condition implies that any inter- and intra-personal comparisons of well-being are meaningless, and the sadistic conclusion itself does not have a great impact on comparing social welfare in the context of social choice problem.

of avoidance of the sadistic conclusion will be reformulated as follows.

Avoidance of the Sadistic Conclusion under Cardinal Full Comparability: $\forall u_N \in U$, $\forall \bar{v} \in \mathbb{R}$, $\exists \bar{s} \in \mathbb{R}$ with $\bar{v} > \bar{s}$, $\forall m, l \in \mathbb{N}$, $(u_N, m*(\bar{v})) \succ (u_N, l*(\bar{s}))$.

The new version of avoidance of the sadistic conclusion requires that adding any number of individuals with high utility to a given utility profile is socially better than adding any number of individuals with *sufficiently* low utility to it. Obviously, the number-dampened stepwise rank-dependent utilitarianism can avoid this new version of avoidance of the sadistic conclusion if a function $f(n)$ is bounded and monotonically increasing. Indeed, when an individual with a utility level less than $u_{[1]}$ is added to a given utility profile u_N, social welfare always decreases. On the contrary, adding an individual with a utility level more than $u_{[n]}$ to u_N, social welfare always increases. These facts imply the following possibility result.[34]

Proposition 4: *Suppose that a function f is bounded and monotonically increasing. Then, the number-dampened stepwise rank-dependent utilitarianism avoids the sadistic conclusion under cardinal full comparability.*

[Proof] Since a function f is bounded and monotonically increasing, there exists a real number γ such that

[34] Asheim and Zuber (2014) discusses various paradoxical properties including the repugnant and the sadistic conclusions, but it is easy to show that a generalized form of stepwise rank-dependent utilitarianism can avoid various problems through appropriate reformulations of these difficulties. For example, it is easy to show that the number-dampened stepwise rank-dependent utilitarianism satisfy the following property:

Avoidance of the Very Sadistic Conclusion: $\exists u_N \in U, \forall m \in \mathbb{N}, \forall \bar{v} \in \mathbb{R}_{++}$ $m*(\bar{v}) \succcurlyeq u_N$.

$\gamma > f(n) \geqq f(1)$ for all n. For all $u_N \in U$, all $\bar{v} \in \mathbb{R}$, all $m, l \in \mathbb{N}$, choose $\bar{s}$ with $\bar{s} < u_{[1]}$ and $f(1)\int_0^1 w(t)u_{[N+M]}(t)dt - \gamma\int_{\frac{l}{n+l}}^1 w(t)u_{[N+L]}(t)dt > \gamma\bar{s}\frac{l}{n+l}\int_0^{\frac{l}{n+l}} w(t)dt$, where $u_{N+M} = (u_N, m*(\bar{v}))$ and $u_{N+L} = (u_N, l*(\bar{s}))$. Then, the following holds true:

$$
\begin{aligned}
f(n+m)\int_0^1 w(t)u_{[N+M]}(t)dt &\geqq f(1)\int_0^1 w(t)u_{[N+M]}(t)dt \\
&> \gamma\left[\bar{s}\frac{l}{n+l}\int_0^{\frac{l}{n+l}} w(t)dt + \int_{\frac{l}{n+l}}^1 w(t)u_{[N+L]}(t)dt\right] \\
&= \gamma\int_0^1 w(t)u_{[N+L]}(t)dt \\
&> f(n+l)\int_0^1 w(t)u_{[N+L]}(t)dt.
\end{aligned}
$$

Therefore, the number-dampened stepwise rank-dependent utilitarianism avoids the sadistic conclusion under cardinal full comparability for all $u_N \in U$, all $\bar{v} \in \mathbb{R}$, all $m, l \in \mathbb{N}$. ■

At the end of this section, let us consider the incompatibility problem between aggregation principle, non-aggregation principle, and replication invariance, which was proved by Fleurbaey and Tungodden (2010). The aggregation principle requires a small loss in a small population should be acceptable in order to improve great benefit in a large population. This principle can be interpreted as one aspect of utilitarian social welfare orderings in general. On the other hand, the non-aggregation principle requires that a small loss among the best-off individuals should be acceptable in order to improve great benefit among the worst-off individuals. Since the number of the best-off individuals who incur a small loss can be any natural number (maximal number is n-1), note that the non-aggregation principle can allow total loss among the best-off individuals to exceed total gain among the worst-off individuals. This principle can be interpreted as a weaker version of maximin social welfare orderings. In fact, Fleurbaey and Tungodden (2010) show that a social welfare ordering that satisfies the axioms of non-aggregation, weak Pareto, and replication invariance must be a refinement of maximin social welfare orderings.[35] This

[35] A binary relation $\succcurlyeq_A$ is a refinement of $\succcurlyeq_B$ if and only if $\succcurlyeq_A$ includes the asymmetric part of $\succcurlyeq_B$.

result implies that there exists no social welfare ordering that satisfies the axioms of aggregation, non-aggregation, weak Pareto, and replication invariance.

One solution for this problem would be to avoid the non-aggregation principle, which is too strong for consideration of well-being distribution. In this case, Pigou-Dalton transfer equity is only required as an equity criterion. Hence, the stepwise rank-dependent utilitarianism or its generalized functional forms should be one of candidates as acceptable social welfare orderings because they can satisfy the desirable axioms of efficiency and consistency of population replication[36].

Another solution is to give a priority to the axioms of non-aggregation and consistency of population replication, while dropping the principle of aggregation, and use the stepwise leximin or the number-dampened stepwise leximin. However, there seems to be no rational reason why the society picks up the extreme form of leximin among the other mild candidates, unless only ordinal full comparability would be admissible.[37] Therefore, all reasonable candidates of social welfare judgment seem to be dependent on the context of interpersonal comparability of individual well-being, just like in Table 1 in the previous section. The impossibility results in the literature are not so serious if the society agrees that some type of interpersonal comparability of individual well-being is admissible.

[36] Depending on the degree of interpersonal comparability of individual well-being, the stepwise rank-dependent Kolm-Pollack or Atkinson-Blackorby-Donaldson social welfare functions can be candidates for acceptable social welfare functions.

[37] The other option may be the class of rank-discount generalized utilitarianisms proposed by Asheim and Zuber (2014). This social welfare ordering satisfies the aggregation principle and a weak version of the non-aggregation principle but violates replication invariance. The author believes that replication invariance is an important axiom but the value judgment as to which axiom should be used for constructing acceptable social welfare functions is not something that can be settled by mathematical logics. Hence, it is inevitable for the society to have, within a certain degree of difference, various opinions on which social welfare orderings are acceptable.

5. *k*-Quantile Mean Comparison Method

As one of applications of stepwise rank-dependent utilitarianisms, this section compares social welfare among eight developed countries by the *k*-quantile mean comparison method. In the *k*-quantile mean comparison method, each country's population is divided into *k* quantiles, and an average well-being level of each quantile is compared in an approximate lexicographic manner. Although it is fully understood that there are major deficits in using income as human well-being, in order to show the advantage of *k*-quantile mean comparison method over the traditional comparison based on GDP per capita, we will compare post-tax income levels among developed countries. Note that it is possible to apply the *k*-quantile mean comparison method based on income levels, because purchasing power obtained from individual or household income seems to be invariant with respect to cardinal full comparability.

Data

The dataset is obtained from the *World Inequality Database* (https://wid.world/) because this group provides highly reliable and finest data including information on the entire 1-100 percentiles of post-tax incomes among many countries. We select eight developed countries (Britain, Denmark, France, Germany, Finland, Norway, Sweden, and the United States) for our analysis, since their detailed information on post-tax income are available. Of course, there is no problem in adding other countries, but we decide to compare the post-tax income of the above countries due to the visibility of graphs. Each country's post-tax income is based on a PPP-adjusted real value. In addition, the post-tax income includes the amount of in-kind transfers and is based on the unit of *equal-split adult* (i.e. income divided equally among spouses).

Although countries are compared on their post-tax income distributions, consumption levels seem to be more appropriate than income levels for measuring individual well-being and social welfare. The big problem is that income levels often are seriously affected by seasonality and life cycle. On the contrary, consumption levels are relatively stable from changes in life cycle or seasonality. Hence, consumption levels are interpreted to reflect properly actual well-being of individuals or households. However, household income is easy to collect, but household consumption is difficult to capture. Also, the commonly called *welfare ratio* (the ratio of annual household income over the minimum living cost based on the concept of absolute poverty) may be one of powerful candidates for welfare measurements, but the calculation of the welfare ratio needs too many information so that we cannot find appropriate data source. Therefore, this study uses only information on post-tax income level of each quantile in each country.

Calculation of the *k*-Quantile Mean Comparison Method

We compare social welfare among the eight developed countries by the *k*-quantile mean comparison method based on their post-tax income distributions. As is clear from the definition of the stepwise rank-dependent utilitarianism, both the number and the width of quantiles are free, and there is no restriction on the weight function to be applied to each quantile. For example, it is possible to compare quantiles with inhomogeneous widths and weights such as the bottom 10% (w_{1st} = 0.45), 11-50% (w_{2nd} = 0.3), 51-90% (w_{3rd} = 0.239), 91-99% (w_{4th} = 0.01), the top 1% (w_{5th} = 0.001), and where $w_{i\text{-th}}$ is a weight for *i*-th quantile. This study simply uses a five-quantile mean comparison method (i.e. each quantile has 20% population) because it does not seem to be a large difference in empirical findings by subdividing quantiles more than necessary. In the case of the simple *k*-quantile comparison method, social welfare does not change for any proportional increase or decrease of the population size because of the property of replication equivalence. Hence, there are obstacles when comparing the United States to countries or regions with extremely small populations such as Monaco and Luxembourg (Note that GDP per capita also has the same problem). To deal with this issue, a generalized form of the stepwise rank-dependent utilitarianism must be specified for assessing the effect of each country’s population size. However, since the population size of the eight countries is sufficiently large, this study does not consider the problem of the generalized form of *k*-quantile mean comparison method. Instead, we simply compute average income for each quantile based on the five-quantile mean comparison method.

Comparison of Average Income in Each Quintile

Figures 1 to 7 shows national income per capita, average income of each quintile, and the top 1% income among the eight countries during 1980-2017. As seen in figure 1, the United States and Norway's national income per capita are higher than those of the other countries. In contrast, a completely different situation can be seen in terms of the *k*-quantile mean comparison method. In fact, figure 2 illustrate that the US income level among the bottom 20% is clearly lower than that of the others. While the middle class in the United States is relatively good among developed countries (figures 4-5), the US income in the top 20% is obviously outstanding from the other countries (figure 6). Among the top income group, the US income in the top 1% is 2-3 times higher than that of the other countries (figure 7). In this sense, the *k*-quantile mean comparison method can directly show us the whole picture of income inequality in the United States, and it can tell us how the US anomality with distributive injustice is proceeding compared to the other countries. If we compare social welfare in an approximate lexicographic manner, the United States is judged to have the lowest welfare among the eight countries. Of course, in the mild case following a value function defined as a weighted sum rather than the approximate lexicographic ordering, social welfare completely depends on the forms of weight functions. To further investigate this issue, let us consider a comparison based on the single-parameter Gini index.

Figure 1. National Income per capita (thousands of USD), 1980-2017

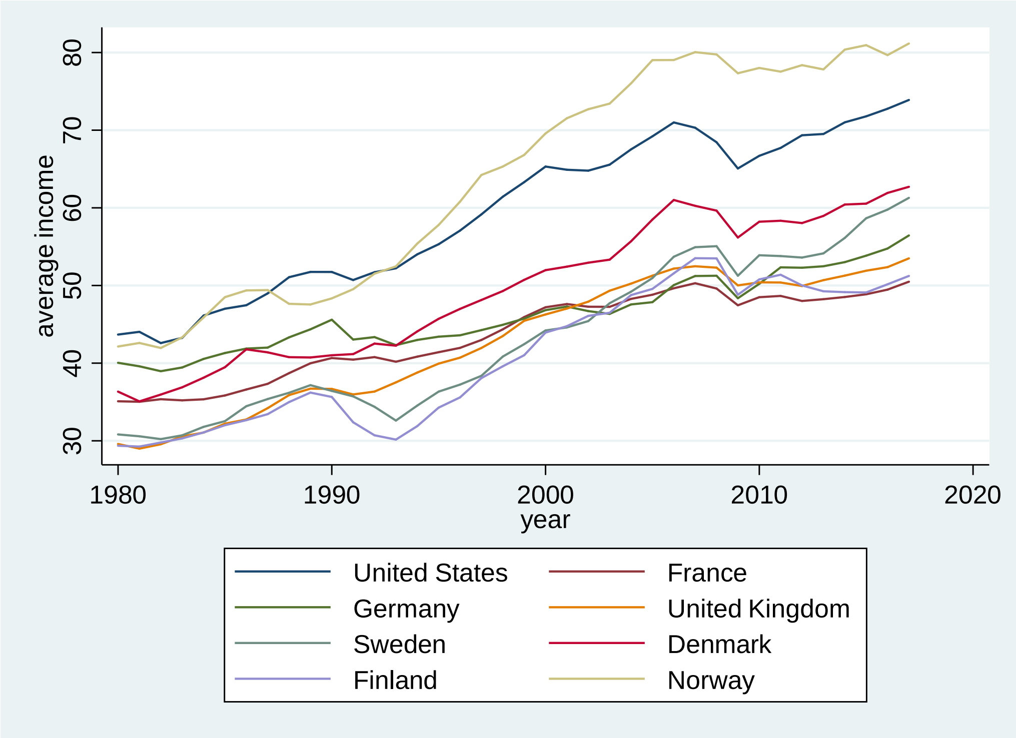


Figure 2. Average Income (thousands of USD) of 1st Quintile (The Bottom 20%), 1980-2017

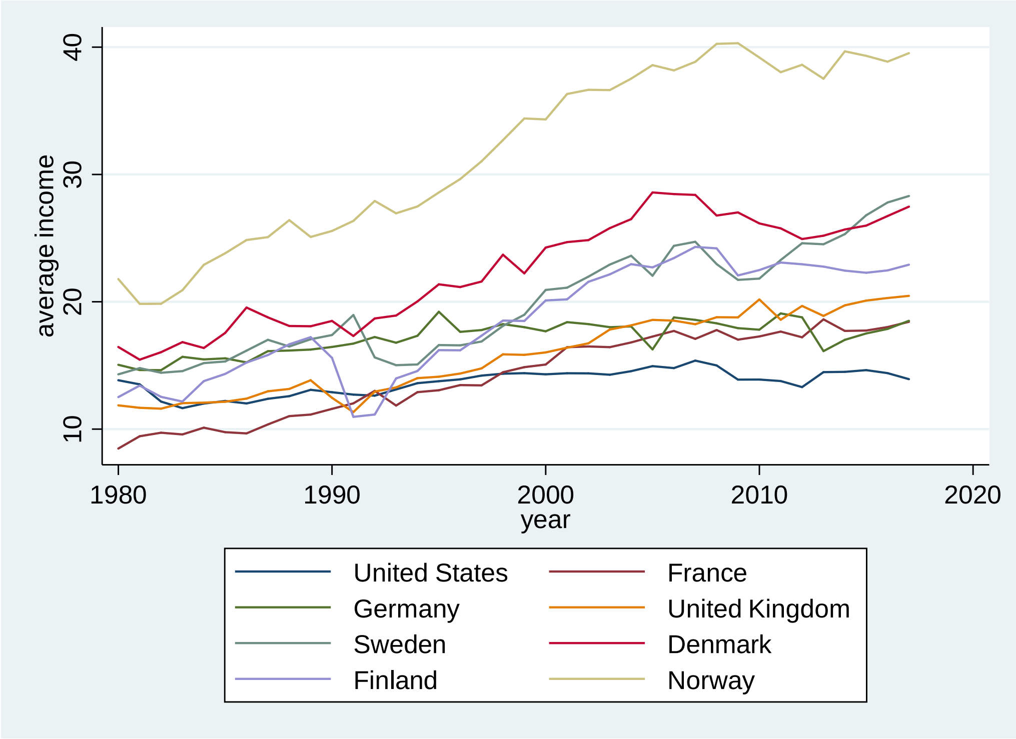


Figure 3. Average Income (thousands of USD) of 2nd Quintile (21-40%), 1980-2017

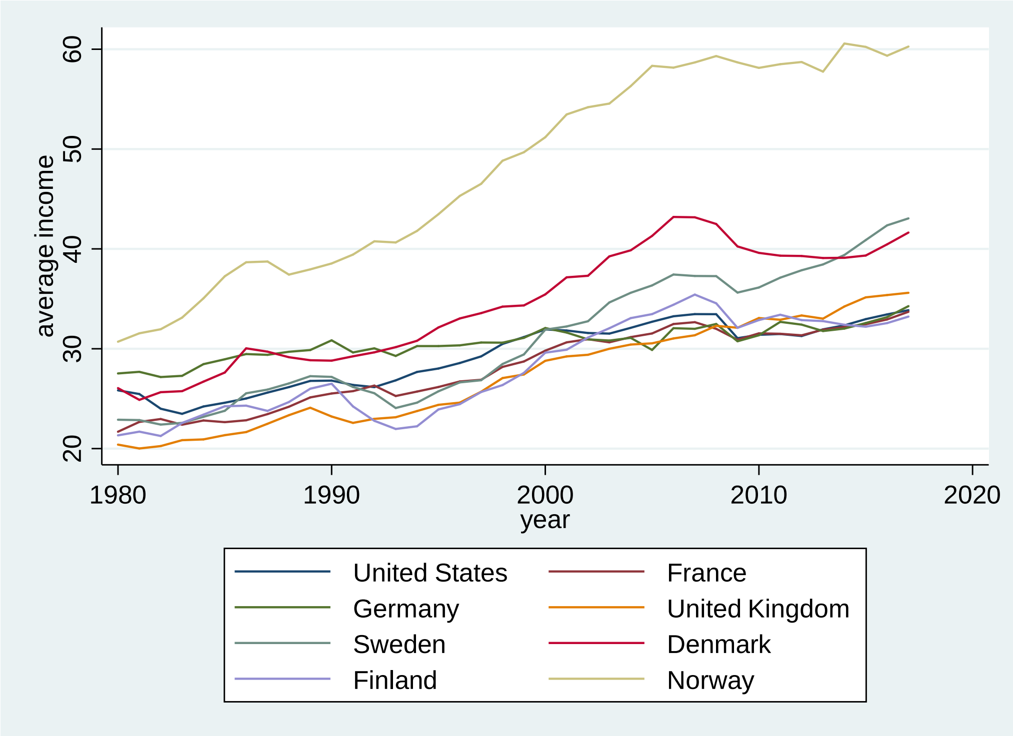


Figure 4. Average Income (thousands of USD) of 3rd Quintile (41-60%), 1980-2017

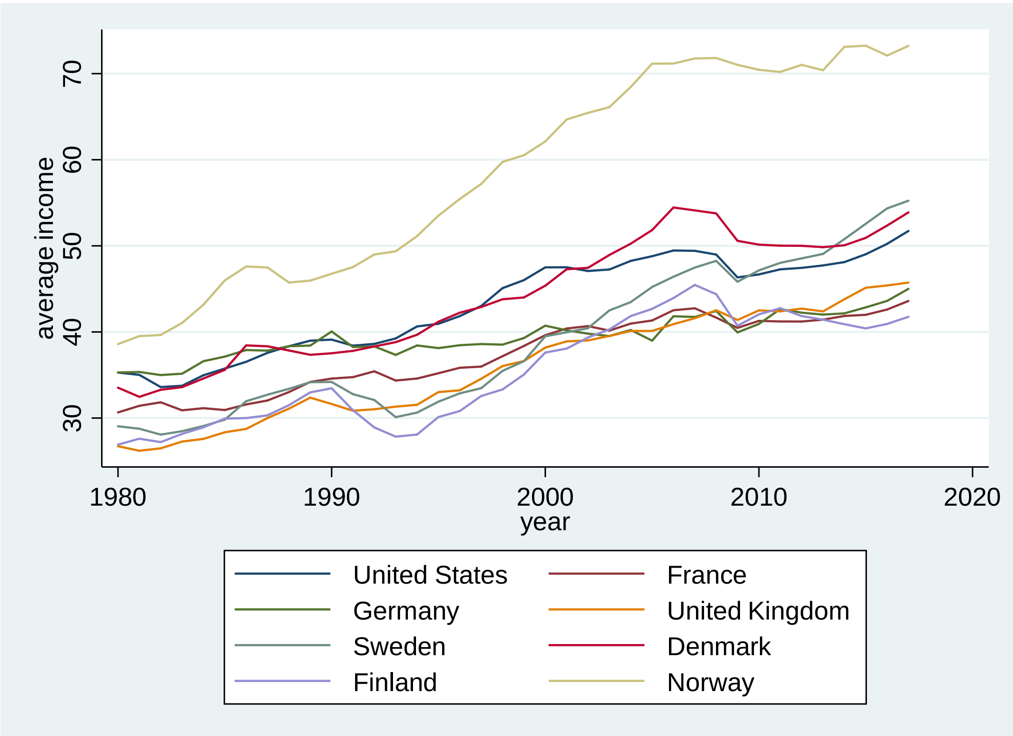


Figure 5. Average Income (thousands of USD) of 4th Quintile (61-80%), 1980-2017

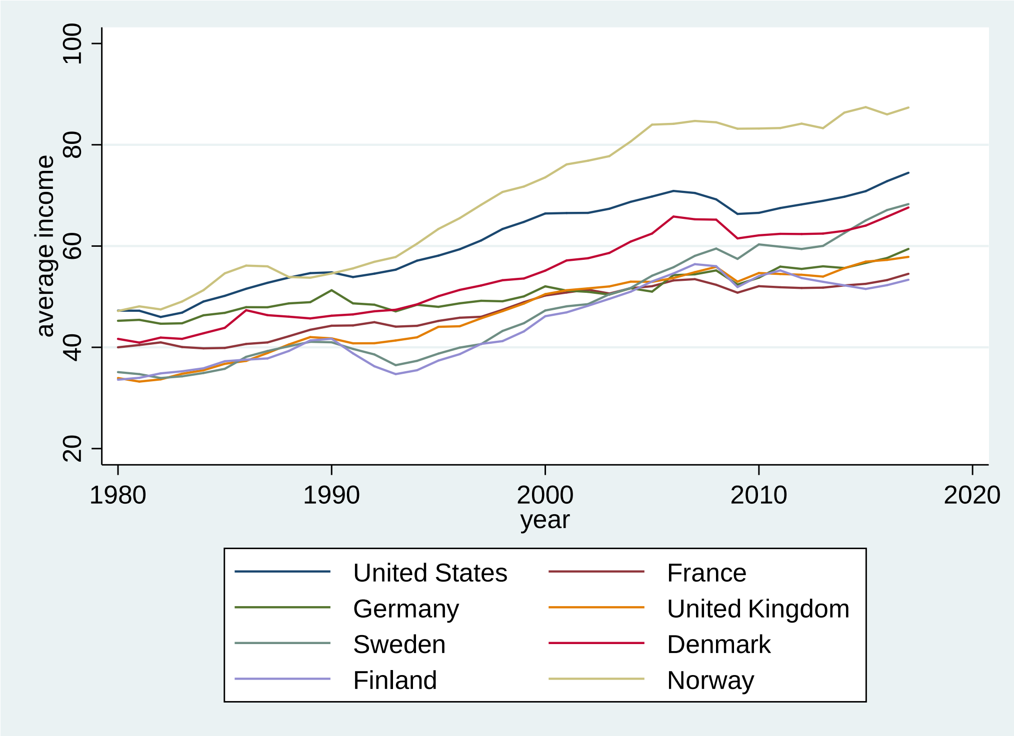


Figure 6. Average Income (thousands of USD) of 5th Quintile (The Top 20%), 1980-2017

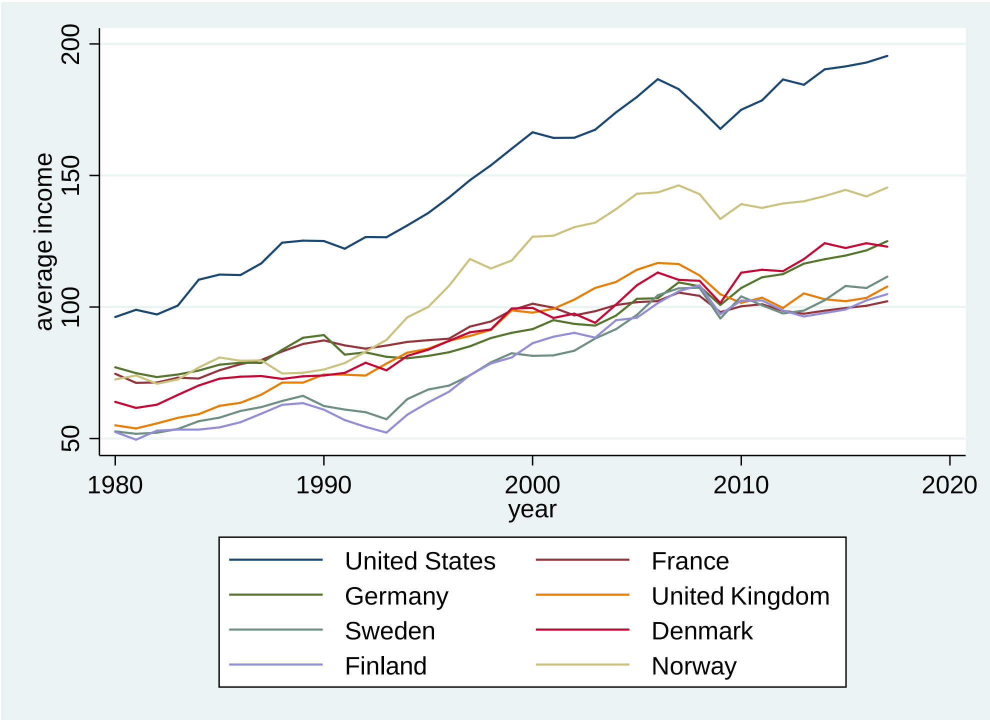
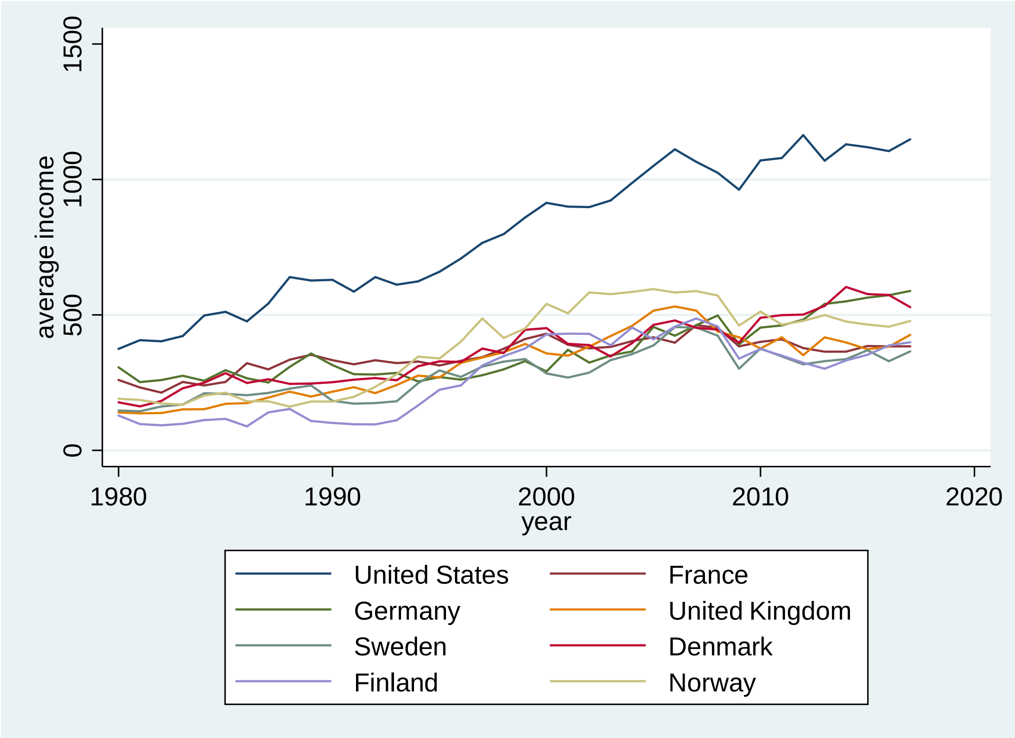


Figure 7. Average Income (thousands of USD) of the Top 1%, 1980-2017



Comparison by Single Parameter Gini Index

This subsection investigates the relationship between the inequality aversion parameter and social welfare measured by the single-parameter Gini index in 2017. The single-parameter Gini social welfare ordering (Donaldson and Weymark 1980) is defined as follows:

$$\forall u_N, v_M \in U, u_N \succcurlyeq v_N \Leftrightarrow \widetilde{u_N} \succcurlyeq \widetilde{v_M} \Leftrightarrow \frac{1}{n^\delta}\sum_{i=1}^{n}[\ i^\delta - \ (i-1)^\delta]\,\widetilde{u_\iota} \geqq \frac{1}{m^\delta}\sum_{i=1}^{m}[\ i^\delta - \ (i-1)^\delta]\,\widetilde{v_\iota},$$

where $\widetilde{u_N}$ is a non-increasing rearrangement of the income vector u_N, that is, $\widetilde{u_1} \geqq \widetilde{u_2} \geqq \cdots \geqq \widetilde{u_n}$ and δ is the inequality aversion parameter of this function. In general, the larger this parameter is, the closer to the maximin social welfare ordering it is. In fact, when δ is zero, it must be a maximax social welfare ordering (comparison of the highest income). When δ is 1, it must be an average utilitarianism (comparison of national income per capita). When δ is 2, it must be an average income × (1 - Gini coefficient). When

δ is infinity, it must be a maximin social welfare ordering (comparison of the lowest income). In the calculation of the single-parameter Gini index, we use the data on average incomes for each 1% due to data availability. This means that the values cannot be the exact ones in the single-parameter Gini index, but it can provide a sufficient approximation.

Next, we compute each value of single-parameter Gini index for each parameter. Figure 8 and Table 2 show the relationship between the single-parameter Gini index and inequality aversion parameter. As shown in figure 8 and table 2, the US social welfare is lower than that of the other countries around δ = 2. If people agree that the Gini coefficient is one of appropriate candidates among various inequality measurements, this suggests that the US national management has failed compared to the other developed countries from the perspective of the single-parameter Gini index despite of its glorious achievements such as high GDP per capita and continuous economic growth.

Figure 8. Logarithmic Single-Parameter Gini Index and Inequality Aversion Parameter in 2017

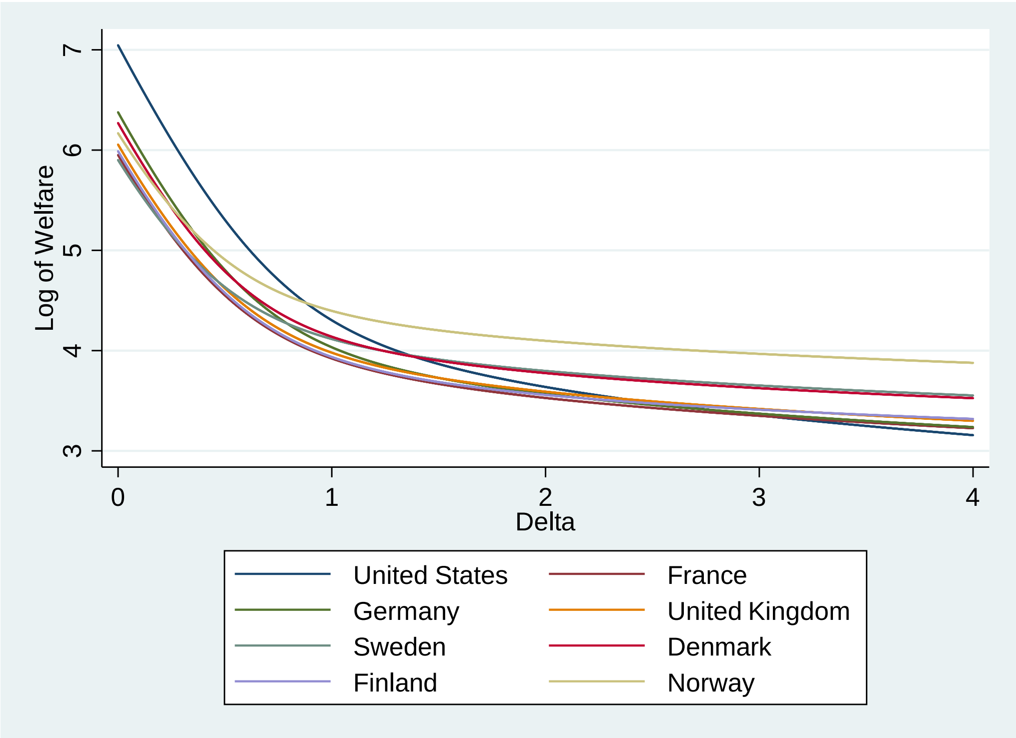


Table 2. Single-Parameter Gini Index (thousands of USD) and Inequality Aversion Parameter in 2017

	$\delta = 0$ (Top 1% Average Income)	$\delta = 1$ (National Income per capita)	$\delta = 2$ (Average Income × (1- Gini))	$\delta = 3$
United States	1148.3	73.9	38.0	28.6
Germany	588.6	56.5	35.4	29.1
Denmark	528.6	62.7	43.6	37.5
Norway	477.6	81.2	60.2	52.9
United Kingdom	426.4	53.5	36.2	30.5
Finland	399.0	51.2	35.1	30.3
France	384.1	50.5	34.0	28.5
Sweden	365.5	61.3	44.6	38.6

At the end of this section, we summarize the differences between the usual practices on policy evaluations and the stepwise rank-dependent utilitarianism. There are at least three popular measures in applied economics such as labor economics and development economics: range; top/bottom ratio; and median income.

Firstly, the *range* is defined as the difference between the highest and the lowest income. Obviously, it measures the maximum value of income gap. By this definition, it has obvious disadvantages: there is no consideration for intermediate income groups except for the top and the bottom; any improvement of income gap among the middle group cannot increase social welfare. Hence, this method violates strong Pareto and Pigou-Dalton transfer equity.

Secondly, the *top/bottom ratio* is defined as the ratio of the average income of the top $\bar{\alpha}$% and the bottom $\underline{\alpha}$% for some $\bar{\alpha}$ and $\underline{\alpha}$ in (0, 100), It measures social income gap based on a relative scale. By the similar way to the range, this also has major disadvantages: there is no consideration for intermediate income groups except for the top and the bottom; any improvement of income gap among the middle group cannot increase social welfare; it is only a ratio and cannot reflect income growth; social welfare is not invariant with respect to cardinal full comparability. Hence, it violates strong Pareto, Pigou-Dalton transfer equity, and cardinal full comparability.

Finally, a *median income* is just a median in the income distribution. By the definition, this also has the similar disadvantages: there is no consideration for other income groups except for the median; any improvement of income gap among the group cannot increase social welfare. Hence, it violates strong Pareto and Pigou-Dalton transfer equity.

Therefore, there are significant problems with the above three popular social indicators, and the *k*-quantile comparison method seems to have great advantages because of its desirable properties. Although there is still a need for social judgment as to how to decide each weight for each quantile in the *k*-quantile mean comparison method, the fact that the class of acceptable social welfare functions is equivalent to the stepwise rank-dependent utilitarianisms seems to be very important. In other words, for any policy evaluation, it is justified from the axiomatic characterization results to divide the income group into some quantiles and analyze the policy effect on the average income of each income group. Depending on the weight of each quantile, it may be possible that the great benefits of the middle group will offset the small loss of the bottom group. The stepwise rank-dependent utilitarianism allows us to directly investigate the policy impact on each income quantile and this seems to be a step forward in the context of policy evaluation practices and measuring social welfare.

6. Concluding Remarks

This study proposes a new class of versatile, efficient, and equitable social welfare orderings for social choice with variable populations. The stepwise rank-dependent utilitarianism is characterized by the desirable axioms: strong Pareto, anonymity, Pigou-Dalton transfer equity, rank-separability, continuity, cardinal full comparability, and replication equivalence. Although this ordering includes the single-parameter Gini index, it retains the maximum degree of freedom in weight selection since any weight vectors are possible. Furthermore, the stepwise rank-dependent utilitarianism can be extended and generalized in a natural way by demanding extended continuity and positive responsiveness to population increment. Notably, the stepwise rank-dependent utilitarianism can provide a simple social welfare ordering in the form of the *k*-quantile mean comparison method. By combining previous results and our findings, theoretical relationships between interpersonal comparability of individual well-being and acceptable aggregation rules would become clear. Suppose that the society could agree to require rank-separability instead of separability.[38] If ordinal full interpersonal comparability were admissible, it would be appropriate to use the stepwise leximin for comparing social welfare among different population sizes. The stepwise rank-dependent Kolm-Pollack social welfare ordering or the stepwise rank-dependent Atkinson-Blackorby-Donaldson social welfare ordering should be used if cardinal partial comparability were admissible. Under cardinal full comparability, it makes sense to use the stepwise rank-dependent utilitarianism. Of course, this study takes only a small step toward the goal of creating a practical approach to compare social welfare with different population sizes. However, the authors are convinced that our generation will be able to provide some practical social welfare orderings characterized by a system of relatively uncontroversial axioms. Although there are still more issues to consider, our generation has a

[38] If we required some type of independence conditions on utility profiles or separability, which are stronger requirements than rank-separability, then any stepwise types of social welfare orderings would vanish and only the class of average social welfare orderings could survive.

bright and hopeful future for normative economics originated from Pigou's welfare economics (Pigou 1920) and Arrow's social choice theory (Arrow 1951; 1963).[39]

The following outlines the remaining challenges for further research.

First, it seems sound to apply the stepwise forms of social welfare orderings to intertemporal social choice problems, but it may be problematic to apply them to the context of intergenerational equity. Although the theory of intergenerational equity compares infinite streams of utilities, it is impossible to predict the streams of utilities over infinite period.[40] Moreover, even if there exists a social welfare ordering extension that satisfies some desirable properties (Bossert, Sprumont and Suzumura 2007), this ordering extension may be inherently incomputable (Zame 2007; Lauwers 2010). In addition, preferences of future generations are essentially unstable, and there are the *non-identity problem* and path-dependency problem in which tastes and values of future generations could be altered by the selections of past generations (Parfit 1984; Suzumura and Tadenuma 2007). In order to deal with the above issues, our era's mathematical tools and models have big problems, so further path-breaking improvement is necessary in all aspects of prediction methods, theoretical models, and intergenerational ethics.

Second, our results can be easily applied to the theory of social choice for assessing risky social situations. Given the celebrated *Harsanyi's aggregation theorem* (Harsanyi 1955), the society must face the trade-off between ex-ante and ex-post equity. As Fleurbaey (2010) shows, if the society could agree to

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[39] This difficult field in economics has numerous impossibility theorems (Suzumura 2001; Sen 1970, 2017). However, as this study shows, it is still possible to construct efficient, equitable, and consistent social welfare orderings. The authors believe that the theory of social choice is full of not only *impossibilities* but various *possibilities* because of its wide range of applications and its importance as a "*science of evaluation*".

[40] Economic models for predicting climate change, economic growth, and financial crises have obvious deficiencies due to vulnerability of model error and the dependency of too-variable parameters. The best model our generation currently knows has many potential errors, even in forecasting for decades. The issues of intergenerational equity should not be analyzed by a model for an infinite period, but by a model for a finite period, and it may be necessary to take measures such as incorporating utilities of future generations after the term period into an objective function as one variable. Alternatively, it may be better to use a method of comparing only the ones that converge to a steady state in possible infinite utility streams.

require weak or strict dominance and ex-post Pareto principle (some restricted versions of ex ante Pareto), then the only option would be ex-post types of social welfare functions. Note that the stepwise forms of social welfare orderings are easily applied to both ex-ante and ex-post types of social welfare functions with risky situations. However, the problem of whether ex-ante or ex-post types of social evaluation should be preferable requires further consideration.

Third, in order to make the *k*-quantile mean comparison method more practical, we need some technical improvements in admissible weighting, appropriateness of the number and the width of quantiles, handling of measurement errors, a treatment of population inflows and outflows, and incompleteness of value judgments. It is essential to accumulate various verifications on the robust results to guarantee that the analysis is consistent with our normative judgments on social inequalities and injustice. Furthermore, when using the *k*-quantile mean comparison method based on household incomes adjusted by purchasing power, it seems necessary to reexamine both the standard consumer theory and usual practice on the aggregated price level and the purchasing power parity in economics. In comparing living standards of home and foreign countries, consumption bundles greatly differ among the poor and the rich in all countries. Hence, there will be cognitive and substantial problems in comparing income levels adjusted by purchasing power to judge collective welfare of different groups in different countries.[41] It may be necessary to devise a new method of comparison of purchasing power and calculation of aggregated price levels.

Fourth, the methods of stepwise rank-dependent utilitarianisms and stepwise leximin can be

[41] Although farmer’s self-consumption and street vendor’s economic transactions often do not appear in official statistics, they are likely to be important sources of income or consumption for low-income groups in developing countries. Hence, comparisons based on simple official statistics would make it difficult to evaluate accurate standards of livings in developing countries. There is no doubt that the capability approach (Sen 1985), which focuses on what people can do or can be, rather than simply comparing incomes or consumptions based on purchasing power, could be the most appropriate method for evaluating human’s well-being. However, the formulation of the capability approach and the attempt to practically measure capabilities will face significant difficulties when trying to consider the aspect of choice opportunities rather than the consequences of choice.

easily applied to the other well-being measurements, such as Better Life Index, multi-dimensional poverty indices, and some inequality indices on ordinal variables. For example, in each item's judgment in the Better Life Index, the *k*-quantile mean comparison method and stepwise leximin, respectively, are applied to cardinal measures such as household incomes and ordinal scales (categorical variables) such as security. Of course, when summarizing all items and evaluating social welfare instead of comparing each item between countries, it would be necessary to develop and construct a class of desirable aggregation rules that properly reflect the results in comparisons of each item.

Finally, considering the problem of optimal population sizes in the context of environmental problems and social welfare programs, the behavior of population size in a social objective function is highly important. Generally speaking, the optimal population size can be obtained by maximizing the generalized form of social welfare function $W(n, V(u_N))$ proposed in this paper. However, this function says nothing on the specific functional form of population size. This implies that *optimality* of population needs further investigations into the extent of whether inequalities of well-being distributions with population increments are *tolerable*. In this sense, social choice theory requires serious consideration in terms of philosophical, empirical, and evolutionary perspectives on the values of human-beings, sustainability, and a desirable society.

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