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Equity Principles and Interpersonal Comparison of Well-being: Old and New Joint Characterizations of Generalized Leximin, Rank-dependent Utilitarian, and Leximin Rules

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Equity Principles and Interpersonal Comparison of Well-being: Old and New Joint Characterizations of Generalized Leximin, Rank-dependent Utilitarian, and Leximin Rules

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Abstract

This paper characterizes new efficient and equitable social welfare orderings when individual well-beings are fully interpersonal comparable. Previous studies show that social welfare orderings satisfying the axioms of strong Pareto, anonymity, separability, and minimal equity are either weak utilitarian or leximin rules. By dropping the separability axiom, this study shows that there are various classes of distribution-sensitive social welfare orderings. In fact, simply imposing rank-separability instead of separability enables a class of social welfare orderings satisfying the axioms of strong Pareto, anonymity, and Pigou-Dalton transfer equity to be a generalized leximin rule (a general distribution-sensitive rule including leximin, rank-dependent utilitarianism, and their lexicographic compositions). This result is proved by a simple method that is intuitive

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I dedicate this paper to the memory of Professor Kotaro Suzumura. Suzumura *sensei* (*sensei* means teacher or professor in Japanese) had given me helpful comments and encouragement on my research for many years. For the last few years, I have been very proud of being able to work on some research projects with Prof. Suzumura. He told me: if you studied seriously and tried to contribute to this difficult field in economics, you would always be connected to the most excellent and sophisticated scholars all over the world, regardless of where you are located. His great soul and noble aspirations led me study normative economics and they have been one of driving forces of my research. I would like to sincerely express my gratitude for everything I gained from interacting intellectually with him. May his great soul be in peace. I also offer my special thanks for the grants I received from Grant-in-Aid for Scientific Research (C) (grant number: 19K01683) and the Joint Research Center for this research. I appreciate thoughtful comments and kind encouragements from Profs. Kotaro Suzumura, Naoki Yoshihara, Reiko Gotoh, Bertil Tungodden, John Weymark, Alex Voorhoeve, and Marc Fleurbaey, which were very helpful in writing this paper.

and easy to understand without the need for advanced mathematical techniques, such as functional analysis and the hyperplane separation theorem, which are often used in typical social choice analyses. Following this new proof, the mechanism by which a class of reasonable social welfare orderings satisfying separability is limited to weak utilitarian and leximin rules can be easily understood and proved. This study also shows the impossibility theorem between the axioms of equity and continuity. Based on the results of previous studies and this paper, theoretical relationships between interpersonal comparability of individual well-being and equality axioms are clarified. That is, if the interpersonal comparability of well-being is a cardinal unit or ratio one, then Paretian and anonymous social welfare orderings are limited to Kolm-Pollack or Atkinson social welfare functions. If it is the ordinal level comparability, the desirable rule must be leximin. If it is the cardinal full comparability, the generalized leximin should be used.

Key Words: Social Welfare Ordering, Joint Characterization, Generalized Leximin
JEL Code: D71, D81

1. Introduction

In all public value judgments, such as economic evaluations of public health, infrastructure planning, redistributive taxation, and welfare policies to reduce poverty, there is a need to aggregate individual preferences and well-beings to assess social welfare. However, given a specific measure of individual well-being, how should we aggregate well-being to obtain a consistent social value judgment? What aggregation method is preferable according to the degree of intrapersonal or interpersonal comparability of individual well-being?

The starting point of welfare economics on this aggregation problem faced with many impossibilities and various difficulties. In the 1930s and 1940s, Bergson and Samuelson proposed the concept of Paretian social welfare function, which aggregates a set of ordinal interpersonal noncomparable utility functions to a real-valued function (Bergson 1938; Samuelson 1947). However, the shape of this social welfare function is unclear except that it is monotonic in each individual utility. Furthermore, which social welfare function fulfill desirable properties has not been determined[1].

Arrow's celebrated impossibility theorem solves this aggregation problem in a nihilistic way (Arrow 1951; 1963). He proves that any aggregation method that maps a profile of *ordinal* individual utilities to a consistent social value judgment (i.e., social *ordering*) must be a dictatorship, if it is required to satisfy an information efficiency condition (i.e., *independence of irrelevant alternatives*) and weak Pareto principle. As long as the individual utility's information is ordinal and interpersonally noncomparable, no social aggregation method satisfies desirable properties[2]. Bergson and Samuelson's idea of a social welfare function turned out to be nothing but a "house of cards[3]."

A turning point was reached in Sen's important analysis of invariance conditions of intra- and interpersonal comparisons of individual well-being. He shows that allowing for the intrapersonal and interpersonal comparability of individual utility, that is, the comparability of ordinal levels, cardinal units, and cardinal ratios, with aggregation methods changes Arrow's impossibility theorem to achieve possibility results (Sen 1970a; 1970b). Many successes were followed by enlightening works (Hammond 1976;

1 See Sen (1970; 2017) and Suzumura (2002) for details on the history of the impossibilities of welfare economics.

2 It is known that weakening Arrow's conditions cannot make a significant difference in the impossibility results as long as individual utilities are based on ordinal and interpersonal noncomparable one. See Sen (1970; 2017).

3 See Suzumura and Samuelson (2005) for excellent surveys and an interview with Samuelson.

d'Aspremont and Gevers 1977; 2002; Sen 1977; Dechamps and Gevers 1978; Maskin 1978; Roberts 1980a; 1980b; Blackorby and Donaldson 1982). These important studies show the following: (i) leximin is the only rule that satisfies desirable axioms in the framework of ordinal-level comparability of individual well-being (Sen 1970a; Hammond 1976; d'Aspremont and Gevers 1977); (ii) simple utilitarian, Kolm-Pollack, Nash, and Atkinson social welfare functions are reasonable social welfare orderings if cardinal partial comparability (i.e., the comparability of cardinal units or ratios) is admissible (Makin 1978; Blackorby and Donaldson 1982; d'Aspremont and Gevers 2002); (iii) weak utilitarian or leximin rules must be adopted if individual well-being is cardinal full comparable and a social welfare ordering has reasonable properties, the axioms of strong Pareto, anonymity, separability, and minimal equity (Dechamps and Gevers 1978)[4].

Do the results discussed in the literature indicate that a class of reasonable social welfare orderings under the cardinal full comparability is limited to only two rules, utilitarian or leximin? A well-known flaw in a simple utilitarian rule is that there is no consideration for distribution in comparing social welfare. The rule considers only the sum of individual utilities. On the other hand, the major drawback of the leximin rule is that it ignores overall welfare loss, and it cannot satisfy continuity because of the excessive consideration for distribution. An aggregation method that lies between these extreme rules is the generalized Gini inequality index (i.e., rank-dependent utilitarian rule), which was developed and proposed by Weymark (1981). According to the generalized Gini inequality index, social welfare is defined as the weighted sum of individual well-beings, in which higher weights are assigned to less well-beings. Therefore, unlike the simple utilitarian rule, it would be able to satisfy not only the equity axiom in distributive justice but also the continuity of social welfare[5]. However, since this social welfare ordering cannot satisfy the separability axiom, it has been eliminated in a system of axioms discussed in the literature.

The purpose of this study is to show various classes of distribution-sensitive social welfare orderings by weakening the demand for separability, which seemed the

[4] See d'Aspremont and Gevers (2002), Blackorby, Bossert and Donaldson (2002), Bossert and Weymark (2004), and Fleurbaey and Hammond (2004) for elegant summaries of theoretical relationships between interpersonal comparability of individual well-being and reasonable social welfare orderings.

[5] See Weymark (1981) for the theoretical properties of the generalized Gini inequality index. Assuming the cardinal full comparability of individual well-being, this social welfare ordering is shown to be characterized by the axioms of strong Pareto, anonymity, rank-separability, and continuity (Ebert 1988a).

least legitimate axiom, while maintaining the indisputable axioms of strong Pareto, anonymity, and full rationality of aggregation rules. In fact, if separability is dropped, and a weaker version of separability, rank-separability, is required, then a class of Paretian, anonymous, and distribution-sensitive social welfare orderings must be a generalized leximin rule that includes generalized Gini inequality indices (i.e., rank-dependent utilitarian) and leximin.

The generalized leximin rule is defined as follows. First, the set of individuals is divided into subgroups according to a hierarchy of well-being profiles from the bottom to the top. Next, each weighted sum of well-being for each subgroup is calculated. Then, the rule lexicographically judges well-being profiles following their sequences of the weighted sum of well-being as defined above. If all subgroups are singletons, it is equivalent to the simple leximin rule. If the subgroup is an entire set of individuals, it is equivalent to the generalized Gini inequality index. Thus, the generalized leximin is a broad class of distribution-sensitive social welfare orderings that can include both the rules of leximin and the generalized Gini inequality index. The proof of this theorem is elementary and intuitive, although previous studies uses advanced mathematical methods, such as a functional analysis and the hyperplane separation theorem to prove many axiomatic characterizations of social welfare orderings. By using the simple proof method demonstrated in this paper, the mechanism by which reasonable social welfare orderings satisfying separability are limited to the weak utilitarian or leximin rules could be easily understood[6]. In addition, this study shows that *only* the extreme equity axiom, which is a stronger version of Hammond equity, contradicts the continuity of social welfare orderings[7]. Hence, continuous social judgments could be consistent with most equity requirements.

The main contributions of this research are summarized as follows. First, this paper finds various distribution-sensitive social welfare orderings satisfying both strong Pareto and anonymity if they are required to satisfy not separability but rank-separability. That is, there is no binary choice between utilitarian rule (no consideration for well-being

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[6] Under the cardinal full comparability, Dechamps and Gevers (1978) shows a beautiful result in which social welfare orderings satisfying the axioms of strong Pareto, anonymity, and separability are either weak utilitarian or leximin rules. However, their proof is extremely complicated and difficult to understand. This study succeeds in providing a simple proof.

[7] Extreme equity requires that social welfare should be strictly increased whenever a transfer improves a well-being gap between two individuals. Generally, social ordering satisfying Hammond equity and continuity exists in the form of null, fixed, and maximin rules.

distribution) and leximin rule (excess consideration for distribution and few interests in total welfare) in seeking desirable aggregation methods. A society can select desirable social welfare orderings from a class of various distributive-sensitive rules. Second, the paper discovers a new class of distribution-sensitive social welfare orderings, called a generalized leximin rule, which includes both rank-dependent utilitarian and leximin, and succeeds in characterizing it. Third, the paper proves that only the extreme equity is incompatible with the continuity, and most equity requirements are consistent with the continuity. Fourth, this study provides an elementary proof that makes it easier to understand how each axiom works in axiomatic characterizations. Moreover, the technique of this proof successfully proves the results of previous studies in an easy-to-understand format. Fifth, by combining these results with the findings in the literature, the theoretical correspondence between equity axioms and the comparability of individual well-beings is clarified.

This paper is structured as follows. The next section describes basic notations and definitions. Section 3 defines the social welfare orderings that are axiomatically characterized in this paper, such as the simple utilitarian rule, leximin rule, rank-dependent utilitarian rule, generalized leximin rule, and their lexicographic compositions. Section 4 shows a set of characterization results in which a class of social welfare orderings satisfying the axioms of strong Pareto, anonymity, and Pigou-Dalton transfer equity must be a generalized leximin (resp. the weak utilitarian or leximin rules) if it satisfies the rank-separability (resp. separability). Finally, Section 5 discusses the remaining issues.

2. Basic Notations and Definitions

This section explains the basic notations, definitions, and axioms used in this paper. Let $N = \{1, \ldots, n\}$ be the set of individuals. The set of all possible well-being vectors is denoted by U^N. Without the loss of generality, suppose that $U^N = \mathbb{R}^N$. For all $u_N \in U^N$, let $u_{[N]} = (u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ be a non-decreasing rearrangement of the well-being vector u_N, that is, $u_{[1]} \leqq u_{[2]} \leqq \ldots \leqq u_{[n]}$. The set of ranks is denoted by $[N] = \{[1], [2], \ldots, [n]\}$. For an arbitrary set X, a binary relation defined on X is an *ordering* if and only if it satisfies completeness and transitivity[8]. Let a *social welfare ordering* $\succcurlyeq$ be defined on U^N. For all $u_N, v_N \in U^N$, $u_N \succcurlyeq v_N$ means that u_N is at least as socially good as v_N. Asymmetric and symmetric parts of a social welfare ordering $\succcurlyeq$ are given by $\succ$ and $\sim$, respectively.

Note that all social welfare orderings are social *ordering* functions because they always generate an ordering defined on the set of well-being profiles[9]. Each individual well-being is assumed to be cardinal full comparable as follows:

Full Interpersonal Comparability of Individual Well-being: $\forall u_N, v_N \in U^N$, $\forall a \in \mathbb{R}$, $\forall b \in \mathbb{R}_{++}$, $u_N \succcurlyeq v_N \Leftrightarrow (a+bu_i)_{i \in N} \succcurlyeq (a+bv_i)_{i \in N}$.

Next, let us define a series of axioms that require social welfare orderings to be satisfied. First, as an axiom of efficiency, the paper requires strong Pareto principle.

Strong Pareto: $\forall u_N, v_N \in U^N$, if $u_N \geqq v_N$, then $u_N \succcurlyeq v_N$. Moreover, if $u_N > v_N$, then $u_N \succ v_N$[10].

Throughout the paper, all social welfare orderings must treat each individual well-being equally, and this requirement is represented by the following anonymity axiom.

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[8] Completeness requires that, for all x, y in X, $x \succcurlyeq y$ or $y \succcurlyeq x$. Transitivity requires that, for all x, y, z in X, ($x \succcurlyeq y$ & $y \succcurlyeq z$) implies $x \succcurlyeq z$.

[9] The framework in this paper implicitly assumes that a social ordering function satisfies *independence of irrelevant alternatives* and *Pareto indifference*. The details of theoretical relationships between welfarism, neutrality, IIA, and Pareto principle are investigated in d'Aspremont and Gevers (1977; 2002).

[10] For all $u_N, v_N \in U^N$, [$u_N \geqq v_N$ iff $u_i \geqq v_i$ for all i] and [$u_N > v_N$ iff $u_i \geqq v_i$ for all i and $u_j > v_j$ for some j].

Anonymity: $\forall$ bijections π on N, $\forall u_N \in U^N$, $u_N \sim u_{\pi(N)}$.

Some results may require the following continuity of social welfare orderings. This axiom demands that both the upper contour set and the lower contour set of social welfare ordering be closed.

Continuity: $\forall u_N \in U^N$, both $\{v_N \in U^N \mid v_N \succcurlyeq u_N\}$ and $\{v_N \in U^N \mid u_N \succcurlyeq v_N\}$ are closed.

Separability requires social welfare orderings to ignore well-being information about indifferent individuals between two profiles. As shown in the proof of the later theorem, this axiom plays a central role in the famous joint characterization theorem, where Paretian and anonymous social welfare orderings must be weak utilitarian, leximin, or leximax rules.

Separability: $\forall u_N, v_N, u'_N, v'_N \in U^N$, if $\exists M \subseteq N$, ($\forall i \in M$, $u_i = u'_i$ & $v_i = v'_i$) and ($\forall j \in N \setminus M$, $u_j = v_j$ & $u'_j = v'_j$), then $u_N \succcurlyeq v_N \Leftrightarrow u'_N \succcurlyeq v'_N$.

The weaker version of separability is called the following rank-separability.

Rank-separability: $\forall u_{[N]}, v_{[N]}, u'_{[N]}, v'_{[N]} \in U^N$, if $\exists [M] \subseteq [N]$, ($\forall i \in [M]$, $u_{[i]} = u'_{[i]}$ & $v_{[i]} = v'_{[i]}$) and ($\forall j \in [N] \setminus [M]$, $u_{[j]} = v_{[j]}$ & $u'_{[j]} = v'_{[j]}$), then $u_{[N]} \succcurlyeq v_{[N]} \Leftrightarrow u'_{[N]} \succcurlyeq v'_{[N]}$.

This axiom requires social welfare orderings to ignore well-being information about the same value of well-being in the same ranks between two profiles. Obviously, separability implies rank-separability if anonymity is required to social welfare orderings. Section 4 demonstrates that simply imposing rank-separability instead of separability yields a variety of classes of distribution-sensitive social welfare orderings.

Finally, let us introduce several axioms of equity. Pigou-Dalton transfer equity states the following: given that well-beings of other persons are fixed, and there is a well-being gap between two individuals, the same amount of well-being transfer that improves the well-being gap will not at least reduce social welfare. This requirement would be a reasonable if each well-being were fully interpersonal comparable and there were no legitimate ethical reason for the well-being gap[11].

[11] In the tradition of normative economic analysis, welfaristic consequentialism has been predominant, and almost all axioms proposed so far are based on consequentialism. However, welfaristic consequentialism has several drawbacks in

Pigou-Dalton Transfer Equity: $\forall u_N, v_N \in U^N$, $\forall \varepsilon \in \mathbb{R}_{++}$, if $\exists i, j \in N$, $v_i - \varepsilon = u_i \geqq u_j = v_j + \varepsilon$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succcurlyeq v_N$.

Next, we define Hammond equity, which is an important concept of equity in characterizing the leximin rule. The axiom demands that, given that well-beings of other persons are fixed, a well-being transfer that reduces a well-being gap between two individuals will not decrease social welfare. Note that this transfer never lessens social welfare no matter how much the rich's well-being is sacrificed to improve slightly the poor's well-being.

Hammond Equity: $\forall u_N, v_N \in U^N$, if $\exists i, j \in N$, $v_i > u_i \geqq u_j > v_j$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succcurlyeq v_N$.

In general, Hammond equity is consistent with the continuity of social welfare ordering. For example, the following three rules satisfy both Hammond equity and continuity: the null rule which sees all well-being profiles as indifferent; the trivial rule which generates a fixed social ordering; the maximin rule which sees only the worst well-being. It is easy to show that the axioms of Hammond equity, continuity, and strong Pareto are incompatible[12]. To investigate the degree of incompatibility between the axioms of equity and continuity, we consider the following extreme equity, which is a strong version of Hammond equity.

Extreme Equity: $\forall u_N, v_N \in U^N$, if $\exists i, j \in N$, $v_i > u_i \geqq u_j > v_j$ and $\forall k \in N \setminus \{i, j\}$, $v_k = u_k$, then $u_N \succ v_N$.

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measuring social welfare. Suppose that there are two individuals with the same abilities, environment, and personality but different well-being levels. This gap is caused only by differences in their labor efforts. Is it important for a society to reduce the gap following Pigou-Dalton transfer equity?--Claiming that there is a social benefit would be controversial. It seems natural that social welfare requires not only welfaristic consequentialism but also non-consequentialism, which considers both the values of procedural justice and consequential fairness. See Pattanaik and Suzumura (1994), Fleurbaey (1994), and Suzumura and Xu (2001) for pioneering studies on non-consequentialist theories in normative economics.

[12] Consider two well-being profiles (1, 2) and (1, 3) in a two-individual economy. By strong Pareto, (1, 3) is strictly better than (1, 2). Hammond equity and strong Pareto implies $(1+\varepsilon, 2)$ is strictly better than (1, 3) for all $\varepsilon \in (0, 1)$. Hence, continuity requires (1, 2) is at least as good as (1, 3). A contradiction.

Among the numerous requirements of equity, only the extreme equity is incompatible with continuity. Let us analyze a relationship between the axioms of equity and continuity in social welfare orderings. It is well known that the leximin rule is a typical example that satisfies extreme equity but not continuity. In contrast, the rank-dependent utilitarian rule is a paragon of social welfare orderings satisfying the axioms of both equity and continuity. These rules seem to show that there is no serious conflict between requirements of equity and continuity. In fact, continuity is incompatible only with extreme equity, and it is compatible with all the other equity axioms in this paper.

Theorem 1: *Suppose $|N| > 3$. Then there exists no social welfare ordering satisfying extreme equity and continuity.*

Proof: Suppose to the contrary that a social welfare ordering $\succcurlyeq$ satisfies extreme equity and continuity. Consider a profile $u_{[N]} = (u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ where $u_{[1]} < u_{[2]} < \ldots < u_{[n]}$. Then, we prove Theorem 1 by showing two claims.

(Claim 1) $\forall [i] \neq n$, $\forall u'_{[i]}$ with $u_{[i]} < u'_{[i]} < u_{[i+1]}$, $(u'_{[i]}, u_{-[i]}) \succcurlyeq u_{[N]}$.
Extreme equity implies that $(u'_{[i]}, u_{[i+1]} - \varepsilon, u_{-[i, i+1]}) \succ u_{[N]}$ for all ε with $u'_{[i]} < u_{[i+1]} - \varepsilon$. By continuity, we have $(u'_{[i]}, u_{-[i]}) \succcurlyeq u_{[N]}$. Hence, Claim 1 holds true. ■

(Claim 2) $\forall [i] \neq 1$, $\forall u'_{[i]}$ with $u_{[i-1]} < u'_{[i]} < u_{[i]}$, $(u'_{[i]}, u_{-[i]}) \succcurlyeq u_{[N]}$.
Extreme equity implies that $(u_{[i-1]} + \varepsilon, u'_{[i]}, u_{-[i-1, i]}) \succ u_{[N]}$ for all ε with $u'_{[i]} > u_{[i-1]} + \varepsilon$. By continuity, we have $(u'_{[i]}, u_{-[i]}) \succcurlyeq u_{[N]}$. Hence, Claim 2 holds true. ■

Claims 1 and 2 mean that $\forall [i] \neq 1, n$, $\forall u'_{[i]}$ with $u_{[i-1]} < u'_{[i]} < u_{[i+1]}$, $(u'_{[i]}, u_{-[i]}) \sim u_{[N]}$. Therefore, $\forall [i], [j] \neq 1, n$ with $[i] < [j]$, $\forall$ sufficiently small numbers $\varepsilon, \delta > 0$, $u_{[N]} \sim (u_{[i]} + \varepsilon, u_{[j]}, u_{-[i,j]}) \sim (u_{[i]} + \varepsilon, u_{[j]} - \delta, u_{-[i,j]})$. However, extreme equity implies $(u_{[i]} + \varepsilon, u_{[j]} - \delta, u_{-[i,j]}) \succ u_{[N]}$. A contradiction. ■

Note that when the number of individuals is three or less, the following social welfare ordering satisfies both continuity and extreme equity.

$$\forall u_N, v_N \in U^N, u_N \succcurlyeq v_N \quad \text{iff } u_{[1]} - u_{[n]} \geqq v_{[1]} - v_{[n]}.$$

By definition, the above rule cannot satisfy strong Pareto[13]. In general, combining strong Pareto with Hammond equity implies extreme equity, but not vice versa[14]. Moreover, no rule satisfies Hammond equity, continuity, and strong Pareto. Hence, Theorem 1 is a stronger result than the fact that Hammond equity, continuity, and strong Pareto are incompatible. Since an impossibility with continuity and equity is limited to extreme equity, there is no need to force big concessions of equity to obtain continuous social orderings.

[13] When the number of individuals is two, a social welfare ordering satisfying extreme equity, continuity and weak Pareto if and only if it is a maximin rule.
[14] Extreme equity implies Hammond equity. Note that Hammond equity and continuity are compatible as the maximin rule indicates.

3. Social Welfare Orderings

This section introduces several social welfare orderings that will be characterized in the next section. The following rules basically satisfy the axioms of strong Pareto, anonymity, and separability (or rank-separability). First, let us introduce three social welfare orderings that will be jointly characterized in the setting of Paretian, anonymous and separable social welfare orderings.

Utilitarian Rule: A social welfare ordering $\succcurlyeq^{\mathbf{U}}$ is a *utilitarian rule* if and only if $\forall u_N, v_N \in U^N, \quad u_N \succcurlyeq^{\mathbf{U}} v_N \Leftrightarrow \sum_{i\in N} u_i \geqq \sum_{i\in N} v_i$.

The utilitarian rule is a method that judges social welfare by summing individual well-beings. If a strict inequality in the aggregated function always implies a strict social relation, and its equality does not necessarily imply a socially indifference relation, let us call such a utilitarian rule *weak utilitarian*.

Leximin Rule: A social welfare ordering $\succcurlyeq^{\mathbf{LM}}$ is a *leximin rule* if and only if $\forall u_N, v_N \in U^N, \quad u_N \succcurlyeq^{\mathbf{LM}} v_N \Leftrightarrow [\forall [i], u_{[i]} = v_{[i]}]$ or $[\exists [j], \forall [i]<[j], u_{[i]} = v_{[i]}$ & $u_{[j]} > v_{[j]}]$.

The leximin rule is a method that judges social welfare by following a lexicographic ordering that sequentially evaluates a hierarchy of well-being from the bottom to the top. It is well-known that this social welfare ordering has various forms of axiomatic characterizations[15].

Leximax Rule: A social welfare ordering $\succcurlyeq^{\mathbf{LX}}$ is a *leximax rule* if and only if $\forall u_N, v_N \in U^N, \quad u_N \succcurlyeq^{\mathbf{LX}} v_N \Leftrightarrow [\forall [i], u_{[i]} = v_{[i]}]$ or $[\exists [j], \forall [i]>[j], u_{[i]} = v_{[i]}$ & $u_{[j]} > v_{[j]}]$.

The leximax rule is a method that judges social welfare by following a lexicographic ordering that sequentially evaluates a hierarchy of well-being from the top to the bottom, which is usually interpreted as an unacceptable rule from the viewpoint of distributive justice. Next, we consider two versions of lexicographic compositions between the utilitarian and leximin rules[16].

[15] See Sen (1970a; 2017) for various axiomatizations of the leximin rule (T.A3*.6, pp. 382-383).
[16] The names of these lexicographic compositions refer to the name of the equity-first

Utilitarian-First and Leximin-Second Rule: A social welfare ordering $\succcurlyeq^{\mathbf{U\text{-}LM}}$ is a *utilitarian-first and leximin-second rule* if and only if $\forall u_N, v_N \in U^N$, $u_N \succcurlyeq^{\mathbf{U\text{-}LM}} v_N \Leftrightarrow [u_N \succ^{\mathbf{U}} v_N]$ or $[u_N \sim^{\mathbf{U}} v_N \ \& \ u_N \succcurlyeq^{\mathbf{LM}} v_N]$.

The utilitarian-first and leximin-second rule judges well-being profiles by the following lexicographic method. First, it judges two profiles following the utilitarian rule. Second, if the utilitarian rule is indifferent between them, then it applies the leximin rule for social judgment. Although it emphasizes the idea of utilitarianism, this rule can be interpreted as distribution-sensitive only if the total well-being sums are the same in the two profiles. This rule and the leximin rule can be jointly characterized by the axioms of strong Pareto, anonymity, separability, strict Pigou-Dalton transfer equity, and strict composite transfer principle, which is shown by Kamaga (2018) using the joint characterization of Dechamps and Gevers (1978). In contrast, the following leximin-first and utilitarian-second rule judges well-being profiles by inverting the above rule. That is, first, the leximin rule is applied to judge the profiles up to a specific rank. Second, if two profiles have the same well-being levels up to the specific rank, then the rule judges them by the simple utilitarian rule.

Leximin-First and Utilitarian-Second Rule: A social welfare ordering $\succcurlyeq^{\mathbf{LM\text{-}U}}$ is a *leximin-first and utilitarian-second rule* if and only if $\exists [j^*]$, $\forall u_N, v_N \in U^N$, $u_N \succcurlyeq^{\mathbf{LM\text{-}U}} v_N \Leftrightarrow [\exists [k] < [j^*], \forall [i] < [k], u_{[i]} \geqq v_{[i]} \ \& \ u_{[k]} > v_{[k]}]$ or $[\forall [i] \leqq [j^*], u_{[i]} = v_{[i]} \ \& \ u_N \succcurlyeq^{\mathbf{U}} v_N]$.

Since this social welfare ordering cannot satisfy separability, it is not derived from the axioms (i.e., strong Pareto, anonymity, and separability) required by Dechamps and Gevers (1978). However, it is an interesting rule because it is partially leximin and thus distribution-sensitive, and it has certain considerations for well-being sums. In the next section, it will be shown that if we abandon separability and demand only rank-separability, a wide class of distribution-sensitive rules, including this rule could survive. Furthermore, let us define a class of social welfare orderings satisfying rank-separability. These rules demonstrate that there are various distribution-sensitive social welfare orderings in addition to the simple leximin rule.

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and efficiency-second rule in Tadenuma (2002). See also Houy and Tadenuma (2009) and Sakamoto (2010; 2013) for results related to the theoretical relationships between lexicographic compositions and their consistency in measuring social welfare.

Rank-dependent Utilitarian Rule (Generalized Gini Inequality Index): A social welfare ordering $\succcurlyeq^{\mathbf{RDU}}$ is a *rank-dependent utilitarian rule* if and only if $\exists (w_{[i]}) \in \mathbb{R}^N_+$ with $w_{[1]} \geqq \ldots \geqq w_{[n]} \geqq 0$ & $\sum_{[i]\in[N]} w_{[i]}=1$, $\forall u_N, v_N \in U^N$, $u_N \succcurlyeq^{\mathbf{RDU}} v_N \Leftrightarrow \sum_{[i]\in[N]} w_{[i]}\, u_{[i]} \geqq \sum_{[i]\in[N]} w_{[i]}\, v_{[i]}$.

The generalized Gini inequality index, or the rank-dependent utilitarian rule, was proposed in a celebrated study by Weymark (1981), and it is known to be a generalization of the Gini coefficient[17]. This rule measures social welfare by a weighted sum of well-beings in which each weight assigned to well-being depends on each rank. As is clear from the definition, if a weight assigned to a lower rank were set to be higher, the rule would become distribution-sensitive. We then can define two versions of lexicographic compositions between the rank-dependent utilitarian and leximin rules in a manner similar to the simple utilitarian and leximin rules.

Rank-dependent Utilitarian-First and Leximin-Second Rule: A social welfare ordering $\succcurlyeq^{\mathbf{RDU\text{-}LM}}$ is a *rank-dependent utilitarian-first and leximin-second rule* if and only if $\forall u_N, v_N \in U^N$, $u_N \succcurlyeq^{\mathbf{RDU\text{-}LM}} v_N \Leftrightarrow [u_N \succ^{\mathbf{RDU}} v_N]$ or $[u_N \sim^{\mathbf{RDU}} v_N \;\&\; u_N \succcurlyeq^{\mathbf{LM}} v_N]$.

Leximin-First and Rank-dependent Utilitarian-Second Rule: A social welfare ordering $\succcurlyeq^{\mathbf{LM\text{-}RDU}}$ is a *leximin-first and rank-dependent utilitarian-second rule* if and only if $\exists [j^*]$, $\forall u_N, v_N \in U^N$, $u_N \succcurlyeq^{\mathbf{LM\text{-}RDU}} v_N \Leftrightarrow [\exists [k] < [j^*], \forall [i] < [k], u_{[i]} \geqq v_{[i]} \;\&\; u_{[k]} > v_{[k]}]$ or $[\forall [i] \leqq [j^*], u_{[i]} = v_{[i]} \;\&\; u_N \succcurlyeq^{\mathbf{RDU}} v_N]$.

Finally, let us define a new rule that includes all the above rules except the leximax rule.

Generalized Leximin Rule: A social welfare ordering $\succcurlyeq^{\mathbf{GLM}}$ is a *generalized leximin rule* if and only if $\exists$ a partition $([M_1], \ldots, [M_k])$ of $[N]$, $\exists$ a set of orderings defined on U^{Mj} with j in $\{1, \ldots, k\}$ $(\succcurlyeq^{M1}, \ldots, \succcurlyeq^{Mk})$, $\forall u_N, v_N \in U^N$, $u_N \succcurlyeq^{\mathbf{GLM}} v_N \Leftrightarrow [\forall [M_i], u_{[Mi]} \sim^{Mi} v_{[Mi]}]$ or $[\exists [j], \forall [i] < [j], u_{[Mi]} \sim^{Mi} v_{[Mi]} \;\&\; u_{[Mj]} \succ^{Mj} v_{[Mj]}]$ where $[\forall [i] < [j], \forall [k_i] \in [M_i], \forall [k_j] \in [M_j], [k_i] < [k_j]]$ & $\forall i$, $\succcurlyeq^{Mi}$ is a refinement[18] of rank-dependent utilitarian rule defined on $[M_i]$.

[17] If every difference between adjacent rank's weights is 2, the generalized Gini inequality index is equivalent to the Gini coefficient.

[18] For all binary relations $\succcurlyeq^{A}$, $\succcurlyeq^{B}$ defined on X, $\succcurlyeq^{A}$ is a refinement of $\succcurlyeq^{B}$ if and only if for all x, y in X, $x \succ^{B} y$ implies $x \succ^{A} y$.

The generalized leximin rule evaluates well-being profiles by following a lexicographic ordering defined on a sequence of weighted sums of subgroups' well-beings. Each subgroup is made of sequential numbers of ranks; and no subgroup exists in which the rank's number is not adjacent to others. Note that if a weighted sum of the lowest subgroup's well-beings were the same in two profiles, one could be better than the other because a binary relation defined on the set of well-being profiles in this subgroup may prefer one over the other.

As the definition clearly states, if the subgroups are all singletons, then it is equivalent to the simple leximin rule. If subgroups to the specific group are all singletons, and the remaining subgroup is the whole complement of them, then it must be either the leximin-first and rank-dependent utilitarian-second or leximin-first and utilitarian second rules. Furthermore, if a subgroup is equal to the entire set of individuals, then it must be a class of rules that includes the rank-dependent utilitarian, simple utilitarian, rank-dependent utilitarian-first and leximin-second, and utilitarian-first and leximin-second rules. Hence, this rule can be interpreted as the generalization of the entire class of distribution-sensitive rules satisfying rank-separability.

4. Old and New Joint Characterizations

This section shows that the generalized leximin rule defined in the previous section is characterized by the axioms of strong Pareto, anonymity, Pigou-Dalton transfer equity, and rank-separability. By dropping separability and requiring rank-separability, we obtain a broad class of Paretian, anonymous, and distribution-sensitive rules, including not only the leximin rule but also rank-dependent utilitarian rule and their lexicographic compositions. The proof of this result can be completed by several elementary claims without using any advanced mathematical results, such as functional analysis and the hyperplane separation theorem. Moreover, the proof can facilitate the understanding of the reason why the famous Dechamps and Gevers' joint characterization theorem holds true. First, let us show that social welfare orderings satisfying the axioms of strong Pareto, anonymity, rank-separability, and Pigou-Dalton transfer equity are limited to a generalized leximin rule.

Theorem 2 (A Characterization of a Generalized Leximin Rule): *A social welfare ordering satisfies the axioms of Full Interpersonal Comparability of Individual Well-being, Strong Pareto, Anonymity, Rank-Separability, and Pigou-Dalton Transfer Equity if and only if it is a generalized leximin.*

Proof: The proof can be completed by using the following five claims.

(Claim 1) $\forall [i] \neq 1$, $\exists !\ \alpha_i \in [0, 1/2]$, $\forall u_N \in U^N$ with $u_{[i]} > u_{[i-1]}, \forall \alpha'_i > \alpha_i$, $((\alpha'_i u_{[i]} + (1 - \alpha'_i)u_{[i-1]})_{[i-1, i]}, u_{-[i-1, i]}) \succ u_N$ & $\forall \alpha'_i < \alpha_i$, $u_N \succ ((\alpha'_i u_{[i]} + (1 - \alpha'_i)u_{[i-1]})_{[i-1, i]}, u_{-[i-1, i]})$.

By strong Pareto, $((u_{[i]})_{[i-1, i]}, u_{-[i-1, i]}) \succ u_N \succ ((u_{[i-1]})_{[i-1, i]}, u_{-[i-1, i]})$ and $\forall \alpha'_i > \alpha''_i$, $((\alpha'_i u_{[i]} + (1 - \alpha'_i)u_{[i-1]})_{[i-1, i]}, u_{-[i-1, i]}) \succ ((\alpha''_i u_{[i]} + (1 - \alpha''_i)u_{[i-1]})_{[i-1, i]}, u_{-[i-1, i]})$ hold true. Then, completeness of a social welfare ordering and Pigou-Dalton transfer equity imply the existence of threshold α_i in [0, 1/2]. Next, we will show the uniqueness of threshold α_i. For all $u_N, v_N \in U^N$ with $u_{[i]} > u_{[i-1]}$, let $a = (v_{[i]} - v_{[i-1]})(u_{[i]} - u_{[i-1]})^{-1}$ and $b = [v_{[i-1]}(u_{[i]} - u_{[i-1]}) - u_{[i-1]}(v_{[i]} - v_{[i-1]})](u_{[i]} - u_{[i-1]})^{-1}$. Obviously, $au_{[i-1]} + b = v_{[i-1]}$ and $au_{[i]} + b = v_{[i]}$. Then, rank-separability and full interpersonal comparability of individual well-being guarantees the uniqueness of threshold α_i. ■

(Claim 2) $\forall [i] \neq 1$, $\forall u_N, v_N \in U^N$ with $u_{-[i-1, i]} = v_{-[i-1, i]}$, $\forall$ threshold α_i, if $\alpha_i u_{[i]} + (1 - \alpha_i) u_{[i-1]} > \alpha_i v_{[i]} + (1 - \alpha_i) v_{[i-1]}$, then $u_N \succ v_N$.

Because of $\alpha_i u_{[i]} + (1 - \alpha_i) u_{[i-1]} > 0.5\alpha_i (u_{[i]} + v_{[i]}) + 0.5(1 - \alpha_i)(u_{[i-1]} + v_{[i-1]}) > \alpha_i v_{[i]} + (1 - \alpha_i) v_{[i-1]}$, $u_N \succ ((0.5\alpha_i (u_{[i]} + v_{[i]}) + 0.5(1 - \alpha_i)(u_{[i-1]} + v_{[i-1]}))_{[i-1, i]}, u_{-[i-1, i]}) \succ v_N$. Transitivity of a social welfare ordering means that $u_N \succ v_N$. ■

From Claims 1 and 2, it turns out that the value $\alpha_i u_{[i]} + (1 - \alpha_i) u_{[i-1]}$ means a representative welfare level of two persons' well-being $(u_{[i-1]}, u_{[i]})$. Let us call a unique number α_i a *threshold* of i and representative welfare level $\alpha_i u_{[i]} + (1 - \alpha_i) u_{[i-1]}$ *critical value* of $[i-1, i]$ in u_N. Then, $\forall u_N, v_N \in U^N$ with $u_{-[i-1, i]} = v_{-[i-1, i]}$, a notation $u_N \approx_{[i-1, i]} v_N$ denotes that the both u_N and v_N have the same critical value of $[i-1, i]$ in them. Using the above notations, Claim 3 is as follows.

(Claim 3) $\forall u_N \in U^N$, $\forall [i] \neq 1$, $\forall$ threshold $\alpha_i \neq 0$, $\forall \varepsilon > 0$, $u_N \approx_{[i-1, i]} (u_{[i-1]} + \frac{\alpha_i}{1-\alpha_i}\varepsilon, u_{[i]} - \varepsilon, u_{-[i-1, i]}) \approx_{[i-1, i]} (u_{[i-1]} - \frac{\alpha_i}{1-\alpha_i}\varepsilon, u_{[i]} + \varepsilon, u_{-[i-1, i]})$

Claim 3 is trivial from the fact that $(1 - \alpha_i)\frac{\alpha_i}{1-\alpha_i}\varepsilon + \alpha_i \varepsilon = 0$. ■

It is easy to expand the concept of critical value of $[i-1, i]$ to the version of any set of ranks. Indeed, strong Pareto and completeness of a social welfare ordering guarantee the uniqueness of a critical value $\sum_{[i]\in[M]} w_{[i]} u_{[i]}$ of $[M]$ in any profile u_N, while a weight vector $(w_{[i]})$ is not uniquely determined (Claim 5 will guarantee the uniqueness of a weight vector). Thus, let us write $u_N \approx_{[M]} v_N$ denotes that the both u_N and v_N have the same critical value of $[M]$ in them. Then, for all $u_N, v_N \in U^N$ with $u_{-[i-1, i]} = v_{-[i-1, i]}$, it is shown that the same critical value of $[i-1, i]$ means the same value of the set of ranks $[M]$ including $[i]$.

(Claim 4) $\forall [i] \neq 1$, $\forall u_N, v_N \in U^N$ with $u_{-[i-1, i]} = v_{-[i-1, i]}$, if $u_N \approx_{[i-1, i]} v_N$ & $[i] \in [M]$, then $u_N \approx_{[M]} v_N$.

Suppose to the contrary that $\exists u_N, v_N \in U^N$ with $u_{-[i-1, i]} = v_{-[i-1, i]}$, $\exists [M]$ containing $[i]$, $u_N \approx_{[i-1, i]} v_N$ & not $u_N \approx_{[M]} v_N$. Without the loss of generality, assume that $u_{[i]} > v_{[i]} > v_{[i-1]} > u_{[i-1]}$. Rank-separability implies that $\forall s_{-[i-1, i]}$, $(u_{[i-1, i]}, s_{-[i-1, i]}) \approx_{[i-1, i]} (v_{[i-1, i]}, s_{-[i-1, i]})$ and not $(u_{[i-1, i]}, s_{-[i-1, i]}) \approx_{[M]} (v_{[i-1, i]}, s_{-[i-1, i]})$. Then, consider two profiles $u'_N = ((u_{[i-1]})_{[1, .., i-1]}, (u_{[i]})_{[i, ..., n]})$ and $v'_N = ((u_{[i-1]})_{[1, .., i-2]}, v_{[i-1]}, v_{[i]}, (u_{[i]})_{[i+1, ..., n]})$. Let the critical values of $[M]$ in u'_N

and v'_N be respectively $\sum_{[j]\in[M]} w_{[j]}\, u'_{[j]}$ and $\sum_{[j]\in[M]} w'_{[j]}\, v'_{[j]}$. Since these critical values are not the same, (i) $\sum_{[j]\in[M]} w_{[j]}\, u'_{[j]} > \sum_{[j]\in[M]} w'_{[j]}\, v'_{[j]}$ or (ii) $\sum_{[j]\in[M]} w_{[j]}\, u'_{[j]} < \sum_{[j]\in[M]} w'_{[j]}\, v'_{[j]}$ holds true. Then, strong Pareto implies $\exists\varepsilon > 0$, (i) the critical value of $[M]$ in $u^*_N = ((u_{[i-1]})_{[1, .., i-1]}, u_{[i]} - \varepsilon, (u_{[i]})_{[i+1, ..., n]})$ is higher than that in v'_N or (ii) the critical value of $[M]$ in $u^{**}_N = ((u_{[i-1]})_{[1, .., i-2]}, u_{[i-1]} + \varepsilon, (u_{[i]})_{[i, ..., n]})$ is lower than that in v'_N. That is, (i) $u^*_N \succ v'_N$ or (ii) $v'_N \succ u^{**}_N$. However, the both cases contradict the facts that $v'_N \succ u^*_N$ & $u^{**}_N \succ v'_N$ because of $\alpha_i\, u_{[i]} + (1 - \alpha_i)\,(u_{[i-1]} + \varepsilon) > \alpha_i v_{[i]} + (1 - \alpha_i) v_{[i-1]} > \alpha_i (u_{[i]} - \varepsilon) + (1 - \alpha_i)\, u_{[i-1]}$. ■

(Claim 5) $\forall u_N \in U^N$, $\forall [M] = \{i, i\text{-}1, \ldots, i\text{-}m\}$ with a sequence of non-zero thresholds $(\alpha_i, \ldots, \alpha_{i-m})$, $u_N \approx_{[M]} ((\sum_{[j]\in[M]} w_{[j]}\, u_{[j]})_{[M]}, u_{-[M]})$ where $w_{[i]} : w_{[i-1]} : \ldots : w_{[i-m]} = 1 : \frac{1-\alpha_i}{\alpha_i} : \ldots : \Pi_{j=i,\, i-1,\, \ldots,\, i-m} \frac{1-\alpha_j}{\alpha_j}$.

By Claims 3 and 4, $u_N \approx_{[M]} (u_{[i-1]} + \frac{\alpha_i}{1-\alpha_i}\varepsilon, u_{[i]} - \varepsilon, u_{-[i-1, i]}) \approx_{[M]} (u_{[i-2]} + \frac{\alpha_i}{1-\alpha_i} \cdot \frac{\alpha_{i-1}}{1-\alpha_{i-1}}\varepsilon, u_{[i-1]}, u_{[i]} - \varepsilon, u_{-[i-2, i-1, i]}) \approx_{[M]} (u_{[i-m]} + \Pi_{j=i,\, i-1,\, \ldots,\, i-m} \frac{\alpha_j}{1-\alpha_j}\varepsilon, u_{[i-m]}, u_{[i]} - \varepsilon, u_{-[i-m, i]})$. This argument means a *leaky* well-being transfer from rank i to rank i-k that keeps the same critical value must be proportional to the ratio $\Pi_{j=i,\, i-1,\, \ldots,\, i-k} \frac{\alpha_j}{1-\alpha_j}$. Thus, the critical value $\sum_{[j]\in[M]} w_{[j]}\, u_{[j]}$ of $[M]$ in u_N is exactly determined in Claim 5. ■

By Claim 5, $\forall u_N \in U^N$, $\forall [M] = \{i, i\text{-}1, \ldots, i\text{-}m\}$ with a sequence of non-zero thresholds $(\alpha_i, \ldots, \alpha_{i-m})$, the critical value of $[M]$ in u_N is given by a rank-dependent utilitarian rule defined on $[M]$. If $\exists [i^*]$ with its threshold $\alpha_{i^*} = 0$, then the critical value of $[M_{i^*}]$ containing $[i^*]$ in u_N is lexicographically preferred to that of $[M_i]$ containing $[i]$ for all $[i] > [i^*]$. This means the social welfare ordering $\succcurlyeq$ is a generalized leximin rule. ■

As Tungodden (2000) and Tungodden and Vallentyne (2005) show, combining the axioms of strong Pareto and perfect equity implies that a social welfare ordering must be a refinement of maximin rule. Perfect equity requires that if two profiles u_N and v_N are Pareto non-comparable and all individuals enjoy the same well-being level in u_N but v_N is not so, then u_N is strictly better than v_N. In the context of a generalized leximin rule, if perfect equity is added to the axioms of Theorem 2, then $[M_1]$ must be [1] in a partition of the generalized leximin rule, that is, the rule must be a refinement of maximin rule.

Note that the generalized leximin rule does not necessarily always prioritize individuals with lower well-being despite the definition of descending weight vector assigned to ranks. For example, consider the following generalized leximin rule. In a three-individual economy, the rule lexicographically judges the following sequence of weighted well-being sums in the two subgroups: $(\sum_{i=1,2} u_{[i]}, \sum_{i=3} u_{[i]})$. Furthermore, this rule judges any profiles by the magnitude of rank 2's well-being whenever the sums of ranks 1and 2's well-beings are the same. To avoid this case, a stronger equity axiom is required[19].

Using a method similar to the proof of Theorem 2, we can consider the problem of a characterization theorem without Pigou-Dalton transfer equity. In fact, if Pigou-Dalton transfer equity is not required, then a class of social welfare orderings satisfying strong Pareto, anonymity, and rank-separability is equivalent to a kind of lexicographic orderings. These orderings are defined on a sequence of weighted sums of well-beings in subgroups according to various priorities that could be far different from the generalized leximin rule. For example, a lexicographic ordering may judge a profile $(u_{[1]}, u_{[2]}, u_{[3]}, u_{[4]}, u_{[5]})$ in a five-individual economy using the following sequence of weighted sums of well-being in three subgroups: $(\sum_{i=2,5} u_{[i]}, \sum_{i=1,4} u_{[i]}, \sum_{i=3} u_{[i]})$. However, this social welfare ordering has no ethical value and a generalization of Theorem 2 is not included in this paper.

Next, if we add continuity to the system of axioms stated above, the generalized leximin rule must be rank-dependent utilitarian. This result is first shown by Ebert (1988a), who uses the advanced technique of functional analysis. However, since the only continuous generalized leximin rule is obviously rank-dependent utilitarian, it is easy to understand and prove Ebert's result.

Theorem 3 (A Characterization of a Rank-dependent Utilitarian Rule): *A social welfare ordering satisfies the axioms of Full Interpersonal Comparability of Individual Well-being, Strong Pareto, Anonymity, Rank-Separability, Pigou-Dalton Transfer Equity, and Continuity if and only if it is a rank-dependent utilitarian.*

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[19] It is shown that some stronger requirements of Pigou-Dalton transfer equity allow a generalized leximin rule to be a sound version in which an individual with lower well-being always takes priority over those with higher well-being even if the sums of well-beings in the subgroup between two profiles are the same.

Let us consider a class of Paretian and anonymous social welfare orderings satisfying separability instead of rank-separability. By Claim 1 in the proof of Theorem 2, it can be easily understood that each rank's threshold of well-being is constant regardless of rank under the separability requirement. Indeed, when separability is required, the possible transfer thresholds defined on any ranks' well-beings are limited to 1, 0.5, or 0 in order to ensure a threshold to be constant for all ranks. If the threshold equals 1, then it is the leximax rule. If the threshold equals 0.5, then it is the weak utilitarian rule. If the threshold equals 0, then it is the leximin rule. Hence, we easily prove Dechamps and Gevers' joint characterization theorem by the above fact and the proof of Theorem 2.

Theorem 4 (A Joint Characterization of a Weak Utilitarian, Leximin and Leximax Rules): *A social welfare ordering satisfies the axioms of Full Interpersonal Comparability of Individual Well-being, Strong Pareto, Anonymity, and Separability if and only if it is either a weak utilitarian, leximin or leximax rules.*

Proof: Since anonymity and separability implies rank-separability, there exists a unique threshold $\alpha \in [0, 1]$ such that $\forall u_N \in U^N$ with $u_{[i]} > u_{[j]}$, $\forall \alpha' > \alpha$, $((\alpha' u_{[i]} + (1 - \alpha')u_{[j]})_{[j, i]}, u_{-[j, i]}) \succ u_N$ & $\forall \alpha' < \alpha$, $u_N \succ ((\alpha' u_{[i]} + (1 - \alpha')u_{[j]})_{[j, i]}, u_{-[j, i]})$ by Claim 1 in the proof of Theorem 2. Note that a threshold α is independent from any ranks since the ranks of $u_{[i]}$ and $u_{[j]}$ can be any possible combinations of ranks due to separability axiom. Then, if $\alpha = 0$ (resp. 1), then it must be the leximin (resp. leximax) rule. Consider a case $\alpha \in (0, 1)$. For all $[i]$ and $[j]$, $w_{[i]} : w_{[j]} = 1 : \left(\frac{1-\alpha}{\alpha}\right)^{i-j}$ due to the logic of Claim 5 in the proof of Theorem 2. Also, since separability requires $[i]$ and $[j]$ should be independent from any differences between ranks, that is, $w_{[i]} : w_{[j]} = 1 : \left(\frac{1-\alpha}{\alpha}\right)^{i-j} = 1 : \left(\frac{1-\alpha}{\alpha}\right)$. Thus, α must be 0.5, which implies that a social welfare ordering is the weak utilitarian rule. ■

The weak utilitarian rule may ignore distributive justice because it could be the *utilitarian-first and lexmax-second rule*. Of course, simple utilitarian and utilitarian-first and leximin-second rules belong to a class of weak utilitarian rules. Hence, the weak utilitarian rule could be slightly distribution-sensitive in the form of a subsidiary criterion of leximin. Then, it is shown that the leximin and utilitarian-first and leximin-second rules are jointly characterized by the additional equity axioms, strict Pigou-Dalton transfer equity and strict composite transfer (Kamaga 2018).

In Theorem 4, the separability axiom excludes a leximin-first and utilitarian-second rule since it ignores rank information, and it cannot allow a well-being transfer of different rank to have different impacts on social welfare. As a result, whenever the separability axiom is required, a class of distribution-sensitive rules is either the leximin or the utilitarian-first and leximin-second rule (i.e. the very weak distribution-sensitive rule). Hence, if a bit of strict equity axiom is imposed on social ordering, then a leximin rule is the only option for social welfare orderings. In the system of axioms used in this paper, combining Hammond equity with strong Pareto implies extreme equity, which leads to the following result immediately.

Theorem 5 (A Characterization of a Leximin Rule): *A social welfare ordering satisfies the axioms of Full Interpersonal Comparability of Individual Well-being, Strong Pareto, Anonymity, Separability, and Hammond Equity if and only if it is a leximin rule.*

Finally, adding continuity to the system of axioms stated in Dechamps and Gevers would eliminate the possibility of leximin and leximax and leave only the utilitarian rule.

Theorem 6 (A Characterization of a Utilitarian Rule): *A social welfare ordering satisfies the axioms of Full Interpersonal Comparability of Individual Well-being, Strong Pareto, Anonymity, Separability, and Continuity if and only if it is a utilitarian rule.*

From Theorems 1-6 and the results in the literature, the theoretical relationships of interpersonal comparability of individual well-being, equity requirements, and candidates of reasonable social welfare orderings (SWOs) are summarized in Table 1. Following the traditional classification and notations (Sen 1970a; d'Aspremont and Gevers 2002), eight types of interpersonal comparability are considered: ordinal non-comparability (Inv($\varphi_i(u_i)$)), cardinal non-comparability (Inv($a_i+b_iu_i$)), ordinal level comparability (Inv($\varphi(u_i)$)), cardinal unit non-comparability (Inv(a_i+bu_i)), cardinal unit comparability (Inv($a+u_i$)), cardinal ratio non-comparability (Inv(b_iu_i)), cardinal ratio comparability (Inv(bu_i)), and cardinal full comparability (Inv($a+bu_i$)), where both $\varphi_i(.)$ and $\varphi(.)$ are monotonic transformation, a_i, a in $\mathbb{R}$, and b_i, b in $\mathbb{R}_{++}$. A notation Inv(.) means that a social welfare ordering must be the same between *original* utility profiles and *rescaled* utility profiles according to the transformation. For example, Inv($a_i+b_iu_i$) means that $\forall u_N, v_N \in U^N$, $\forall a_i \in \mathbb{R}$, $\forall b_i \in \mathbb{R}_{++}$, $u_N \succcurlyeq v_N \Leftrightarrow (a_i+b_iu_i)_{i\in N} \succcurlyeq (a_i+b_iv_i)_{i\in N}$.

Under the ordinal/ cardinal non-comparability (Inv($\varphi_i(u_i)$) or Inv($a_i+b_iu_i$)), Arrow's impossibility theorem holds true and no candidates exists as a reasonable social welfare ordering (Arrow 1951; 1963; Sen 1970a).

Under the ordinal level comparability (Inv($\varphi(u_i)$)), strict inequalities between two individual's well-beings are significant, but both the ratios and the differences between them have no significance. Therefore, well-beings should be regarded as a simple ordinal scale, and it is reasonable that an informational basis of equity requirement is focused only on the strict inequalities in well-beings between individuals[20]. In this case, a meaningful concept of equity is either Hammond equity or extreme equity. Since no social welfare ordering satisfies the axioms of extreme equity and continuity (Theorem 1), the leximin rule, which is the only social welfare function satisfying the axioms of Hammond equity, strong Pareto, anonymity, and separability, would be reasonable (Hammond 1976; d'Aspremont and Gevers 1977).

However, under the cardinal unit or ratio non-comparability (Inv(a_i+bu_i) or Inv(b_iu_i)), *intrapersonal* comparisons of well-being are admissible, but any *interpersonal* comparisons are meaningless. Therefore, strict inequalities between two individuals' well-being are meaningless, and it is not possible to define a concept of equity that considers well-being gaps between individuals. Under this comparability, the utilitarian or Nash social welfare functions are the only class of rules satisfying strong Pareto, anonymity, separability, and continuity (d'Aspremont and Gevers 2002). Hence, the only option is to give up any equity requirements and use either the utilitarian or Nash social welfare functions as a social welfare ordering because any interpersonal comparisons of well-being are impossible.

On the contrary, under the cardinal unit or ratio comparability (Inv($a+u_i$) or Inv(bu_i)), the differences or the ratios between two individuals' well-beings are significant, so it is possible to apply strong concepts of equity, such as Pigou-Dalton transfer equity,

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[20] Morreau and Weymark (2013) analyze a theoretical framework in which generalized social welfare functionals could distinguish a *genuine change* in individual well-beings from a *representational change* due to the use of different measurement scales. They show a necessary and sufficient condition for *scale-dependent welfarism* that is similar to the standard neutrality theorem.

to measure social welfare. In this case, when the axioms of strong Pareto, anonymity, separability, continuity, and Pigou-Dalton transfer equity are imposed, the options are Kolm-Pollack or Atkinson social welfare functions (Blackorby and Donaldson 1982)[21]. Moreover, when separability is dropped and rank-separability is imposed, Ebert (1988b) shows that *rank-dependent* Kolm-Pollack and *rank-dependent* Atkinson social welfare functions are available by using the functional analysis techniques that are similar to the proof in Blackorby and Donaldson (1982).

Under the cardinal full comparability (Inv($a+bu_i$)), both the differences and the ratios between two individuals' well-beings are significant, so Pigou-Dalton transfer equity can be applied for measuring social welfare. In this strong assumption, if the axioms of strong Pareto, anonymity, separability, and Pigou-Dalton transfer equity are imposed, the only options are the weak utilitarian or leximin rules (Dechamps and Gevers 1978; Theorem 4). However, when rank-separability is imposed instead of separability, it is possible to open a wide path that leads to a variety of distribution-sensitive social welfare orderings, which is called the generalized leximin (Theorem 2). As long as there is no reason to maintain separability, a society could use various social welfare orderings that take distributive justice seriously.

Finally, this paper shows the theoretical relationship of the axiomatization of the generalized leximin rule and Dechamps and Gevers' joint characterization theorem by a simple proof method. It is expected that similar relationships will be obtained in the case of Atkinson and Kolm-Pollack social welfare orderings. That is, the following conjectures hold.

Conjecture 1: *A social welfare ordering satisfies the axioms of Strong Pareto, Anonymity, Separability, and Inv*($a+u_i$) *if and only if it is either the weak Kolm-Pollack, leximin or leximax rules.*

Conjecture 2: *A social welfare ordering satisfies the axioms of Strong Pareto, Anonymity, Separability, and Inv*(bu_i) *if and only if it is either the weak Atkinson, leximin or leximax rules.*

[21] Kolm-Pollack social welfare function is a generalization of simple utilitarianism and Atkinson social welfare function is a generalization of Nash social welfare function. Note that Atkinson social welfare functions are defined only on the n-dimensional *non-negative* real space. Blackorby and Donaldson (1982) show a generalized Atkinson social welfare function that is defined on the n-dimensional real space.

Furthermore, if separability is replaced with rank-separability in Conjectures 1 and 2, it is expected that *Inv(a+*u_i*) implies a generalized leximin rule based on rank-dependent Kolm-Pollack social welfare functions* (Conjecture 3) and *Inv(b*u_i*) implies a generalized leximin rule based on rank-dependent Atkinson social welfare functions* (Conjecture 4). To prove Conjectures 1-4, several parts of the proof in this paper would be useful.

Invariance	**Sensible Equity**	**Continuous SWO**	**Non-Continuous SWO**
Inv($\varphi_i(u_i)$) **Inv($a_i+b_iu_i$)**	n.a.	n.a.	n.a. (Arrow's Impossibility Theorem)
Inv($\varphi(u_i)$)	HE or XE	n.a.	Leximin (=SP+A+SEP+HE)
Inv(a_i+bu_i)	n.a.	Utilitarian (=SP+A+SEP+C)	-
Inv($a+u_i$)	PD	Kolm-Pollack (=SP+A+SEP+C) RD Kolm-Pollack (=SP+A+R-SEP+C+PD)	Conjectures 1 & 3
Inv(b_iu_i)	n.a.	Nash (=SP+A+SEP+C)	-
Inv(bu_i)	PD	Atkinson (=SP+A+SEP+C) RD Atkinson (=SP+A+R-SEP+C+PD)	Conjectures 2 & 4
Inv($a+bu_i$)	PD	Utilitarian (=SP+A+SEP+C) RD Utilitarian (=SP+A+R-SEP+C+PD)	Weak Utilitarian or Leximin (=SP+A+SEP+PD) Generalized Leximin (=SP+A+R-SEP+PD)

Table 1: Theoretical Relationships of Interpersonal Comparability, Equity of Well-being, and Reasonable SWOs.

Each abbreviation is defined as follows: HE: Hammond Equity; XE: Extreme Equity; PD: Pigou-Dalton Transfer Equity; SP: Strong Pareto; A: Anonymity; SEP: Separability; R-SEP: Rank-Separability; C: Continuity; RD: Rank-Dependent.

5. Concluding Remarks

This study shows that a social welfare ordering satisfies the axioms of strong Pareto, anonymity, and rank-separability in the setting of cardinal full comparability of individual well-being if and only if it is the generalized leximin rule. By an elementary proof method, this paper also succeeds in proving Dechamps and Gevers' celebrated joint characterization theorem, in which a class of social welfare orderings satisfying the separability must be either a weak utilitarian, leximn or leximax rules. In addition, by showing that only the extreme equity is incompatible with continuity, the paper demonstrates that a trade-off between the axioms of equity and continuity would be superficial.

By combining these results with previous findings, the theoretical correspondence between interpersonal comparability of well-being and reasonable social welfare orderings is clarified. When only the cardinal unit or ratio non-comparability of individual well-being is allowed, there is no meaning in interpersonal comparisons of well-being because it is not possible to consistently measure well-being gap between two individuals. Hence, the concept of equity cannot be considered, and the utilitarian or Nash social welfare functions are the only choices. When the ordinal level comparability of individual well-being is allowed, Hammond equity is the only meaningful concept of equity because an absolute value of difference between two individuals' well-beings makes no sense and the relation of inequality between them only makes sense as an ordinal measure. Hence, the leximin is the only reasonable option as an efficient and equitable social welfare ordering. When the cardinal unit or ratio comparability of individual well-being is allowed, Kolm-Pollack or Atkinson social welfare functions are candidates. When the cardinal full comparability of individual well-being is allowed, as shown in this paper, it is appropriate to use the generalized leximin as a reasonable social welfare ordering.

Several remaining issues should be addressed for further research.

First, this study finds a variety of distribution-sensitive social welfare orderings by using rank-separability. However, the following question remains: What rules would be reasonable if there were no requirement of rank-separability? Based on the proof demonstrated in this study, if rank-separability is not required, each rank's weight differs in each profile of individual well-beings. In other words, given a profile of individual well-beings, a class of Paretian, anonymous, and *continuous* social welfare orderings must belong to a class of rank-weighted utilitarian rules in which a rank-weight vector could vary on an upper boundary of a regular n-dimensional simplex. However, what kinds of rank-weight vectors have legitimacy and consistency? To ensure acceptable social welfare orderings, a new persuasive axiom is needed.

Second, this study characterizes social welfare orderings based on the premise that all individual well-beings are fully interpersonally comparable. However, there is no common agreement on the index that is the most suitable for measuring individual well-being and living standards. The candidates for measuring well-being include multidimensional poverty indices (Alkire and Foster 2011; Alkire et al. 2015), the equivalent income approach (Fleurbaey 2005; Fleurbaey and Blanchet 2013), and the consensus approach (Sakamoto 2018). These indices are designed to satisfy the ordinal level comparability. Moreover, the capability approach (Sen 1985) may have the potential to solve the measurement problem of individual well-being. However, this approach needs to be greatly improved to define practical measures of unobserved capability sets, which vary greatly among persons according to external conditions, such as sexuality, health, ethnicity, political stability, and religions. Furthermore, the measurement of happiness (i.e., subjective well-being), which is often used as a cardinal measure rather than an ordinal measure, is popular. It is necessary to deepen our understanding of these approaches and to achieve common agreement on individual well-being measures for different and various situations.

Third, it is necessary to scrutinize how the results of this study differ in various contexts of normative economics, such as models with uncertainty, variable populations, and intergenerational equity. For example, in the problem of social choice with variable populations, there may be a reason for wariness regarding the simple extension of rank-separability. The simplest extension of rank-dependent utilitarian rules is a class of rules that have a specific rank-weight vector defined on the set of the maximal population in this model, in which weighted individual well-being is summed following this vector. However, it seems ethically dubious that a society must evaluate using the same weight in a value of the 100th individual's well-being in a 10,000-population economy and that of the 100th individual's well-being in a 100-population economy. A similar problem could occur in the contexts of intergenerational equity and uncertainty. Prudent consideration is needed.

Fourth, there is an open question about the degree of completeness of a generalized leximin in the setting of interpersonal comparability between full and partial levels of individual well-being. Similar to Sen's famous result (Sen 1970b), comparable regions of the profiles of individual well-beings could be expanded in the generalized leximin rule whenever the degree of interpersonal comparability increases. However, unlike the simple utilitarian rule, there is no guarantee that a one-to-one correspondence would be obtained between the interpersonal comparability of cardinal individual welfare and the completeness of social welfare. Hence, it is necessary to determine how the comparable set changes[22].

Finally, in using a generalized leximin rule as a social welfare ordering to measure social welfare, we must resolve the practical problem of determining the weight vector assigned to ranks of well-being. This problem may be solved not by an axiomatic requirement[23] but in a political process of a compromised consensus across a society. That is, the problem of social decision-making, which determines the level of *tolerable inequality* in a society, may not be mathematically induced by axiomatic methods. It may be solved by analyzing the normative behaviors of humans. Hence, this problem might involve empirical, experimental, and cultural issues that might be determined by observing emotional and negative reactions to inequality, oppression, and injustice in a fair and informative situation.

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[22] In Sen's theorem, there exits a one-to-one correspondence between an incomparable set of well-being profiles and the incompleteness of the simple utilitarian rule. Hence, the mathematical relationship is very clear. However, in the case of the generalized leximin rule, the rank of individual well-being becomes unclear when the set of incomparable areas becomes large, so the relationship of one-to-one correspondence is lost, and it seems to be complicated.

[23] For example, Bossert and Kamaga (2020) axiomatically characterize a linear combination of minimum and mean utilities as a kind of rank-dependent utilitarian rules. However, the axiom used in their study requires that a well-being gap at higher welfare levels should be further increased in order to reduce a gap at lower welfare levels. Because many social welfare orderings can avoid such situations and the axiom is not ethically appealing, there seems to be no reason that social assessments should be based solely on minimum and average utilities.

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