Multi-Profile Intergenerational Social Choice*

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Abstract. In an infinite-horizon setting, Ferejohn and Page showed that Arrow's axioms and stationarity lead to a dictatorship by the first generation. Packel strengthened this result by proving that no collective choice rule generating complete social preferences can satisfy unlimited domain, weak Pareto and stationarity. We prove that a domain restriction can be imposed and completeness can be dropped without affecting the incompatibility. We propose a more suitable stationarity axiom and show that a social welfare function on a specific domain satisfies this modified version if and only if it is a chronological dictatorship. *Journal of Economic Literature* Classification No.: D71.

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1 Introduction

As is well-known, the validity of Arrow's celebrated general impossibility theorem (Arrow, 1951; 1963) hinges squarely on the finiteness of population. Fishburn (1970), Sen (1979) and Suzumura (2000) presented their respective method of proving Arrow's theorem and highlighted the crucial role played by the assumption of finiteness of population. Kirman and Sondermann (1972) and Hansson (1976) cast a new light on the structure of an Arrovian social welfare function with an infinite population, revealing the structure of decisive coalitions for an Arrow social welfare function as *ultrafilters.* In their analysis, however, there was no explicit consideration of a sequential relationship among the members of an infinite population. It was a pioneering analysis due to Ferejohn and Page (1978) that introduced time explicitly into the analysis. Time flows only unidirectionally, and two members t and t' of the society, to be called generation t and generation t', respectively, are such that generation t' appears in the society only after generation t appears in the society if and only if $t < t'$ holds. As a result of introducing this time structure of infinite population, Ferejohn and Page (1978) also opened a new gate towards marrying Arrovian social choice theory and the theory of evaluating infinite intergenerational utility streams, which was initiated by Koopmans (1960) and Diamond (1965). This paper is an attempt to reexamine the Ferejohn and Page analysis of intergenerational social choice theory in a multi-profile setting.

Starting out with Hansson's (1976) result that any social welfare function satisfying Arrow's (1951; 1963) axioms must be such that the set of decisive coalitions is an ultrafilter, Ferejohn and Page (1978) proposed a stationarity condition in an infinite-horizon multi-profile social choice model and showed that if a social welfare function satisfying Arrow's conditions and stationarity exists, generation one must be a dictator. As they noted themselves, the question whether such a social welfare function exists at all was left open by their analysis; what they showed was that *if* such a function exists, it must be dictatorial with generation one being the dictator. Packel (1980) answered the question Ferejohn and Page left open by establishing a strong impossibility result: even without independence of irrelevant alternatives and without assuming social preferences to be transitive, no collective choice rule can satisfy the remaining axioms, not even dictatorships of any kind.

In this paper, we prove that the negative implications of their stationarity condition are even more far-reaching: there exist no collective choice rules satisfying unlimited domain, weak Pareto and stationarity. The same conclusion holds if individual preferences are restricted to those that are history-independent. No restrictions whatsoever are imposed on social preferences—they need not be reflexive, complete or transitive. By dropping reflexivity and completeness, we strengthen Packel's impossibility result even further.

Packel's (1980) approach to resolve the impossibility consisted of restricting the domain of a social welfare function to profiles where generation one's preferences are themselves stationary. This allowed him to obtain possibility results in that setting. In contrast, we think that the natural way to formulate a domain restriction in the intertemporal context is to assume that the preferences of any generation are restricted to depend on the outcome for this generation only. In that case, there do exist Arrow social welfare functions satisfying stationarity but they are all such that generation one has even more dictatorial power than established in the Ferejohn-Page result. Adding Pareto indifference as a requirement leads again to an impossibility. We conclude that the version of stationarity employed by Ferejohn and Page (1978) is too demanding and has some counter-intuitive features. In response, we propose what we suggest is a more suitable multi-profile version of stationarity and characterize the lexicographic dictatorship where the generations are taken into consideration in chronological order. The main conclusion is that, although the infinite-population version of Arrow's social choice problem permits, in principle, non-dictatorial rules, these additional possibilities all but vanish even if an alternative stationarity axiom is imposed. The relationship between the Ferejohn and Page analysis and our extensions thereof, on the one hand, and the Koopmans-Diamond analysis of the evaluation of infinite intergenerational utility streams and their subsequent extensions, on the other, is discussed in the Concluding Remarks.

2 Infinite-Horizon Social Choice

Suppose there is a set of per-period alternatives X containing at least three elements, that is, $|X| \geq 3$ where $|X|$ denotes the cardinality of X. These per-period alternatives could be consumption bundles, for example, but we do not restrict attention to one particular interpretation. We identify the population with a sequence of generations indexed by the positive integers $\mathbb{N}$. Let X^∞ be the set of all streams of per-period alternatives $\mathbf{x} = (x_1, x_2, \ldots, x_t, \ldots)$ where, for each $t \in \mathbb{N}$, $x_t \in X$ is the period-t alternative experienced by generation t. We also refer to x_t as the factor of $\mathbf{x}$ relevant for generation t.

That the set of feasible per-period alternatives is the same for all generations appears to be a rather restrictive assumption; for example, technological progress is likely to generate dramatically different feasible sets of consumption bundles several decades into the future. The above more restrictive formulation is chosen because it is needed in order to define Ferejohn and Page's (1978) version of stationarity and their relevant results using this property; this is the case because, to apply their axiom, some generations must be able to assess not only their own per-period alternatives but those of other generations as well. However, the new approach we develop in our analysis can easily accommodate a framework where the per-period feasible sets may be period-dependent. The reason is that our proposed model is based on axioms that do not require a generation t to be capable of comparing per-period alternatives other than those relevant for t.

The set of all binary relations on X^∞ is denoted by $\mathcal{B}$, and $\mathcal{C}$ is the set of all complete relations on X^∞. Furthermore, the set of all orderings on X^∞ is denoted by $\mathcal{R}$, where an ordering is a reflexive, complete and transitive relation. A social relation is an element R of $\mathcal{B}$. We assume that each generation $t \in \mathbb{N}$ has an ordering $R_t \in \mathcal{R}$. A (preference) profile is a stream $\mathbf{R} = (R_1, R_2, \ldots, R_t, \ldots)$ of orderings on X^∞. The set of all such profiles is denoted by $\mathcal{R}^\infty$. Let $t \in \mathbb{N}$. For $\mathbf{x} \in X^\infty$, we define

$$\mathbf{x}_{\geq t} = (x_t, x_{t+1}, \ldots) \in X^\infty$$

and, analogously, for $\mathbf{R} \in \mathcal{R}^\infty$,

$$\mathbf{R}_{\geq t} = (R_t, R_{t+1}, \ldots) \in \mathcal{R}^\infty.$$

Two subsets of the *unlimited domain* $\mathcal{R}^\infty$ will be of importance in this paper. We define the *forward-looking domain* $\mathcal{R}^\infty_F \subseteq \mathcal{R}^\infty$ by letting, for all $\mathbf{R} \in \mathcal{R}^\infty$, $\mathbf{R} \in \mathcal{R}^\infty_F$ if, for each $t \in \mathbb{N}$, there exists an ordering Q_t on X^∞ such that, for all $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; \mathbf{x}_{\geq t} Q_t \mathbf{y}_{\geq t}.$$

Analogously, the *selfish domain* $\mathcal{R}^\infty_S \subseteq \mathcal{R}^\infty_F \subseteq \mathcal{R}^\infty$ is obtained by letting, for all $\mathbf{R} \in \mathcal{R}^\infty$, $\mathbf{R} \in \mathcal{R}^\infty_S$ if, for each $t \in \mathbb{N}$, there exists an ordering $\succeq_t$ on X such that, for all $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; x_t \succeq_t y_t.$$

For a relation $R \in \mathcal{B}$, the *asymmetric part* $P(R)$ of R is defined by

$$\mathbf{x} P(R) \mathbf{y} \;\Leftrightarrow\; [\mathbf{x} R \mathbf{y} \text{ and } \neg\, \mathbf{y} R \mathbf{x}]$$

for all $\mathbf{x}, \mathbf{y} \in X^\infty$. The *symmetric part* $I(R)$ of R is defined by

$$\mathbf{x} I(R) \mathbf{y} \;\Leftrightarrow\; [\mathbf{x} R \mathbf{y} \text{ and } \mathbf{y} R \mathbf{x}]$$

for all $\mathbf{x}, \mathbf{y} \in X^\infty$. Furthermore, for all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $R \in \mathcal{B}$, $R|_{\{\mathbf{x},\mathbf{y}\}}$ is the restriction of R to the set $\{\mathbf{x}, \mathbf{y}\}$.

In the infinite-horizon context studied in this paper, a *collective choice rule* is a mapping $f : \mathcal{D} \to \mathcal{B}$, where $\mathcal{D} \subseteq \mathcal{R}^\infty$ with $\mathcal{D} \neq \emptyset$ is the domain of f. The interpretation is that, for a profile $\mathbf{R} \in \mathcal{D}$, $f(\mathbf{R})$ is the social ranking of streams in X^∞. If $f(\mathcal{D}) \subseteq \mathcal{C}$, f is a *complete collective choice rule.* If $f(\mathcal{D}) \subseteq \mathcal{R}$, f is a *social welfare function.*

Arrow (1951; 1963) imposed the axioms of *unlimited domain, weak Pareto* and *independence of irrelevant alternatives* and showed that, in the case of a finite population, the resulting social welfare functions are *dictatorial:* there exists an individual such that, whenever this individual strictly prefers one alternative over another, this strict preference is reproduced in the social ranking, irrespective of the preferences of other members of society. The axioms relevant in our context are defined as follows.

Unlimited domain. $\mathcal{D} = \mathcal{R}^\infty$.

Forward-looking domain. $\mathcal{D} = \mathcal{R}^\infty_F$.

Selfish domain. $\mathcal{D} = \mathcal{R}^\infty_S$.

Weak Pareto. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$,

$$\mathbf{x} P(R_t) \mathbf{y} \; \forall t \in \mathbb{N} \;\Rightarrow\; \mathbf{x} P(f(\mathbf{R})) \mathbf{y}.$$

Pareto indifference. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$,

$$\mathbf{x} I(R_t) \mathbf{y} \; \forall t \in \mathbb{N} \;\Rightarrow\; \mathbf{x} I(f(\mathbf{R})) \mathbf{y}.$$

Independence of irrelevant alternatives. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$R_t|_{\{\mathbf{x},\mathbf{y}\}} = R'_t|_{\{\mathbf{x},\mathbf{y}\}} \; \forall t \in \mathbb{N} \;\Rightarrow\; f(\mathbf{R})|_{\{\mathbf{x},\mathbf{y}\}} = f(\mathbf{R}')|_{\{\mathbf{x},\mathbf{y}\}}.$$

Let $f : \mathcal{D} \to \mathcal{R}$ be a social welfare function and let $\mathbf{x}, \mathbf{y} \in X^\infty$ be distinct. A set $T \subseteq \mathbb{N}$ (also referred to as a *coalition*) is *decisive for* $\mathbf{x}$ *over* $\mathbf{y}$ *for* f (in short, T is $d_f(\mathbf{x}, \mathbf{y})$) if, for all $\mathbf{R} \in \mathcal{D}$,

$$\mathbf{x} P(R_t) \mathbf{y} \; \forall t \in T \;\Rightarrow\; \mathbf{x} P(f(\mathbf{R})) \mathbf{y}.$$

Furthermore, a set $T \subseteq \mathbb{N}$ is *decisive for* f if T is $d_f(\mathbf{x}, \mathbf{y})$ for all distinct $\mathbf{x}, \mathbf{y} \in X^\infty$. Clearly, $\mathbb{N}$ is decisive for any social welfare function f satisfying weak Pareto. If there is a generation $t \in \mathbb{N}$ such that $\{t\}$ is decisive for f, generation t is a *dictator for f.* Let $\mathcal{T}(f)$ denote the set of all decisive coalitions for a social welfare function f.

Hansson (1976) has shown that if a social welfare function f satisfies unlimited domain, weak Pareto and independence of irrelevant alternatives, then $\mathcal{T}(f)$ must be an *ultrafilter.* Recall that an ultrafilter on $\mathbb{N}$ is a collection $\mathcal{U}$ of subsets of $\mathbb{N}$ such that

1. $\emptyset \notin \mathcal{U}$;
2. $\forall T, T' \subseteq \mathbb{N}$, $[[T \in \mathcal{U}$ and $T \subseteq T'] \;\Rightarrow\; T' \in \mathcal{U}]$;
3. $\forall T, T' \in \mathcal{U}$, $T \cap T' \in \mathcal{U}$;
4. $\forall T \subseteq \mathbb{N}$, $[T \in \mathcal{U}$ or $\mathbb{N} \setminus T \in \mathcal{U}]$.

The conjunction of properties 1 and 4 implies that $\mathbb{N} \in \mathcal{U}$ and, furthermore, the conjunction of properties 1 and 3 implies that the disjunction in property 4 is exclusive—that is, T and $\mathbb{N} \setminus T$ cannot both be in $\mathcal{U}$.

An ultrafilter $\mathcal{U}$ is *principal* if there exists a $t \in \mathbb{N}$ such that, for all $T \subseteq \mathbb{N}$, $T \in \mathcal{U}$ if and only if $t \in T$. Otherwise, $\mathcal{U}$ is a *free* ultrafilter. It can be verified easily that if $\mathbb{N}$ is replaced with a finite set, then the only ultrafilters are principal and, therefore, Hansson's theorem reformulated for finite populations reduces to Arrow's (1951; 1963) theorem—that is, there exists an individual (or a generation) t which is a dictator. In the infinite-population case, a set of decisive coalitions that is a principal ultrafilter corresponds to a dictatorship just as in the finite case. Unlike in the finite case, however, there also exist free ultrafilters but they cannot be defined explicitly; the proof of their existence relies on non-constructive methods such as the axiom of choice. These free ultrafilters are non-dictatorial.

3 Stationarity

None of the above-defined axioms invoke the intertemporal structure imposed by our intergenerational interpretation. In contrast, the following *stationarity* property proposed by Ferejohn and Page (1978) is based on the unidirectional nature of time. The intuition underlying stationarity is that if two streams of per-period alternatives agree in the first period, their relative social ranking is unchanged if this common first-period alternative is eliminated. To formulate a property of this nature in a multi-profile setting, the profile under consideration for each of the two comparisons must be specified. In Ferejohn and

Page's (1978) and Packel's (1980) contributions, the same profile is employed before and after the first-period alternative is eliminated. It seems to us that this leads to a rather demanding requirement because the preferences of the first generation continue to be taken into consideration even though the alternative relevant for this generation has been eliminated. Ferejohn and Page's (1978) stationarity axiom, which is due originally to Koopmans (1960) in a related but distinct context, is defined as follows.

Stationarity. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$, if $x_1 = y_1$, then

$$\mathbf{x} f(\mathbf{R}) \mathbf{y} \;\Leftrightarrow\; \mathbf{x}_{\geq 2} f(\mathbf{R}) \mathbf{y}_{\geq 2}.$$

Ferejohn and Page's (1978) result establishes that if a social welfare function f satisfies unlimited domain, weak Pareto, independence of irrelevant alternatives and stationarity, then generation one is a dictator for f. Packel (1980, Theorem 1) settled an open question posed by Ferejohn and Page (1978, p.273) in the negative by showing that there does not exist any complete collective choice rule that satisfies unlimited domain, weak Pareto and stationarity. Neither transitivity nor independence of irrelevant alternatives are needed to establish this impossibility result.

Our first result strengthens Packel's (1980) impossibility theorem. In particular, we show that, in addition to transitivity, reflexivity and completeness can be dropped and, moreover, the impossibility persists even on the forward-looking domain.

Theorem 1 *There exists no collective choice rule that satisfies forward-looking domain, weak Pareto and stationarity.*

Proof. Suppose f is a collective choice rule that satisfies the axioms of the theorem statement. Let $x, y \in X$ and let, for each generation t, $\succeq_t$ be an ordering on X such that $y \succ_t x$ for all odd t and $x \succ_t y$ for all even t. Define a forward-looking profile $\mathbf{R}$ as follows. For all $\mathbf{x}, \mathbf{y} \in X^\infty$, let

$$\mathbf{x} P(R_1) \mathbf{y} \;\Leftrightarrow\; x_1 \succ_1 y_1 \;\text{ or }\; [x_1 = y_1 \;\text{ and } x_3 \succ_1 y_3].$$

Now let $\mathbf{x} R_1 \mathbf{y}$ if and only if $\neg[\mathbf{y} P(R_1) \mathbf{x}]$. For all $t \in \mathbb{N} \setminus \{1\}$ and for all $\mathbf{x}, \mathbf{y} \in X^\infty$, let

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; x_t \succeq_t y_t.$$

Clearly, the profile thus defined is in $\mathcal{R}_F^\infty$. Now consider the streams

$$\begin{aligned} \mathbf{x} &= (x, y, x, y, x, y, \ldots) = (x, \mathbf{y}), \\ \mathbf{y} &= (y, x, y, x, y, x, \ldots) = (y, \mathbf{x}), \\ \mathbf{z} &= (x, x, y, x, y, x, \ldots) = (x, \mathbf{x}). \end{aligned}$$

Thus, $\mathbf{x}_{\geq 2} = \mathbf{y}$ and $\mathbf{z}_{\geq 2} = \mathbf{x}$. We have $\mathbf{z}P(R_t)\mathbf{x}$ for all $t \in \mathbb{N}$ and, by weak Pareto, $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. Stationarity implies $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. But $\mathbf{y}P(R_t)\mathbf{x}$ for all $t \in \mathbb{N}$, and we obtain a contradiction to weak Pareto. ■

Clearly, replacing forward-looking domain with unlimited domain does not affect the validity of the above theorem.

The impossibility can be resolved by replacing forward-looking domain with selfish domain. For example, the social welfare function f defined by letting, for all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{R}_S^\infty$, $\mathbf{x}f(\mathbf{R})\mathbf{y}$ if and only if

$$[x_\tau I(\succeq_1) y_\tau \ \forall\, \tau \in \mathbb{N}] \ \text{ or } \ [\exists t \in \mathbb{N} \ \text{ such that } \ [x_\tau I(\succeq_1) y_\tau \ \forall \tau < t \ \text{ and } \ x_t P(\succeq_1) y_t]]$$

satisfies selfish domain, weak Pareto, independence of irrelevant alternatives and stationarity. However, it does not satisfy Pareto indifference. More generally, replacing forward-looking domain with selfish domain and adding Pareto indifference in Theorem 1 produces another impossibility.

Theorem 2 *There exists no collective choice rule that satisfies selfish domain, weak Pareto, Pareto indifference and stationarity.*

Proof. Suppose f is a collective choice rule that satisfies the axioms of the theorem statement. Let $x, y, z \in X$ and let, for each generation t, $\succeq_t$ be an ordering on X such that $z \succ_t x \sim_t y$ for all odd t and $x \sim_t z \succ_t y$ for all even t. Define a selfish profile $\mathbf{R}$ as follows. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $t \in \mathbb{N}$, let

$$\mathbf{x}R_t\mathbf{y} \ \Leftrightarrow \ x_t \succeq_t y_t.$$

Clearly, the profile thus defined is in $\mathcal{R}_S^\infty$. Now consider the streams

$$\begin{aligned}
\mathbf{x} &= (z, x, z, x, z, x, \ldots), \\
\mathbf{y} &= (x, y, x, y, x, y, \ldots), \\
\mathbf{z} &= (z, z, x, z, x, z, x, \ldots) = (z, \mathbf{x}), \\
\mathbf{w} &= (z, x, y, x, y, x, y, \ldots) = (z, \mathbf{y}).
\end{aligned}$$

Thus, $\mathbf{z}_{\geq 2} = \mathbf{x}$ and $\mathbf{w}_{\geq 2} = \mathbf{y}$. We have $\mathbf{z}I(R_t)\mathbf{w}$ for all $t \in \mathbb{N}$ and, by Pareto indifference, $\mathbf{z}I(f(\mathbf{R}))\mathbf{w}$. Stationarity implies $\mathbf{x}I(f(\mathbf{R}))\mathbf{y}$. But $\mathbf{x}P(R_t)\mathbf{y}$ for all $t \in \mathbb{N}$, and we obtain a contradiction to weak Pareto. ■

In view of these impossibilities, we require less stringent intertemporal conditions than stationarity in order to obtain existence results in a multi-profile intergenerational infinite-horizon setting. This route is explored in the following section.

4 Multi-Profile Stationarity

In Ferejohn and Page's (1978) stationarity axiom, the same profile $\mathbf{R}$ is applied in both comparisons even though the period-one alternative is no longer present. This seems to us to be rather counter-intuitive and, consequently, we propose the following version that takes this point into consideration by eliminating the first-period factor not only from the alternatives but also from the profile. When combined with selfish domain, this appears to be a natural version of the axiom.

Multi-profile stationarity. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$, if $x_1 = y_1$, then

$$\mathbf{x} f(\mathbf{R}) \mathbf{y} \;\Leftrightarrow\; \mathbf{x}_{\geq 2} f(\mathbf{R}_{\geq 2}) \mathbf{y}_{\geq 2}.$$

Unlike stationarity, multi-profile stationarity does not require generation t to be able to compare per-period alternatives other than those relevant for period t itself. Because this is the case for all other axioms as well, the results of this section remain true if the per-period sets of alternatives are period-dependent, thus providing a more realistic framework. For simplicity of presentation, we do not state these alternative versions explicitly and leave it to the reader to verify that if X is replaced with X_t for each $t \in \mathbb{N}$, all arguments continue to go through, provided that each X_t contains at least three elements.

We now examine the implications of our multi-profile stationarity axiom. In particular, it allows us to characterize the *chronological dictatorship.* This variant of a lexicographic dictatorship consults generation one first but, in the case of indifference, then moves on to consult *generation two* regarding the ranking of two streams, and so on. Thus, there still is a strong dictatorship component but it is not as extreme as that generated by stationarity—and it is compatible with Pareto indifference. Moreover, the chronological dictatorship is a social welfare function and not merely a collective choice rule.

The chronological dictatorship f^{CD} is defined as follows. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{R}_S^\infty$, $\mathbf{x} f^{CD}(\mathbf{R}) \mathbf{y}$ if and only if

$$[x_\tau I(\succeq_\tau) y_\tau \; \forall\, \tau \in \mathbb{N}] \;\text{ or }\; [\exists t \in \mathbb{N} \;\text{ such that }\; [x_\tau I(\succeq_\tau) y_\tau \; \forall \tau < t \;\text{ and }\; x_t P(\succeq_t) y_t]].$$

We begin by proving a version of Hansson's (1976) theorem that applies to the selfish domain. The following preliminary result will be of convenience in establishing Hansson's theorem on our domain. It is an adaptation of Sen's (1995, p.4) *field expansion lemma* to our framework. Note that the lemma is true under many domain assumptions; however, the domain cannot be completely arbitrary because the profiles we use have to be in the domain.

Lemma 1 *Let f be a social welfare function that satisfies selfish domain, weak Pareto and independence of irrelevant alternatives. Let $\mathbf{x}, \mathbf{y} \in X^\infty$ be distinct and let $T \subseteq \mathbb{N}$. If T is $d_f(\mathbf{x}, \mathbf{y})$, then $T \in \mathcal{T}(f)$.*

Proof. Let f be a social welfare function that satisfies the three requisite axioms, let $\mathbf{x}, \mathbf{y} \in X^\infty$ be distinct and let $T \subseteq \mathbb{N}$ be $d_f(\mathbf{x}, \mathbf{y})$. We have to establish that T is $d_f(\mathbf{z}, \mathbf{w})$ for any choice of distinct alternatives $\mathbf{z}$ and $\mathbf{w}$. Thus, we have to show that T is:

(i) $d_f(\mathbf{z}, \mathbf{w})$ for all distinct $\mathbf{z}, \mathbf{w} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(ii) $d_f(\mathbf{x}, \mathbf{z})$ for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(iii) $d_f(\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(iv) $d_f(\mathbf{z}, \mathbf{x})$ for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(v) $d_f(\mathbf{y}, \mathbf{z})$ for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(vi) $d_f(\mathbf{y}, \mathbf{x})$.

(i) By selfish domain, we can consider a profile $\mathbf{R} \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} \mathbf{z}P(R_t)\mathbf{x}P(R_t)\mathbf{y}P(R_t)\mathbf{w} \qquad & \forall t \in T, \\ \mathbf{y}P(R_t)\mathbf{w} \text{ and } \mathbf{z}P(R_t)\mathbf{x} \qquad & \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

By weak Pareto, $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$ and $\mathbf{y}P(f(\mathbf{R}))\mathbf{w}$. Because T is $d_f(\mathbf{x}, \mathbf{y})$, we have $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. By transitivity, $\mathbf{z}P(f(\mathbf{R}))\mathbf{w}$. Because of independence of irrelevant alternatives, this social preference cannot depend on individual preferences over pairs of alternatives other than $\mathbf{z}$ and $\mathbf{w}$. The ranking of $\mathbf{z}$ and $\mathbf{w}$ is not specified for individuals outside of T and, thus, T is $d_f(\mathbf{z}, \mathbf{w})$.

(ii) Selfish domain allows us to consider a profile $\mathbf{R} \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} \mathbf{x}P(R_t)\mathbf{y}P(R_t)\mathbf{z} \qquad & \forall t \in T, \\ \mathbf{y}P(R_t)\mathbf{z} \qquad & \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

Because T is $d_f(\mathbf{x}, \mathbf{y})$, we have $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. By weak Pareto, $\mathbf{y}P(f(\mathbf{R}))\mathbf{z}$. By transitivity, $\mathbf{x}P(f(\mathbf{R}))\mathbf{z}$ and it follows as in the proof of (i) that T is $d_f(\mathbf{x}, \mathbf{z})$.

(iii) Let $\mathbf{R} \in \mathcal{R}_S^\infty$ be such that

$$\begin{aligned} \mathbf{z}P(R_t)\mathbf{x}P(R_t)\mathbf{y} \qquad & \forall t \in T, \\ \mathbf{z}P(R_t)\mathbf{x} \qquad & \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

By weak Pareto, $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. Because T is $d_f(\mathbf{x}, \mathbf{y})$, we have $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. By transitivity, $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$ and it follows as in the proof of (i) and (ii) that T is $d_f(\mathbf{z}, \mathbf{y})$.

(iv) Let $\mathbf{R} \in \mathcal{R}_S^\infty$ be such that

$$\begin{aligned} \mathbf{z}P(R_t)\mathbf{y}P(R_t)\mathbf{x} \qquad & \forall t \in T, \\ \mathbf{y}P(R_t)\mathbf{x} \qquad & \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

By (iii), $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$. Weak Pareto implies $\mathbf{y}P(f(\mathbf{R}))\mathbf{x}$ and, by transitivity, we obtain $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. As in the earlier cases, it follows that T is $d_f(\mathbf{z}, \mathbf{x})$.

(v) Let $\mathbf{R} \in \mathcal{R}_S^\infty$ be such that

$$\begin{aligned} \mathbf{y}P(R_t)\mathbf{x}P(R_t)\mathbf{z} \qquad & \forall t \in T, \\ \mathbf{y}P(R_t)\mathbf{x} \qquad & \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

By weak Pareto, $\mathbf{y}P(f(\mathbf{R}))\mathbf{x}$. By (ii), we have $\mathbf{x}P(f(\mathbf{R}))\mathbf{z}$. By transitivity, $\mathbf{y}P(f(\mathbf{R}))\mathbf{z}$ and it follows that T is $d_f(\mathbf{y}, \mathbf{z})$.

(vi) Let $\mathbf{R} \in \mathcal{R}_S^\infty$ be such that

$$\mathbf{y}P(R_t)\mathbf{z}P(R_t)\mathbf{x} \qquad \forall t \in T.$$

By (v), $\mathbf{y}P(f(\mathbf{R}))\mathbf{z}$ and by (iv), $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. By transitivity, $\mathbf{y}P(f(\mathbf{R}))\mathbf{x}$ and it follows that T is $d_f(\mathbf{y}, \mathbf{x})$. ■

Our version of Hansson's (1976) theorem is formulated for the selfish domain.

Theorem 3 *If a social welfare function f satisfies selfish domain, weak Pareto and independence of irrelevant alternatives, then $\mathcal{T}(f)$ is an ultrafilter.*

Proof. Suppose f satisfies selfish domain, weak Pareto and independence of irrelevant alternatives. We need to show that $\mathcal{T}(f)$ has the four properties of an ultrafilter.

1. If $\emptyset \in \mathcal{T}(f)$, we obtain $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$ and $\mathbf{y}P(f(\mathbf{R}))\mathbf{x}$ for any two alternatives $\mathbf{x}, \mathbf{y} \in X^\infty$ and for any profile $\mathbf{R} \in \mathcal{R}_S^\infty$, which is impossible. Thus, $\emptyset \notin \mathcal{T}(f)$.

2. This property follows immediately from the definition of decisiveness.

3. Suppose $T, T' \in \mathcal{T}(f)$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^\infty$ be pairwise distinct and, by selfish domain, let $\mathbf{R} \in \mathcal{R}_S^\infty$ be such that

$$\begin{aligned} \mathbf{x}P(R_t)\mathbf{y} \text{ and } \mathbf{x}P(R_t)\mathbf{z} &\qquad \forall t \in T \setminus T', \\ \mathbf{z}P(R_t)\mathbf{x}P(R_t)\mathbf{y} &\qquad \forall t \in T \cap T', \\ \mathbf{y}P(R_t)\mathbf{x} \text{ and } \mathbf{z}P(R_t)\mathbf{x} &\qquad \forall t \in T' \setminus T. \end{aligned}$$

Because T is decisive, we have $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. Because T' is decisive, we have $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. By transitivity, $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$. This implies that $T \cap T'$ is $d_f(\mathbf{z}, \mathbf{y})$ because the preferences of individuals outside of $T \cap T'$ over $\mathbf{z}$ and $\mathbf{y}$ are not specified. By Lemma 1, $T \cap T' \in \mathcal{T}(f)$.

4. Let $T \subseteq \mathbb{N}$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^\infty$ be pairwise distinct and, using selfish domain, let $\mathbf{R} \in \mathcal{R}_S^\infty$ be such that

$$\begin{aligned} \mathbf{x}P(R_t)\mathbf{y} \text{ and } \mathbf{x}P(R_t)\mathbf{z} &\qquad \forall t \in T, \\ \mathbf{x}P(R_t)\mathbf{y} \text{ and } \mathbf{z}P(R_t)\mathbf{y} &\qquad \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

If $\mathbf{x}P(f(\mathbf{R}))\mathbf{z}$, T is $d_f(\mathbf{x}, \mathbf{z})$ because the preferences of individuals outside of T over $\mathbf{x}$ and $\mathbf{z}$ are not specified. Lemma 1 implies that $T \in \mathcal{T}(f)$.

If $\neg(\mathbf{x}P(f(\mathbf{R}))\mathbf{z})$, we have $\mathbf{z}f(\mathbf{R})\mathbf{x}$ by completeness. Furthermore, $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$ by weak Pareto. Transitivity implies $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$. Because the preferences of those in T over $\mathbf{z}$ and $\mathbf{y}$ are not specified, $\mathbb{N} \setminus T$ is $d_f(\mathbf{y}, \mathbf{z})$ and, by Lemma 1, $\mathbb{N} \setminus T \in \mathcal{T}(f)$. ■

The next step towards our characterization result consists of showing that Ferejohn and Page's (1978) dictatorship result is true even on a selfish domain *and* with multi-profile stationarity instead of stationarity.

Theorem 4 *If a social welfare function f satisfies selfish domain, weak Pareto, independence of irrelevant alternatives and multi-profile stationarity, then generation one is a dictator for f.*

Proof. Suppose f satisfies selfish domain, weak Pareto, independence of irrelevant alternatives and multi-profile stationarity. By Theorem 3, $\mathcal{T}(f)$ is an ultrafilter. Let x and y be two distinct elements of X and let $\succeq$ be an ordering on X such that $xP(\succeq)y$. By selfish domain, we can define a profile $\mathbf{R} \in \mathcal{R}_S^\infty$ by letting, for all $t \in \mathbb{N}$ and for all $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\mathbf{x}R_t\mathbf{y} \;\Leftrightarrow\; x_t \succeq y_t.$$

Now consider the streams

$$\begin{aligned}
\mathbf{x} &= (x, y, x, y, x, y, \ldots) = (x, \mathbf{y}),\\
\mathbf{y} &= (y, x, y, x, y, x, \ldots) = (y, \mathbf{x}),\\
\mathbf{z} &= (x, x, y, x, y, x, \ldots) = (x, \mathbf{x}),\\
\mathbf{w} &= (y, y, x, y, x, y, \ldots) = (y, \mathbf{y}).
\end{aligned}$$

We have $\mathbf{x}P(R_t)\mathbf{y}$ for all odd t and $\mathbf{y}P(R_t)\mathbf{x}$ for all even t. Because of property 4 of an ultrafilter, one of the two sets $\{2, 4, 6, \ldots\}$ and $\{1, 3, 5, \ldots\}$ must be decisive. If $\{2, 4, 6, \ldots\}$ is decisive, we have

$$\mathbf{y}P(f(\mathbf{R}))\mathbf{x} \;\text{ and }\; \mathbf{y}P(f(\mathbf{R}_{\geq 2}))\mathbf{x}. \tag{1}$$

By multi-profile stationarity,

$$\mathbf{x}f(\mathbf{R})\mathbf{z} \;\Leftrightarrow\; \mathbf{x}_{\geq 2}f(\mathbf{R}_{\geq 2})\mathbf{z}_{\geq 2} \;\Leftrightarrow\; \mathbf{y}f(\mathbf{R}_{\geq 2})\mathbf{x}$$

and, thus, $\mathbf{x}P(f(\mathbf{R}))\mathbf{z}$ by (1). Analogously, multi-profile stationarity implies

$$\mathbf{w}f(\mathbf{R})\mathbf{y} \;\Leftrightarrow\; \mathbf{w}_{\geq 2}f(\mathbf{R}_{\geq 2})\mathbf{y}_{\geq 2} \;\Leftrightarrow\; \mathbf{y}f(\mathbf{R}_{\geq 2})\mathbf{x}$$

and (1) implies $\mathbf{w}P(f(\mathbf{R}))\mathbf{y}$. By transitivity, $\mathbf{w}P(f(\mathbf{R}))\mathbf{z}$. We have $\mathbf{z}P(R_t)\mathbf{w}$ for all $t \in \{1, 2, 4, 6, \ldots\}$ and, because $\{2, 4, 6, \ldots\}$ is decisive and $\{2, 4, 6, \ldots\} \subseteq \{1, 2, 4, 6, \ldots\}$, property 2 of an ultrafilter implies that $\{1, 2, 4, 6, \ldots\}$ is decisive. Thus, $\mathbf{z}P(f(\mathbf{R}))\mathbf{w}$, a contradiction. Therefore, $\{1, 3, 5, \ldots\}$ must be decisive and, thus,

$$\mathbf{x}P(f(\mathbf{R}))\mathbf{y} \;\text{ and }\; \mathbf{x}P(f(\mathbf{R}_{\geq 2}))\mathbf{y}. \tag{2}$$

By multi-profile stationarity,

$$\mathbf{z}f(\mathbf{R})\mathbf{x} \;\Leftrightarrow\; \mathbf{z}_{\geq 2}f(\mathbf{R}_{\geq 2})\mathbf{x}_{\geq 2} \;\Leftrightarrow\; \mathbf{x}f(\mathbf{R}_{\geq 2})\mathbf{y}$$

and, thus, $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$ by (2). Analogously, multi-profile stationarity implies

$$\mathbf{y}f(\mathbf{R})\mathbf{w} \;\Leftrightarrow\; \mathbf{y}_{\geq 2}f(\mathbf{R}_{\geq 2})\mathbf{w}_{\geq 2} \;\Leftrightarrow\; \mathbf{x}f(\mathbf{R}_{\geq 2})\mathbf{y}$$

and (2) implies $\mathbf{y}P(f(\mathbf{R}))\mathbf{w}$. By transitivity, $\mathbf{z}P(f(\mathbf{R}))\mathbf{w}$. Because $\mathbf{z}P(R_t)\mathbf{w}$ for all $t \in \{1, 2, 4, 6, \ldots\}$ and $\mathbf{w}P(R_t)\mathbf{z}$ for all $t \in \{3, 5, 7, \ldots\}$, $\{3, 5, 7, \ldots\}$ cannot be decisive. By property 4 of an ultrafilter, it follows that $\{1, 2, 4, 6, \ldots\}$ is decisive.

We have thus established that $\{1, 3, 5, \ldots\}$ and $\{1, 2, 4, 6, \ldots\}$ are decisive and, by property 3 of an ultrafilter, $\{1\} = \{1, 3, 5, \ldots\} \cap \{1, 2, 4, 6, \ldots\}$ is decisive, which means generation one is a dictator. ■

The final result of this paper characterizes f^{CD}.

Theorem 5 *A social welfare function f satisfies selfish domain, weak Pareto, Pareto indifference, independence of irrelevant alternatives and multi-profile stationarity if and only if $f = f^{CD}$.*

Proof. That f^{CD} satisfies the required axioms can be verified by the reader. To prove the converse implication, suppose f satisfies the required axioms. It is sufficient to show that, for all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{R}_S^\infty$,

$$\mathbf{x} I(f^{CD}(\mathbf{R}))\mathbf{y} \;\Rightarrow\; \mathbf{x} I(f(\mathbf{R}))\mathbf{y} \tag{3}$$

and

$$\mathbf{x} P(f^{CD}(\mathbf{R}))\mathbf{y} \;\Rightarrow\; \mathbf{x} P(f(\mathbf{R}))\mathbf{y}. \tag{4}$$

(3) follows immediately from Pareto indifference. To prove (4), suppose $t \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in X^\infty$ and $\mathbf{R} \in \mathcal{R}_S^\infty$ are such that

$$x_\tau I(\succeq_\tau) y_\tau \;\forall \tau < t \;\text{ and }\; x_t P(\succeq_t) y_t.$$

If $t = 1$, let $\mathbf{z} = \mathbf{y}$; if $t \geq 2$, let $\mathbf{z} = (x_1, \ldots, x_{t-1}, \mathbf{y}_{\geq t})$. By Pareto indifference, $\mathbf{y} I(f(\mathbf{R}))\mathbf{z}$. Transitivity implies

$$\mathbf{x} f(\mathbf{R})\mathbf{y} \;\Leftrightarrow\; \mathbf{x} f(\mathbf{R})\mathbf{z}.$$

Together with the application of multi-profile stationarity $t-1$ times, we obtain

$$\mathbf{x} f(\mathbf{R})\mathbf{y} \;\Leftrightarrow\; \mathbf{x} f(\mathbf{R})\mathbf{z} \;\Leftrightarrow\; \mathbf{x}_{\geq t} f(\mathbf{R}_{\geq t})\mathbf{z}_{\geq t} = \mathbf{y}_{\geq t}. \tag{5}$$

By Theorem 4, the relative ranking of $\mathbf{x}_{\geq t}$ and $\mathbf{y}_{\geq t}$ according to $\mathbf{R}_{\geq t}$ is determined by the strict preference for $\mathbf{x}$ over $\mathbf{y}$ according to the first generation in the profile $\mathbf{R}_{\geq t}$ (which is generation t in $\mathbf{R}$), so that $\mathbf{x}_{\geq t} P(f(\mathbf{R}_{\geq t}))\mathbf{y}_{\geq t}$ and, by (5), $\mathbf{x} P(f(\mathbf{R}))\mathbf{y}$. ■

5 Concluding Remarks

In concluding this paper, it may be worthwhile to clarify the relationship between the multi-profile intergenerational social choice theory developed in this paper, on the one hand, and the theory of evaluating infinite intergenerational utility streams, which capitalizes on the Koopmans (1960) analysis of impatience and the Diamond (1965) impossibility theorem on the existence of continuous evaluation orderings over the set of infinite utility streams satisfying the Sidgwick (1907) anonymity principle and the Pareto efficiency principle, on the other. Among many contributions that appeared after Diamond (1965), those which are most relevant in the present context include Asheim, Mitra and Tungodden (2007), Basu and Mitra (2003; 2007), Bossert, Sprumont and Suzumura (2007), Hara, Shinotsuka, Suzumura and Xu (2008) and Svensson (1980). Although these two lines of inquiry are related in the sense that both are concerned with aggregating generational evaluations of their well-beings into the overall social evaluation, they contrast sharply in at least two respects. In the first place, the latter investigation is *welfaristic* in the sense of basing the overall social evaluation on the infinite-generational utility streams, whereas the former exercise is free from such an early commitment to this informational basis. In the second place, while the latter approach hinges squarely on the continuity assumption even in a vestigial form, the former has nothing to do with any continuity assumption on social welfare orderings. More substantially, the Sidgwick anonymity principle, which plays a crucial role in establishing the Diamond impossibility theorem and related work, has nothing to do with our impossibility theorems. The same observation also applies to the Hammond (1976) equity axiom, which plays an important role in some recent developments in the theory of evaluating infinite-generational utility streams. Since continuity is a requirement which is rather technical in nature, to get rid of the dependence on this assumption may be counted as a virtue rather than a vice. Although the Sidgwick anonymity principle and the Hammond equity principle have an obvious intuitive appeal, it is fortunate that we need not go against these appealing axioms in defending our approach. It can surely be added to the list of axioms but all that is thereby obtained is another set of Arrow-type impossibility results, some of which will even contain redundancies.

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