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Technical Progress, Capital Accumulation, and Distribution

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Abstract

We study the effects of innovations on income distribution in capitalist economies characterised by a drive to accumulate. Consistent with the basic intuitions of Marx's theory of technical change, we show that there is no obvious relation between ex-ante profitable innovations and the income distribution that actually emerges in equilibrium, and individually rational choices of technique do not necessarily lead to optimal outcomes. Innovations may even cause the disappearance of *all* equilibria. Methodologically, it is not possible to fully understand the 'creative destruction' induced by innovations without capturing the dialectic between individual choices and aggregate outcomes, and the complex network of relations typical of capitalist economies.

JEL: O33; D33; B51.

Keywords: technical change, income distribution, profit rate.

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1 Introduction

What is the relation between technical change and the distribution of income and wealth in capitalist economies? More specifically, is there a relation between innovation and the dynamics of profitability? Both questions have been central in the theorising of classical, as well as early neoclassical, economists.

According to Ricardo, Mill, and Walras, for example, the equilibrium rate of profit (or rate of interest) tends to fall because of decreasing returns to scale due to the scarcity of a primary production factor, such as land. Famously, Marx [12, 13] rejected this explanation and argued instead that the profit rate tends to fall whenever capital-using and labour-saving innovations – the kind of innovations that have characterised capitalist economies – are adopted. Even if such innovations are cost-reducing, the argument goes, once they are generally adopted they tend to lower the labour content of goods, and thus – in Marx's theory – prices and profits, which leads the profit rate to fall provided the real wage and the labour share are constant.

The classicals', and early neoclassicals', law of the falling rate of profit has been put into question. Starting at least from Schumpeter [21], countless authors have emphasised the role of innovation as the main driver of economic growth and as the key countervailing force avoiding rates of profit and interest to fall, thanks to the creation of temporary monopolies and abnormal profits.[1]

The most direct, and significant, criticism of Marx's theory of the falling rate of profit, however, has been developed by Nobuo Okishio [14] in a seminal paper. The so-called *Okishio Theorem* (henceforth, OT) shows that if the real wage rate is fixed at the (historically and culturally determined) subsistence level, then any cost-reducing technical change will necessarily increase the equilibrium profit rate. In other words, under a set of assumptions consistent with both Marx's and Schumpeter's theoretical framework, OT proves that the Marxian theory of the falling rate of profit is invalid.

This result has sparked substantial controversy and a vast literature. Some critics have rejected OT on exegetical grounds, arguing that the assumptions, definitions, or indeed the very mathematical approach adopted by Okishio are not Marx's. Others have provided 'refutations' of OT by constructing more or less plausible scenarios in which *some* profit rate is shown

[1] More recent contributions in a Schumpeterian vein include, among the many others, Romer [16], Grossman and Helpman [8], Aghion and Howitt [3], and Acemoglu [1, 2].

to fall even if the wage rate remains constant.[2] Another strand of literature has shown that capital-using labour-saving innovations can lead to a falling profit rate if the wage *share* remains constant.[3]

The theoretical relevance of the conditions under which the profit rate is shown to fall in this literature is sometimes unclear. Perhaps more importantly, however, these contributions suffer from two general limitations, which they share with Okishio's [14] original analysis: *both* individual choices concerning the development and adoption of new techniques, *and* the structure of the economy, including the complex web of connections that characterise capitalist economies, are severely underspecified. Innovations are usually assumed to simply appear and, if they are cost-reducing, to be automatically and universally adopted. The economy is assumed to reach an equilibrium which is defined as simply requiring a uniform profit rate across sectors.

The problem with this conceptualisation is not its simplicity: all theoretical models – formal or otherwise – inevitably abstract from some features of social reality. The problem is that it misses some key features of the process of technical change in capitalist economies. First, at a general conceptual level, Marx has a dialectical view of capitalist economies in which aggregate outcomes are often the unintended consequence of individual actions. What is optimal ex ante for individuals is not necessarily optimal ex post at either the individual or the aggregate level. His theory of technical progress and of the effects of technical change on profitability is fundamentally based on this view of the complex relation between the micro and macro levels.

Second, and related, in both Marx's and Schumpeter's theories, innovations are not smoothly and automatically adopted in the economy even when they are cost-reducing at current prices. Technical progress has an inherently disruptive nature – it involves a process of 'creative destruction', in Schumpeter's words – in that consolidated practices are abandoned and old equilibria disappear. Innovations have general equilibrium effects that simple comparative statics exercises cannot really capture: technical progress leads to changes in equilibrium prices, which in turn feedback on individual decisions in potentially unpredictable ways.

Interestingly, in later contributions, Okishio [15] himself has raised some doubts on the relevance of OT as a characterisation of the long run tendencies

[2] The literature is too vast for a comprehensive list of references. For a (partial) survey, see Veneziani [23, 24].

[3] See, for example, Franke [7] and the references therein.

of capitalist economies. On the one hand, he has argued that the assumption of a fixed (subsistence) wage seems empirically unrealistic and theoretically restrictive. For, in the dynamic process of capital accumulation, the real wage rate would increase unless population grew faster than capital. Further, Okishio [15] has acknowledged that OT is essentially a comparative statics result, which compares two equilibrium rates of profits (before and after the introduction of a new technique), but does not consider the complex transition from the old equilibrium to the new one.

In this paper, we build on Okishio's [15] insights and provide an alternative framework to analyse the interaction of innovation, capital accumulation, and factor income distribution. To be specific, we set up a dynamic general equilibrium model in which, in every period, agents exchange goods and services on a number of interrelated markets. Propertyless agents simply supply labour, while capitalists activate optimal production activities and adopt production techniques yielding the maximum rate of profit. Unlike in standard neoclassical models, production takes time. Hence, capital and labour are traded at the beginning of each production period, while consumption goods are exchanged after production has taken place.

We adopt a classical view of the functioning of capitalism and assume the economy to be driven by an accumulation motive,[4] and conceive of capital as a vector of reproducible commodities (rather than as a single aggregate factor, as in standard growth theory). A production technique, in this framework, is a blueprint describing how to combine a vector of produced inputs with labour in order to produce outputs. At the beginning of each production period, the production set consists of a set of known blueprints which agents can choose from in order to activate production. When innovations do emerge, they expand the production set by generating new blueprints that may be used first at the beginning of the next available production period.

As in the literature on OT, we do not explicitly consider R&D activities and the process of generating innovations. Unlike in the literature on OT, however, we provide an explicit, thorough analysis both of individual choices after a new technique is discovered, *and* of the general equilibrium effects of innovation on both prices and income distribution. We design a thought experiment by supposing the economy to start out on a balanced growth path characterised by an equilibrium price vector and an optimally chosen

[4] "Accumulate, accumulate! That is Moses and the prophets!" (Marx [12], Ch.24, Sect. 3).

technique that – absent any perturbations – would persist over time. What are the effects of profitable, cost reducing innovations in this context? *If* (i) a new equilibrium exists in which (ii) the new technique is adopted and (iii) the wage rate remains unchanged, then OT continues to hold (Theorem 2). Aside from this rather special case, however, our findings are much more nuanced and perhaps surprising, and at a broad theoretical level, they vindicate some of Marx's original intuitions.

First, innovations may have negative welfare effects in capitalist economies: while all cost-reducing capital-using and labour-saving innovations improve labour productivity (as measured by the Leontief employment multipliers), there exist profitable innovations which, if universally adopted in equilibrium, would unambiguously worsen labour productivity (Proposition 2).

Second, the distributive implications of technical progress depend on the general equilibrium effects of technical change. If innovations maintain, or create, a reserve army of the unemployed, then technical change leads to an increase in the equilibrium profit rate as predicted by OT (Theorem 6). This conclusion does not hold in general, though. If a new equilibrium with full employment of productive factors is reached, then the effect of innovation on distribution is a priori unclear, as there exist (infinitely) many profit rates and wage rates that can be supported in equilibrium. Indeed, it is even possible for either the equilibrium wage rate or the profit rate to *decrease* compared to the equilibrium with the old technology (Theorem 5). The actual distributional outcome depends on the equilibrium selection mechanism.[5]

Perhaps more surprisingly, a cost-reducing change of technique may yield a decrease in the equilibrium profit rate. If the new technique significantly increases labour productivity while it makes the present capital stock abundant relative to population, then its introduction drives the equilibrium profit rate to zero (Theorems 7 and 9). Contrary to OT, and much like in Marx's original intuition, an innovation that is profitable for individual capitalists at current prices yields, after it is universally adopted, a change in the equilibrium price vector – and consequently in individual behaviour – eventually leading the equilibrium profit rate to decrease. Indeed, the equilibrium profit

[5] This result is reminiscent of the well known indeterminacy of the functional distribution of income in Sraffa's [22] system of production prices. Nonetheless, the indeterminacy in Theorem 5 is quite different: it is the result of the inter-period equilibrium transition triggered by innovations, and it obtains under a more general equilibrium notion which includes the standard excess demand conditions for all markets. For recent analyses of indeterminacy in Sraffian models, see Mandler [11] and Yoshihara and Kwak [26].

rate falls (albeit not necessarily to zero) even if the new technique worsens labour productivity, though in this case the mechanism is subtler and less intuitive, as the innovation is *not* adopted: it has a pure general equilibrium effect leading to capitalists to opt for an older technique (Theorem 8).

Third, innovations can be highly disruptive and the process of 'creative destruction' is anything but smooth, as Schumpeter [21] emphasised. Unlike in the standard literature on OT, which assumes that any new technique improving profitability is adopted, we show that there exist cost-reducing, capital-saving and labour-using innovations that destroy the existing equilibrium and yet are *not* adopted in the new equilibrium (Theorem 3). Innovations may paradoxically lead *older* techniques to become profitable again, due to changes in equilibrium prices (Theorem 8). Even more radically, innovations may lead to the disappearance of equilibrium altogether: the process of creative destruction entails a disequilibrium dynamics.[6]

Finally, our analysis suggests that the falling rate of profit cannot be fully understood if capital is conceived of as a single, homogeneous factor of production, as in standard neoclassical models. In section 7 below, we highlight an interesting, novel connection between technical changes leading to a falling rate of profit and the so-called *reswitching of techniques* and *capital reversing* identified in classical capital theory (Sraffa [22]), which can be analysed only if capital is understood as a vector of produced inputs as argued in the Cambridge capital controversy.

The rest of the paper is structured as follows. Sections 2 and 3 present the economy and the equilibrium concept. Section 4 introduces the notion of profitable technical change and derives a generalisation of OT. Section 5 introduces a classification of the types of profitable changes of techniques. Section 6 examines various types of equilibrium transitions triggered by such changes of techniques. Section 7 characterises the conditions under which technical change leads to a falling rate of profit. Section 8 concludes.

[6] In his analysis of choice of technique in linear economies with joint production, Bidard [4] defines an algorithm which ensures the convergence to an optimal technique in a given class of production sets. This suggests that, as in our model, outside of that class the algorithm may not converge leading to a disequilibrium dynamics. The mechanism underlying the non-existence of equilibrium is, however, different: in our model joint production is ruled out and non-convergence derives instead from the interaction between individually optimal choice of technique and the general equilibrium effects of innovations.

2 The economy

Consider a closed economy with n commodities produced and consumed. In order to focus on the relation between technical change, optimal behaviour, and profitability we assume that the set of commodities is constant over time.

2.1 Technology, innovation, and knowledge

At the beginning of each production period $t = 1, 2, \ldots$, there is a finite set of production techniques $\mathcal{B}_t$, the *base set*, which consists of Leontief techniques (A_t, L_t), where A_t is a $n \times n$ nonnegative, productive, and indecomposable matrix of material input coefficients, and L_t is a $1 \times n$ positive vector of labour coefficients. The set of all available production techniques at t, $\mathcal{P}_t$, consists of all convex combinations of the techniques in $\mathcal{B}_t$, so that $\mathcal{B}_t \subseteq \mathcal{P}_t$.

The base set $\mathcal{B}_t$ contains the blueprints that can be used at t to produce the n goods – the technically feasible ways of combining inputs in order to produce outputs.[7] The stock of knowledge does not depreciate: once a production technique is discovered, it remains available for agents to use. But knowledge can be accumulated. Formally, $\mathcal{B}_{t-1} \subseteq \mathcal{B}_t$ holds in general, and *technical progress* takes place between period $t-1$ and period t if and only if $\mathcal{B}_{t-1} \subset \mathcal{B}_t$ and $(A_t^*, L_t^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$: (A_t^*, L_t^*) is an *innovation*, which is available at the beginning of t.[8] Because we are interested in the effects of innovation on profitability in competitive market economies, we suppose that, at the beginning of t, information both about $\mathcal{B}_{t-1}$ and about any new technique $(A_t^*, L_t^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ is available to all agents in the economy.

At all t, for any $(A_t, L_t) \in \mathcal{B}_t$, let $\delta(A_t) = \frac{1}{1+\Pi(A_t)}$ denote the Frobenius eigenvalue of A_t. By the assumptions on $\mathcal{B}_t$, $\Pi(A_t) > 0$ and $\delta(A_t) < 1$.

2.2 Agents

We study some fundamental dynamic laws of capitalist economies characterised by a drive to accumulate, and assume that agents aim to maximise

[7] This is a generalisation of Jones's [9] model of 'ideas'.

[8] In discrete time models, there is always a degree of arbitrariness concerning the timing of decisions, and the economy analysed in this paper is no exception. Yet our key insights are robust to small perturbations in the timing of decisions. The assumption that only one new technique can emerge in a period is for simplicity and yields no loss of generality.

their wealth subject to reaching a minimum consumption standard.[9]

Let $\mathbb{R}$ (resp., $\mathbb{R}_+$) denote the sets of (resp., nonnegative) real numbers. In period t, let the set of agents be denoted by $\mathcal{N}_t$, with cardinality N_t and generic element ν. At the beginning of each t, each agent $\nu \in \mathcal{N}_t$ is endowed with a (possibly zero) vector of produced goods determined by past decisions, $\omega_{t-1}^{\nu} \in \mathbb{R}_+^n$, whose distribution in the economy is given by $\Omega_{t-1} = \left(\omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_t} \in \mathbb{R}_+^{nN_t}$, and with one unit of labour.[10] Let $\mathbf{0} = (0, \ldots, 0)$.

We follow Roemer [19, 20] in making the time structure of production explicit: "Production takes time. One enters inputs today and gets outputs tomorrow. Furthermore, capitalists, facing [today's prices], are constrained in their choice of activity levels by the value of their capital ... There is no credit market, and they must pay for inputs today" (Roemer [19], p.18). Within each period of production, market exchanges take place at two points in time: productive inputs are traded and labour contracts are signed at the beginning of the period; outputs are traded and wages are paid at the end.

Therefore let $p_t^b \in \mathbb{R}_+^n$ be the vector of beginning-of-period commodity prices, and for every agent $\nu \in \mathcal{N}_t$, let $(p_t^{e\nu}, w_t^{e\nu}) \in \mathbb{R}_+^{n+1}$ be agent ν's expectation about the vector of commodity prices and the nominal wage rate ruling at the end of period t, denoted by (p_t, w_t). Because agents have the same preferences and possess the same information, in what follows we assume them to have identical expectations and drop the superscript ν for simplicity. Formally, $(p_t^{e\nu}, w_t^{e\nu}) = (p_t^{e\mu}, w_t^{e\mu}) = (p_t^e, w_t^e)$ for all $\nu, \mu \in \mathcal{N}_t$.

Given $\left(p_t^b, p_t^e, w_t^e\right)$, at the beginning of each t, every agent $\nu \in \mathcal{N}_t$ chooses her labour supply, l_t^{ν}, and uses wealth, $W_t^{\nu} = p_t^b \omega_{t-1}^{\nu}$, either to buy goods δ_t^{ν} (spending $p_t^b \delta_t^{\nu}$) for sale at the end of the period or to finance production. In the latter case, each agent chooses a production technique, $(A_t^{\nu}, L_t^{\nu}) \in \mathcal{P}_t$, which is activated at level x_t^{ν} by investing (part of) W_t^{ν} to finance the operating costs of the activities she activates, $p_t^b A_t^{\nu} x_t^{\nu}$, and by hiring workers $L_t^{\nu} x_t^{\nu}$, which are paid ex post the expected amount $w_t^e L_t^{\nu} x_t^{\nu}$.[11] Thus, expected gross revenue at the end of t is $p_t^e x_t^{\nu} + w_t^e l_t^{\nu} + p_t^e \delta_t^{\nu}$, which is used to pay wages

[9]The model is a dynamic extension of Roemer's [20] accumulating economy with a labour market.

[10]In every period t, we take the distribution of endowments as exogenously given and abstract from all issues related to bequests and the endowment of newly born agents. This is without any loss of generality and none of our results depends on it.

[11]The model can allow agents to operate production activities with their own capital as self-employed producers. Given the convexity of the optimisation programme MP_t^{ν} below, this makes no difference to our results.

and finance accumulation, $p^e_t \omega^\nu_t$, subject to purchasing a consumption bundle $b \in \mathbb{R}^n_+$, $b > \mathbf{0}$,[12] per unit of labour performed.[13]

Let $\triangle \equiv \left\{ p \in \mathbb{R}^n_+ \mid pb = 1 \right\}$: $\triangle$ is the set of normalised price vectors. Formally, in every period t, given $\left(p^b_t, p^e_t, w^e_t\right) \in \triangle^2 \times \mathbb{R}_+$, agents are assumed to choose (A^ν_t, L^ν_t), $\xi^\nu_t \equiv (x^\nu_t, l^\nu_t, \delta^\nu_t)$, and ω^ν_t to solve:[14]

$$MP^\nu_t : \max_{(A^\nu_t, L^\nu_t);\xi^\nu_t;\omega^\nu_t} p^e_t \omega^\nu_t$$

subject to

$$\begin{aligned}
\left[p^e_t - w^e_t L^\nu_t\right] x^\nu_t + w^e_t l^\nu_t + p^e_t \delta^\nu_t &= p^e_t b l^\nu_t + p^e_t \omega^\nu_t && (1)\\
p^b_t A^\nu_t x^\nu_t + p^b_t \delta^\nu_t &= p^b_t \omega^\nu_{t-1}, && (2)\\
0 &\leqq l^\nu_t \leqq 1, && (3)\\
(A^\nu_t, L^\nu_t) &\in \mathcal{P}_t; && (4)\\
x^\nu_t, \delta^\nu_t, \omega^\nu_t &\in \mathbb{R}^n_+. && (5)
\end{aligned}$$

In other words, we focus on a temporary resource allocation problem whereby agents choose an optimal plan in each production period. The analysis of the transition process sparked by technical progress is developed in section 4 below, where we explicitly consider the change in production techniques occurring after the emergence of an innovation.

Finally, let $v_t \equiv L_t(I - A_t)^{-1}$ denote the standard vector of employment multipliers. In the rest of the paper, we assume that for all $(A_t, L_t) \in \mathcal{B}_t$, $1 > v_t b$ holds: this is a basic condition for the productiveness of the economy.

3 Equilibrium

An *accumulation economy at period t* is a set of agents, $\mathcal{N}_t$, a set of production techniques, $\mathcal{B}_t$, a consumption bundle, b, and a distribution of productive endowments, Ω_{t-1}, and is denoted as $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.Let $x_t \equiv \sum_{\nu \in \mathcal{N}_t} x^\nu_t$, and let a similar notation hold for δ_t, ω_t, and l_t. Similar to Roemer [19, 20], the equilibrium notion of this economy can be defined.

[12] Vector inequalities: for all $x, y \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$.

[13] Given our analytical focus on the relation between technical change and profitability, we do not explicitly analyse the agents' consumption choices and treat b as a parameter. This is without significant loss of generality.

[14] Constraints (1) and (2) are written as equalities without loss of generality.

Definition 1: A *competitive equilibrium* (CE) for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ is a vector $\left(p_t^b, p_t, w_t\right) \in \triangle^2 \times \mathbb{R}_+$ and associated $((A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ such that:
(a) $((A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu)$ solves MP_t^ν, for all $\nu \in \mathcal{N}_t$ (individual optimality);
(b) $\sum_{\nu \in \mathcal{N}_t} A_t^\nu x_t^\nu + \delta_t \leqq \omega_{t-1}$ (social feasibility of production);
(c) $\sum_{\nu \in \mathcal{N}_t} L_t^\nu x_t^\nu = l_t$ (labour market);
(d) $x_t + \delta_t \geqq \sum_{\nu \in \mathcal{N}_t} bL_t^\nu x_t^\nu + \omega_t$ with $x_t > \mathbf{0}$ (commodity markets);
(e) $p_t = p_t^e = p_t^b$ and $w_t = w_t^e$ (realised expectations).

In other words, at a CE, (a) all agents optimise; (b) aggregate capital is sufficient for production plans; (c) the labour market is in equilibrium; (d) the total supply of commodities is sufficient for consumption and accumulation plans; and (e) agents' expectations are realised ex post, $(p_t^e, w_t^e) = (p_t, w_t)$. For the sake of notational simplicity, because at a CE expectations are realised and $p_t = p_t^b = p_t^e$, we shall write the price vector as $(p_t, w_t) \in \triangle \times \mathbb{R}_+$.

Several points should be noted about Definition 1. First, the concept of CE is a temporary equilibrium notion which focuses on each period in isolation. The dynamic evolution of the economy can thus be conceived of as a sequence of temporary equilibria. Second, it focuses on non-trivial allocations with a positive gross output vector, $x_t > \mathbf{0}$. This is without loss of generality because agents will optimally activate all sectors if the profit rate is positive; and even if the profit rate is zero, $x_t > \mathbf{0}$ can always be the product of optimal choices, consistent with such a CE.

Third, following Roemer [19, 20], we assume that agents have *stationary expectations*, $p_t^e = p_t^b$. Although there are infinitely many ways of predicting end-of-period prices, we assume agents to adopt the simplest possible heuristics and suppose that beginning-of-period commodity prices will also rule at the end of the period. Thus, we are endowing agents with an extremely weak form of rationality in expectation formation in line with a large literature on bounded rationality and behavioural economics.

It is now possible to derive some preliminary results. First of all, it is immediate to prove that for any $(p_t, w_t) \in \triangle \times \mathbb{R}_+$, if $((A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu)$ solves MP_t^ν then (A_t^ν, L_t^ν) must yield the maximum profit rate. Formally,

$$(A_t^\nu, L_t^\nu) \in \arg \max_{(A,L) \in \mathcal{P}_t} \max_{i=1,\ldots,n} \pi_{it} = \frac{p_{it} - p_t A_i - w_t L_i}{p_t A_i},$$

where p_{it} is the i-th entry of p_t and A_i is the i-th column of A.[15] For any

[15] The set $\arg \max_{(A,L) \in \mathcal{P}_t} \max_{i=1,\ldots,n} \pi_{it}$ is non-empty, as $\mathcal{B}_t$ is finite.

$(p_t, w_t) \in \triangle \times \mathbb{R}_+$, let the maximum profit rate associated with a technique $(A, L) \in \mathcal{P}_t$ be $\pi_t(A, L) = \max_{i=1,...,n} \pi_{it}$, and let $\pi_t^{\max} = \max_{(A,L)\in\mathcal{P}_t} \pi_t(A, L)$ denote the maximum profit rate attainable with the existing techniques.[16]

In principle, in equilibrium there may be various optimal production techniques. However, as they will all yield the same (maximum) rate of profit, in what follows we assume without loss of generality that all agents who activate some production process opt for the same (A_t, L_t), and drop the agent's superscript from production techniques.

Next, by constraints (1)-(2), it immediately follows that, at a CE the following equation holds for all $\nu \in \mathcal{N}_t$:

$$p_t\omega_t^\nu = [p_t - p_tA_t - w_tL_t]\, x_t^\nu + (w_t - p_tb)\, l_t^\nu + p_t\omega_{t-1}^\nu, \tag{6}$$

Then, Lemma 1 derives some properties of the optimal solution to MP_t^ν.[17]

Lemma 1: Let $\left((p_t, w_t), ((A_t, L_t); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. For all $\nu \in \mathcal{N}_t$: if $\pi_t^{\max} > 0$, then $p_tA_tx_t^\nu = p_t\omega_{t-1}^\nu$ and if $w_t > p_tb$, then $l_t^\nu = 1$.

Lemma 2 proves that equilibrium prices are strictly positive and competition leads to the equalisation of sectoral profit rates in equilibrium.

Lemma 2: Let $\left((p_t, w_t), ((A_t, L_t); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Then $\pi_t^{\max} \geqq 0$, $p_t = (1 + \pi_t^{\max})\, p_tA_t + w_tL_t$, $p_t > \mathbf{0}$, and $w_t \geqq p_tb$.

Lastly, Theorem 1 derives some key properties of competitive equilibria.[18]

Theorem 1: Let $\left((p_t, w_t), ((A_t, L_t); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.
(i) If $\pi_t^{\max} > 0$ and $w_t > p_tb$, then $N_t = l_t = L_tA_t^{-1}\omega_{t-1}$.
(ii) If $N_t > L_tA_t^{-1}\omega_{t-1}$, then $w_t = p_tb$;
(iii) If $N_t < L_tA_t^{-1}\omega_{t-1}$, then $\pi_t^{\max} = 0$.

[16] It is immediate to show that for all $(p_t, w_t) \in \triangle \times \mathbb{R}_+$ such that $w_t \leqq p_tb$, $1 > v_tb$ implies $\pi_t(A_t, L_t) > 0$ and, a fortiori, $\pi_t^{\max} > 0$.

[17] The results in this section follow rather straightforwardly from MP_t^ν and Definition 1 and their proofs are therefore omitted. (See the Addendum.)

[18] Theorem 1 only provides necessary conditions for the existence of a CE: this is all we need in the analysis of technical change in the next sections. A complete characterisation of the necessary and sufficient conditions for the existence of equilibrium is provided in the Addendum.

Finally, given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with aggregate capital stock $\omega_{t-1} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, we define the *set of feasible production techniques* (A, L) such that, given ω_{t-1}, all agents can reach subsistence b:

$$\mathcal{B}_t(\omega_{t-1}; b) \equiv \left\{(A, L) \in \mathcal{B}_t \mid \exists A^{-1} \text{ s.t. } A^{-1}\omega_{t-1} > \mathbf{0} \text{ and } (I - bL) A^{-1}\omega_{t-1} \geqq \mathbf{0}\right\}.$$

In other words, if $(A, L) \in \mathcal{B}_t(\omega_{t-1}; b)$ is adopted, then there exists a profile of actions $(x_t^\nu;\ l_t^\nu; \delta_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ satisfying Definitions 1(b)-(d).

4 Technical progress and technical change

We are interested in the effects of technical progress on distribution. When do innovations induce changes in production methods? More precisely, in the context of our model: when does the appearance of a new technique force a change in the equilibrium of the economy over time? How do innovations alter the equilibrium distribution of income between profits and wages?

Recall that within every given period t the production set $\mathcal{B}_t$, population $\mathcal{N}_t$, and aggregate capital stocks ω_{t-1} are constant, and they are given in the determination of equilibrium prices and actions in t.[19] Demographic changes, accumulation, and technical innovations may, however, occur after the end of period t, potentially leading to a new equilibrium, with different prices and different production methods adopted.

Suppose, for example, that at $t-1$ there was an excess supply of labour pushing the wage down to the subsistence level and leading to the adoption of a certain technique of production (Theorem 1(ii)). Given the high profits, this may lead to overaccumulation such that in period t a new equilibrium emerges in which the economy is labour constrained, wages increase, and producers adopt a different production technique (Theorem 1(iii)).

In this paper, we abstract from changes in equilibrium prices and actions induced by accumulation or demographic factors, and focus on *inter-period changes of technique* from (A_{t-1}, L_{t-1}), which is chosen in period $t-1$, to another technique available in $\mathcal{B}_t$ due to technical progress, and their effect on the equilibrium distribution of income.

In order to abstract from other factors, and analyse the equilibrium effects of inter-period change of technique due to technical progress, we shall

[19] This reflects the fact that the production of commodities – including capital stocks – takes time and the assumption of a closed economy with no external source of labour.

consider a subset of equilibria such that, *absent any innovation*, equilibrium prices would be invariant across two periods. Formally:

Definition 2: Let $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1})_{\nu \in \mathcal{N}_{t-1}}\right)$ be a CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Then, the CE is *persistent* if and only if there exists a profile of individual actions in period t, $(\xi^{\nu}_t; \omega^{\nu}_t)_{\nu \in \mathcal{N}_t}$, such that $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi^{\nu}_t; \omega^{\nu}_t)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$ with $\omega_{t-1} = \sum_{\nu \in \mathcal{N}_{t-1}} \omega^{\nu}_{t-1}$.

In other words, if a CE for the period $t-1$ economy is persistent, then neither equilibrium prices (p_{t-1}, w_{t-1}) nor the production technique (A_{t-1}, L_{t-1}) need to vary in period t, as long as no technical progress takes place between the two periods. Therefore, the notion of persistent CE is primarily an analytical device to examine the effect of technical progress in isolation, and it describes a potentially counterfactual allocation that would emerge at t if the economy had the same production set as at $t-1$, $\mathcal{B}_t = \mathcal{B}_{t-1}$.

Absent technical progress, the conditions for the persistence of a CE are not particularly strong, as they basically require capital accumulation to appropriately adjust to changes in demographic conditions. Formally:[20]

Proposition 1: Let $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1})_{\nu \in \mathcal{N}_{t-1}}\right)$ be a CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Then, given $\mathcal{N}_t = \{1, \ldots, N_t\}$,
(i) if $\pi^{\max}_{t-1} > 0$ and $w_{t-1} > p_{t-1}b$, then this CE is persistent if and only if $A^{-1}_{t-1}\omega_{t-1} > \mathbf{0}$ and $N_t = L_{t-1}A^{-1}_{t-1}\omega_{t-1}$;
(ii) if $\pi^{\max}_{t-1} > 0$ and $w_{t-1} = p_{t-1}b$, then this CE is persistent if and only if $A^{-1}_{t-1}\omega_{t-1} > \mathbf{0}$ and $N_t \geqq L_{t-1}A^{-1}_{t-1}\omega_{t-1}$;
(iii) if $\pi^{\max}_{t-1} = 0$ and $w_{t-1} > p_{t-1}b$, then this CE is persistent if and only if there exists $\delta \geqq \mathbf{0}$ such that $A^{-1}_{t-1}(\omega_{t-1} - \delta) > \mathbf{0}$ and $N_t = L_{t-1}A^{-1}_{t-1}(\omega_{t-1} - \delta)$.

The concept of CE allows us to analyse the emergence of what may be thought of as Schumpeterian innovations: new techniques that create unforeseen profit opportunities, disrupt existing production processes, and cause fundamental shifts in the distribution of income. To see this, suppose that the economy is at a persistent CE in period $t-1$ and technical progress

[20] The conditions in Proposition 1 would be satisfied if, for example, the economy was on a balanced growth path with a given (A_{t-1}, L_{t-1}) and the growth rate of capital was equal to the growth rate of population.

occurs after production has taken place in $t-1$, but before productive inputs are bought and production starts in period t. Under what conditions will the emergence of an innovation $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ alter incentives and lead agents to deviate from the persistent CE?

Clearly, if $\pi_{t-1}(A^*, L^*) \leqq \pi_{t-1}^{\max} = \pi_{t-1}(A_{t-1}, L_{t-1})$ holds, then (p_{t-1}, w_{t-1}) and (A_{t-1}, L_{t-1}) would still constitute a CE in period t. This motivates our focus on innovations that are profitable in the following sense:

Definition 3: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \ \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique. Inter-period change of technique from (A, L) to (A^*, L^*) is *profitable* if and only if the following condition holds:

$$\left(1 + \pi_{t-1}^{\max}\right) p_{t-1} A^* + w_{t-1} L^* \leq \left(1 + \pi_{t-1}^{\max}\right) p_{t-1} A + w_{t-1} L.$$

In other words, a new technique is profitable if at prices (p_{t-1}, w_{t-1}), a producer can expect *extra profits* by switching from (A, L) to (A^*, L^*). Thus, Definition 3 characterises a necessary condition for the persistent CE to disappear: if the change from (A, L) to (A^*, L^*) is profitable, then capitalists will never adopt (A, L) at (p_{t-1}, w_{t-1}).

Three points are worth noting about Definition 3. First, the premise that the economy is at a persistent CE in period $t-1$ is crucial. If the CE were not persistent, then Definition 3 would not capture a relevant condition for innovation to disrupt behaviour, as the economy may move to a different equilibrium because of demographic factors and/or due to capital accumulation. Similarly, the fact that the new technique would have been profitable at last period's prices would be immaterial for today's decisions.

Second, and most important for our analysis, Definition 3 per se does not tell us anything, a priori, about the effect of technical progress on distribution and profits. For, on the one hand, the condition in Definition 3 is not sufficient to guarantee that the new production technique will indeed be adopted in equilibrium. If it is profitable to switch from (A, L) to (A^*, L^*), then the old technique will be abandoned, but this in turn means that equilibrium prices will not, in general, remain the same. After the change in prices, nothing guarantees that (A^*, L^*) will yield the highest profit rate in $\mathcal{P}_t$, and therefore, it will not necessarily be used at the new CE.[21]

[21] We shall return to this issue below in more detail.

On the other hand, even if the new technique (A^*, L^*) is indeed optimal at the new equilibrium prices (p_t^*, w_t^*) in period t, and therefore yields the highest profit rate, $\pi_t^{*\max}$, among all techniques in $\mathcal{P}_t$, $\pi_t^{*\max}$ may be higher or lower than the profit rate $\pi_{t-1}^{\max}$ associated with the old technique at the equilibrium prices in $t-1$. Therefore it is unclear whether technical progress has a positive effect on profitability, as Schumpeter suggested, or rather it drives the tendency of equilibrium profit rates to fall, as Marx argued.

In his seminal contribution, Okishio [14] has shown that if the wage rate is fixed at the subsistence level, then the equilibrium profit rate always increases, thus casting doubts on the law of the falling rate of profit. Theorem 2 extends the Okishio Theorem (OT) by assuming that wages are paid ex post and by allowing the wage rate to be higher than the subsistence level.

Theorem 2: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique such that inter-period change of technique from (A, L) to (A^*, L^*) is profitable. If $\left((p_t^*, w_t^*), \left((A^*, L^*); \xi_t^{\nu}; \omega_t^{\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $w_t^* = w_{t-1}$, then $\pi_t^{*\max} > \pi_{t-1}^{\max}$.

Proof: By Lemma 2, $p_{t-1} = \left(1 + \pi_{t-1}^{\max}\right) p_{t-1} A + w_{t-1} L$ and $p_t^* = \left(1 + \pi_t^{*\max}\right) p_t^* A^* + w_{t-1} L^*$. Since the change of technique is profitable, $p_{t-1} \geq \left(1 + \pi_{t-1}^{\max}\right) p_{t-1} A^* + w_{t-1} L^*$ holds. Then, we have

$$p_{t-1} = w_{t-1} L \left[I - \left(1 + \pi_{t-1}^{\max}\right) A\right]^{-1} \geq w_{t-1} L^* \left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1}. \quad (7)$$

where $\left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1} > \mathbf{0}$. Post-multiplying both sides of equation (7) by $b > \mathbf{0}$ and recalling that $p_{t-1} \in \Delta$, we obtain

$$1 = w_{t-1} L \left[I - \left(1 + \pi_{t-1}^{\max}\right) A\right]^{-1} b > w_{t-1} L^* \left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1} b.$$

Since $p_t^* = w_t^* L^* \left[I - \left(1 + \pi_t^{*\max}\right) A^*\right]^{-1}$, $w_t^* = w_{t-1}$, and $p_t^* \in \Delta$, we also have

$$1 = w_{t-1} L^* \left[I - \left(1 + \pi_t^{*\max}\right) A^*\right]^{-1} b > w_{t-1} L^* \left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1} b.$$

The result then follows noting that $L^* \left[I - (1 + \pi) A^*\right]^{-1}$ is increasing in π. ■

Theorem 2 is far from obvious. As Roemer ([17], p.409) put it: "Clearly if a capitalist introduces a cost-reducing technical change his short-run rate of profit rises. This, however, produces a disequilibrium; what the theorem says is that after prices have readjusted to equilibrate the rate of profit again, the new rate of profit will be higher than the old rate." Theorem 2 is actually more general than standard versions of OT in that we adopt a general equilibrium concept which implies, but does not reduce to, the equalisation of sectoral profit rates.

5 Technical progress and equilibrium profits

One of the main advantages of the capitalist economic system, according to both Marx and Schumpeter, is that profit-seeking behaviour and capitalist competition induce permanent technical innovations, which in turn result in a steady improvement in human welfare thanks to labor productivity increases. Is this claim true in general? Is it possible to prove that any cost-reducing technical change is actually progressive? In order to examine this issue, we follow Roemer [17] and define two types of technical changes that are relevant to understand innovation in capitalist economies.[22]

Definition 4: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1};\ \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique. Inter-period change of technique from (A, L) to (A^*, L^*) is:
(i) *capital-using and labour-saving* (CU-LS) if and only if $A^* \geq A$ and $L^* \leq L$, and there exists at least one sector i such that $A^*_i \geq A_i$ and $L^*_i < L_i$;
(ii) *capital-saving and labour-using* (CS-LU) if and only if $A^* \leq A$ and $L^* \geq L$, and there exists at least one sector i such that $A^*_i \leq A_i$ and $L^*_i > L_i$.

Two features of Definition 4 are worth stressing. First, different types of inter-period change of technique are defined in physical, rather than monetary terms in order to abstract from the general equilibrium effects of technical change on prices. Second, only technical changes that are weakly monotonic in all produced inputs are considered. Although this may seem restrictive in an n-good space, it is in line with the definitions used in the literature (and in policy debates), and with intuitive notions of the mechanisation process that has characterised much of capitalist development.

The next Definition classifies different types of changes of technique based on their effect on labour multipliers and on labour productivity.

[22] See also Flaschel et al. [6] and Yoshihara and Veneziani [27].

Definition 5: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}, \mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique. Inter-period change of technique from (A, L) to (A^*, L^*) is:
(a) *progressive* if and only if $v^* < v$;
(b) *neutral* if and only if $v^* = v$;
(c) *regressive* if and only if $v^* > v$.

In other words, the adoption of a new technique is progressive if it leads to a uniform *decrease* in employment multipliers, and therefore to an *increase* in labour productivity. Regressive technical changes have the opposite effect.[23]

Given Definitions 4 and 5, we can derive a generalisation of some standard results in the literature.[24]

Proposition 2 (Roemer [17])**:** Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu};\ \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique. Suppose that inter-period change of technique from (A, L) to (A^*, L^*) is profitable. Then, (i) if it is CU-LS, then it is progressive; while (ii) if it is CS-LU with $v(A - A^*) \leq (L^* - L)$, then it is regressive.

Proposition 2 states that not all forms of 'creative destruction' have unambiguously beneficial effects. While cost-reducing innovations that substitute capital for labour expand the economy's production possibility frontier, other types of profitable technical change may be incompatible with the improvement of human welfare in terms of a decrease in toil and effort.

Although Proposition 2 provides some insights on the welfare effects of technical progress, it only tells part of the story, because it does not say anything about its distributive effects. In the rest of this section, we start to address this issue by characterising equilibria in which innovations are adopted. In order to derive the next results, we impose more structure on technical progress and focus on technical changes whose main effect is on labour, rather than on capital inputs. Formally:

Definition 6: Let inter-period change of technique from (A, L) to (A^*, L^*)

[23] Definition 5 focuses on changes of technique that modify all employment multipliers in the same direction. As Roemer ([17], p.410) notes, this is without any loss of generality if one considers changes of technique of the type described in Definition 4.

[24] The proof of Proposition 2 is a straightforward extension of the proof of Theorems 1-2 in Flaschel et al [6] and is therefore omitted. (See Addendum.)

take place in sector i such that $A_i \neq A_i^*$. Then, the change of technique is *labour inelastic* if and only if $|L_i - L_i^*| > |LA^{-1}(A_i - A_i^*)|$.

Definition 6 identifies innovations whose main effect is on labour, rather than produced inputs. The intuition is straightforward in a one-good economy: an inter-period change of technique is labour inelastic if the percentage change in produced input is smaller than the percentage change in labour input. In an n-good economy, $(A_i - A_i^*)$ is the change in the *vector* of commodity inputs necessary to produce one unit of good i. Definition 6 uses the linear operator LA^{-1} to transform the units of physical goods into labour: $LA^{-1}(A_i - A_i^*)$ represents the amount of direct labour demand necessary for the operation of the variational commodity inputs. Then, Definition 6 states that a change of technique is labour inelastic if and only if the change in the profile of commodity inputs measured in labour units is smaller than the change of direct labour input necessary to produce one unit of good i.

We can now derive the following result:

Theorem 3: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \ \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique, and let inter-period change of technique from (A, L) to (A^*, L^*) be profitable and labour inelastic. Then:

(i) if the change of technique is CU-LS and it results in a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, then $w_t^* = p_t^* b$ and $\pi_t^{*\max} > \pi_{t-1}^{\max}$ whenever $A^* x_t^* = \omega_{t-1}$; otherwise, $\pi_t^{*\max} = 0$;

(ii) if the change of technique is CS-LU and it results in a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, then $\pi_t^{*\max} = 0 < \pi_{t-1}^{\max}$;

(iii) if the change of technique is CS-LU and regressive, then there is no CE in t in which (A^*, L^*) is adopted.

Proof: As $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ is a persistent CE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$, Proposition 1 implies that $N_t = LA^{-1}\omega_{t-1}$, and there exist $(\xi_t^{\nu})_{\nu \in \mathcal{N}_t} = (x_t^{\nu}; 1; \mathbf{0})_{\nu \in \mathcal{N}_t}$ and $(\omega_t^{\nu})_{\nu \in \mathcal{N}_t}$ such that $x_t > \mathbf{0}$ with $Ax_t = \omega_{t-1}$, and $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_t^{\nu}; \ \omega_t^{\nu})_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.

Part (i). For $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$, let there be no $x' \in \mathbb{R}_+^n$ such that $A^* x' = \omega_{t-1}$ holds. As $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, there exists an aggregate production activity vector $x_t^* \in \mathbb{R}_+^n$ such that

$A^*x_t^* \leq \omega_{t-1}$ holds. In this case, as $p_t^* > \mathbf{0}$ by Lemma 2, we have $p_t^* A^* x_t^* < p_t^* \omega_{t-1}$. Then, by Lemma 1, $\pi_t^{*\max} = 0$ holds.

Let the change of technique from (A, L) to (A^*, L^*) take place, and let $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ in which there exists an aggregate production activity vector $x_t^* \in \mathbb{R}_+^n$ such that $A^* x_t^* = \omega_{t-1}$ holds. Because this change of technique is CU-LS, $A^* x_t \geq A x_t = \omega_{t-1}$ and $L^* x_t < L x_t = N_t$. Therefore, since $A^* x_t^* = \omega_{t-1}$, we obtain $A^*(x_t - x_t^*) \geq \mathbf{0}$. We consider two cases.

Case 1: $0 < x_t^* \leqq x_t$.

Clearly, $L^* x_t^* < L x_t = N_t$, so that $w_t^* = p_t^* b$ follows from Lemma 1 and Definition 1(c).

Let $\pi' \in [0, \Pi(A^*))$ be such that $\rho(\pi') \equiv w_{t-1} L^* [I - (1 + \pi') A^*]^{-1} b = 1$. To see that such π' exists, observe that $lim_{\pi' \to \Pi(A^*)} \rho(\pi') = \infty$, while $\rho(0) = w_{t-1} v^* b$ and $1 = p_{t-1} b \geqq w_{t-1} v b > w_{t-1} v^* b$ holds, where the latter inequality follows from Proposition 2(i). Therefore, as $\rho(\pi')$ is a continuous function, the existence of $\pi' \in [0, \Pi(A^*))$ follows from the intermediate value theorem.

Then, setting $p' \equiv w_{t-1} L^* [I - (1 + \pi') A^*]^{-1}$, we have $(p', w_{t-1}) \in \triangle \times \mathbb{R}_+$ with $p' = (1 + \pi') p' A^* + w_{t-1} L^* > \mathbf{0}$. Using the same argument as in the proof of Theorem 2, it can be proved that $\pi' > \pi_{t-1}^{\max}$. However, since $p_t^* = (1 + \pi_t^{*\max}) p_t^* A^* + w_t^* L^*$ and $w_t^* = p_t^* b < w_{t-1}$, $\pi_t^{*\max} > \pi'$ holds, which implies $\pi_t^{*\max} > \pi_{t-1}^{\max}$.

Case 2: $x_t^* \nleqq x_t$.

We only need to show $L^* x_t^* < L x_t$, for then the argument for case 1 can be used to conclude the proof of part (i). Suppose, by way of contradiction, that $L^* x_t^* \geqq L x_t = N_t$. By Definition 1(c), this implies $L^* x_t^* = N_t$. Then, as $A^* x_t^* = \omega_{t-1}$,

$$\begin{aligned}
LA^{-1}A^* x_t^* &= L x_t = N_t \text{ by } A x_t = \omega_{t-1} \\
\Leftrightarrow LA^{-1}(A + \triangle_A) x_t^* &= N_t \text{ where } \triangle_A \equiv (A^* - A) \\
\Leftrightarrow L x_t^* + LA^{-1} \triangle_A x_t^* &= N_t \\
\Rightarrow L^* x_t^* + LA^{-1} \triangle_A x_t^* &< N_t \text{ by } L^* \leq L \text{ and } x_t^* > \mathbf{0}.
\end{aligned}$$

As $L^* x_t^* = N_t$, the last inequality implies $LA^{-1} \triangle_A x_t^* = LA^{-1}(A^* - A) x_t^* < 0$. Because technical change is labour inelastic, it follows that $(L^* - L) x_t^* < LA^{-1}(A^* - A) x_t^* = LA^{-1} A^* x_t^* - L x_t^*$, which implies $L^* x_t^* < LA^{-1} A^* x_t^*$. The desired contradiction then follows noting that $L^* x_t^* = N_t$ and by $LA^{-1} A^* x_t^* = N_t$. Therefore we conclude that $L^* x_t^* < L x_t$ holds, as sought.

Part (ii). Let the change of technique from (A, L) to (A^*, L^*) be profitable and CS-LU, and let $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Suppose, ad absurdum, that $\pi_t^{*\max} > 0$. By Lemma 1, at the optimal solution $\xi_t^{*\nu}$ it must be $\delta_t^{*\nu} = \mathbf{0}$ and $p_t^* A^* x_t^{*\nu} = p_t^* \omega_{t-1}^{\nu}$, for all $\nu \in \mathcal{N}_t$. Since $p_t^* > \mathbf{0}$, then by Definition 1(b), $A^* x_t^* = \omega_{t-1} = A x_t$. Hence $L x_t = L A^{-1} \omega_{t-1} = N_t$ and $\omega_{t-1} = A^* x_t^*$ imply $L A^{-1} A^* x_t^* = N_t$. Therefore

$$\begin{aligned} L x_t^* + L A^{-1} \triangle_A x_t^* &= N_t \text{ where } \triangle_A \equiv (A^* - A) \\ \Rightarrow L^* x_t^* + L A^{-1} \triangle_A x_t^* &> N_t \text{ by } L^* \geq L \text{ and } x_t^* > \mathbf{0}, \\ \Rightarrow L x_t + L A^{-1} \triangle_A x_t^* &> N_t \text{ by } L^* x_t^* \leqq L x_t = N_t. \end{aligned}$$

As $L x_t = N_t$, the last inequality implies $L A^{-1} \triangle_A x_t^* = L A^{-1} (A^* - A) x_t^* > 0$. Because technical change is labour inelastic, $(L^* - L) x_t^* > L A^{-1} (A^* - A) x_t^* = N_t - L x_t^*$, which implies $L^* x_t^* > N_t = L x_t$, a contradiction.

Part (iii). Suppose, contrary to the statement, that $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Then, by the proof of Part (ii), $\pi_t^{*\max} = 0$ holds. Thus, $(p_t^*, w_t^*) = \left(\frac{v^*}{v^* b}, \frac{1}{v^* b}\right)$ holds. As (A^*, L^*) is optimal at prices (p_t^*, w_t^*), it follows that $v^* \leqq v^* A + L$. Thus, $v^* \leqq v$ holds. This contradicts the fact that technical change is regressive. Therefore, the new technique (A^*, L^*) cannot be optimally chosen in a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. ■

A key assumption in Theorem 3 is that the CE in period $t-1$ is characterised by the full employment of labour ($w_{t-1} > p_{t-1} b$) and capital ($\pi_{t-1}^{\max} > 0$), and that it is persistent. This implies that aggregate capital stocks accumulated in $t-1$ would be sufficient to guarantee the full employment of labour at t at the same price vector (p_{t-1}, w_{t-1}) and using the same production technique (A, L), if no new technique emerged and $\mathcal{B}_t = \mathcal{B}_{t-1}$.

Suppose, however, that a new technique $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ does appear, possibly as the product of successful R&D, right after period $t-1$. If the change from (A, L) to (A^*, L^*) is profitable and CU-LS, then Theorem 3(i) shows that if the new technique is adopted in equilibrium, then it leads the profit rate to increase and the wage rate to fall to the subsistence level, due to the emergence of an excess supply of labour and unemployment.[25]

If technical change is profitable and CS-LU, then the effect on factor income distribution is less clear-cut. If the new technique is adopted in

[25] This is the Marxian "industrial reserve army of the unemployed". Together, Theorem 1, Proposition 1, and Theorem 3 may be interpreted as illustrating Marx's [12] general law of capitalist accumulation.

equilibrium, then the equilibrium profit rate *falls to zero*, as the shift to the new technique makes aggregate capital abundant relative to the labour force (Theorem 3(ii)). If, however, the CS-LU change of technique is regressive, then the dynamic transition from the CE at $t-1$ does not lead the economy to a new equilibrium associated with this new technique at t (Theorem 3(iii)).

Next, Theorem 4 characterises equilibria with a new technique when the aggregate capital stock at t is *not* sufficient to allow for the full employment of labour using the old production technique ($N_t > LA^{-1}\omega_{t-1}$):

Theorem 4: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $Ax_{t-1} = \omega_{t-2}$ and $Lx_{t-1} < N_{t-1}$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be a new technique, and inter-period change of technique from (A, L) to (A^*, L^*) be profitable. Suppose it results in a CE $\left((p_t^*, w_t^*), \left((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. If the change of technique is (i) CU-LS and labour inelastic, then $w_t^* = p_t^* b$ and $\pi_t^{*\max} > \pi_{t-1}^{\max}$ whenever $A^* x_t^* = \omega_{t-1}$; otherwise, $\pi_t^{*\max} = 0$. If it is (ii) CS-LU with sufficiently small $(A - A^*, L^* - L)$, then $w_t^* = p_t^* b$ and $\pi_t^{*\max} > \pi_{t-1}^{\max}$.

Proof: As the CE in period $t-1$ is persistent, Proposition 1 implies that $N_t \geqq LA^{-1}\omega_{t-1}$, and there exist $\xi_t^{\nu} = (x_t^{\nu}; 1; \mathbf{0})_{\nu \in \mathcal{N}_t}$ and $(\omega_t^{\nu})_{\nu \in \mathcal{N}_t}$ such that $x_t > \mathbf{0}$ with $Ax_t = \omega_{t-1}$ and $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_t^{\nu}; \omega_t^{\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.

Part (i). Consider a CU-LS and labour inelastic change of technique that is adopted at a CE $\left((p_t^*, w_t^*), \left((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Suppose, first, that $A^* x^* = \omega_{t-1}$. Then, using an argument similar to that used in the proof of Theorem 3(i):

$$\begin{aligned} Lx_t^* + LA^{-1}\triangle_A x_t^* &= Lx_t \text{ where } \triangle_A \equiv (A^* - A) \\ \Rightarrow L^* x_t^* + LA^{-1}\triangle_A x_t^* &< Lx_t \text{ by } L^* \leq L \text{ and } x_t^* > \mathbf{0}. \end{aligned}$$

Suppose $L^* x_t^* \geqq Lx_t$. Then, the above inequalities imply $LA^{-1}\triangle_A x_t^* < 0$, and since technical change is labour inelastic, it follows that $(L^* - L)\, x_t^* < LA^{-1}\triangle_A x_t^* < 0$. As $LA^{-1}\triangle_A x_t^* = Lx_t - Lx_t^*$, $(L^* - L)\, x_t^* < Lx_t - Lx_t^*$ implies that $L^* x_t^* < Lx_t$, yielding the desired contradiction. Thus, $L^* x_t^* < Lx_t \leqq N_t$. Therefore, Theorem 1(ii) implies $w_t^* = p_t^* b$, which in turn implies $\pi_t^{*\max} > \pi_{t-1}^{\max}$ by Theorem 2.

Suppose, next, that $A^* x_t^* \leq \omega_{t-1} = Ax_t$. Then, as $p_t^* > \mathbf{0}$ by Lemma 2, we have $p_t^* A^* x_t^* < p_t^* \omega_{t-1}$, and so $\pi_t^{*\max} = 0$ holds by Lemma 1.

Part (ii). Since $(A - A^*, L^* - L)$ is sufficiently small, $L^* A^{*-1} \omega_{t-1}$ is sufficiently close to $LA^{-1}\omega_{t-1}$, which implies that $N_t > L^* A^{*-1} \omega_{t-1}$ holds. Then, $w_t^* = p_t^* b$ follows from Theorem 1(ii), and by Theorem 2, $\pi_t^{*\max} > \pi_{t-1}^{\max}$. ■

Theorems 3 and 4 provide a comprehensive analysis of the effects of innovation on the equilibrium income distribution.[26] More precisely, they characterise the new factor income distribution between the capitalist class and the working class, *if* the economy moves to a new equilibrium with the new technique. Yet, they are silent on whether the different types of technical change actually lead to a new equilibrium in which the new technique is adopted. To this topic we turn next.

6 The transition to new equilibria

In this section, and the next, we analyse whether innovations lead to a new equilibrium in which a new technique is adopted. In the standard literature on OT (e.g. Okishio [14]; Roemer [17, 18, 19]; Franke [7]; Flaschel et al [6]), a profitable new technique is assumed to be adopted, as it leads to disequilibrium profits that are eventually competed away. In our framework, the interaction of technical progress, labour market conditions and maximising behaviour leads to a more complex, and arguably more realistic picture.

For each $(A, L) \in \mathcal{B}_t$, let

$$F(\pi; (A, L)) = \begin{cases} \frac{1}{L[I-(1+\pi)A]^{-1}b} & \text{if } \pi \in [0, \Pi(A)), \\ 0 & \text{if } \pi = \Pi(A). \end{cases}$$

Let $F^{-1}(\cdot; (A, L))$ be the inverse function of $F(\cdot; (A, L))$, which is well-defined since $F(\cdot; (A, L))$ is strictly decreasing at every $\pi \in [0, \Pi(A)]$.

The *wage-profit curve associated with* (A, L) can be defined as follows:

$$\pi w(A, L) \equiv \left\{(\pi, w) \in \mathbb{R}_+^2 \mid w = F(\pi; (A, L)) \text{ for } \pi \in [0, \Pi(A)]\right\}.$$

The *wage-profit frontier associated with* $\mathcal{B}_t$ is the envelope of the various

[26] In both theorems, we focus on persistent CEs such that at the beginning of t, if the period $t-1$ optimal technique (A, L) was adopted, then $N_t \geqq LA^{-1}\omega_{t-1}$ would hold. In other words, we rule out dynamic paths that end up with an excess supply of capital. This is theoretically reasonable, given our focus on the growth path of capitalist economies.

wage-profit curves and can be defined as follows:

$$\begin{aligned}\pi w(\mathcal{B}_t) \quad \equiv \quad & \left\{(\pi, w) \in \mathbb{R}^2_+ \mid \exists (A, L) \in \mathcal{B}_t : (\pi, w) \in \pi w(A, L)\right. \\ & \left.\& \ \ \forall (A', L') \in \mathcal{B}_t, \forall (\pi', w') \in \pi w(A', L') : w' = w \Rightarrow \pi' \leqq \pi\right\}.\end{aligned}$$

The concepts of wage-profit curve and wage-profit frontier provide the analytical tools to examine optimal choice of technique and the interaction between technical progress and distribution. For in equilibrium only techniques that lie on $\pi w(\mathcal{B}_t)$ will be adopted. Formally:

Lemma 3: A technique $(A^*, L^*) \in \mathcal{B}_t$ with $p^* = (1 + \pi^*) p^* A^* + w^* L^*$ for some $(p^*, w^*) \in \triangle \times \mathbb{R}^2_+$ minimises production costs at the rate of profit π^* if and only if $(\pi^*, w^*) \in \pi w(\mathcal{B}_t)$).

Proof: See Kurz and Salvadori ([10]; Theorem 5.1). ■

In other words, a technique (A^*, L^*) is adopted if no other technique in $\mathcal{B}_t$ allows for a wage rate higher than w^* for $\pi = \pi^*$.

For each (A, L), the intercepts of $\pi w(A, L)$ on the vertical axis and on the horizontal axis are, respectively, the points $\left(0, \frac{1}{vb}\right)$ and $(\Pi(A), 0)$. Therefore, for any $(A, L), (A', L') \in \mathcal{B}_t$, if $v > v'$ and $A \leq A'$, then $\left(0, \frac{1}{vb}\right) \leq \left(0, \frac{1}{v'b}\right)$ and $(\Pi(A), 0) \geq (\Pi(A'), 0)$ and the wage-profit curves $\pi w(A, L), \pi w(A', L')$ intersect at least once, and quite possibly more than once, given the rather complicated polynomial form of wage-profit curves.

Finally, given a wage-profit curve $\pi w(A, L)$ and given $(\pi, w) \in \mathbb{R}^2_+$, let $\pi w(A, L; (\pi, w)) \equiv \{(\pi', w') \in \pi w(A, L) \mid (\pi', w') \geq (\pi, w)\}$.

6.1 Full employment

In the analysis of the interaction between technical progress and the equilibrium income distribution, it is natural to start from innovations which allow for the full employment of all factors of production. Theorem 5 shows that in this special case, the distributive effects of technical change are difficult to predict and may not be Pareto-improving.

Theorem 5: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^\nu_{t-1}; \omega^\nu_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique with $N_t = L^* A^{*-1} \omega_{t-1}$. Let inter-period change of technique from (A, L) to (A^*, L^*) be profitable. Then, there exists a non-empty $\pi w\left(A^*, L^*; \left(\pi^{\max}_{t-1}, w_{t-1}\right)\right)$

such that for any $(\pi', w') \in \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$, there exists a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $w_t^* = w'$ and $\pi_t^{*\max} = \pi'$. Furthermore, if $N_t = LA^{-1}\omega_{t-1}$, $\pi_{t-1}^{\max} > 0$, and $w_{t-1} > p_{t-1}b$, then there exist CEs with either $\pi_t^{*\max} < \pi_{t-1}^{\max}$ or $w_t^* < w_{t-1}$.

Proof:
Case 1. Let $N_t = LA^{-1}\omega_{t-1}$.

1. Because $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_{t-1}^\nu; \omega_{t-1}^\nu)_{\nu \in \mathcal{N}_{t-1}}\right)$ is a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$, $(A, L) \in \arg\max_{(A', L') \in \mathcal{P}_{t-1}} \pi_{t-1}(A', L')$. Then, by Lemma 3, $\left(\pi_{t-1}^{\max}, w_{t-1}\right) \in \pi w(\mathcal{B}_{t-1})$ with $\pi_{t-1}^{\max} = \pi_{t-1}(A, L)$. Moreover, since the change from (A, L) to (A^*, L^*) is profitable at (p_{t-1}, w_{t-1}), by Proposition 5.1 in Kurz and Salvadori [10], there exists a technique $(A^{**}, L^{**}) \in \mathcal{P}_t$ such that for some $(p^{**}, w^{**}) \in \triangle \times \mathbb{R}_+$ with $w^{**} > w_{t-1}$, $p^{**} = \left(1 + \pi_{t-1}^{\max}\right) p^{**}A^{**} + w^{**}L^{**}$.[27] However, because (A, L) is the most profitable technique in $\mathcal{P}_{t-1}$ at (p_{t-1}, w_{t-1}) and $\{(A^*, L^*)\} = \mathcal{B}_t \backslash \mathcal{B}_{t-1}$, then $(A^{**}, L^{**}) = (A^*, L^*)$ holds, without loss of generality.

Clearly, $(\pi_{t-1}, w^{**}) \in \pi w(A^*, L^*)$ and $(\pi_{t-1}, w^{**}) \in \pi w(\mathcal{B}_t) \backslash \pi w(\mathcal{B}_{t-1})$. Then, noting that $\pi w(A^*, L^*)$ and $\pi w(\mathcal{B}_{t-1})$ are both downward sloping, it follows that there is a continuum of points $(\pi^*, w^*) \in \pi w(A^*, L^*)$ with $(\pi^*, w^*) \geq \left(\pi_{t-1}^{\max}, w_{t-1}\right)$, and for any such point, $(\pi^*, w^*) \in \pi w(\mathcal{B}_t)$ holds.

Insert Figure 1 around here.

2. Consider any $(\pi', w') \in \pi w(\mathcal{B}_t)$ such that $(\pi', w') \geq \left(\pi_{t-1}^{\max}, w_{t-1}\right)$. By Lemma 3, there is a $p^* \in \triangle$ such that $p^* = w'L^*\left[I - (1 + \pi')A^*\right]^{-1} > \mathbf{0}$ and (A^*, L^*) is optimal at (p^*, w'). Hence, since $N_t = L^*A^{*-1}\omega_{t-1}$ and $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, there exists a suitable assignment of capital stocks $(\omega_t^\nu)_{\nu \in \mathcal{N}_t}$ and production activities $(\xi_t^\nu)_{\nu \in \mathcal{N}_t} = (x_t^\nu; l_t^\nu; \delta_t^\nu)_{\nu \in \mathcal{N}_t}$ such that $\sum_{\nu \in \mathcal{N}_t} x_t^\nu = A^{*-1}\omega_{t-1}$, $p^*A^*x_t^\nu = p^*\omega_{t-1}^\nu$ for all $\nu \in \mathcal{N}_t$, and $(l_t^\nu, \delta_t^\nu) = (1, \mathbf{0})$ that solve MP_t^ν for all $\nu \in \mathcal{N}_t$. Then, $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu;\ \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $p_t^* = p^*$, $w_t^* = w'$, and $\pi_t^{*\max} = \pi'$.

3. If $\pi_{t-1}^{\max} > 0$, and $w_{t-1} > p_{t-1}b$ then the argument in step 1 can be used to show that there exists a sufficiently small $\varepsilon > 0$ such that if, for some $(\pi^*, w^*) \in \pi w(A^*, L^*)$, *either* $0 < \pi_{t-1}^{\max} - \pi^* < \varepsilon$, *or* $0 < w_{t-1} - w^* < \varepsilon$, then $(\pi^*, w^*) \in \pi w(\mathcal{B}_t)$ still holds. The argument in step 2 can then be applied to prove the existence of a CE.

[27] For a statement of Proposition 5.1 in Kurz and Salvadori [10] see the Addendum.

Case 2. Let $N_t < LA^{-1}\omega_{t-1}$.

By Proposition 1, at $t-1$ it must be $\pi_{t-1}^{\max} = 0$ and $w_{t-1} = \frac{1}{vb}$. However, by applying the same reasoning as in Case 1, for any $(\pi^*, w^*) \in \pi w(A^*, L^*)$ with $(\pi^*, w^*) \geq \left(\pi_{t-1}^{\max}, w_{t-1}\right)$, $(\pi^*, w^*) \in \pi w(\mathcal{B}_t)$ holds. Therefore, $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right) \subsetneq \pi w(\mathcal{B}_t)$ is non-empty. Moreover, as $N_t = L^* A^{*-1}\omega_{t-1}$ holds, for any $(\pi', w') \in \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$, there exists a CE $\left((p_t^*, w_t^*), ((A^*, L^*); x_t^\nu;\ l_t^\nu; \delta_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $w_t^* = w'$ and $\pi_t^{*\max} = \pi'$.

Case 3. Let $N_t > LA^{-1}\omega_{t-1}$.

By Proposition 1, at $t-1$ it must be $\pi_{t-1}^{\max} > 0$ and $w_{t-1} = p_{t-1}b$. Again, by applying the same reasoning as in Case 1, $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right) \subsetneq \pi w(\mathcal{B}_t)$ is non-empty. Thus, for any $(\pi', w') \in \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$, there exists a CE $\left((p_t^*, w_t^*), ((A^*, L^*); x_t^\nu;\ l_t^\nu; \delta_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $w_t^* = w'$ and $\pi_t^{*\max} = \pi'$. ■

Theorem 5 suggests that when a profitable change of technique guarantees the full employment of both labour and capital, then a new equilibrium emerges at t in which the new technique is indeed adopted. The effect of innovation on distribution is not clear a priori, however, because of the (infinitely) many profit rates and wage rates that can be supported in equilibrium. Interestingly, technical progress may even make either capitalists or workers *strictly worse off* as there exist equilibria at t with either $\pi_t^{*\max} < \pi_{t-1}^{\max}$ or $w_t^* < w_{t-1}$. The distributional outcome will depend on the actual equilibrium selection mechanism.[28] Yet, by Theorem 2, we know that if $w_t^* = w_{t-1}$, then $\pi_t^{*\max} > \pi_{t-1}^{\max}$ and so, absent a strong distributional shift towards labour, technical change will tend to increase profitability.

6.2 Technological unemployment

Theorem 3(i) characterises the implications of profitable CU-LS technical change on distribution *if* the new technique is adopted. It is not clear, however, whether the technological shift toward the new technique will indeed take place in equilibrium: while the new technique is profitable at the equilibrium prices ruling at $t-1$, the very introduction of the new technique is likely to cause disequilibrium in commodity markets and in the labour market, which in turn would cause prices to change.

[28] One possible solution to this indeterminacy is to consider some form of bargaining over distributive outcomes. See, e.g., Cogliano et al. [5] and Yoshihara and Kaneko [25].

Theorem 6 derives the conditions under which profitable, CU-LS technical change leads to a new CE in which the newly discovered technique is adopted:

Theorem 6: Let $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu})_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max} > 0$ and sufficiently small $w_{t-1} - p_{t-1}b \geqq 0$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique. If inter-period change of technique from (A, L) to (A^*, L^*) is profitable, CU-LS, and labour inelastic, then it results in a new CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ with $\xi_t^{*\nu} = (x_t^{*\nu}; l_t^{*\nu}; \mathbf{0})_{\nu \in \mathcal{N}_t}$, $w_t^* = p_t^* b$, and $\pi_t^{*\max} > \pi_{t-1}^{\max}$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

Proof: 1. Let $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu})_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max} > 0$ and sufficiently small $w_{t-1} - p_{t-1}b \geqq 0$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Then, by Proposition 1, $N_t \geqq LA^{-1}\omega_{t-1}$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique and let the change of technique from (A, L) to (A^*, L^*) be profitable. Then, as in step 1 of the proof of Theorem 5, it follows that for any $(\pi^*, w^*) \in \pi w(A^*, L^*)$ with $(\pi^*, w^*) \geq (\pi_{t-1}^{\max}, w_{t-1})$, $(\pi^*, w^*) \in \pi w(\mathcal{B}_t)$.

2. Since both $\pi w(\mathcal{B}_{t-1})$ and $\pi w(A^*, L^*)$ are strictly downward sloping, there exists a sufficiently small $\epsilon > 0$ such that for any $(\pi', w') \in \pi w(\mathcal{B}_{t-1})$ with $\epsilon \geqq w_{t-1} - w' \geqq 0$, there exists $w'^* > w_{t-1}$ with $(\pi', w'^*) \in \pi w(A^*, L^*)$ Then, as in step 1 of the proof of Theorem 5, it follows that for any $(\pi^*, w^*) \in \pi w(A^*, L^*)$ with $(\pi^*, w^*) \geq (\pi', w')$, $(\pi^*, w^*) \in \pi w(\mathcal{B}_t)$. Thus, there exists $\pi'^* > \pi'$ such that $(\pi'^*, w') \in \pi w(A^*, L^*) \cap \pi w(\mathcal{B}_t)$, and, by Lemma 3, there is a price vector $p' \in \triangle$ such that $p' = w'L^*[I - (1 + \pi'^*)A^*]^{-1} > \mathbf{0}$ and (A^*, L^*) is optimal with respect to (p', w'). If $\epsilon \geqq w_{t-1} - p_{t-1}b \geqq 0$, then we can set $w' = 1 = p_{t-1}b$, and (A^*, L^*) is optimal at (p_t^*, w_t^*) with $w_t^* = p_t^* b = 1$ and $w_t^* L^* = p_t^*[I - (1 + \pi_t^{*\max})A^*]$.

3. Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, $x_t^* \equiv A^{*-1}\omega_{t-1} > \mathbf{0}$. Then the argument used in the proof of Theorem 3(i) for the case with $A^* x_t^* = \omega_{t-1}$ can be used to prove that $N_t \geqq LA^{-1}\omega_{t-1} > L^* A^{*-1}\omega_{t-1}$. Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, $(I - bL^*)A^{*-1}\omega_{t-1} \geqq \mathbf{0}$ holds, so that for $x_t^* \equiv A^{*-1}\omega_{t-1}$, we have $x_t^* = bLx_t^* + \omega_t$ with $\omega_t \equiv x_t^* - bLx_t^* \geqq \mathbf{0}$. Then, there exists a suitable assignment of capital stocks $(\omega_t^{*\nu})_{\nu \in \mathcal{N}_t}$ and production activities $(\xi_t^{*\nu})_{\nu \in \mathcal{N}_t} = (x_t^{*\nu}; l_t^{*\nu}; \mathbf{0})_{\nu \in \mathcal{N}_t}$ such that $\sum_{\nu \in \mathcal{N}_t} x_t^{*\nu} = x_t^*$, $p_t^* A^* x_t^{*\nu} = p_t^* \omega_{t-1}^{\nu}$, and $(l_t^{*\nu}, \delta_t^{*\nu}) \in [0,1] \times \{\mathbf{0}\}$ for all $\nu \in \mathcal{N}_t$ such that $\sum_{\nu \in \mathcal{N}_t} l_t^{*\nu} = L^* x_t^* < N_t$. Thus, $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ is a CE with $w_t^* = p_t^* b = 1$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Finally, as $\pi w(\mathcal{B}_{t-1})$ and $\pi w(A^*, L^*)$ are downward sloping, we can see that $\pi_t^{*\max} > \pi_{t-1}^{\max}$ holds. ■

Theorem 6 shows that a profitable, CU-LS innovation may indeed be adopted in equilibrium at t, provided w_{t-1} is sufficiently low. If this is not the case, however, the new technique may not be adopted. To see this, suppose that change of technique from (A, L) to (A^*, L^*) is CU-LS, labour inelastic, and profitable at (p_{t-1}, w_{t-1}), and $N_t = LA^{-1}\omega_{t-1}$ holds. Then, $N_t > L^* A^{*-1}\omega_{t-1}$, as per the proof of Theorem 3(i). Thus, if $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ is a CE, then it must be $w_t^* = p_t^* b = 1$ by Theorem 1(ii).

Insert Figure 2 around here.

Yet, while (A^*, L^*) yields higher profits than (A, L) in a neighbourhood of (p_{t-1}, w_{t-1}), it does not necessarily maximise the profit rate at (p_t^*, w_t^*) if $w_t^* = 1$ is much lower than w_{t-1}. In this case, it is possible for (A, L) to be optimal at (p_t^*, w_t^*), and there may be a CE with prices (p_t^*, w_t^*) and actions $((A, L); (x_t^\nu; 1; \mathbf{0}); \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ with $Ax_t = \omega_{t-1}$. Figure 3 describes this situation.

Insert Figure 3 around here.

The above argument can be summarised by the following corollary:

Corollary 2: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^\nu; \omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max} > 0$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique. Let inter-period change of technique from (A, L) to (A^*, L^*) be profitable, CU-LS, and labour inelastic. Then, there exists a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ if and only if $(\pi_t^{*\max}, w_t^* = 1) \in \pi w(\mathcal{B}_t) \cap \pi w(A^*, L^*)$.

Corollary 2 characterises the conditions under which what may be deemed a market failure occurs: if the condition in Corollary 2 is violated, there exists no equilibrium in which a new technique is adopted in equilibrium *even if* it is profitable and increases labour productivity.

Indeed, in this case innovations may cause an even deeper failure and disrupt the functioning of capitalist economies in a more surprising and counterintuitive way: technical progress may cause the economy to reach *no* equilibrium at t. To see this, reconsider the previous example and suppose that there is another technique $(A^{**}, L^{**}) \in \mathcal{B}_{t-1} \backslash \mathcal{B}_{t-1}(\omega_{t-1}, b)$ such that $v^{**} > v$. Thus, $\mathcal{B}_t = \{(A, L), (A^*, L^*), (A^{**}, L^{**})\}$. Suppose further that (A^{**}, L^{**}) is uniquely optimal at prices $p_t^* = w_t^* L^* \left[I - (1 + \pi_t^{*\max}) A^*\right]^{-1}$ and $w_t^* = p_t^* b = 1$, and that $\pi w(A, L)$ is not on the envelope $\pi w(\mathcal{B}_t)$. Then, there may exist no CE in the economy.

To see this, note, first, that as $\pi w\left(A,L\right)$ is not on the envelope $\pi w\left(\mathcal{B}_t\right)$, (A,L) will never be activated in equilibrium. Next, we rule out the possibility that agents activate convex combinations of the other two techniques. Let $\mu \in [0,1]$ be the weight assigned to activity (A^*, L^*) in such combinations.

We proceed by contradiction. Suppose that $\mu = 1$ and only (A^*, L^*) is activated in equilibrium. Then, by Theorem 1(ii), the corresponding equilibrium prices would be (p_t^*, w_t^*). However, as (A^{**}, L^{**}) is optimal at (p_t^*, w_t^*), (A^*, L^*) would not be chosen in equilibrium.

Next, suppose that $\mu = 0$ and only (A^{**}, L^{**}) is adopted in equilibrium. Because $(A^{**}, L^{**}) \in \mathcal{B}_{t-1} \backslash \mathcal{B}_{t-1}\left(\omega_{t-1}, b\right)$, there is no $x' > \mathbf{0}$ such that $A^{**}x' = \omega_{t-1}$.[29] Hence, by Lemmas 1 and 2 the equilibrium price vector must be $p_t^{**} = \frac{1}{v^{**}b}v^{**}$ and $w_t^{**} = \frac{1}{v^{**}b}$. Yet, because $\frac{1}{v^{**}b} < \frac{1}{vb} < \frac{1}{v^*b}$ by $v^{**} > v$ and Proposition 2(i), (A^*, L^*) yields a higher profit rate than (A^{**}, L^{**}) at (p_t^{**}, w_t^{**}). Thus, (A^{**}, L^{**}) cannot be optimally chosen in equilibrium.

Finally, suppose that in equilibrium agents activate both production techniques with weight $\mu \in (0,1)$. This implies that the equilibrium price vector (p', w') with $p' = w'[I - (1+\pi')\,A^{-1}]$ corresponds to a point (π', w') on the envelope $\pi w\left(\mathcal{B}_t\right)$ where the wage curves $\pi w\left(A^*, L^*\right)$ and $\pi w\left(A^{**}, L^{**}\right)$ intersect. By construction, $(\pi', w') > (0,1)$. Therefore, by Lemma 1 in equilibrium $\mu L^* A^{*-1}\omega_{t-1} + (1-\mu)\,L^{**}A^{**-1}\omega_{t-1} = N_t$ must hold. As $L^*A^{*-1}\omega_{t-1} < N_t$, it follows that $L^{**}A^{**-1}\omega_{t-1} > N_t$ and there exists a unique $\mu \in (0,1)$ such the labour market is in equilibrium.

Because $(A^{**}, L^{**}) \in \mathcal{B}_{t-1} \backslash \mathcal{B}_{t-1}\left(\omega_{t-1}, b\right)$, either $A^{**-1}\omega_{t-1} \not> \mathbf{0}$ or $(I - bL)A^{**-1}\omega_{t-1} \ngeq \mathbf{0}$ holds. But then, there exist a range of values of $\mu \in (0,1)$ such that either $\mu A^{*-1}\omega_{t-1} + (1-\mu)\,A^{**-1}\omega_{t-1} \not> \mathbf{0}$ or $\mu A^{*-1}\omega_{t-1} + (1-\mu)\,A^{**-1}\omega_{t-1} \ngeq N_t b$. In either case, if the unique $\mu \in (0,1)$ that clears the labour market falls within the relevant range, Definition 1(d) is violated.

As a result, the economy with $\mathcal{B}_t = \left\{(A,L),(A^*,L^*),(A^{**},L^{**})\right\}$ may not arrive at any equilibrium. Figure 4 describes such a situation.

Insert Figure 4 around here.

[29] The case with $x^{**} \ngeq bL^{**}x^{**}$ for $x^{**} \equiv A^{**-1}\omega_{t-1} > \mathbf{0}$ can be ignored as it violates Definition 1(d) and therefore cannot be part of an equilibrium.

7 The falling rate of profit

In the previous section, we have shown that – once the general equilibrium effects of technical change are taken into account – the distributive effects of innovations are not obvious. Absent a significant shift in bargaining power towards workers, however, innovations – and especially labour saving technical progress – tend to increase equilibrium profits. These results would seem to confirm the main intuition of OT and provide a further obituary on Marx's theory of the falling rate of profit. In this section, we show that, at a general level, this conclusion would be unwarranted – or would at least need to be qualified – and there are indeed some ex ante profitable innovations that may lead to a decrease in the equilibrium rate of profit.

Our first result characterises the conditions under which profitable CS-LU change of technique leads to a falling rate of profit:

Theorem 7: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique. Let inter-period change of technique from (A, L) to (A^*, L^*) be profitable, CS-LU, and labour inelastic. Then, there exists a CE $\left((p_t^*, w_t^*), \left((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ with $\pi_t^{*\max} = 0$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ if and only if

$$(A^*, L^*) \in \arg \min_{(A', L') \in \mathcal{P}_t} L' (I - A')^{-1} b. \tag{8}$$

Proof: ($\Leftarrow$) Let $\left((p_t^*, w_t^*), \left((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ be a CE with $\pi_t^{*\max} = 0$. By construction, at prices (p_t^*, w_t^*) with $p_t^* = w_t^* v^*$ and $w_t^* = \frac{1}{v^*b}$, $\pi_t^*(A', L') \leqq 0 = \pi_t^*(A^*, L^*)$ holds for all $(A', L') \in \mathcal{B}_t$. Suppose, contrary to the statement, that for some $(A', L') \in \mathcal{B}_t$, $\frac{1}{v'b} > \frac{1}{v^*b}$. Then, by Proposition 5.2 in Kurz and Salvadori [10]), it follows that (A', L') yields a higher profit rate than (A^*, L^*) in at least some sectors at (p_t^*, w_t^*), a contradiction.[30]

($\Rightarrow$) 1. Let $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$. Then $\frac{1}{v'b} \leqq \frac{1}{v^*b}$ holds for all $(A', L') \in \mathcal{B}_t$. Therefore by Lemma 3 it follows that (A^*, L^*) is optimal at $p_t^* = w_t^* v^* > \mathbf{0}$ and $w_t^* = \frac{1}{v^*b}$.

Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, $x_t^* \equiv A^{*-1}\omega_{t-1} > \mathbf{0}$, and the argument used in the proof of Theorem 3(ii) can be adapted to prove that $N_t =$

[30] For a statement of Proposition 5.2 in Kurz and Salvadori [10] see the Addendum.

$LA^{-1}\omega_{t-1} < L^*A^{*-1}\omega_{t-1}$. Then, let $k \equiv \frac{N_t}{L^*A^{*-1}\omega_{t-1}} < 1$. For all $\nu \in \mathcal{N}_t$, consider $(\xi_t^{*\nu};\omega_t^{*\nu})$ such that $p_t^*A^*x_t^{*\nu} = p_t^*k\omega_{t-1}^{\nu}$, $p_t^*\delta_t^{*\nu} = p_t^*(1-k)\omega_{t-1}^{\nu}$, $l_t^{*\nu} = 1$, and $\omega_t^{*\nu}$ is such that equation (6) holds. Any such $(\xi_t^{*\nu};\omega_t^{*\nu})$ is optimal at (p_t^*, w_t^*), given (A^*, L^*), and Definition 1(a) is satisfied for any profile $((A^*, L^*);\xi_t^{*\nu};\omega_t^{*\nu})_{\nu\in\mathcal{N}_t}$.

2. Among the set of profiles in step 1, choose $(\xi_t^{*\nu};\omega_t^{*\nu})_{\nu\in\mathcal{N}_t}$ such that $x_t^* = A^{*-1}k\omega_{t-1}$, $\delta_t^* = (1-k)\omega_{t-1}$ and $\omega_t^* = x_t^* + \delta_t^* - bL^*x_t^*$. Then it immediately follows that Definition 1(b) holds. By the definition of k and $l_t^{*\nu} = 1$ all $\nu \in \mathcal{N}_t$, $l_t^* = N_t = L^*x_t^*$ and Definition 1(c) also holds.

3. Because $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, it follows that $(I - bL^*)A^{*-1}\omega_{t-1} \geqq \mathbf{0}$. Then, by construction, we have $(I - bL^*)A^{*-1}k\omega_{t-1} \geqq \mathbf{0}$. Therefore $(I - bL^*)x_t^* \geqq \mathbf{0}$ which in turn implies $x_t^* + \delta_t^* \geqq bL^*x_t^*$. The latter inequality proves that Definition 1(d) holds, as $\omega_t^* = x_t^* + \delta_t^* - bL^*x_t^*$. Finally, $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$ implies $(I - bL^*)x_t^* \geqq \mathbf{0}$, and so $x_t^* \geqq N_tb > \mathbf{0}$.

In summary, $\left((p^*, w_t^*), ((A^*, L^*);\xi_t^{*\nu};\omega_t^{*\nu})_{\nu\in\mathcal{N}_t}\right)$ constitutes a CE with $\pi_t^{*\max} = 0$. ■

Theorem 7 shows the existence of innovations that are profitable from the viewpoint of an individual capitalist but which, if adopted universally, lead the equilibrium rate of profit to fall. From a broad theoretical perspective, this result contradicts OT and may therefore be dubbed the *Anti-Okishio Theorem*. It rigorously derives the conditions under which individually rational actions lead to collectively suboptimal outcomes, an intuition which is at the core of Marx's theory of technical change. But Theorem 7 also shows some interesting and perhaps surprising connections between the theory of the falling rate of profit and some central insights of classical capital theory.

As an illustration, and without any loss of generality, consider the simplest possible case of technical change, whereby only one technique is known in period $t-1$, so that $\mathcal{B}_{t-1} = \{(A, L)\}$ and $\mathcal{B}_t = \{(A, L), (A^*, L^*)\}$. Under the conditions of Theorem 7, the wage-profit curve of the new technique, $\pi w(A^*, L^*)$, dominates the wage-profit curve of (A, L), $\pi w(A, L)$, at least in a neighbourhood of points $\left(0, \frac{1}{v^*b}\right)$ and $(\Pi(A^*), 0)$,[31] as well as in the non-empty subset $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$.[32]

[31] The former follows noting that if (8) holds, then $\frac{1}{v'b} \leqq \frac{1}{v^*b}$ for all $(A', L') \in \mathcal{B}_t$ and $\pi w(A^*, L^*)$ coincides with the wage-profit frontier $\pi w(\mathcal{B}_t)$ in a neighbourhood of $(\pi_t^{*\max}, w_t^*) = \left(0, \frac{1}{v^*b}\right)$. The latter follows noting that $A^* \leq A$ implies $\Pi(A^*) > \Pi(A)$.

[32] Because technical change is profitable, an argument similar to that used for Theorem 5 shows that the set $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$ is non-empty and coincides with $\pi w(\mathcal{B}_t)$.

Then, there are two scenarios in which the profit rate will fall. In the first, $\pi w\left(A^{*}, L^{*}\right)$ completely dominates $\pi w\left(A, L\right)$ as shown in Figure 5.

Insert Figure 5 around here.

In this case, technical change is profitable at *any* prices and yet, according to Theorem 7 the adoption of $\left(A^{*}, L^{*}\right)$ leads the equilibrium profit rate to drop to zero. This is quite a strong – and perhaps surprising – result from a theoretical viewpoint, but it is possibly of limited empirical relevance, because innovations that are profitable at any prices are rare.

Alternatively, if $\pi w\left(A^{*}, L^{*}\right)$ does not completely dominate $\pi w\left(A, L\right)$, and given that the former dominates the latter in at least three regions, the two curves must intersect at least twice, as shown in Figure 6.

Insert Figure 6 around here.

Figure 6 describes a situation in which a *reswitching of techniques* (Kurz and Salvadori [10], p.148) occurs: because $\frac{1}{vb} < \frac{1}{v^{*}b}$, close to the vertical axis the wage-profit frontier coincides with the wage-profit curve of the technique $\left(A^{*}, L^{*}\right)$, which is therefore optimal for a sufficiently small (or zero) profit rate. Further, as $\left(A^{*}, L^{*}\right)$ is the optimal technique at $\pi^{*} = 0$, the corresponding wage rate, $w^{*} = \frac{1}{v^{*}b}$ is higher than the wage rate, $w = \frac{1}{vb}$, associated with $\pi = 0$ under $\left(A, L\right)$. In this case, as well-known in the literature on the Cambridge capital controversy, the capital-labour ratio of $\left(A^{*}, L^{*}\right)$ is higher than that of $\left(A, L\right)$ when the values of capital are measured by means of the commodity price vectors corresponding to each of the two switching points, and so $\left(A^{*}, L^{*}\right)$ is a more capital-intensive technique than $\left(A, L\right)$.

As the profit rate increases, a switching point arrives after which the frontier coincides with $\pi w\left(A, L\right)$ and the more labour-intensive technique $\left(A, L\right)$ becomes optimal. However, since $\Pi\left(A^{*}\right) > \Pi\left(A\right)$ another switching point exists after which, as the profit rate *increases* further, the *capital intensive* technique $\left(A^{*}, L^{*}\right)$ becomes optimal again – a phenomenon known in the literature as *capital reversing* (Kurz and Salvadori [10], pp.447-451).

In other words, setting aside the empirically less relevant case of an innovation unambiguously dominating older techniques, the above arguments show that there exists an interesting relation between capital theory – and the phenomena known as reswitching of techniques and capital reversing, – and the theory of the falling rate of profit.[33]

[33] The above arguments may perhaps also explain why the possibility of profitable in-

How robust is the insight of Theorem 7? Does the equilibrium profit rate fall as a result of profitable, CS-LU technical change if condition (8) is not satisfied, or – more strongly – if technical change is regressive? This is not obvious. By Theorem 3(ii), if technical change is CS-LU and regressive, the new technique will not be adopted in equilibrium, even if it is profitable. In this case, either an equilibrium emerges in which an old technique is adopted, or no equilibrium exists – as in the case discussed in section 6.2. Theorem 8 addresses the first scenario.

Theorem 8: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi^{\max}_{t-1} > 0$ and $w_{t-1} > p_{t-1}b$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique. Let inter-period change of technique from (A, L) to (A^*, L^*) be profitable, CS-LU, labour inelastic, and regressive. Let there be a technique $(A^{**}, L^{**}) \in \left[\arg\min_{(A',L')\in\mathcal{B}_{t-1}} L'(I - A')^{-1} b\right] \cap \mathcal{B}_{t-1}(\omega_{t-1}, b)$ such that $(A^{**}, L^{**}) \neq (A^*, L^*)$.[34] Then, there exists a CE $\left((p^{**}_t, w^{**}_t), \left((A^{**}, L^{**}); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu\in\mathcal{N}_t}\right)$ with $\pi^{**\max}_t < \pi^{\max}_{t-1}$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ if and only if $N_t \leqq L^{**}A^{**-1}\omega_{t-1}$.

Proof: ($\Rightarrow$) Assume $N_t \leqq L^{**}A^{**-1}\omega_{t-1}$. We prove that there is a CE $\left((p^{**}_t, w^{**}_t), \left((A^{**}, L^{**}); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu\in\mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $\pi^{**\max}_t < \pi^{\max}_{t-1}$.

Consider (A^*, L^*). Observe that $(\pi^{*\max}_t, w^*_t) = \left(0, \frac{1}{v^*b}\right) \in \pi w(A^*, L^*)$ with $p^* = (1 + \pi^*_t) p^*A^* + w^*_t L^*$ where $p^* = \frac{v^*}{v^*b}$. Because technical change from (A, L) to (A^*, L^*) is regressive, it follows that $vb < v^*b$ and by the definition of A^{**}, $v^{**}b < v^*b$. Therefore, at $\left(\frac{1}{v^*b}v^*, \frac{1}{v^*b}\right)$, technical change from (A^*, L^*) to (A^{**}, L^{**}) is profitable. Hence, as in step 1 of the proof of Theorem 5, we can find a price vector $(p'_t, w'_t) \in \triangle \times \mathbb{R}_+$ with $w'_t = \frac{1}{v^*b}$ and $p'_t = (1 + \pi'^{\max}_t) p'_t A^{**} + w'_t L^{**}$ for some $\pi'^{\max}_t > \pi^{*\max}_t = 0$.

Since $(\pi'^{\max}_t, w'_t), \left(\pi^{\max}_{t-1}, w_{t-1}\right) \in \pi w(\mathcal{B}_{t-1})$, and $\pi w(\mathcal{B}_{t-1})$ is downward sloping, $w'_t > w_{t-1}$ implies $\pi'^{\max}_t < \pi^{\max}_{t-1}$. (Note that $w'_t \leqq w_{t-1}$ is impossible because the change from (A, L) to (A^*, L^*) is profitable at (p_{t-1}, w_{t-1}).)

Suppose $N_t = L^{**}A^{**-1}\omega_{t-1}$. Since $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$, then as shown in the proof of Theorem 5, there exists a CE $\left((p^{**}_t, w^{**}_t), \left((A^{**}, L^{**}); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu\in\mathcal{N}_t}\right)$ with $(p^{**}_t, w^{**}_t) = (p'_t, w'_t)$ and so $\pi^{**\max}_t = \pi'^{\max}_t < \pi^{\max}_{t-1}$.

novations leading to a falling rate of profit is usually excluded in standard neoclassical growth models. For, reswitching of techniques and capital reversing emerge when capital is modelled as a bundle of reproducible commodities, as in this paper, rather than as a single homogeneous aggregate, as in standard neoclassical macro models.

[34] Note that (A^{**}, L^{**}) could be equal to (A, L).

Suppose $N_t < L^{**}A^{**-1}\omega_{t-1}$. Since $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$, then as shown in the proof of Theorem 7, there exists a CE $\left((p_t^{**}, w_t^{**}), ((A^{**}, L^{**}); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ with $(p_t^{**}, w_t^{**}) = \left(\frac{1}{v^{**}b}v^{**}, \frac{1}{v^{**}b}\right)$ and so $\pi_t^{**\max} = 0 < \pi_{t-1}^{\max}$.

($\Leftarrow$) Let $\left((p_t^{**}, w_t^{**}), ((A^{**}, L^{**}); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $\pi_t^{**\max} < \pi_{t-1}^{\max}$. Suppose, by way of contradiction, that $N_t > L^{**}A^{**-1}\omega_{t-1}$. Then at the CE it must be $w_t^{**} = 1 \leqq w_{t-1}$, which contradicts the assumption that $\pi_t^{**\max} < \pi_{t-1}^{\max}$. ■

Theorem 8 suggests that the insight of Theorem 7 is indeed robust: there exist a range of scenarios in which the emergence of individually profitable innovations leads to a decline in the equilibrium rate of profit. The mechanism highlighted in Theorem 8, however, is rather different and the result provides an original perspective on the debates on the falling rate of profit. For it shows that technical progress may indeed lead to a decline in profitability because of the general equilibrium effects of innovations even though, unlike in Theorem 7, *the new technique is not adopted in equilibrium.*

The main effect of innovations, in Theorem 8, is to disrupt consolidated production activities. The appearance of the new, profitable technique (A^*, L^*) leads agents to abandon old production methods, moving the economy away from equilibrium. The new technique is not optimal at any CE, however, because it implies a CS-LU and regressive type of technical change, and therefore is not adopted. One may imagine an equilibrating process of trial and error in which the economy deviates from the original price system (p_{t-1}, w_{t-1}) and eventually settles on another equilibrium in which a previously suboptimal technique, (A^{**}, L^{**}), is adopted.[35] If capital becomes relatively abundant and $N_t < L^{**}A^{**-1}\omega_{t-1}$, then the profit rate falls to zero. However, and perhaps more surprisingly, Theorem 8 proves that there is a decrease in the equilibrium profit rate even if the economy moves to an equilibrium with full employment of labour and capital, $N_t = L^{**}A^{**-1}\omega_{t-1}$, although in this case the new equilibrium profit rate is positive.

Two additional comments are in order. First, note that the existence of (A^{**}, L^{**}) is not a very restrictive condition. For, because the change from (A, L) to (A^*, L^*) is regressive, then there must exist $(A^{**}, L^{**}) \neq (A^*, L^*)$ such that $(A^{**}, L^{**}) \in \arg\min_{(A', L') \in \mathcal{B}_t} L'(I - A')^{-1} b$.

[35] Interestingly, although technical change to (A^*, L^*) would be CS-LU, the production activity that is actually adopted in equilibrium is more capital intensive than the original technique (A, L), where the value of capital is evaluated based on the commodity prices corresponding to the switching point of these techniques on $\pi w(\mathcal{B}_{t-1})$.

Second, Theorems 7 and 8 hold under the assumption of full employment of labour at the persistent CE in $t-1$. What if, instead, there is a sufficiently big industrial reserve army of the unemployed? It can be shown that if $N_t > LA^{-1}\omega_{t-1}$, then a profitable CS-LU change of technique will be adopted in equilibrium, and lead to an increase in the profit rate, provided it is gradual.[36] This scenario could obtain, for example, in a developing economy in which aggregate capital is still low relative to the labour force.

Theorems 7 and 8 prove that CS-LU changes of technique may cause the profit rate to fall. Is this the *only* scenario that may lead to a decrease in the rate of profit? Not really. Theorem 9 proves that if general equilibrium effects are considered, then the profit rate may fall even in the standard case of CU-LS change of technique.[37]

Theorem 9: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max} > 0$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be such that $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$. If inter-period change of technique from (A, L) to (A^*, L^*) is profitable and CU-LS, then it results in a new CE $\left((p_t^*, w_t^*), \left((A^*, L^*); (x_t^{*\nu}; 1; \delta_t^{*\nu}); \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ with $\pi_t^{*\max} = 0$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ if and only if (1) there exists $x^* > \mathbf{0}$ such that $(I - bL^*)\, x^* \geqq A^* x^* - \omega_{t-1}$ with $A^* x^* \leq \omega_{t-1}$ and $L^* x^* = N_t$; and (2) $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$.

8 Conclusions

Our results paint a much more complex picture of the effects of innovations than in standard neoclassical theory *and* in the literature on OT. At a broad theoretical level, they vindicate the basic intuitions of Marx's theory of technical change. There is no obvious relation between ex-ante profitable innovations and the (functional) distribution of income that emerges in equilibrium after technical change is implemented. If technical change leads to an equilibrium with full employment of productive factors, the distribution of income is undetermined, and it is even possible for either the profit rate or the wage rate to decrease. More generally, individually rational choices of technique do not necessarily lead to ex post optimal outcomes. There exist

[36] For a formal statement, see the Addendum.

[37] The proof of Theorem 9 is similar to that of Theorems 7 and 8 – noting that $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$ leads to $\pi_t^{*\max} = 0$ as shown in Theorem 3(i) – and is therefore omitted. (See the Addendum.)

a set of innovations that are profitable – at current prices – for individual capitalists which lead either to socially sub-optimal outcomes characterised by a decrease in labour productivity, or to a falling rate of profit, when they are universally adopted thus causing equilibrium prices to change.

Methodologically, our analysis suggests that the distributive effects of technical progress cannot be fully understood in models that do not capture the dialectic between individual choices and aggregate outcomes, and the complex network of relations that characterise capitalist economies. This is a major limitation of most of the literature on OT, which does not explicitly analyse the general equilibrium effects of innovations and usually *assumes* profitable new techniques to be adopted. In contrast, a general equilibrium approach to technical change allows us to model some aspects of the Schumpeterian process of creative destruction. Innovations disrupt consolidated production practices and move an economy away from its original equilibrium. Indeed, we have shown that they may even cause the disappearance of *all* equilibria and lead the economy to a persistent disequilibrium dynamics.

This methodological insight is, we believe, robust and our theoretical approach provides a rich framework for the analysis of innovations. In closing, we briefly mention three possible extensions of our analysis. First, we have focused only on *process* innovations – new ways of combining inputs in the production of a given set of goods. It would be interesting to investigate the distributive effects of *product* innovations – the invention of new goods and services. Second, given our focus on the effect of the appearance of innovations on the functional distribution of income, we have not explicitly modelled the process of discovering new techniques. Yet, from the general equilibrium perspective adopted in our paper, it would be interesting to endogenise R&D activities and investment, and then examine how the decisions of R&D investors interact with the choices of capitalists in productive sectors in driving changes in the equilibrium income distribution (for a preliminary analysis in an one-good model, see Cogliano et al. [5]). Finally, we have followed the classical literature on OT by focusing on economies with homogeneous labour. It would be worth extending our analysis to more complex models with heterogeneous labour inputs: in addition to allowing for a richer picture of production processes, and of innovations, this would also provide a more nuanced analysis of the distributive effects of innovations which goes beyond the stark two-class framework of the canonical classical model by including cleavages within the working class (for example, between high-skilled and low-skilled workers).

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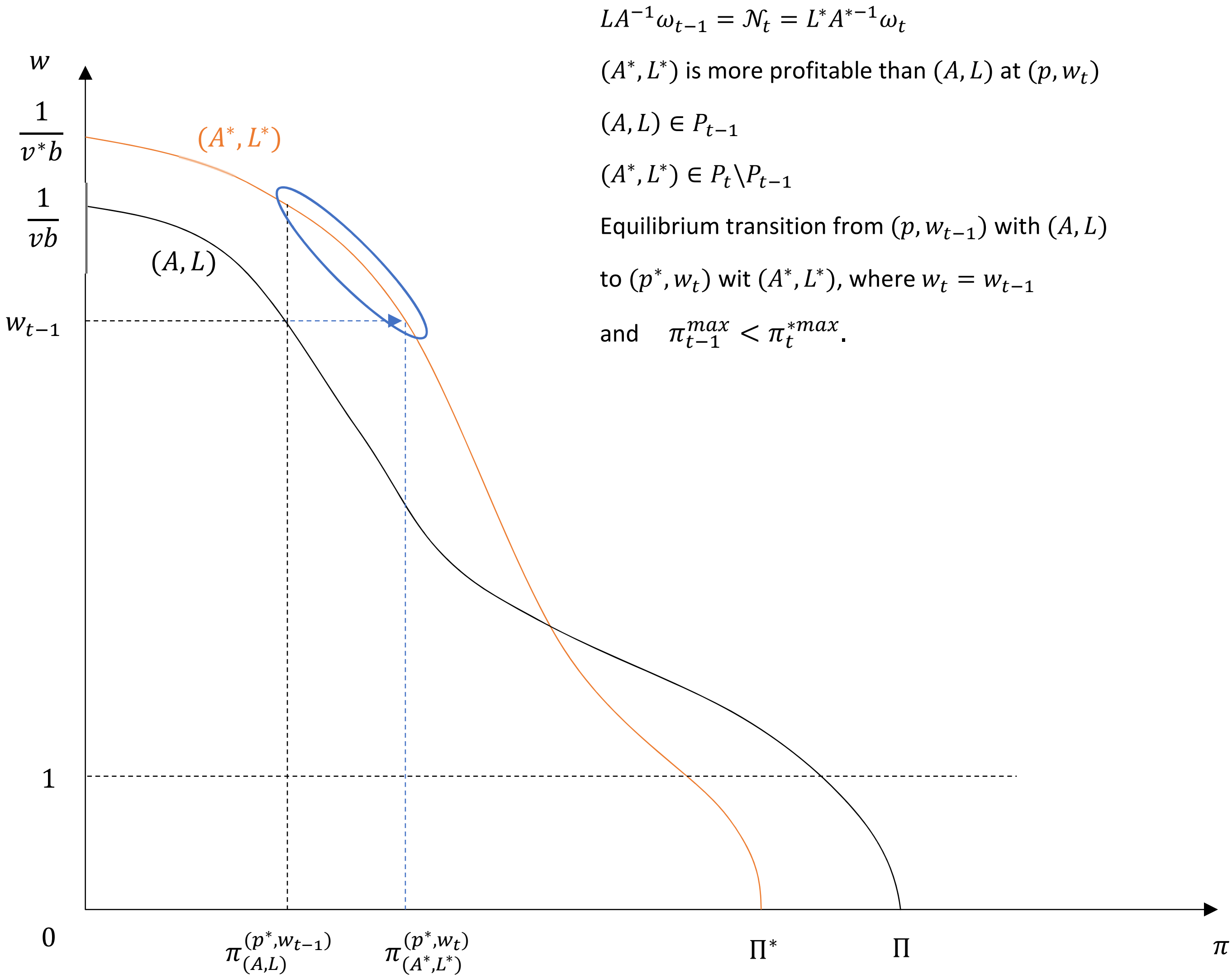

$LA^{-1}\omega_{t-1} = \mathcal{N}_t = L^{*}A^{*-1}\omega_t$
(A^{*}, L^{*}) is more profitable than (A, L) at (p, w_t)
$(A, L) \in P_{t-1}$
$(A^{*}, L^{*}) \in P_t \backslash P_{t-1}$
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p^{*}, w_t) wit (A^{*}, L^{*}), where $w_t = w_{t-1}$
and $\pi_{t-1}^{max} < \pi_t^{*max}$.
w
$\frac{1}{v^{*}b}$
$\frac{1}{vb}$
(A^{*}, L^{*})
(A, L)
w_{t-1}
1
0
$\pi_{(A,L)}^{(p^{*},w_{t-1})}$
$\pi_{(A^{*},L^{*})}^{(p^{*},w_t)}$
Π^{*}
Π
π

Figure 1

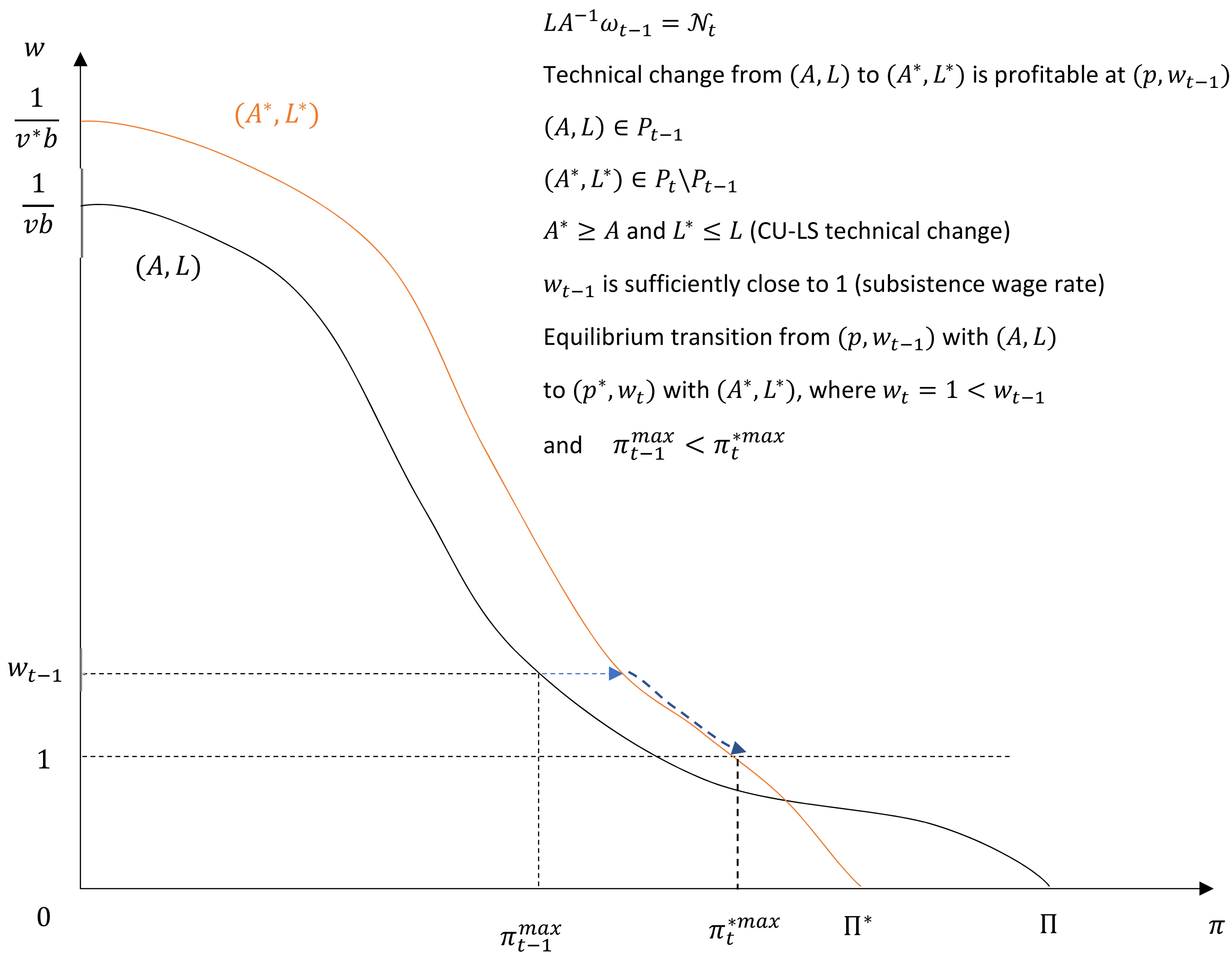

$LA^{-1}\omega_{t-1} = \mathcal{N}_t$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in P_{t-1}$
$(A^*, L^*) \in P_t \backslash P_{t-1}$
$A^* \geq A$ and $L^* \leq L$ (CU-LS technical change)
w_{t-1} is sufficiently close to 1 (subsistence wage rate)
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p^*, w_t) with (A^*, L^*), where $w_t = 1 < w_{t-1}$
and $\pi_{t-1}^{max} < \pi_t^{*max}$
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
(A^*, L^*)
(A, L)
w_{t-1}
1
0
π_{t-1}^{max}
π_t^{*max}
Π^*
Π
π

Figure 2

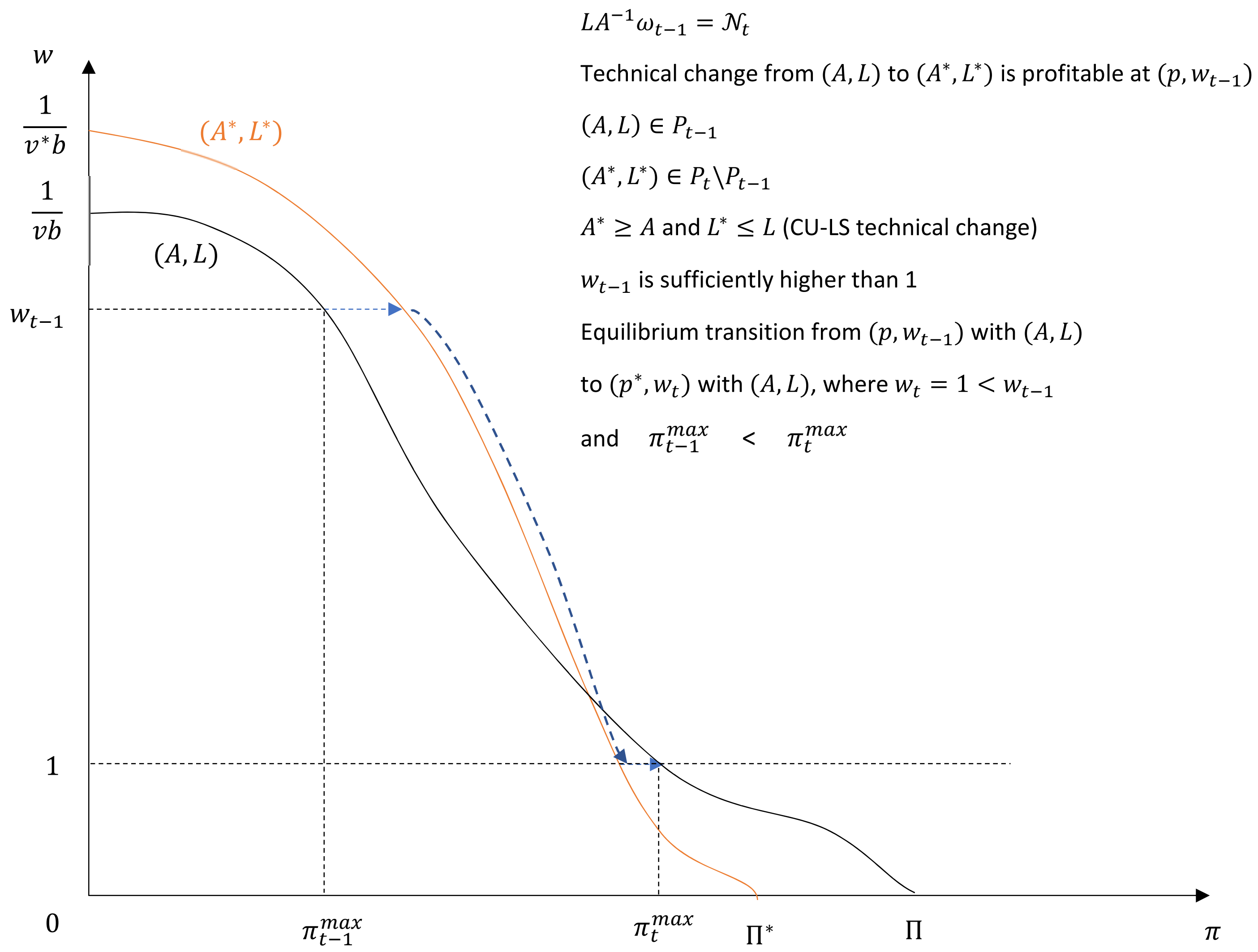
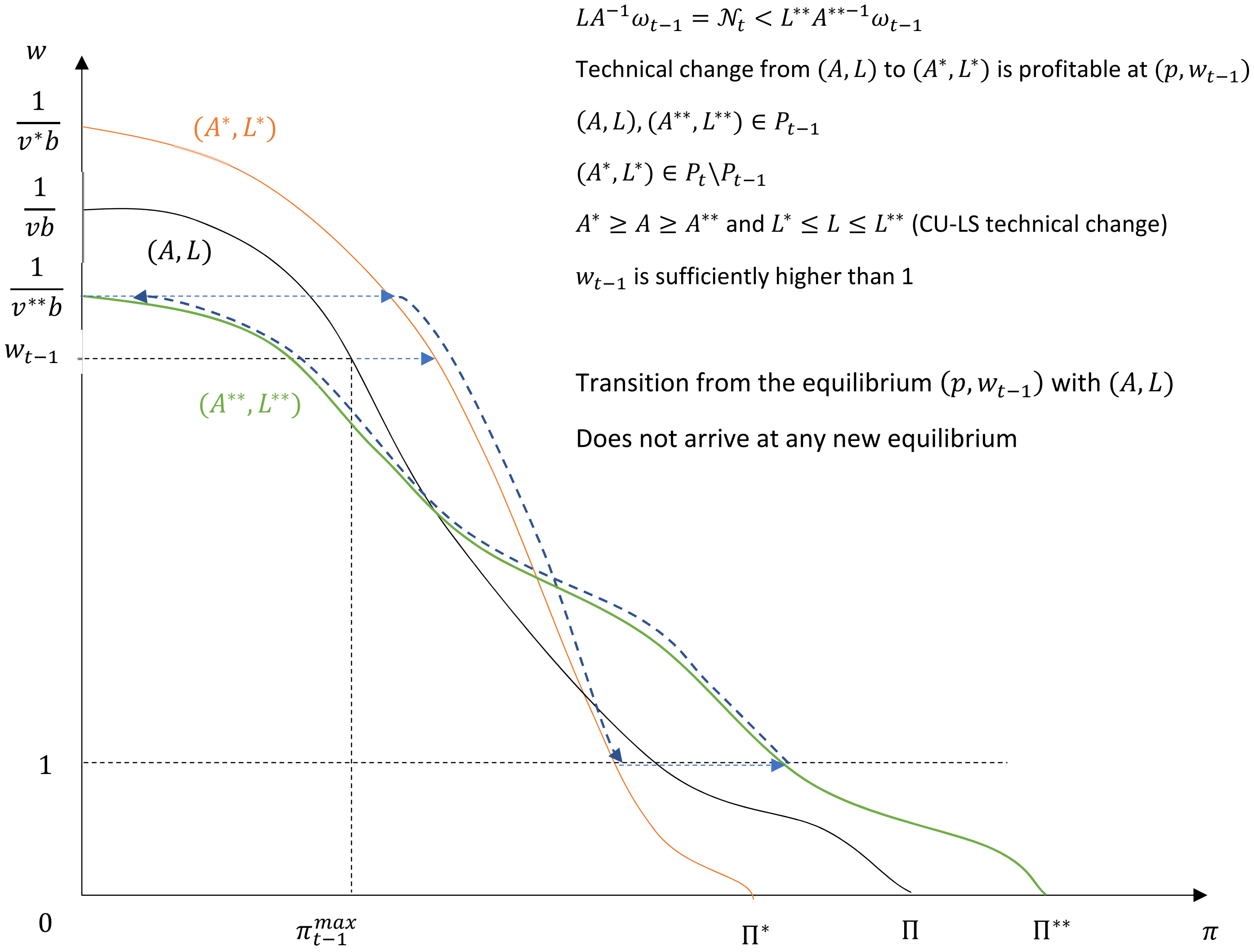
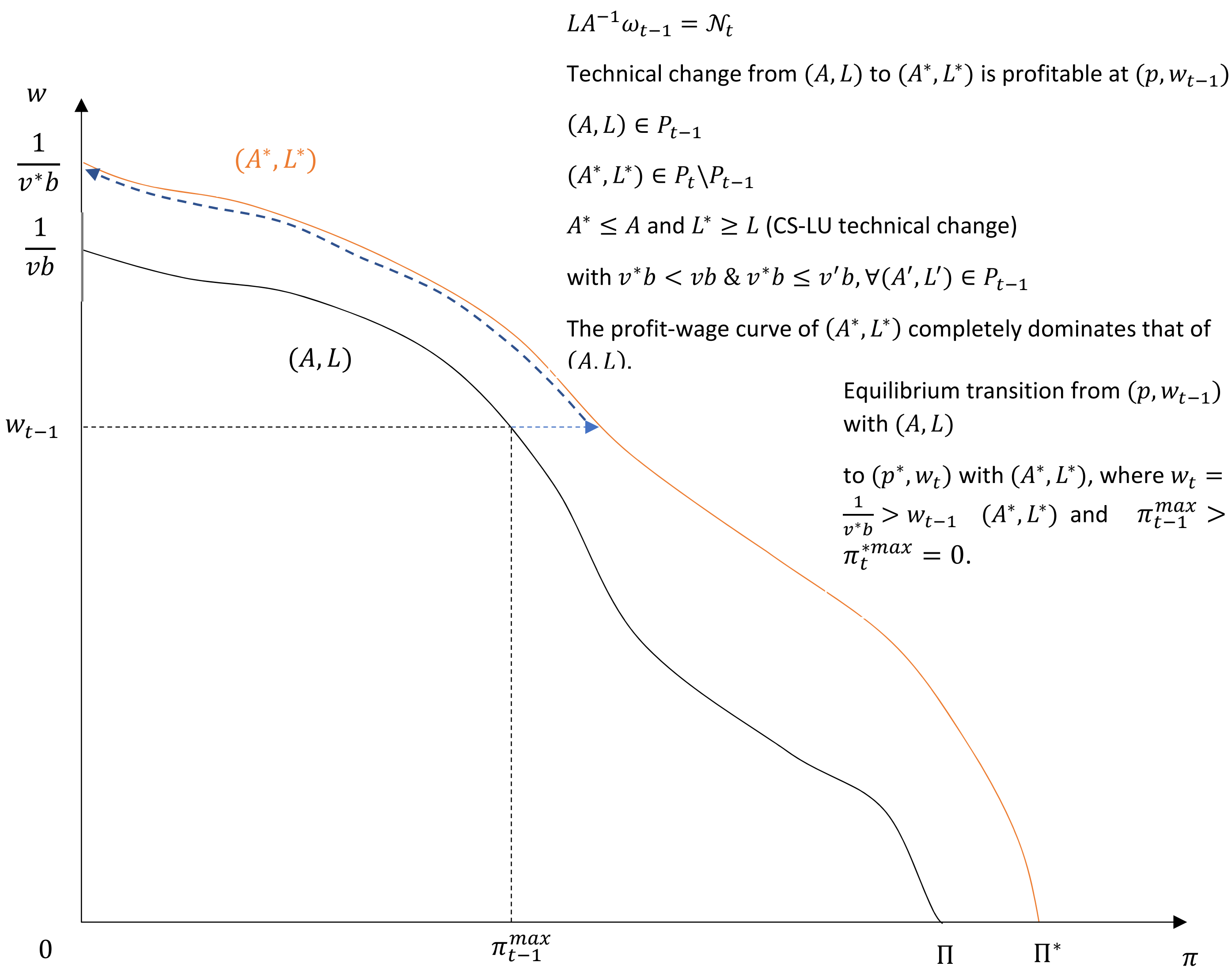

$LA^{-1}\omega_{t-1} = \mathcal{N}_t$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in P_{t-1}$
$(A^*, L^*) \in P_t \backslash P_{t-1}$
$A^* \geq A$ and $L^* \leq L$ (CU-LS technical change)
w_{t-1} is sufficiently higher than 1
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p^*, w_t) with (A, L), where $w_t = 1 < w_{t-1}$
and π_{t-1}^{max} < π_t^{max}
w
$\frac{1}{v^* b}$
$\frac{1}{vb}$
w_{t-1}
1
0
(A^*, L^*)
(A, L)
π_{t-1}^{max}
π_t^{max}
Π^*
Π
π

Figure 3



$LA^{-1}\omega_{t-1} = \mathcal{N}_t < L^{**}A^{**-1}\omega_{t-1}$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L), (A^{**}, L^{**}) \in P_{t-1}$
$(A^*, L^*) \in P_t \backslash P_{t-1}$
$A^* \geq A \geq A^{**}$ and $L^* \leq L \leq L^{**}$ (CU-LS technical change)
w_{t-1} is sufficiently higher than 1
Transition from the equilibrium (p, w_{t-1}) with (A, L)
Does not arrive at any new equilibrium
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
$\frac{1}{v^{**}b}$
w_{t-1}
(A^*, L^*)
(A, L)
(A^{**}, L^{**})
1
0
π_{t-1}^{max}
Π^*
Π
Π^{**}
π

Figure 4



$LA^{-1}\omega_{t-1} = \mathcal{N}_t$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in P_{t-1}$
$(A^*, L^*) \in P_t \backslash P_{t-1}$
$A^* \leq A$ and $L^* \geq L$ (CS-LU technical change)
with $v^*b < vb$ & $v^*b \leq v'b, \forall (A', L') \in P_{t-1}$
The profit-wage curve of (A^*, L^*) completely dominates that of (A, L).
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p^*, w_t) with (A^*, L^*), where $w_t = \frac{1}{v^*b} > w_{t-1}$ (A^*, L^*) and $\pi_{t-1}^{max} > \pi_t^{*max} = 0$.
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
(A^*, L^*)
(A, L)
w_{t-1}
0
π_{t-1}^{max}
Π
Π^*
π

Figure 5

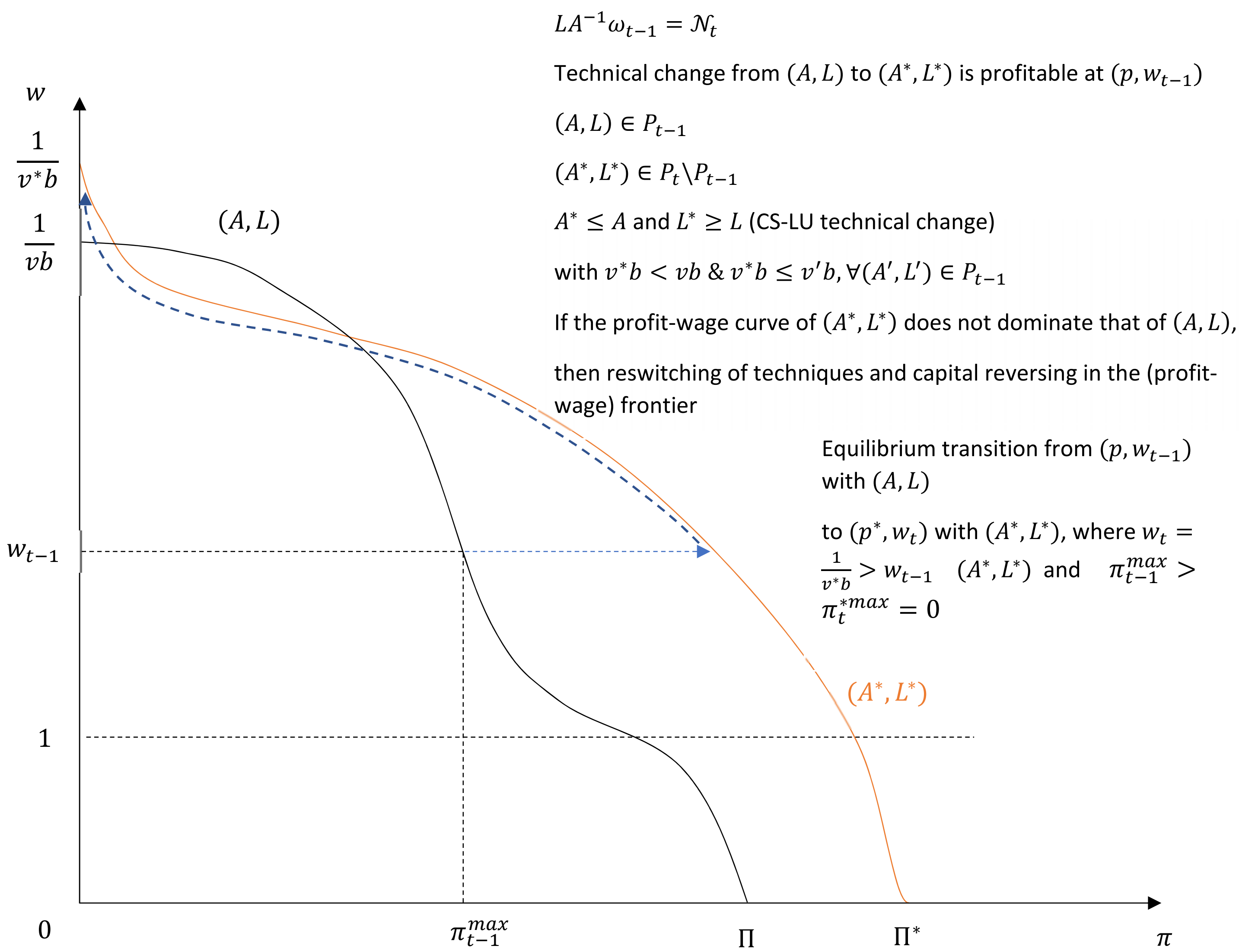

$LA^{-1}\omega_{t-1} = \mathcal{N}_t$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in P_{t-1}$
$(A^*, L^*) \in P_t \backslash P_{t-1}$
$A^* \leq A$ and $L^* \geq L$ (CS-LU technical change)
with $v^*b < vb$ & $v^*b \leq v'b, \forall (A', L') \in P_{t-1}$
If the profit-wage curve of (A^*, L^*) does not dominate that of (A, L),
then reswitching of techniques and capital reversing in the (profit-wage) frontier
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p^*, w_t) with (A^*, L^*), where $w_t = \frac{1}{v^*b} > w_{t-1}$ (A^*, L^*) and $\pi_{t-1}^{max} > \pi_t^{*max} = 0$
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
(A, L)
w_{t-1}
1
(A^*, L^*)
0
π_{t-1}^{max}
Π
Π^*
π

Figure 6

Addendum for the paper: "*Technical progress, capital accumulation, and distribution*"

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November 27, 2019

Abstract

Section 1 contains the proofs omitted in the paper. Section 2 reports some results by Kurz and Salvadori [2] used in the proofs of Theorems 5 and 7 in the paper. Section 3 shows an example of nonexistence of equilibrium after profitable, CS-LU, L-labour inelastic, and regressive technical change. Section 5 provides a formal statement of the distributive implications of technical progress in developing economies. Section 6 provides the proof of existence of equilibrium.

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1 Proofs

Proof of Lemma 1: The result follows immediately from equation (6). ■

Proof of Lemma 2: At a CE, as $x_t > \mathbf{0}$ holds, $\pi_t^{\max} \geqq 0$ must hold. Indeed, if $\pi_t^{\max} < 0$, then $x_t^\nu = \mathbf{0}$ constitutes an optimal action at the CE for every ν. Moreover, since only sectors yielding the maximum profit rate are activated at the solution to MP_t^ν, $x_t > \mathbf{0}$ implies that $p_t = (1+\pi_t^{\max}) p_t A_t + w_t L_t$ holds. Then, by the productiveness and the indecomposability of A_t, $p_t = \pi_t^{\max} p_t A_t (I - A_t)^{-1} + w_t L_t (I - A_t)^{-1} > \mathbf{0}$ holds. Finally, $x_t > \mathbf{0}$ implies that $L_t x_t > 0$ and therefore by Definition 1(c) and (6), it must be $w_t \geqq p_t b$. ■

Proof of Theorem 1: *Part (i).* By Lemma 1, $p_t A_t x_t^\nu = p_t \omega_{t-1}^\nu$ and $l_t^\nu = 1$ for all $\nu \in \mathcal{N}_t$. Then, $p_t A_t x_t = p_t \omega_{t-1}$ holds, and by Definition 1(c), $L_t x_t = l_t = N_t$. By Lemma 2, $p_t > \mathbf{0}$. Therefore, $p_t A_t x_t = p_t \omega_{t-1}$ and Definition 1(b), imply $A_t x_t = \omega_{t-1}$. Since $x_t = A_t^{-1} \omega_{t-1}$, then $N_t = L_t A_t^{-1} \omega_{t-1}$ holds.

Part (ii). Suppose, contrary to the statement, that $w_t > p_t b$. Then, by Lemma 1, $l_t^\nu = 1$, all $\nu \in \mathcal{N}_t$. But, then noting that $A_t x_t \leqq \omega_{t-1}$ by Definition 1(b), $N_t > L_t A_t^{-1} \omega_{t-1}$, implies that $L_t x_t < l_t$ holds, contradicting Definition 1(c).

Part (iii). Suppose, contrary to the statement, that $\pi_t^{\max} > 0$. Then, by Lemma 1, $p_t A_t x_t^\nu = p_t \omega_{t-1}^\nu$, for all $\nu \in \mathcal{N}_t$, and so $p_t A_t x_t = p_t \omega_{t-1}$. Therefore since by Lemma 2 $p_t > \mathbf{0}$ holds, Definition 1(b) implies $A_t x_t = \omega_{t-1}$. But, since $N_t < L_t A_t^{-1} \omega_{t-1}$, $L_t x_t > l_t$ holds, contradicting Definition 1(c). ■

Proof of Proposition 2: By Lemma 2, since the change of technique is profitable, $\left(1+\pi_{t-1}^{\max}\right) p_{t-1} A + w_{t-1} L = p_{t-1} \geq \left(1+\pi_{t-1}^{\max}\right) p_{t-1} A^* + w_{t-1} L^*$, which implies $w_{t-1} L = p_{t-1} - \left(1+\pi_{t-1}^{\max}\right) p_{t-1} A$ and $w_{t-1} L^* \leq p_{t-1} - \left(1+\pi_{t-1}^{\max}\right) p_{t-1} A^*$.

(i) Therefore, if the change of technique is CU-LS:

$$w_{t-1} (L - L^*) \geq \left(1+\pi_{t-1}^{\max}\right) p_{t-1} (A^* - A) \geqq p_{t-1} (A^* - A) \geq \mathbf{0}.$$

Then, since $\frac{p_{t-1}}{w_{t-1}} \geqq v$, $(L - L^*) \geq v (A^* - A) \geq \mathbf{0}$ holds. This implies $L + vA \geq L^* + vA^* \Leftrightarrow v \geq L^* + vA^*$. Therefore, $v (I - A^*) \geq L^*$, which implies $[v (I - A^*) - L^*] (I - A^*)^{-1} > \mathbf{0}$ by $(I - A^*)^{-1} > \mathbf{0}$. Thus, $v > v^*$ holds.

(ii) In contrast, if the change of technique is CS-LU and $v (A - A^*) \leq (L^* - L)$, then $\mathbf{0} \leq v (A - A^*) \leq (L^* - L)$ holds. Then, $v = vA + L \leq vA^* + L^*$. Thus, $[v (I - A^*) - L^*] \leq \mathbf{0}$. Then, by multiplying $(I - A^*)^{-1} > \mathbf{0}$ from the right, we have $v - v^* < \mathbf{0}$. ■

2 Auxiliary results from Kurz and Salvadori [2]

Proposition 5.1 (Kurz and Salvadori [2]): Let $(\pi, w) \in \pi w(A, L)$ with $p = (1+\pi) pA + wL$ for some $p \in \triangle$. If there is a technique $(A', L') \in \mathcal{B}_t$ such that the change from (A, L) to (A', L') is profitable at (p, w), then there exists a technique $(A^*, L^*) \in \mathcal{B}_t$ such that for some $(p^*, w^*) \in \triangle \times \mathbb{R}_+$ with $w^* > w$, $p^* = (1+\pi) p^* A^* + w^* L^*$.

Proposition 5.2 (Kurz and Salvadori [2]): Let $(A, L), (A', L') \in \mathcal{B}_t$ be such that $(\pi, w) \in \pi w(A, L)$ with $p = (1+\pi) pA + wL$ for some $p \in \triangle$, $(\pi, w') \in \pi w(A', L')$ with $p' = (1+\pi) p'A' + w'L'$ for some $p' \in \triangle$, and $w' > w > 0$. Then, there exists a technique $(A^*, L^*) \in \mathcal{B}_t$ such that $p \geq (1+\pi) pA^* + wL^*$.

3 Non-existence of equilibrium after profitable, CS-LU, L-labour inelastic, and regressive technical change

Note that the second scenario is available in a situation that in addition to the present technique $(A, L) \in \mathcal{B}_{t-1}$ and the new discovered technique $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$, there is another technique $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}$ which was not used in the previous period such that $N_t > L^{**} A^{**-1} \omega_{t-1}$. In this case, even if the third alternative technique (A^{**}, L^{**}) is the most progressive one in the sense of $(A^{**}, L^{**}) \in \arg\min_{(A',L') \in \mathcal{B}_{t-1}} L' (I - A')^{-1} b \cap \mathcal{B}_{t-1}(\omega_{t-1}, b)$, it could not necessarily constitute an equilibrium. More precisely, unless a reswitching structure between the two wege-profit curves $\pi w(A, L)$ and $\pi w(A^{**}, L^{**})$ is observed, the economy may not reach to any equilibrium in period t, as Figure 7 describes.

Figure 7 around here.

In Figure 7, the change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1}). Therefore, the economy shifts to a price system associated with (A^*, L^*) as an optimal technique. However, since this change is CS-LU, $N_t < L^* A^{*-1} \omega_{t-1}$ so that the economy shifts to the price system $\left(\frac{1}{v^* b} v^*, \frac{1}{v^* b}\right)$ with zero profit rate. However, given that price system, (A^{**}, L^{**}) is more profitable so that the change from (A^{*},L^{*}) to (A^{**},L^{**}) occurs at $\left(\frac{1}{v^{*}b}v^{*},\frac{1}{v^{*}b}\right)$, which also leads to a further change of price system to $(p^{**},w^{**})\in\triangle\times\mathbb{R}_{+}$ with $w^{**}=\frac{1}{v^{*}b}$ and $p^{**}=\left(1+\pi^{p^{**},w^{**}}\right)p^{**}A^{**}+w^{**}L^{**}$ for some $\pi^{p^{**},w^{**}}>0$. However, since $N_{t}>L^{**}A^{**-1}\omega_{t-1}$ and $w^{**}>1$, (p^{**},w^{**}) cannot be an equilibrium price system so that the economy shifts to a price system with the subsistence wage. However, as Figure 6 describes, $\pi w\left(A^{*},L^{*}\right)$ covers the wage-profit frontier at around the subsistence wage level, so that the change from (A^{**},L^{**}) to (A^{*},L^{*}) occurs at such a price system. Then, since $N_{t}<L^{*}A^{*-1}\omega_{t-1}$ so that the economy shifts to the price system $\left(\frac{1}{v^{*}b}v^{*},\frac{1}{v^{*}b}\right)$ with zero profit rate. Thus, the same movement would repeat, so that it would not arrive at any equilibrium in this situation.

4 Proof of Theorem 9

Theorem 9: Let $\left((p_{t-1},w_{t-1}),\left((A,L);\xi_{t-1}^{\nu};\omega_{t-1}^{\nu}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ be a persistent CE with $\pi_{t-1}^{\max}>0$ for $E(\mathcal{N}_{t-1};\mathcal{B}_{t-1};b;\Omega_{t-2})$. Let $(A^{*},L^{*})\in\mathcal{B}_{t}\backslash\mathcal{B}_{t-1}$ be a new technique such that $(A^{*},L^{*})\notin\mathcal{B}_{t}\left(\omega_{t-1},b\right)$. If inter-period change of technique from (A,L) to (A^{*},L^{*}) is profitable and CU-LS, then it results in a new CE $\left((p_{t}^{*},w_{t}^{*}),((A^{*},L^{*});\xi_{t}^{*\nu};\omega_{t}^{*\nu})_{\nu\in\mathcal{N}_{t}}\right)$ with $\xi_{t}^{*\nu}=(x_{t}^{*\nu};1;\delta_{t}^{*\nu})_{\nu\in\mathcal{N}_{t}}$ and $\pi_{t}^{*\max}=0$ for $E(\mathcal{N}_{t};\mathcal{B}_{t};b;\Omega_{t-1})$ if and only if (1) there exists $x^{*}>\mathbf{0}$ such that $(I-bL^{*})\,x^{*}\geqq A^{*}x^{*}-\omega_{t-1}$ with $A^{*}x^{*}\leq\omega_{t-1}$ and $L^{*}x^{*}=N_{t}$; and (2) $(A^{*},L^{*})\in\arg\min_{(A',L')\in\mathcal{B}_{t}}L'\left(I-A'\right)^{-1}b$. Moreover, there is no other CE associated with (A^{*},L^{*}) for $E(\mathcal{N}_{t};\mathcal{B}_{t};b;\Omega_{t-1})$.

Proof of Theorem 9: (*Only if* part) Let $\left((p_{t}^{*},w_{t}^{*}),((A^{*},L^{*});\xi_{t}^{*\nu};\omega_{t}^{*\nu})_{\nu\in\mathcal{N}_{t}}\right)$ be a CE with $\xi_{t}^{*\nu}=(x_{t}^{*\nu};1;\delta_{t}^{*\nu})_{\nu\in\mathcal{N}_{t}}$ and $\pi_{t}^{*\max}=0<\pi_{t-1}^{\max}$ for $E(\mathcal{N}_{t};\mathcal{B}_{t};b;\Omega_{t-1})$ be a CE after the profitable, CU-LS, and labor inelastic technical change. As $\pi_{t}^{*\max}=0$, $(p_{t}^{*},w_{t}^{*})=\left(\frac{1}{v^{*}b}v^{*},\frac{1}{v^{*}b}\right)$ holds, so that $(A^{*},L^{*})\in\arg\min_{(A',L')\in\mathcal{B}_{t}}L'\left(I-A'\right)^{-1}b$ should hold. Moreover, it follows that $x_{t}^{*}\equiv\sum_{\nu\in\mathcal{N}_{t}}x_{t}^{*\nu}>\mathbf{0}$ and $L^{*}x_{t}^{*}=N$ by Definition 1(c)-(d). Noting $(A^{*},L^{*})\notin\mathcal{B}_{t}\left(\omega_{t-1},b\right)$, suppose $A^{*}x_{t}^{*}=\omega_{t-1}$. Then, it implies that $(I-bL^{*})\,x_{t}^{*}\ngeqq\mathbf{0}$. However, by Definition 1(d), $(I-bL^{*})\,x_{t}^{*}+\delta_{t}^{*}\geqq\mathbf{0}$ must hold for $\delta_{t}^{*}\equiv\sum_{\nu\in\mathcal{N}_{t}}\delta_{t}^{*\nu}$, which implies that $\delta_{t}^{*}\geq\mathbf{0}$. Then, $A^{*}x_{t}^{*}+\delta_{t}^{*}\geq\omega_{t-1}$, which constradicts Definition 1(b). Therefore, $A^{*}x_{t}^{*}\leq\omega_{t-1}$ should hold, and $\delta_{t}^{*}=\omega_{t-1}-A^{*}x_{t}^{*}$. Thus, by Definition 1(d), $(I-bL^{*})\,x_{t}^{*}\geqq A^{*}x_{t}^{*}-\omega_{t-1}$.

(*If* part) Let there exist $x^{*}>\mathbf{0}$ such that $(I-bL^{*})\,x^{*}\geqq A^{*}x^{*}-\omega_{t-1}$ with $A^{*}x^{*}\leq\omega_{t-1}$ and $L^{*}x^{*}=N_{t}$; and let $(A^{*},L^{*})\in\arg\min_{(A',L')\in\mathcal{B}_{t}}L'\left(I-A'\right)^{-1}b$.

Then, as shown in the proof of Theorem 7, (A^*, L^*) is optimal at $(p_t^*, w_t^*) \equiv \left(\frac{1}{v^*b}v^*, \frac{1}{v^*b}\right)$. In this case, as $w_t^* > 1$, $l_t^{*\nu} = 1$ is the optimal labour supply for each $\nu \in \mathcal{N}_t$ at (p_t^*, w_t^*). Let $\delta^* \equiv \omega_{t-1} - A^*x^* \geq \mathbf{0}$. Moreover, let $\omega_t^* \equiv x^* + \delta^* - Nb \geqq \mathbf{0}$. Then, consider a suitable feasible assignment, $(x^{*\nu}, \delta^{*\nu}, \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}$, of $(x^*, \delta^*, \omega_t^*)$ among all agents in $\mathcal{N}_t$, in order to meet $x_t^{*\nu} = \frac{p_t^*\omega_{t-1}^{*\nu}}{p_t^*\omega_{t-1}^*}x^*$, $\delta_t^{*\nu} = \frac{p_t^*\omega_{t-1}^{*\nu}}{p_t^*\omega_{t-1}^*}\delta^*$, and $p_t^*\omega_t^{*\nu} = p_t^*\omega_{t-1}^{*\nu} + w_t^* - p_t^*b$ for each $\nu \in \mathcal{N}_t$. Then, $(x^{*\nu}, 1, \delta^{*\nu}; \omega_t^{*\nu})$ constitutes an optimal action for each $\nu \in \mathcal{N}_t$ at (p_t^*, w_t^*). Thus, as the aggregation of the profile $(x^{*\nu}, 1, \delta^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}$ satisfy Definitions 1-(b), 1-(c), and 1-(d), the profile $\left((p_t^*, w_t^*), ((A^*, L^*); (x^{*\nu}, 1, \delta^{*\nu}; \omega_t^{*\nu}))_{\nu \in \mathcal{N}_t}\right)$ constitutes a CE. As $(p_t^*, w_t^*) = \left(\frac{1}{v^*b}v^*, \frac{1}{v^*b}\right)$, $\pi_t^{*\max} = 0$ holds.

Finally, to show that no other CE associated with (A^*, L^*), suppose that there exists a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ with $\pi_t^{*\max} > 0$. Then, by Lemma 1 and Lemma 2, $A^*x_t^* = \omega_{t-1}$. Then, as $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$, it implies that $(I - bL^*)x_t^* \ngeqq \mathbf{0}$. Then, as shown in (Only if part), we derive a contradiction from Definition 1(b)-(d). Thus, no such CE exists. ■

5 CS-LU Technical Change in Developing Economies

Consider a developing economy in which the social endowments of capital stocks accumulated in the past are still very low relative to the size of population. In this case, it is natural to assume that a persistent CE is characterised by $N_t > LA^{-1}\omega_{t-1}$ and ask whether the premise of Theorem 4 can be satisfied. This is particularly relevant if a CS-LU change of technique is considered, as in the next result.

Theorem 10: Let $\left((p, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ associated with $N_t > LA^{-1}\omega_{t-1}$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \setminus \mathcal{B}_{t-1}$ be a new technique. Let inter-period change of technique from (A, L) to (A^*, L^*) be profitable and CS-LU with sufficiently small $(A - A^*, L^* - L)$. Then, there exists a CE $\left((p^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ with $w_t^* = 1$ and $\pi_t^{*\max} > \pi_{t-1}^{\max}$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

Proof: Straightforward adaptation of the proof of Theorem 6. ■

Theorem 10 shows that the premise of Theorem 4(ii) is satisfied: if there is a sufficiently big industrial reserve army of the unemployed, then a profitable, gradual, CS-LU change of technique will indeed be adopted in equilibrium, and lead to an increase in the profit rate, even if this change of technique is regressive.

Both the assumption $N_t > LA^{-1}\omega_{t-1}$, and the characteristics of the new equilibrium described in Theorem 9 are quite realistic in developing economies, in which aggregate labour is abundant relative to the level of accumulated capital stock. These economies may wish to import the advanced technology (a more capital-intensive technique) from advanced economies, but their aggregate capital endowments are often insufficient to adopt capital-intensive techniques. In this case, developing economies may modify such advanced technology into a slightly more labour-intensive one, as in the case of the Japanese economy just after the Meiji Revolution around the mid 19th century (see, e.g., Allen [1]).

6 The existence of a persistent competitive equilibrium

In this appendix, we analyse the existence of persistent CEs for an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with a set of agents $\mathcal{N}_t$, a set of base techniques $\mathcal{B}_t$, a subsistence vector b, and a distribution of physical endowments Ω_{t-1}, in period t. As specified in section 6 of the main text, the wage-profit frontier $\pi w(\mathcal{B}_t)$ is specified from the set $\mathcal{B}_t$. Likewise, for each $(A, L) \in \mathcal{B}_t$, the wage-profit curve $\pi w(A, L)$ associated with (A, L) is specified. Then, let

$$\overline{\mathcal{B}}_t \equiv \{(A, L) \in \mathcal{B}_t \mid \exists (\pi, w) \in \pi w(A, L) \cap \pi w(\mathcal{B}_t) \text{ s.t. } (\pi, w) \geq (0, 1)\}.$$

6.1 The existence of persistent CEs with full employment of all factors of production

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ at period t with $\mathcal{N}_{t+1}$, a set of endowments of produced inputs is defined as follows:

$$C_t^* \equiv \left\{ \omega \in \mathbb{R}_+^n \mid \exists x^* > \mathbf{0} \ \& \ (A, L) \in \overline{\mathcal{B}}_t \text{:} \begin{array}{c} Ax^* = \omega, \ (I - bL)\, x^* \geq \mathbf{0}, \ Lx^* = N_t, \\ A^{-1}(x^* - N_t b) > \mathbf{0}, \ LA^{-1}(x^* - N_t b) = N_{t+1} \end{array} \right\}.$$

We will show that $\omega_{t-1} \in C_t^*$ is the necessary and sufficient condition for the existence of persistent CEs with full employment of all productive factors. First of all, let us show that the set C_t^* is well-defined. To show it, let us define

$$\overline{\mathcal{B}}_t\left(N_t, N_{t+1}\right) \equiv \left\{(A, L) \in \overline{\mathcal{B}}_t \mid L\left[I - \frac{N_{t+1}}{N_t}A\right]^{-1} b = 1\right\}.$$

Then:

Theorem A.1: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t with $\mathcal{N}_{t+1}$. Then, if $\overline{\mathcal{B}}_t\left(N_t, N_{t+1}\right) \neq \varnothing$, then $C_t^* \neq \varnothing$.

Proof. As $\overline{\mathcal{B}}_t\left(N_t, N_{t+1}\right) \neq \varnothing$, let $(A, L) \in \overline{\mathcal{B}}_t\left(N_t, N_{t+1}\right)$. Let $(1+g) \equiv \frac{N_{t+1}}{N_t}$. Then, $L\left[I - (1+g)A\right]^{-1} b = 1$ holds. The last equation implies that there exists $p \in \triangle$ such that $p \equiv L\left[I - (1+g)A\right]^{-1} > \mathbf{0}$. Therefore, $p = p\left[(1+g)A + bL\right]$ holds. The last equations imply that the Frobenius eigenvalue of the matrix $\left[(1+g)A + bL\right]$ is 1 associated with the unique Frobenius eigenvector $p > \mathbf{0}$. Then, there exists the Frobenius eigenvector $x^* > \mathbf{0}$ such that $x^* = \left[(1+g)A + bL\right]x^*$ unique up to $Lx^* = N_t$. Then, $(1+g)Ax^* = x^* - N_t b$ holds. As $(1+g)Ax^* > \mathbf{0}$ by the indecomposability of A and $x^* > \mathbf{0}$, $x^* - N_t b = (I - bL)x^* > \mathbf{0}$. Moreover, $A^{-1}\left(x^* - N_t b\right) = A^{-1}(1+g)Ax^* = (1+g)x^* > \mathbf{0}$. Finally,

$$LA^{-1}(1+g)Ax^* = LA^{-1}\left(x^* - N_t b\right)$$

which is equivalent to

$$(1+g)Lx^* = LA^{-1}\left(x^* - N_t b\right) \Leftrightarrow \frac{N_{t+1}}{N_t}N_t = LA^{-1}\left(x^* - N_t b\right).$$

Thus, by $\omega \equiv Ax^*$, we can see that $\omega \in C_t^*$. ■

Note that for each $(A, L) \in \mathcal{B}_t$, there exists $\pi \in (0, \Pi(A))$ such that $L\left[I - (1+\pi)A\right]^{-1} b = 1$ holds, due to the intermediate value theorem. Then, if $\overline{\mathcal{B}}_t$ contains sufficiently many alternative Leontief techniques, then it would be likely to exist $(A, L) \in \overline{\mathcal{B}}_t$ such that for some $\pi = \frac{N_{t+1}}{N_t} = -1$, $L\left[I - (1+\pi)A\right]^{-1} b = 1$ holds. Therefore, it would be likely that the set C_t^* is non-empty.

Theorem A.2: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t with $\mathcal{N}_{t+1}$. Then, there exists a persistent CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ such that $A^* x^* = \omega_{t-1}$ and $L^* x^* = N_t$ if and only if $\omega_{t-1} \in C_t^*$.

Proof. (*Only if* part) Let $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ be a persistent CE such that $A^* x_t^* = \omega_{t-1}$ and $L^* x_t^* = N_t$. As $(\pi_t^{*\max}, w_t^*) \geq (0, 1)$ holds in this case, $(A^*, L^*) \in \overline{\mathcal{B}}_t$ holds. In this CE, $\omega_t = x_t^* - N_t b$ holds. As the CE is persistent, $\omega_t \geq \mathbf{0}$ holds, which implies that $x_t^* - N_t b = (I - bL) x_t^* \geq \mathbf{0}$. Moreover, by Proposition 1(i), $A^{*-1} \omega_t > \mathbf{0}$ holds, which implies that $A^{*-1}(x_t^* - N_t b) > \mathbf{0}$. Finally, by Proposition 1(i), $L^* A^{*-1} \omega_t = N_{t+1}$. Thus, $L^* A^{*-1}(x_t^* - N_t b) = N_{t+1}$ holds. In conclusion, $\omega_{t-1} \in C_t^*$ holds.

(*If* part) Let $\omega_{t-1} \in C_t^*$ hold. Then, there exist $x^* > \mathbf{0}$ and $(A, L) \in \overline{\mathcal{B}}_t$ such that $Ax^* = \omega_{t-1}$, $(I - bL) x^* \geq \mathbf{0}$, $Lx_t^* = N_t$, $A^{-1}(x^* - N_t b) > \mathbf{0}$, and $LA^{-1}(x^* - N_t b) = N_{t+1}$. As $(A, L) \in \overline{\mathcal{B}}_t$, there exists $(\pi_t^{*\max}, w_t^*) \geq (0, 1)$ such that for $p_t^* \equiv w_t^* L \left[I - (1 + \pi_t^{*\max}) A\right]^{-1} > \mathbf{0}$,

$$p_t^* = (1 + \pi_t^{*\max}) p_t^* A + w_t^* L \leqq (1 + \pi_t^{*\max}) p_t^* A' + w_t^* L' \text{ for any } (A', L') \in \mathcal{B}_t.$$

Then, for each $\nu \in \mathcal{N}_t$, a suitable optimal action profile $(\xi_t^\nu; \omega_t^\nu)$ with $\xi_t^\nu = (x_t^{\nu *}, 1, 0)$ can be specified so as to satisfy $\sum_{\nu \in \mathcal{N}_t} x_t^{\nu *} = x^*$ and $\sum_{\nu \in \mathcal{N}_t} \omega_t^\nu = \omega_t \equiv x^* - N_t b \geq \mathbf{0}$. Thus, $\left((p_t^*, w_t^*), ((A, L); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE with the full employment of all productive factors. Finally, $A^{-1} \omega_t > \mathbf{0}$ and $LA^{-1} \omega_t = N_{t+1}$ follow from $A^{-1}(x^* - N_t b) > \mathbf{0}$ and $LA^{-1}(x^* - N_t b) = N_{t+1}$ respectively, and $\omega_t = x^* - N_t b$. Thus, Proposition 1 implies that this CE is persistent. ■

6.2 The existence of persistent CEs with unemployment of labour

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, let

$$\overline{\mathcal{B}}_t^* \equiv \{(A, L) \in \mathcal{B}_t \mid \exists (\pi, w) \in \pi w(A, L) \cap \pi w(\mathcal{B}_t) \text{ s.t. } \pi > 0 \;\&\; w = 1\}.$$

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ at period t with $\mathcal{N}_{t+1}$, a set of endowments of produced inputs is defined as follows:

$$C_t^{**} \equiv \left\{ \omega \in \mathbb{R}_+^n \mid \exists x^* > \mathbf{0} \;\&\; (A, L) \in \overline{\mathcal{B}}_t^* \colon \begin{array}{c} Ax^* = \omega, \; (I - bL) x^* \geq \mathbf{0}, \; Lx^* < N_t, \\ A^{-1}(I - bL) x^* > \mathbf{0}, \; LA^{-1}(I - bL) x^* < N_{t+1} \end{array} \right\}.$$

We will show that $\omega_{t-1} \in C_t^{**}$ is the necessary and sufficient condition for the existence of persistent CEs with unemployment of labour.

First, we will show that C_t^{**} is well-defined:

Theorem A.3: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t with $\mathcal{N}_{t+1}$. Then, $C_t^{**} \neq \varnothing$.

Proof. Let $(A, L) \in \overline{\mathcal{B}}_t^*$. Then, by the intermediate value theorem, there exists $g_{(A,L)} > 0$ such that $L\left[I - \left(1 + g_{(A,L)}\right) A\right]^{-1} b = 1$. The last equation implies that there exists $p \in \triangle$ such that $p \equiv L\left[I - \left(1 + g_{(A,L)}\right) A\right]^{-1} > \mathbf{0}$. Therefore, $p = p\left[\left(1 + g_{(A,L)}\right) A + bL\right]$ holds. The last equations imply that the Frobenius eigenvalue of the matrix $\left[\left(1 + g_{(A,L)}\right) A + bL\right]$ is 1 associated with the unique Frobenius eigenvector $p > \mathbf{0}$. Then, there exists the Frobenius eigenvector $x^* > \mathbf{0}$ such that $x^* = \left[\left(1 + g_{(A,L)}\right) A + bL\right] x^*$ with $Lx^* < \frac{N_{t+1}}{1+g_{(A,L)}}$. Then, $\left(1 + g_{(A,L)}\right) Ax^* = x^* - bLx^* > \mathbf{0}$ holds by $x^* > \mathbf{0}$ and the indecomposability of A. Moreover, $A^{-1}\left(I - bL\right) x^* = A^{-1}\left(1 + g_{(A,L)}\right) Ax^* = \left(1 + g_{(A,L)}\right) x^* > \mathbf{0}$ holds. Finally, we have

$$LA^{-1}\left(1 + g_{(A,L)}\right) Ax^* = LA^{-1}\left(x^* - bLx^*\right)$$

which is equivalent to $\left(1 + g_{(A,L)}\right) Lx^* = LA^{-1}\left(x^* - bLx^*\right)$. Then, $\left(1 + g_{(A,L)}\right) Lx^* < N_{t+1}$ as $Lx^* < \frac{N_{t+1}}{1+g_{(A,L)}}$. Thus, $LA^{-1}\left(x^* - bLx^*\right) < N_{t+1}$. Then, by $\omega \equiv Ax^*$, we can see that $\omega \in C_t^{**}$. ■

Theorem A.4: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t with $\mathcal{N}_{t+1}$. Then, there exists a persistent CE $\left(\left(p_t^*, w_t^*\right), \left(\left(A^*, L^*\right); \xi_t^\nu; \omega_t^\nu\right)_{\nu \in \mathcal{N}_t}\right)$ such that $A^* x^* = \omega_{t-1}$ and $L^* x^* < N_t$ if and only if $\omega_{t-1} \in C_t^{**}$.

Proof. (*Only if* part) Let $\left(\left(p_t^*, w_t^*\right), \left(\left(A^*, L^*\right); \xi_t^\nu; \omega_t^\nu\right)_{\nu \in \mathcal{N}_t}\right)$ be a persistent CE such that $A^* x_t^* = \omega_{t-1}$ and $L^* x_t^* < N_t$. As $\pi_t^{*\max} > 0$ and $w_t^* = 1$ hold in this case, $\left(A^*, L^*\right) \in \overline{\mathcal{B}}_t^*$ holds. In this CE, $\omega_t = x_t^* - bL^* x_t^*$ holds. As the CE is persistent, $\omega_t \geq \mathbf{0}$ holds, which implies that $\left(I - bL\right) x_t^* \geq \mathbf{0}$. Moreover, by Proposition 1(ii), $A^{*-1}\omega_t > \mathbf{0}$ holds, which implies that $A^{*-1}\left(x_t^* - bL^* x_t^*\right) > \mathbf{0}$. Finally, as this CE is persistent with unemployment of labour, it follows from Proposition 1(ii) that $L^* A^{*-1}\omega_t < N_{t+1}$. Thus, $L^* A^{*-1}\left(x_t^* - bL^* x_t^*\right) < N_{t+1}$ holds. In conclusion, $\omega_{t-1} \in C_t^{**}$ holds.

(*If* part) Let $\omega_{t-1} \in C_t^{**}$ hold. Then, there exist $x^* > \mathbf{0}$ and $(A, L) \in \overline{\mathcal{B}}_t^*$ such that $Ax^* = \omega_{t-1}$, $\left(I - bL\right) x^* \geq \mathbf{0}$, $Lx^* < N_t$, $A^{-1}\left(I - bL\right) x^* > \mathbf{0}$, and

$LA^{-1}(I-bL)x^* < N_{t+1}$. As $(A,L) \in \overline{\mathcal{B}}^*_t$, there exist $\pi_t^{*\max} > 0$ and $w_t^* = 1$ such that for $p_t^* \equiv w_t^* L\left[I - (1+\pi_t^{*\max})A\right]^{-1} > \mathbf{0}$,

$$p_t^* = (1+\pi_t^{*\max})\, p_t^* A + w_t^* L \leqq (1+\pi_t^{*\max})\, p_t^* A' + w_t^* L' \text{ for any } (A',L') \in \mathcal{B}_t.$$

Then, for each $\nu \in \mathcal{N}_t$, a suitable optimal action profile $(\xi_t^\nu; \omega_t^\nu)$ with $\xi_t^\nu = \left(x_t^{\nu *}, \frac{Lx^*}{N_t}, 0\right)$ can be specified so as to satisfy $\sum_{\nu \in \mathcal{N}_t} x_t^{\nu *} = x^*$ and $\sum_{\nu \in \mathcal{N}_t} \omega_t^\nu = \omega_t \equiv x^* - bLx^*$. Thus, $\left((p_t^*, w_t^*), ((A,L); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE with unemployment of labour. Finally, $A^{-1}\omega_t > \mathbf{0}$ and $LA^{-1}\omega_t < N_{t+1}$ follow from $A^{-1}(I-bL)x^* > \mathbf{0}$ and $LA^{-1}(x^* - bLx^*) < N_{t+1}$ respectively, noting that $\omega_t = x^* - bLx^*$. Thus, Proposition 1(ii) implies that this CE is persistent. ■

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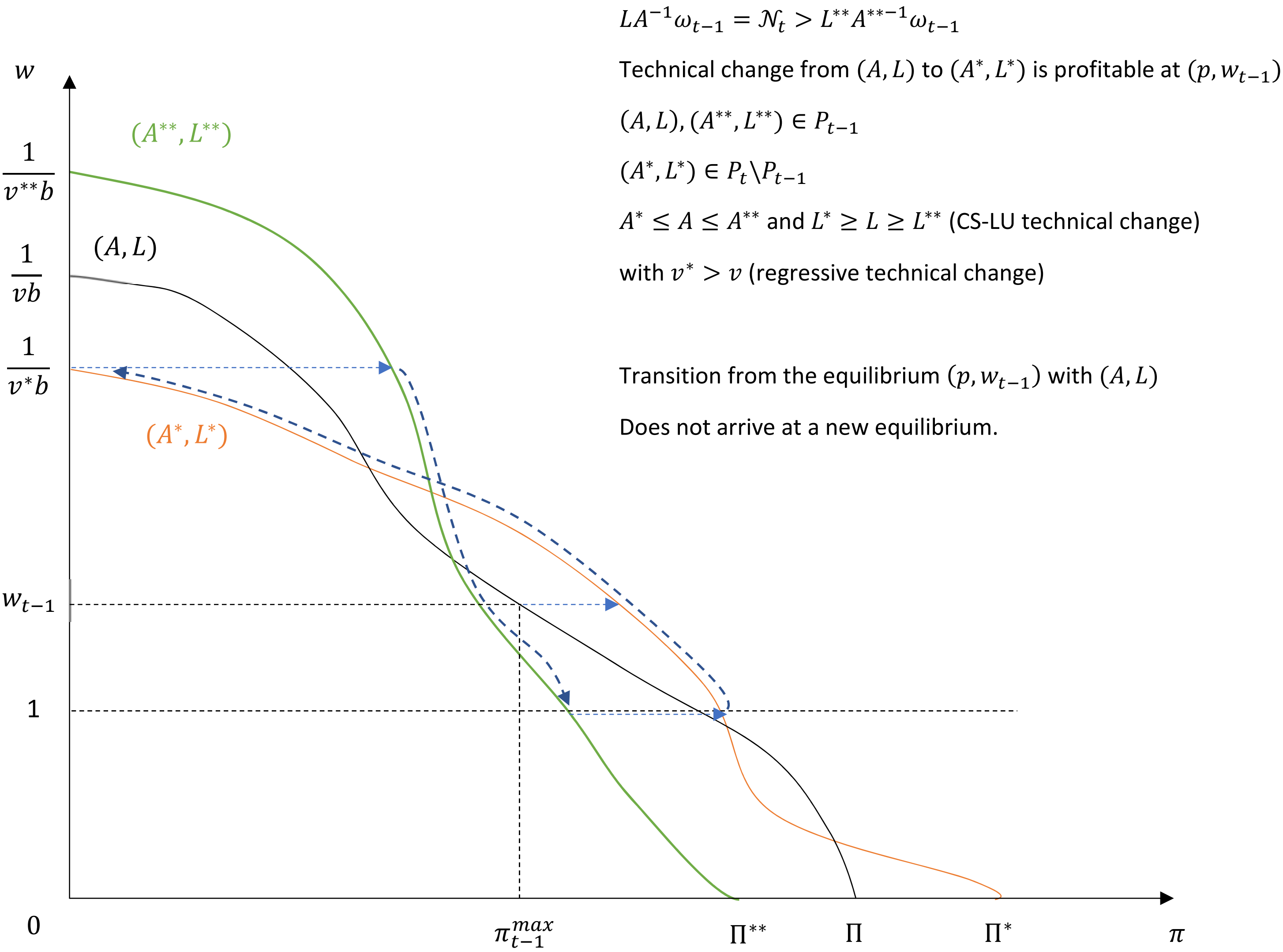
$LA^{-1}\omega_{t-1} = \mathcal{N}_t > L^{**}A^{**-1}\omega_{t-1}$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L), (A^{**}, L^{**}) \in P_{t-1}$
$(A^*, L^*) \in P_t \backslash P_{t-1}$
$A^* \leq A \leq A^{**}$ and $L^* \geq L \geq L^{**}$ (CS-LU technical change)
with $v^* > v$ (regressive technical change)
Transition from the equilibrium (p, w_{t-1}) with (A, L)
Does not arrive at a new equilibrium.
w
(A^{**}, L^{**})
$\frac{1}{v^{**}b}$
(A, L)
$\frac{1}{vb}$
$\frac{1}{v^*b}$
(A^*, L^*)
w_{t-1}
1
0
π_{t-1}^{max}
Π^{**}
Π
Π^*
π

Figure 7