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Unequal Exchange in the Free Trade Equilibrium

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Abstract

As in Roemer (1982, chapter 1), this paper considers a Marxian Heckscher–Ohlin model of subsistence international economies with a simple Leontief production technique and examines the existence and characterization of free trade equilibria involving the unequal exchange of labor (UEL). The paper provides an almost complete characterization of the domain of economies in which free trade equilibria with incomplete specialization exist. Moreover, the necessary and sufficient condition for a free trade equilibrium to involve UEL is identified.

JEL classification: D63; D51

Keywords: Unequal exchange of labor; Marxian Heckscher–Ohlin international economies; international division of labor

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1 Introduction

A model of international trade is called a Heckscher–Ohlin international economy if every nation can access a common production technology, but when the initial endowments of the productive factors differ among nations. This setting implies that any (advanced) knowledge of production technology can be dispersed among nations and thus becomes common knowledge in the long run. However, some nations cannot access the production activity corresponding to such knowledge because they lack sufficient wealth to purchase a profile of the relevant capital goods.

In this paper, we define a model of Heckscher–Ohlin international economies with N nations and n commodities and then examine whether a free trade equilibrium among rich and poor nations involves exploitation as the unequal exchange of labor (UEL). Since the Heckscher–Ohlin model does not have international markets of productive factors, our question here is beyond classical Marxian theory and Okishio (1963)–Morishima (1973) exploitation theory, which address the existence of exploitation in the production process based on the employment relations between capital and labor.

The main literature relevant to international UEL has often applied a Ricardian model of international trade, because one of the essential sources of UEL has typically been considered to be the disparity in wage rates among nations due to the disparity in the production technologies available to them (see Marx, 1954, chapter 20). However, this disparity in the available production technologies tends to be a transitional phenomenon unless there is serious disparity in wealth endowments, since knowledge of production technologies per se would disperse—at least in the long run. Therefore, if the issue of international UEL is taken as a universal long-run feature of free trade equilibria, it would be sensible to formulate the subject in a Heckscher–Ohlin-type international trade model.

Research by the *dependence school* typically takes capital mobility across nations as an indispensable factor to generate UEL. For instance, Emmanuel (1972) discusses that under the center-periphery structure of international economies, where the disparity in wage rates between developed capitalist nations and underdeveloped nations is institutionalized, the free trade of commodities with international capital mobility results in, first, the transfer of surplus labor from poor nations with lower capital–labor ratios to richer nations with higher capital–labor ratios, and second, the impoverishment of poor nations and the enrichment of wealthy nations.

Samuelson (1976) criticizes the work by Emmanuel (1972) by arguing that the Emmanuel theory of generating UEL is inconsistent with the mutual gain from the trade. However, his criticism can refute only the second claim of the Emmanuel theory and not the first claim that illustrates a mechanism for generating UEL. Indeed, the generation of UEL would be compatible with the mutual gain from the trade; as Marx (1968, chapter 20, (e)) argues, "a richer country exploits a poorer one, even when the latter benefits from the exchange." Moreover, as suggested by Roemer (1982, chapter 1), in contrast to the Emmanuel theory of unequal exchange, neither the institutionalized disparity of wage rates nor international capital mobility is essential for the generation of UEL. Indeed, Roemer (1982, chapter 1) provides a numerical example of free trade equilibria with UEL under the international immobility of capital and labor.

For these reasons, we restrict our analysis to examining an essential mechanism for generating international UEL by adopting a Heckscher–Ohlin-type trade model, in which neither labor nor capital mobility among nations is available. Note that, however, our Heckscher–Ohlin-type model has two distinct features from its neoclassical counterpart. First, the neoclassical Heckscher–Ohlin model does not discuss the production of capital goods by treating *capital* as a primary factor.[1] By contrast, we model capital as a bundle of reproducible commodities and explicitly analyze the production process of capital goods. Moreover, in contrast to neoclassical Heckscher–Ohlin theory, we also analyze the international trade of capital goods as well as of consumption goods.

Second, we explicitly take the time structure of production. Specifically, the production of every commodity takes one period, meaning that to operate a production activity, each nation must be able to finance, at the beginning of the period, its requirement for inputs from its own financial capital accumulated in past periods. Adopting such a structure allows us to discuss that while each nation produces and internationally trades capital goods (for the production of consumption and capital goods in a future period), its pro-

[1]Some studies of Heckscher–Ohlin trade theory define capital as a bundle of reproducible goods. Please see Burmeister (1978), Chen (1992), Nishimura and Shimomura (2002, 2006), and Bond, Iwasa, and Nishimura (2011, 2012). However, the structure of the models developed in these works intrinsically differs from that developed in this paper, in that these papers simply consider models with one (pure) capital good and one (pure) consumption good. Such models do not generate any difficulty related to the Cambridge controversies on capital theory.

duction activity is constrained by its endowed financial capital accumulated from past economic activities.

We call our model of international economies with these two features, which are indispensable for the analysis of UEL, the *Marxian Heckscher–Ohlin model*. In the neoclassical Heckscher–Ohlin model, it is easy to confirm that given that all nations consume the same consumption bundles in the free trade equilibrium, a wealthier nation specialized in a more capital-intensive activity supplies fewer labor hours than a poorer nation specialized in a more labor-intensive activity. However, the neoclassical Heckscher–Ohlin model is unsuitable for discussing UEL, since it does not include the amount of labor supplied to the production process of capital goods. In the Marxian Heckscher–Ohlin model, by contrast, the amount of labor inputs necessary to produce capital goods is included in the calculation of the social necessary labor time that each nation can receive from its national income.

In this paper, we consider a model of Marxian Heckscher–Ohlin international economies by assuming a simple Leontief production technique and by using a simple form of any nation's welfare function. This common welfare function represents every nation's primary concern about citizens' enjoyment of free hours (or leisure time), once they are guaranteed a common subsistence consumption bundle that every citizen of every nation must consume to survive in each period. Such a model is called a *subsistence economy* and was first introduced by Roemer (1982, chapter 1).[2]

While the subject of this paper is essentially identical to that of Roemer (1982, chapter 1), its main results show some significant extensions. Although Roemer (1982, chapter 1) discusses and shows the existence of free trade equilibria with UEL, he only provides a numerical example. By contrast, this paper characterizes the domain of subsistence economies in which free trade equilibria with as well as without UEL exist. To show this domain, the paper considers two cases. The first case refers to subsistence economies with excessive social endowments of capital stocks. In this case, no free trade equilibrium has UEL (see Theorem 4). The second case refers to subsistence economies with non-excessive social endowments of capital stocks. In this case, if no essential technical difference among sectors exists, there is no free trade equilibrium with UEL, while free trade equilibria without UEL exist whenever the initial endowments of financial capital are equalized (see The-

[2] Another variation of the Marxian Heckscher–Ohlin model is also discussed by Roemer (1983), where every nation's common welfare objective is to maximize its national income.

orem 2). On the contrary, if an essential technical difference exists among sectors, there is a large class of initial endowments of financial capital, under each of which a free trade equilibrium with UEL exists (see Theorem 3). Finally, unlike Roemer (1982, chapter 1), any equilibrium price vector in those economies can be characterized as being associated with an equal interest rate and an equal wage rate among nations (see Theorem 1).

The remainder of the paper is organized as follows. Section 2 presents the basic model and an equilibrium notion. Section 3 defines the formulation of exploitation as UEL. Section 4 discusses the existence and characterization of free trade equilibria with as well as without UEL. Finally, section 5 concludes.

2 A Basic Model

Let $\mathcal{N}$ be the set of agents (nations), with cardinality N; there are $n \geqq 2$ commodities. An economy comprises a set of agents, $\mathcal{N} = \{1, .., N\}$, with a generic element $\nu \in \mathcal{N}$, and n types of (purely private) commodities are transferable in the market. The production technology, commonly accessible by any agent, is represented by a Leontief production technique, (A, L), where A is an $n \times n$ non-negative square matrix of the material input coefficients and L is a $1 \times n$ positive vector of the labor input coefficients. Here, A is assumed to be productive and indecomposable.[3] For the sake of simplicity, let us assume that for each production period, the maximal amount of labor supply by every agent is equal to unity and there is no difference in labor skills (human capital) among agents. Let $b \in \mathbb{R}^n_{++}$ be the *subsistence consumption bundle* that every citizen in every nation must consume for his/her survival in one period of production, regardless of whether he/she supplies labor. For the sake of simplicity, each nation also has the same population size, normalized to unity. Let $\overline{\omega} \in \mathbb{R}^n_{++}$ be the world endowments of material capital goods at the beginning of the initial period of production.

Assume $\overline{\omega} \geqq A[I-A]^{-1}(Nb)$ and $L[I-A]^{-1}(Nb) < N$.[4] Note that $A[I-A]^{-1}(Nb)$ represents the minimal level of capital stocks necessary for

[3]Let K be the index set of A's dimension. Then, A is said to be *decomposable* if there is a pair of I and J such that $K = I \cup J$, $I \cap J = \varnothing$, $I, J \neq \varnothing$, and $a_{ij} = 0$ for $i \in I, j \in J$. If A is *indecomposable*, then it has at least one non-zero off-diagonal entry in every row and column.

[4]For all vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, 2, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, 2, \ldots, n)$.

the survival of the economy. Both assumptions imply that the production of the aggregate amount of subsistence consumption bundles is technologically feasible in this economy. Moreover, assume $\overline{\omega} \leq A[I-A]^{-1}(Nb) + Nb$, which implies a *non-free lunch* in the initial period. Every national economy has the common consumption space $C \equiv \left\{c \in \mathbb{R}^n_+ \mid c \geqq b\right\} \times [0,1]$ and the common welfare function $u : C \to \mathbb{R}$, defined as follows: for each $(c,l) \in C$,

$$u(c,l) = 1 - l.$$

That is, no nation is concerned about an increase in consumption goods beyond the subsistence level, b, but nations evaluate their social welfare in terms of the increase in free hours (leisure time), once b is guaranteed. An *international economy* is thus defined by the profile $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$, which we call a *subsistence (international) economic environment.*

In addition, we explicitly take the time structure of production. Hence, the capital goods available in the present period of production cannot exceed the amount of capital goods accumulated until the end of the preceding period of production. Moreover, the time structure of production is given as follows:

(1) Given the market prices $p_{t-1} \geq \mathbf{0}$ at the beginning of period t, each nation, $\nu \in \mathcal{N}$, purchases, under the constraint of its wealth endowment, $p_{t-1}\omega^\nu_t$, capital goods Ax^ν_t as production inputs in the present period. Each nation also purchases the commodities δ^ν_t to sell, for speculative purposes, at the end of the present period;

(2) Each nation is engaged in the production activity of period t by inputting labor, Lx^ν_t, and the purchased capital goods, Ax^ν_t;

(3) The production activity is completed and x^ν_t is produced as an output at the end of this period. Then, in goods markets with market prices $p_t \geq \mathbf{0}$, each nation earns the revenue $(p_t x^\nu_t + p_t\delta^\nu_t)$ derived from the output x^ν_t as well as the speculative commodity bundle δ^ν_t. The nation uses this revenue to purchase the bundle b for consumption at the end of this period and the capital stock ω^ν_{t+1} for production in the next period. Therefore, the wealth endowment carried over to the next period, $t+1$, is $p_t\omega^\nu_{t+1}$.

A model of international trade endowed with the above-mentioned time structure is called a *Marxian Heckscher–Ohlin model of international trade* herein. Note that given the profile of the wage rate and interest rate in ν's domestic markets in period t, $(w_t^\nu, r_t^\nu) \geq (0,0)$, each ν distributes exhaustively the net revenue from the production activity $p_t x_t^\nu - p_{t-1} A x_t^\nu$ as the total wage $w_t^\nu L x_t^\nu$ and the total interest income $r_t^\nu p_{t-1} A x_t^\nu$ through the domestic market.

Given a price system, $\left(\{p_{t-1}, p_t\}; (w_t^\nu, r_t^\nu)_{\nu \in \mathcal{N}}\right)$, in period t, each nation, $\nu \in \mathcal{N}$, solves the following program (MP_t^ν):

$$
\begin{aligned}
(MP_t^\nu) \quad & \min_{x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu \in \mathbb{R}_+^n} \; l_t^\nu \\
\text{subject to } \; p_t x_t^\nu + p_t \delta_t^\nu &\geqq p_t b + p_t \omega_{t+1}^\nu; \\
p_t x_t^\nu - p_{t-1} A x_t^\nu &= w_t^\nu L x_t^\nu + r_t^\nu p_{t-1} A x_t^\nu; \\
l_t^\nu &= L x_t^\nu \leqq 1; \\
p_{t-1} \delta_t^\nu + p_{t-1} A x_t^\nu &= p_{t-1} \omega_t^\nu; \\
p_t \omega_{t+1}^\nu &\geqq p_{t-1} \omega_t^\nu.
\end{aligned}
$$

We denote the set of solutions to the optimization program (MP_t^ν) of each nation, ν, in period t by $\mathcal{O}_t^\nu\left(\{p_{t-1}, p_t\}; (w_t^\nu, r_t^\nu)_{\nu \in \mathcal{N}}\right)$.

For the sake of simplicity, we focus on the subset of equilibria in which prices remain constant over time, i.e., $p_t = p_{t+1} = p^*$. In this case, nations are indifferent to the selection of the speculative commodity bundle δ_t^ν whenever the budget constraint is met. Moreover, in this case, any $\omega_{t+1}^\nu \in \mathbb{R}_+^n$ satisfying $p^* \omega_{t+1}^\nu = p^* \omega_t^\nu$ is an optimal selection, and then $p^* x_t^{*\nu} - p^* A x_t^{*\nu} = p^* b$ holds at $\left(x_t^{*\nu}, \delta_t^\nu, \omega_{t+1}^\nu\right) \in \mathcal{O}_t^\nu\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right)$ for any $\delta_t^\nu, \omega_{t+1}^\nu \in \mathbb{R}_+^n$ satisfying $p^* \delta_t^\nu + p^* A x_t^{*\nu} = p^* \omega_t^\nu = p^* \omega_{t+1}^\nu$. Because of these, we can remove the elements $\delta_t^\nu, \omega_{t+1}^\nu$ from $\mathcal{O}_t^\nu\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right)$.

Definition 1: A *reproducible solution* (RS) for a subsistence economy $\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu \in \mathcal{N}} \rangle$ in period t is a price vector $\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right)$ and an associated profile of actions $(x_t^{*\nu})_{\nu \in \mathcal{N}}$ such that:
(i) for each $\nu \in \mathcal{N}$, $x_t^{*\nu} \in \mathcal{O}_t^\nu\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right)$ holds for all t;
(ii) $Nb + \sum_{\nu \in \mathcal{N}} \omega_{t+1}^\nu \leqq \sum_{\nu \in \mathcal{N}} x_t^{*\nu}$ for all t;
(iii) $A\left(\sum_{\nu \in \mathcal{N}} x_t^{*\nu}\right) \leqq \sum_{\nu \in \mathcal{N}} \omega_t^\nu$ for all t;
(iv) $\sum_{\nu \in \mathcal{N}} \omega_{t+1}^\nu \geqq \sum_{\nu \in \mathcal{N}} \omega_t^\nu$.

Definition 1 states that in an RS, taking the price system $\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right)$, every nation chooses its own optimal action in each period (condition (i)); aggregate gross outputs are sufficient to meet the aggregate demand of subsistence consumption bundles in each period and the aggregate capital stock invested for the next period (condition (ii)); the aggregate activities of production are feasible under the stock of capital goods in each period (condition (iii)); and the aggregate capital endowment $\sum_{\nu \in \mathcal{N}} \omega_t^\nu$ in each period is at least reproduced and carried over for the production of the next period (condition (iv)). Note that according to conditions (ii), (iii), and (iv) of Definition 1, $\sum_{\nu \in \mathcal{N}} x_t^{*\nu} - A\left(\sum_{\nu \in \mathcal{N}} x_t^{*\nu}\right) \geqq Nb$ holds (which is called condition (ii')). Therefore, in every period t, aggregate net outputs are sufficient to meet the aggregate demand of subsistence consumption bundles.

We next show that any allocation at an RS is Pareto efficient. As a preliminary step, let us define:

Definition 2: Given a subsistence economy without a labor or capital market, $\langle \mathcal{N}, (A, L, u), \omega_t \rangle$, $\left((x_t^\nu)_{\nu \in \mathcal{N}}, \omega_{t+1} \right) \in \mathbb{R}_+^{nN} \times \mathbb{R}_+^n$ is a *feasible allocation in period* t if and only if:
(1) $\sum_{\nu \in \mathcal{N}} x_t^\nu \geqq Nb + \omega_{t+1}$;
(2) $A\left(\sum_{\nu \in \mathcal{N}} x_t^\nu\right) \leqq \omega_t$;
(3) $\omega_{t+1} \geqq \omega_t$; and
(4) $Lx_t^\nu \in [0,1]$ $(\forall \nu \in \mathcal{N})$.

Definition 3: Given a subsistence economy without a labor or capital market, $\langle \mathcal{N}, (A, L, u), \omega_t \rangle$, $\left((x_t^\nu)_{\nu \in \mathcal{N}}, \omega_{t+1} \right) \in \mathbb{R}_+^{nN} \times \mathbb{R}_+^n$ is a *Pareto efficient allocation in period* t if and only if it is feasible, and there is no other feasible allocation $\left((x'^\nu)_{\nu \in \mathcal{N}}, \omega'_{t+1} \right) \in \mathbb{R}_+^{nN} \times \mathbb{R}_+^n$ such that $L\left(\sum_{\nu \in \mathcal{N}} x_t'^\nu\right) < L\left(\sum_{\nu \in \mathcal{N}} x_t^\nu\right)$.

Given $\overline{\omega} \geqq A[I-A]^{-1}(Nb)$, note that any Pareto efficient allocation $\left((x_t^\nu)_{\nu \in \mathcal{N}}, \omega_{t+1} \right)$ in period t is characterized by $\omega_{t+1} = \overline{\omega}$, $\left(\sum_{\nu \in \mathcal{N}} x_t^\nu\right) = [I-A]^{-1}(Nb)$, and $L\left(\sum_{\nu \in \mathcal{N}} x_t^\nu\right) = Nvb$, where $v \equiv L[I-A]^{-1}$.

Proposition 1: Given an economy $\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu \in \mathcal{N}} \rangle$ with $\sum_{\nu \in \mathcal{N}} \omega_0^\nu = \overline{\omega}$, let $\langle p; (w_t^\nu, r_t^\nu)_{\nu \in \mathcal{N}}, (x_t^\nu)_{\nu \in \mathcal{N}} \rangle$ be an RS in period t. Then, $(x_t^\nu)_{\nu \in \mathcal{N}}$ is Pareto efficient.

Proof. As shown by Roemer (1982, chapter 1), $0 < Lx_t^\nu$ holds for any $\nu \in \mathcal{N}$, since there is no labor market. From the definition of (MP_t^ν), $p\omega_{t+1}^\nu = p\omega_t^\nu$ and $px_t^\nu - pAx_t^\nu = pb$ hold, as argued when (MP_t^ν) is defined. Therefore, $p[I-A]\left(\sum_{\nu \in \mathcal{N}} x_t^\nu\right) = Npb$ and $p\omega_{t+1} = p\omega_t$ hold. According to Definition 1, $\sum_{\nu \in \mathcal{N}} x_t^\nu - A\left(\sum_{\nu \in \mathcal{N}} x_t^\nu\right) \geqq Nb$, meaning that $\sum_{\nu \in \mathcal{N}} x_t^\nu \geqq [I-A]^{-1}(Nb) > \mathbf{0}$ because $[I-A]^{-1} > \mathbf{0}$. Then, from $px_t^\nu - pAx_t^\nu = pb$ for any $\nu \in \mathcal{N}$,

$p[I-A] > \mathbf{0}$. (Indeed, if $p_j - pA\mathbf{e}_j \leqq 0$, where $\mathbf{e}_j$ denotes the j-th unit vector, for some commodity j, then $x^{\nu}_{jt} = 0$ holds for any $\nu \in \mathcal{N}$ by optimality, which is a contradiction.) Thus, $p > \mathbf{0}$ because $[I-A]^{-1} > \mathbf{0}$. Then, $\omega_{t+1} = \omega_t = \overline{\omega}$ holds according to Definition 1(iv), and from $p[I-A]\left(\sum_{\nu\in\mathcal{N}} x^{\nu}_t\right) = Npb$, $[I-A]\left(\sum_{\nu\in\mathcal{N}} x^{\nu}_t\right) = Nb$ holds, meaning that $\left(\sum_{\nu\in\mathcal{N}} x^{\nu}_t\right) = [I-A]^{-1}(Nb)$ and $L\left(\sum_{\nu\in\mathcal{N}} x^{\nu}_t\right) = Nvb$. This finding implies that $(x^{\nu}_t)_{\nu\in\mathcal{N}}$ is Pareto efficient. □

As shown in the above proof, $\omega_{t+1} = \omega_t$ holds for any RS.

In what follows, we devote special attention to the subset of RSs with *incomplete specialization* in which each $\nu \in \mathcal{N}$ produces all commodities:[5]

Definition 4: An *RS with incomplete specialization* for a subsistence economy $\langle \mathcal{N}, (A, L, u), (\omega^{\nu}_0)_{\nu\in\mathcal{N}}\rangle$ in period t is an RS $\langle p^*; (w^{\nu*}_t, r^{\nu*}_t)_{\nu\in\mathcal{N}}, (x^{*\nu}_t)_{\nu\in\mathcal{N}}\rangle$ such that for each $\nu \in \mathcal{N}$, $x^{*\nu}_t \in \mathbb{R}^n_{++}$ holds.

At an RS with incomplete specialization, $p^* = (1 + r^{\nu*}_t)\, p^*A + w^{\nu*}_t L$ holds for every $\nu \in \mathcal{N}$ because $x^{*\nu}_t > \mathbf{0}$. Therefore, $p^* > \mathbf{0}$ since $L > \mathbf{0}$.

The following theorem provides the factor price equalization in subsistence economies with Leontief production techniques.

Theorem 1 [Factor Price Equalization Theorem]: Given an economy $\langle \mathcal{N}, (A, L, u), (\omega^{\nu}_0)_{\nu\in\mathcal{N}}\rangle$, let $\langle p^*; (w^{\nu*}_t, r^{\nu*}_t), (x^{*\nu}_t)_{\nu\in\mathcal{N}}\rangle$ be an RS with incomplete specialization. Suppose that there is at least one pair of $i, j = 1, \ldots, n$ such that $i \neq j$ and $\frac{p^*A\mathbf{e}_i}{L_i} \neq \frac{p^*A\mathbf{e}_j}{L_j}$, where $\mathbf{e}_i$ denotes the i-th unit vector. Then, $(w^{\nu*}_t, r^{\nu*}_t) = \left(w^{\nu'*}_t, r^{\nu'*}_t\right)$ for all $\nu, \nu' \in \mathcal{N}$.

Proof. Note that at an RS with incomplete specialization, we have for each $\nu \in \mathcal{N}$,

$$p^*[I-A] = r^{\nu*}_t p^*A + w^{\nu*}_t L.$$

[5] For a typical international economic environment with two commodities, the equilibrium notion of incomplete specialization is naturally defined such that each nation produces both commodities. When considering the case with three or more types of commodities, we may have two extensions of the incomplete specialization notion: one extension is that each nation produces all commodities and the other is that each nation produces at least two types of commodities. In this paper, we adopt the former extension.

From the proof of Proposition 1, we know that $p^* [I - A] > \mathbf{0}$ and $p^* > \mathbf{0}$, which also implies that $p^* A > \mathbf{0}$ by indecomposability of A. Then, we obtain that, for each $\nu, \nu' \in \mathcal{N}$,

$$\left(r_t^{\nu *} - r_t^{\nu' *} \right) p^* A + \left(w_t^{\nu *} - w_t^{\nu' *} \right) L = \mathbf{0}.$$

Take $i, j = 1, \ldots, n$ such that $i \neq j$ and $\frac{p^* A \mathbf{e}_i}{L_i} \neq \frac{p^* A \mathbf{e}_j}{L_j}$. The above system of equations implies that for each $\nu, \nu' \in \mathcal{N}$,

$$\left(r_t^{\nu *} - r_t^{\nu' *}, w_t^{\nu *} - w_t^{\nu' *} \right) \begin{bmatrix} p^* A \mathbf{e}_i & p^* A \mathbf{e}_j \\ L_i & L_j \end{bmatrix} = (0, 0),$$

where L_i denotes the i-th element of $L > \mathbf{0}$. Since $\frac{p^* A \mathbf{e}_i}{L_i} \neq \frac{p^* A \mathbf{e}_j}{L_j}$ for these i, j, we have $p^* A \mathbf{e}_i \cdot L_j - p^* A \mathbf{e}_j \cdot L_i \neq 0$. Then, the matrix $\begin{bmatrix} p^* A \mathbf{e}_i & p^* A \mathbf{e}_j \\ L_i & L_j \end{bmatrix}$ is non-singular, and hence the row vectors $(p^* A \mathbf{e}_i, p^* A \mathbf{e}_j)$ and (L_i, L_j) are linearly independent. Thus, $(r_t^{\nu *}, w_t^{\nu *}) = \left(r_t^{\nu' *}, w_t^{\nu' *} \right)$. Note that this result follows with respect to each $\nu, \nu' \in \mathcal{N}$. Therefore, by fixing i, j, we have that for each $\nu'' \in \mathcal{N} \setminus \{\nu'\}$,

$$\left(r_t^{\nu *} - r_t^{\nu'' *}, w_t^{\nu *} - w_t^{\nu'' *} \right) \begin{bmatrix} p^* A \mathbf{e}_i & p^* A \mathbf{e}_j \\ L_i & L_j \end{bmatrix} = \mathbf{0},$$

which implies $(r_t^{\nu *}, w_t^{\nu *}) = \left(r_t^{\nu'' *}, w_t^{\nu'' *} \right)$. Thus, $(r_t^{\nu *}, w_t^{\nu *}) = \left(r_t^{\nu' *}, w_t^{\nu' *} \right)$ for all $\nu, \nu' \in \mathcal{N}$. $\square$

Note that the above factor price equalization holds, since no technical choice among multiple alternative Leontief production techniques is allowed. However, if such a technical choice were introduced, factor price equalization may not generally hold, as Metcalfe and Steedman (1972, 1973) discuss.

3 Exploitation as UEL

By noting that condition (ii) of Definition 1 is reduced to (ii′), as shown in the previous section, the notion of labor exploitation in subsistence international economies is formally defined as follows:

Definition 5: For any subsistence economy $\left\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu \in \mathcal{N}} \right\rangle$, let

$\left\langle p^*; (w_t^{\nu *}, r_t^{\nu *})_{\nu \in \mathcal{N}}, (x_t^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$ be an RS in period t. Then, the amount of socially necessary labor required to produce b as a net output is

$$\frac{1}{N} L \left(\sum_{\nu \in \mathcal{N}} x_t^{*\nu} \right) = vb = L \left[I - A \right]^{-1} b,$$

where $v = L \left[I - A \right]^{-1}$ is called the *labor value vector*. Moreover, for each nation $\nu \in \mathcal{N}$, the supply of labor hours to earn revenue $p^* b$ for its own survival is $Lx^{*\nu}$, which implies

$$\nu \text{ is an } \textit{exploiting} \text{ nation} \Longleftrightarrow Lx^{*\nu} < vb;$$
$$\nu \text{ is an } \textit{exploited} \text{ nation} \Longleftrightarrow Lx^{*\nu} > vb.$$

Denote the sets of exploiters and exploited respectively by $\mathcal{N}^{ter}$ and $\mathcal{N}^{ted}$.

Definition 6 [Roemer (1982, Definitions 1.3 and 1.4)]**:** For any subsistence economy, $\left\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu \in \mathcal{N}} \right\rangle$, an RS in period t, $\left\langle p^*; (w_t^{\nu *}, r_t^{\nu *})_{\nu \in \mathcal{N}}, (x_t^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$, is *inegalitarian* if and only if $\mathcal{N}^{ter} \neq \varnothing$ and $\mathcal{N}^{ted} \neq \varnothing$.

Thus, if an RS in period t is inegalitarian, it involves UEL. By contrast, we can state that for any subsistence economy $\left\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu \in \mathcal{N}} \right\rangle$, an RS in period t, $\left\langle p^*; (w_t^{\nu *}, r_t^{\nu *})_{\nu \in \mathcal{N}}, (x_t^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$, is *egalitarian* if and only if $Lx_t^{*\nu} = Lx_t^{*\nu'}$ for all $\nu, \nu' \in \mathcal{N}$.

Definition 5 presents a standard Okishio (1963)–Morishima (1973) form of exploitation as UEL in subsistence economies with simple Leontief production techniques. On the contrary, if a more general class of production economies is considered, many alternative definitions of exploitation have been proposed other than the Okishio–Morishima form, as discussed by Veneziani and Yoshihara (2014a, 2014b), Yoshihara (2010), and Yoshihara and Veneziani (2009). However, all such alternative exploitation forms are reduced to Definition 5 within the restricted class of subsistence economies with simple Leontief production techniques. Therefore, the following analysis on the existence and characterization of inegalitarian RSs is free from debate on the proper definitions of labor exploitation.

4 Existence and Characterization of Free Trade Equilibria with UEL

Consider the existence problem of *inegalitarian RS*s. In the following discussion, without loss of generality, we remove any subscript "t" whenever RSs are presented.

As discussed in the proof of Proposition 1, the equilibrium price vector p should be positive with $p - pA > \mathbf{0}$ at an RS, and its associated social production activity vector is $x^* \equiv (I - A)^{-1} (Nb) > \mathbf{0}$. Let $\frac{1}{1+R}$ with $R > 0$ be the Frobenius eigenvalue of the productive and indecomposable matrix A. Note that $R > 0$ follows from the productiveness of A.

Let us begin with characterizing the domain of economies in which only egalitarian RSs exist.

Lemma 1: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$, let a price vector $p^r = (1 + r) p^r A + wL > \mathbf{0}$ be associated with its unique equal interest rate $r > 0$. Then, p^r is the unique Frobenius eigenvector of A associated with the Frobenius eigenvalue $\frac{1}{1+R}$ such that p^r is proportional to the vector of labor values v if and only if $p^r A$ and L are linearly dependent. By contrast, let $\overline{p} = (1 + R) \overline{p} A > \mathbf{0}$ be the unique Frobenius eigenvector of A associated with the Frobenius eigenvalue $\frac{1}{1+R}$. Then, for any $r' \in [0, R]$, $\overline{p}$ is the price vector associated with the equal interest rate r', that is, $\overline{p} = (1 + r') \overline{p} A + \overline{w} L$ holds for some $\overline{w} > 0$ and is proportional to the vector of labor values v if and only if $\overline{p} A$ and L are linearly dependent.

Proof. Let $p^r = (1 + r) p^r A + wL > \mathbf{0}$ be such that $p^r A$ and L are linearly dependent. This finding implies that there exists $\varsigma > 0$ such that $p^r A = \varsigma L$. Therefore, $p^r (I - A) = (r\varsigma + w) L$. Thus, $p^r = (r\varsigma + w) v$, which implies that p^r is proportional to the vector of labor values v. In addition, it follows that $p^r (I - A) = (r + w\varsigma^{-1}) p^r A$, meaning that $p^r (I - (1 + r + w\varsigma^{-1}) A) = \mathbf{0}$. Therefore, since A is indecomposable, $p^r > \mathbf{0}$ is the Frobenius eigenvector of A unique up to scale, and $[1 + (r + w\varsigma^{-1})]^{-1}$ can be the Frobenius eigenvalue of A. By contrast, if $p^r A$ and L are linearly independent, then the vectors p^r and $p^r A$ must be linearly independent. Then, it is impossible to have $p^r = (1 + R) p^r A$ for some $(1 + R) > 0$, which implies that p^r can never be the Frobenius eigenvector of A.

Let $\overline{p} = (1+R)\,\overline{p}A > \mathbf{0}$ be the unique Frobenius eigenvector of A associated with the Frobenius eigenvalue $\frac{1}{1+R}$ such that $\overline{p}A$ and L are linearly dependent. Therefore, there exists $\varsigma > 0$ such that $\overline{p}A = \varsigma L$. Then, for any $r' \in [0, R)$, $(R - r')\,\overline{p}A = (R - r')\,\varsigma L$ holds, meaning that $\overline{p} = (1 + r')\,\overline{p}A + \overline{w}L$ for $\overline{w} \equiv (R - r')\,\varsigma > 0$. Since A is indecomposable, $L\,(I - (1 + r')\,A)^{-1} > \mathbf{0}$ exists such that $\overline{p} = \overline{w}L\,(I - (1 + r')\,A)^{-1}$ holds. Moreover, since $\overline{p}\,(I - A) = R\overline{p}A = R\varsigma L$, $\overline{p} = R\varsigma v$ holds, so that $\overline{p}$ is proportional to the vector of labor values v. By contrast, if $\overline{p}A$ and L are linearly independent, it is impossible to have $\overline{p} = (1 + r')\,\overline{p}A + \overline{w}L$ for some $r' \in [0, R)$ and some $\overline{w} > 0$, since $\overline{p}$ and $\overline{p}A$ are linearly dependent by definition. $\square$

Lemma 1 suggests that if in an economy with the Leontief production technique (A, L), the unique Frobenius eigenvector $\overline{p}$ of A is linearly independent of the vector L, then for any equilibrium price vector $p^r \equiv L\,(I - (1 + r)\,A)^{-1}$ associated with an equal interest rate $r \in (0, R)$, $p^r A$ and L are linearly independent.

Lemma 1 has some interesting implications. First, it classifies subsistence economies with Leontief production techniques into two types. One type is economies in which the unique Frobenius eigenvector $\overline{p}$ of A and L are linearly dependent. In such a case, if every nation establishes a positive wage rate in its domestic factor market under the international equilibrium, a common capital–labor ratio is established among all sectors, evaluated by the corresponding equilibrium prices of commodities. Thus, in this type of economy, essentially no technical difference among sectors exists in that the capital–labor ratios are common among sectors under any equilibrium. The other type is economies in which the unique Frobenius eigenvector $\overline{p}$ of A and L are linearly independent. In this type of economy, technical differences among sectors exist in that the capital–labor ratios are not identical among sectors under any equilibrium.

Second, in combination with Theorem 1, Lemma 1 offers the following observation. According to Lemma 1, if the production technique (A, L) reveals that its unique Frobenius eigenvector $\overline{p}$ and L are linearly dependent, then no price vector is associated with an equal positive interest rate, except in cases of labor value pricing (i.e., when the price vector is proportional to the vector of labor value).

Let us examine the existence and characterization of RSs in economies with no essential technical difference among sectors. That is, let us take any subsistence economy with a Leontief production technique such that its unique Frobenius eigenvector and its labor coefficient vector are linearly dependent. Let $\triangle(\overline{\omega}) \equiv \left\{p \in \mathbb{R}^n_+ \mid p \cdot \overline{\omega} = 1\right\}$ and $\triangle(W) \equiv \left\{(W^\nu)_{\nu \in N} \in \mathbb{R}^N_{++} \mid \sum_{\nu \in \mathcal{N}} W^\nu = 1\right\}$. Then,

Theorem 2: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega}\rangle$ with $\overline{\omega} = A[I-A]^{-1}(Nb)$, at most only an *egalitarian RS* exists under the equal initial endowments of financial capital and no inegalitarian RS exists under any initial endowments of financial capital if and only if the unique Frobenius eigenvector $\overline{p} > \mathbf{0}$ of A and $L > \mathbf{0}$ are linearly dependent.

Proof. Given the technology (A, L), let us consider $\overline{p}(I - (1+R)A) = \mathbf{0}$, where $\frac{1}{1+R}$ is the unique Frobenius eigenvalue of A and $\overline{p} > \mathbf{0}$ is its associated eigenvector uniquely up to scale. Therefore, let us suppose that $\overline{p} \in \triangle(\overline{\omega})$. By definition, the row vectors $\overline{p}$ and $\overline{p}A$ are linearly dependent.

Then, if the two row vectors $\overline{p}A$ and L are linearly dependent, which is derived from the linear dependency of $\overline{p}$ and L, then a pair of $\overline{p}$ and any allocation $(x^\nu)_{\nu \in \mathcal{N}}$ satisfying $(I-A)\left(\sum_{\nu \in \mathcal{N}} x^\nu\right) = Nb$ and $Lx^\nu = vb$ can constitute an egalitarian RS in an economy with equal initial endowments of financial capital $(W^\nu)_{\nu \in \mathcal{N}} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$, and no other RS is in an economy with any initial endowments of financial capital. This situation occurs because $\overline{p}(I-A)$ and $\overline{p}A$ are linearly dependent and thus the hyperplanes $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x = \overline{p}b\right\}$ and $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}Ax = \frac{\overline{p}\overline{\omega}}{N}\right\}$ coincide. Therefore, if wealth endowments are unequal, there is at least one nation $\nu \in \mathcal{N}$ such that $W^\nu < \frac{1}{N}$, meaning that this agent's set $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}Ax \leqq W^\nu\right\}$ of capital-constrained feasible activities is included in the strictly lower contour set $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x < \overline{p}b\right\}$, which implies that this nation has no feasible production activity. By contrast, since $\overline{p}(I-A)$ and L are linearly dependent, the hyperplane $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x = \overline{p}b\right\}$ and an indifferent surface $\left\{x \in \mathbb{R}^n_+ \mid Lx = L(I-A)^{-1}b\right\}$ coincide. Therefore, any point in $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x = \overline{p}b\right\}$ can constitute a solution to the optimization program (MP^ν_t) for any nation under the equal initial endowments of financial capital $(W^\nu)_{\nu \in \mathcal{N}} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right) \in \triangle(W)$, which implies that any nation can realize $Lx^\nu = vb$ as its optimal labor supply. Therefore, $\left(\overline{p}, (x^\nu)_{\nu \in \mathcal{N}}\right) \in \triangle(\overline{\omega}) \times \mathbb{R}^{nN}_+$ with $(I-A)\left(\sum_{\nu \in \mathcal{N}} x^\nu\right) = Nb$ and $Lx^\nu = vb$ for all $\nu \in \mathcal{N}$ can constitute an egalitarian RS in an economy with $(W^\nu)_{\nu \in \mathcal{N}} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$, and there is no inegalitarian RS in such an economy.

Moreover, if $\overline{p}$ and L are linearly dependent, Lemma 1 implies that $\overline{p}$ can be any equilibrium price vector associated with any equal interest rate $r \in [0, R)$, which is proportional to the labor value vector v. In combination with the previous analysis, this finding further implies that in an economy with the linear dependency of $\overline{p}$ and L, the only available types of RSs are egalitarian associated with the equal initial endowments of financial capital, regardless of whether the associated equal interest rate is positive.

Next, let $\overline{p}$ and L be linearly independent. Then, $\overline{p}A$ and L are linearly independent, since $\overline{p}$ and $\overline{p}A$ are linearly dependent by definition. In this case, no RS corresponds to the price system $\overline{p}$ because no nation's optimal solution can constitute a feasible allocation. First, if wealth endowments are unequal, there is at least one nation $\nu \in \mathcal{N}$ such that $W^{\nu} < \frac{1}{N}$, meaning that this agent's set of capital-constrained feasible activities, $\left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}Ax \leqq W^{\nu}\right\}$, is included in the strictly lower contour set

$$\left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}\,(I - A)\,x < \overline{p}b\right\} = \left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}Ax < \frac{1}{N}\right\}.$$

Thus, there is no RS in such a case. Second, even if wealth endowments are presumed to be equal, every nation is faced with the common set of feasible activities $\left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}\,(I - A)\,x = \overline{p}b\right\}$, which is not identical to the indifference surface $\left\{x \in \mathbb{R}_{+}^{n} \mid Lx = L\,(I - A)^{-1}\,b\right\}$, meaning that every nation ν would choose the same activity $x'^{\nu} = \arg\min_{x \in \mathbb{R}_{+}^{n};\ \overline{p}(I-A)x = \overline{p}b} Lx$ to minimize its own labor supply. Note that $Lx'^{\nu} < vb$ for any $\nu \in \mathcal{N}$, since while $(I - A)^{-1}\,b > \mathbf{0}$ holds, the solution of the program $\min_{x \in \mathbb{R}_{+}^{n};\ \overline{p}(I-A)x = \overline{p}b} Lx$ should be a boundary point of $\mathbb{R}_{+}^{n}$, which implies that $(I - A)\,x'^{\nu} \ngeqq b$ holds for any $\nu \in \mathcal{N}$. Thus, the aggregate net output does not coincide with Nb.

Let us consider a case that, given $\overline{p}$ and L are linearly independent, $\widetilde{p}\,(I - (1 + r)\,A) - wL = \mathbf{0}$ for some $\widetilde{p} \in \triangle\,(\overline{\omega})$, some $r \in [0, R)$, and some $w > 0$. Note that such a price vector $(\widetilde{p}, w, r)$ exists because of the productiveness and indecomposability of A. If $\widetilde{p}A$ and L are linearly dependent, then from Lemma 1, $\widetilde{p}$ is identical to the Frobenius eigenvector of A uniquely up to scale, meaning that $\overline{p}A$ and L are linearly dependent, which is a contradiction. Thus, $\widetilde{p}A$ and L are linearly independent. Then, according to Lemma 1, $\widetilde{p}$ cannot be proportional to v. This finding implies that $r > 0$ must hold, since $\widetilde{p}\,(I - A) - wL = \mathbf{0}$ implies $\frac{\widetilde{p}}{w} = v$, where w is determined to fulfill the gap between $\widetilde{p} \in \triangle\,(\overline{\omega})$ and v. Then, since $\overline{p}$ and L are linearly independent, Theorem 3 shows that, given the suitable assignment of

$\overline{\omega}$ among nations, $(\widetilde{p}, w, r)$ can constitute an inegalitarian RS, which implies that the desired result is obtained. □

Theorem 2, combined with Lemma 1, suggests that in any subsistence economy with no essential technical difference among sectors, the only available type of free trade equilibrium is that of egalitarian RSs realized under the equal initial distribution of financial capital and, moreover, that the equilibrium prices of all such RSs are characterized by labor value pricing.

Note that Theorem 2 also suggests that the international trade of commodities could be conducted among nations under egalitarian RSs, although no nation can enjoy a strict gain from trade under such RSs. In such RSs, every nation can choose an autarkic economy in that, if preferred, it would self-produce and consume the net output b by investing equally distributed financial capital and equalized labor supply vb, which is, for every nation, indifferent to its own production activities currently implemented under such RSs. Therefore, no nation has a strong rationale to shift from the autarkic activity to the free trade equilibrium.

In the following argument, our main concern focuses on economies with technical differences among sectors, in which the existence and characterization of RSs are examined. Let us take any subsistence economy with a Leontief production technique (A, L) such that its unique Frobenius eigenvector and its labor coefficient vector are linearly independent. From Lemma 1, this case is equivalent to the case that for any $r \in (0, R)$, its associated price vector $p^r = L\left(I - (1+r)A\right)^{-1}$ has the property that p^r and L are linearly independent, which is equivalent to the property that $p^r(I - A)$ and $p^r A$ are linearly independent. Therefore, Theorem 1 suggests that in free trade equilibria, every nation's factor prices must be equalized, meaning that to examine the existence of equilibrium price vectors in free trade equilibria, it is sufficient to focus only on the types of $p^r = L\left(I - (1+r)A\right)^{-1}$.

Lemma 2: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A\left[I - A\right]^{-1}(Nb)$, for any price vector $p > \mathbf{0}$ associated with its unique equal profit rate $r \in [0, R)$, if there is no $\varsigma > 0$ such that $p(I - A) = \varsigma pA$, it follows that x^* is a solution to the following program:

$$\min_{Lx \in [0,N]} Lx, \text{ subject to } p(I - A)x \geqq pNb;\ pAx \leqq p\overline{\omega}. \ (*)$$

Proof. Let $p \in \mathbb{R}^n_{++}$ be a price vector such that there exists $r \in [0, R)$ with $p = L\left(I - (1+r)A\right)^{-1}$, where $0 < \frac{1}{1+R} < 1$ is the unique Frobenius eigenvalue associated with A. Then, define $Y_1(p) \equiv \left\{x \in \mathbb{R}^n_+ \mid p(I-A)x \geqq pNb\right\}$ and $Y_2(p) \equiv \left\{x \in \mathbb{R}^n_+ \mid pAx \leqq p\varpi\right\}$. Note that $p(I-A)x^* = pNb$ and $pAx^* = p\varpi$, thus $x^* \in Y_1(p) \cap Y_2(p)$.

If $r = 0$, then p is proportional to v, which implies that there is some $\varsigma > 0$ such that $p(I-A) = \varsigma v(I-A) = \varsigma L$, meaning that the hyperplane $\left\{x \in \mathbb{R}^n_+ \mid p(I-A)x = pNb\right\}$ and the indifferent surface $\left\{x \in \mathbb{R}^n_+ \mid Lx = Nvb\right\}$ coincide. Therefore, if $r = 0$, x^* is an optimal solution, since any activity $x \in \mathbb{R}^n_+$ with $Lx < Lx^* = Nvb$ implies $p(I-A)x < pNb$.

Next, consider $r > 0$. In this case, if $x \in Y_1(p) \cap Y_2(p)$ is $pAx \leqq p\varpi$ and $p(I-A)x > pNb$, then for some small positive vector $\varepsilon > \mathbf{0}$, $pA(x-\varepsilon) < p\varpi$, $p(I-A)(x-\varepsilon) \geqq pNb$, and $L(x-\varepsilon) < Lx$ hold. Thus, x cannot be an optimal solution. Therefore, if $x \in Y_1(p) \cap Y_2(p)$ is an optimal solution to the program (*), then $p(I-A)x = pNb$ holds. Suppose that x^* is not an optimal solution to the program (*). Then, there should be another activity vector $x' \in Y_1(p) \cap Y_2(p)$ such that $Lx' < Lx^*$. Since $p(I-A)x' = pNb = p(I-A)x^*$, $pAx' \leqq p\varpi = pAx^*$, and $Lx' < Lx^*$, it follows that $[p(I-A) - rpA - L](x' - x^*) > 0$. However, since $p(I-A) - rpA - L = \mathbf{0}$ by definition, the aforementioned inequality is impossible. Thus, there is no such x', and x^* is a solution to the program (*). $\square$

Lemma 3: Given an economy $\langle \mathcal{N}, (A, L, u), \varpi \rangle$ with $\varpi = A[I-A]^{-1}(Nb)$, let a price vector $p > 0$ be associated with its unique equal interest rate $r \in [0, R)$ such that there is no $\varsigma > 0$ satisfying $p(I-A) = \varsigma pA$. Moreover, let $x^\nu \in \mathbb{R}^n_+$ be such that $p(I-A)x^\nu = pb$. Then, there exists $\omega^\nu \equiv Ax^\nu$ such that $x^\nu$is a solution to the following program:

$$\min_{Lx \in [0,1]} Lx, \text{ subject to } p(I-A)x \geqq pb;\ pAx \leqq p\omega^\nu. \ (^{**})$$

Proof. Let $x^\nu \in \mathbb{R}^n_+$ be such that $p(I-A)x^\nu = pb$ and let $\omega^\nu \equiv Ax^\nu$. Let $Y_1^\nu(p) \equiv \left\{x \in \mathbb{R}^n_+ \mid p(I-A)x \geqq p(I-A)x^\nu\right\}$ and $Y_2^\nu(p) \equiv \left\{x \in \mathbb{R}^n_+ \mid pAx \leqq pAx^\nu\right\}$. Because of this supposition, the intersection $Y_1^\nu(p) \cap Y_2^\nu(p)$ has its interior set $int\left(Y_1^\nu(p) \cap Y_2^\nu(p)\right)$. Then, as shown in the proof of Lemma 2, for any $x' \in \left(Y_1^\nu(p) \cap Y_2^\nu(p)\right) \setminus \{x^\nu\}$, if x' is a solution to the program (**), then $p(I-A)x' = p(I-A)x^\nu$ and $pAx' \leqq pAx^\nu$. Suppose $Lx' < Lx^\nu$. Then, $[p(I-A) - rpA - L](x' - x^\nu) > 0$, which contradicts $p(I-A) - rpA - L =$

$\mathbf{0}$. Thus, $Lx^{\nu} = Lx'$ holds, since x' is a solution to the program (**), which implies x^{ν} is a solution to the program (**). $\square$

Given $\overline{\omega} = A\left[I - A\right]^{-1}(Nb)$, let

$$\overline{\Omega} \equiv \left\{ \left(\omega^{\nu}\right)_{\nu \in N} \in \mathbb{R}_{+}^{nN} \mid \sum_{\nu \in \mathcal{N}} \omega^{\nu} = \overline{\omega} \ \& \ LA^{-1}\omega^{\nu} \in [0,1] \quad (\forall \nu \in \mathcal{N}) \right\}.$$

Lemma 4: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A\left[I - A\right]^{-1}(Nb)$, let a price vector $p > \mathbf{0}$ be associated with its unique equal profit rate $r \in [0, R)$, such that there is no $\varsigma > 0$ satisfying $p(I - A) = \varsigma pA$. Then, there exists a suitable profile $\left(\omega^{\nu}\right)_{\nu \in \mathcal{N}} \in \overline{\Omega}$ such that $LA^{-1}\omega^{\nu} \in [0,1]$ and $p(I - A)A^{-1}\omega^{\nu} = pb$ for any $\nu \in \mathcal{N}$, and $\left(p, \left(A^{-1}\omega^{\nu}\right)_{\nu \in N}\right)$ constitutes an RS. In particular, if $A^{-1}\omega^{\nu'} \neq (I - A)^{-1} b$ for some $\nu' \in \mathcal{N}$, then $\left(p, \left(A^{-1}\omega^{\nu}\right)_{\nu \in N}\right)$ constitutes an inegalitarian RS if and only if $r > 0$.

Proof. From the supposition about the price vector, $p = L\left(I - (1 + r)A\right)^{-1}$ and there is no $\varsigma > 0$ such that $p(I - A) = \varsigma pA$ holds. Then, from Lemma 2, $\frac{x^*}{N}$ is a solution of $\min_{Lx \in [0,1]} Lx$ such that $p(I - A)x \geqq pb$ and $pAx \leqq \frac{p\overline{\omega}}{N}$. Take any profile $\left(\omega^{\nu}\right)_{\nu \in N} \in \overline{\Omega}$ such that $LA^{-1}\omega^{\nu} \in [0,1]$ and $p(I - A)A^{-1}\omega^{\nu} = pb$ for any $\nu \in \mathcal{N}$ and $A^{-1}\omega^{\nu'} \neq \frac{x^*}{N}$ for some $\nu' \in \mathcal{N}$. Then, for each $\nu \in \mathcal{N}$, let $x^{\nu} \equiv A^{-1}\omega^{\nu}$. Lemma 3 implies that for each $\nu \in \mathcal{N}$, x^{ν} is a solution of $\min_{Lx \in [0,1]} Lx$ such that $p(I - A)x \geqq pb$ and $pAx \leqq p\omega^{\nu}$. Since $\sum_{\nu \in N} x^{\nu} = A^{-1}\overline{\omega} = x^*$, $\left(p, \left(x^{\nu}\right)_{\nu \in N}\right)$ constitutes an RS.

Moreover, noting $\frac{x^*}{N} = (I - A)^{-1} b$, let us consider $A^{-1}\omega^{\nu'} \neq \frac{x^*}{N}$ for some $\nu' \in \mathcal{N}$. Then, owing to the setting of $p(I - A)A^{-1}\omega^{\nu'} = pb$, which is equivalent to $p(I - A)A^{-1}\omega^{\nu'} = p(I - A)\frac{x^*}{N}$, the property $A^{-1}\omega^{\nu'} \neq \frac{x^*}{N}$ implies that $p\omega^{\nu'} \neq \frac{p\overline{\omega}}{N}$ holds. Without loss of generality, let $p\omega^{\nu'} < \frac{p\overline{\omega}}{N}$. Then, there exists $p\omega^{\nu''} > \frac{p\overline{\omega}}{N}$ for another $\nu'' \in N$. Since $pb = rp\omega^{\nu} + Lx^{\nu}$ for each $\nu \in \mathcal{N}$, $Lx^{\nu'} > Lx^{\nu''}$ holds if and only if $r > 0$. This finding implies that $\left(p, \left(x^{\nu}\right)_{\nu \in N}\right)$ is an inegalitarian RS if and only if $r > 0$. $\square$

Let $\overline{\omega} = A\left[I - A\right]^{-1}(Nb)$. For each $r \in (0, R)$, let $p^r \equiv L\left(I - (1 + r)A\right)^{-1}$. Then, for each given $\theta \in [0,1]$, consider a non-negative and non-zero vector

$x \in \mathbb{R}_{+}^{n}$ to solve the following system of equations:

$$\left\{ \begin{array}{c} p^{r}Ax = \theta p^{r}\overline{\omega}; \\ p^{r}\left(I - A\right)x = p^{r}b; \\ Lx \in \left[0,1\right]. \end{array} \right.$$

Denote the set of solutions for this system of equations by $X^{r}\left(\theta\right)$ with the generic element $x^{r}\left(\theta\right)$. Then, define the following program:

$$\min_{\theta \in [0,1]} \theta \text{, subject to } X^{r}\left(\theta\right) \neq \varnothing. \text{ (***)}$$

Note that $\cup_{\theta \in [0,1]} X^{r}\left(\theta\right)$ is non-empty, since for $\theta = \frac{1}{N}$, $x\left(\theta\right) = \left(I - A\right)^{-1} b$ is the solution.

Let $\theta^{r,b} \equiv \frac{p^{r}b}{p^{r}\overline{\omega}}$. Then, the system of equations is reduced to the form $p^{r}\left(I - A\right)x = p^{r}b = p^{r}Ax$ with $Lx \in [0,1]$. Let $\overline{X}\left(\theta^{r,b}\right)$ be the set of solutions satisfying $p^{r}\left(I - A\right)x = p^{r}b = p^{r}Ax$. The set $\overline{X}\left(\theta^{r,b}\right)$ is non-empty and compact. Since $pb > 0$, $x = \mathbf{0} \notin \overline{X}\left(\theta^{r,b}\right)$. Note that for any $\theta < \theta^{r,b}$, no $x \in \mathbb{R}_{+}^{n}$ satisfies $p^{r}Ax = \theta p^{r}\overline{\omega}$ and $p^{r}\left(I - A\right)x = p^{r}b$, since in such a case, the set of non-negative vectors satisfying $p^{r}Ax = \theta p^{r}\overline{\omega}$ is contained by the strictly lower contour set of the hyperplane $p^{r}\left(I - A\right)x = p^{r}b$. By contrast, for any $\theta \geqq \theta^{r,b}$, there is a non-empty set $\overline{X}\left(\theta\right) \subseteq \mathbb{R}_{+}^{n}$ such that for any $x \in \overline{X}\left(\theta\right)$, $p^{r}Ax = \theta p^{r}\overline{\omega}$ and $p^{r}\left(I - A\right)x = p^{r}b$ hold.

Since each $\overline{X}\left(\theta\right)$ is compact, we can find the solution to the program $\min_{x \in \overline{X}(\theta)} Lx$ whenever $\overline{X}\left(\theta\right) \neq \varnothing$. Therefore, the program (***) can be reduced to the following form:

$$\min_{\theta \in \left[\theta^{r,b},1\right]} \theta \text{, subject to } \min_{x \in \overline{X}(\theta)} Lx \leqq 1. \text{ (****)}$$

Since $\min_{x \in \overline{X}(\theta)} Lx$ is decreasing with respect to $\theta \in \left[\theta^{r,b},1\right]$ and $\min_{x \in \overline{X}\left(\frac{1}{N}\right)} Lx = vb < 1$, there exists $\theta^{r} \in \left[\theta^{r,b}, \frac{1}{N}\right]$, which is the solution to the program (****).

Then, we can define $\underline{\theta} \equiv \inf_{r \in (0,R)} \theta^{r}$. Let

$$\triangle_{\underline{\theta}}\left(W\right) \equiv \left\{ \left(W^{\nu}\right)_{\nu \in N} \in \triangle\left(W\right) \mid \min_{\nu \in \mathcal{N}} W^{\nu} > \underline{\theta} \right\}.$$

Now, we are ready to show the existence and characterization of inegalitarian RSs.

Theorem 3: Let an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A[I-A]^{-1}(Nb)$ be such that the unique Frobenius eigenvector of A and $L > \mathbf{0}$ are linearly independent. Then, for any profile $(W^\nu)_{\nu \in \mathcal{N}} \in \triangle_{\underline{\theta}}(W)$ of the initial endowments of financial capital, there exist $(p^*, w^*, r^*) \in \triangle(\overline{\omega}) \times \mathbb{R}_{++} \times \mathbb{R}_+$ and $(\omega^\nu)_{\nu \in \mathcal{N}} \in \overline{\Omega}$ by which $\langle p^*; (w^*, r^*), (x^{*\nu})_{\nu \in \mathcal{N}} \rangle$ constitutes an RS with incomplete specialization for the subsistence private ownership economy $\langle \mathcal{N}, (A, L, u), (\omega^\nu)_{\nu \in \mathcal{N}} \rangle$, such that the following property is satisfied: this RS is inegalitarian if and only if the profile $(W^\nu)_{\nu \in \mathcal{N}}$ is unequal initial endowments of financial capital and the equilibrium price vector $p^* > \mathbf{0}$ is associated with a positive equal interest rate $r^* > 0$.

Proof. Given the technology (A, L), let us consider $\overline{p}(I - (1+R)A) = \mathbf{0}$, where $\frac{1}{1+R}$ is the unique Frobenius eigenvalue of A and $\overline{p} > \mathbf{0}$ is its associated eigenvector unique up to scale. Therefore, let us suppose that $\overline{p} \in \triangle(\overline{\omega})$. By definition, the row vectors $\overline{p}$ and $\overline{p}A$ are linearly dependent. From this supposition, it follows that $\overline{p}$ and L are linearly independent. Then, $\overline{p}A$ and L are linearly independent. From Lemma 1, $\overline{p}$ cannot be a positive price vector associated with a non-negative positive interest rate $r \in [0, R)$. In addition, because of the uniqueness of the Frobenius eigenvector of indecomposable A, no positive price vector $p \in \triangle(\overline{\omega})$ associated with an equal interest rate $r \in [0, R)$ can be the Frobenius eigenvector, since the Frobenius eigenvector $\overline{p} > \mathbf{0}$ cannot be associated with a non-negative interest rate $r \in [0, R)$. Therefore, according to Lemma 1, for any such p, pA and L are linearly independent.

Let us consider $r = 0$, meaning that $\widetilde{p}(I - A) - wL = \mathbf{0}$ for some $\widetilde{p} \in \triangle(\overline{\omega})$. Then, as before, $\widetilde{p}$ is proportional to the labor value vector v, $\frac{\widetilde{p}}{\widetilde{w}} = v$, where $\widetilde{w}$ is determined to fulfill the gap between $\widetilde{p} \in \triangle(\overline{\omega})$ and v. Hence, there exists an RS even under unequal initial endowments of financial capital and such an RS is always egalitarian. Indeed, in this case, since $\widetilde{p}(I - A)$ and L are linearly dependent according to $\widetilde{p}(I - A) - wL = \mathbf{0}$, the hyperplane $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I - A)x = \widetilde{p}b\right\}$ and indifference surface $\left\{x \in \mathbb{R}^n_+ \mid Lx = vb\right\}$ coincide. Hence, for each nation $\nu \in \mathcal{N}$, the intersection of $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I - A)x = \widetilde{p}b\right\}$ and $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}Ax \leqq W^\nu\right\}$ constitutes the set of optimal activities at the price $\widetilde{p} \in \triangle(\overline{\omega})$. Therefore, for any $(x^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_+$ with $(I - A)\left(\sum_{\nu \in \mathcal{N}} x^\nu\right) = Nb$, $\left(\widetilde{p}, (x^\nu)_{\nu \in \mathcal{N}}\right)$ can constitute an RS if and only if there exists an assignment $(W^\nu)_{\nu \in \mathcal{N}}$ such that

$$x^\nu \in \left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I - A)x = \widetilde{p}b\right\} \cap \left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}Ax \leqq W^\nu\right\}.$$

This finding also implies that $Lx^{\nu} = vb < 1$ for any $\nu \in \mathcal{N}$, from the identity of $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}\,(I-A)\,x = \widetilde{p}b\right\}$ with $\left\{x \in \mathbb{R}^n_+ \mid Lx = vb\right\}$, regardless of whether $(W^{\nu})_{\nu\in\mathcal{N}}$ is unequal. For instance, if $(W^{\nu})_{\nu\in\mathcal{N}}$ is equalized, then $x^{\nu} = (I-A)^{-1}\,b$ is the unique optimal solution for any agent $\nu \in \mathcal{N}$. If $(W^{\nu})_{\nu\in\mathcal{N}} \in \triangle(W)$ represents an unequal distribution, but it meets the property that $\min_{\nu\in\mathcal{N}} W^{\nu} \geqq \min_{\omega\leqq\overline{\omega};\ LA^{-1}\omega=vb} \widetilde{p}\omega$, then there exists a suitable assignment $(\omega^{\nu})_{\nu\in\mathcal{N}}$ of $\overline{\omega}$ such that $\widetilde{p}\omega^{\nu} = W^{\nu}$ for any $\nu \in \mathcal{N}$ and for some $(x^{\nu})_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$, $x^{\nu} = A^{-1}\omega^{\nu}$. By definition, $\widetilde{p}Ax^{\nu} = \widetilde{p}\omega^{\nu}$ and $Lx^{\nu} = vb$ for all $\nu \in \mathcal{N}$, which also implies $\widetilde{p}\,(I-A)\,x^{\nu} = \widetilde{p}b$ for all $\nu \in \mathcal{N}$. Thus, this RS is egalitarian, although its initial distribution of financial capital is unequal. In such an equilibrium, international division of labor is generated by the difference in the capital–labor ratios among nations. Because every nation supplies the same amount of labor vb, $W^{\nu} > W^{\nu'}$ implies that ν is specialized to a more capital-intensive production activity than ν' is.

Let us consider $r \in (0, R)$, which allows us to find a unique price vector $p^r = L\,(I - (1+r)\,A)^{-1} > \mathbf{0}$ and $p^r A$ and L are linearly independent, according to Lemma 1. By definition, $(W^{\nu})_{\nu\in\mathcal{N}} \in \triangle_{\underline{\theta}}(W)$ implies that there exists $r^* \in (0, R)$ such that $\min_{\nu\in\mathcal{N}} W^{\nu} \geqq \theta^{r^*}$ and for some $p^{r^*} = L\,(I - (1+r^*)\,A)^{-1}$, there exists $x\,(r^*) \in \mathbb{R}^n_{++}$ such that $p^{r^*}Ax\,(r^*) = \theta^{r^*}p^{r^*}\overline{\omega}$, $p^{r^*}\,(I-A)\,x\,(r^*) = p^{r^*}b$, and $Lx\,(r^*) \in [0,1]$. Then, there exist $p^* \equiv \frac{1}{L(I-(1+r^*)A)^{-1}\overline{\omega}}L\,(I-(1+r^*)\,A)^{-1} \in \triangle(\overline{\omega})$ and $(\omega^{\nu})_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$ such that $p^*\omega^{\nu} = W^{\nu}$ for any $\nu \in \mathcal{N}$. Let us define $w^* > 0$ to fulfill the gap between p^* and p^{r^*} as $p^* = w^*p^{r^*}$.

Since $\min_{\nu\in\mathcal{N}} W^{\nu} > \underline{\theta}$, there exists $(x^{*\nu})_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_{++}$ such that for each $\nu \in \mathcal{N}$, $p^*Ax^{*\nu} = p^*\omega^{\nu}$, $p^*\,(I-A)\,x^{*\nu} = p^*b$, and $Lx^{*\nu} \leqq 1$. From Lemma 3, for a profile $(\omega^{*\nu})_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_{++}$ with $\omega^{*\nu} \equiv Ax^{*\nu}$ for each $\nu \in \mathcal{N}$, $x^{*\nu}$ is a solution to the program (**). Then, since $p^*\omega^{\nu} = p^*\omega^{*\nu}$ for each $\nu \in \mathcal{N}$, $x^{*\nu}$ is also a solution to the following optimization program:

$$\min_{Lx\in[0,1]} Lx, \text{ subject to } p^*\,(I-A)\,x \geqq p^*b;\ p^*Ax \leqq p^*\omega^{\nu}.$$

Since $x^{*\nu} = A^{-1}\omega^{*\nu}$ for each $\nu \in \mathcal{N}$, Lemma 4 applies, meaning that $\left\langle p^*; (w^*, r^*)\,, (x^{*\nu})_{\nu\in\mathcal{N}}\right\rangle$ is an inegalitarian RS if and only if $(W^{\nu})_{\nu\in\mathcal{N}}$ is an unequal distribution. $\square$

Theorem 3 implies that in economies with a technical difference among sectors, an RS exists in a broad class of initial distributions of financial capital. This result is in sharp contrast to the case of economies with no technical difference among sectors, in which only the equal distribution of financial capital allows the existence of RSs. In addition, owing to the broadly available domain of initial distributions, most RSs are characterized as free trade equilibria derived from the unequal initial distribution of financial capital. More interestingly, such RSs are characterized as having international division of labor because of the unequal distribution of financial capital: relatively rich nations are more specialized to more capital-intensive production activities, while relatively poor nations are more specialized to more labor-intensive production activities, in that for any $\nu, \nu' \in \mathcal{N}$ with $W^{\nu} > W^{\nu'}$, $\frac{p^* A x^{*\nu}}{L x^{*\nu}} > \frac{p^* A x^{*\nu'}}{L x^{*\nu'}}$ holds under the RS $\left\langle p^*; (w^*, r^*), (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$.

Such RSs are also characterized as involving UEL whenever their associated interest rates are positive. In particular, relatively rich nations supply fewer labor hours than the socially necessary labor hours vb to produce the subsistence vector b, meaning that they are exploiting, while relatively poor nations supply more labor hours than the socially necessary labor hours vb, meaning that they are exploited. In addition, such RSs do not involve UEL whenever interest rates are zero.

Finally, in those RSs with a zero interest rate, international division of labor occurs because of the unequal endowment of financial capital, even though labor supply is equalized among nations. Moreover, RSs with a zero interest rate are also characterized by labor value pricing.

Since all the arguments presented above assume that $\overline{\omega} = A[I-A]^{-1}(Nb)$, let us examine the case of $\overline{\omega} \geq A[I-A]^{-1}(Nb)$. Note that if $\overline{\omega} \ngeqq A[I-A]^{-1}(Nb)$, then there is no RS in subsistence economies. Therefore, the only remaining task to characterize the class of RSs is to check the case of $\overline{\omega} \geq A[I-A]^{-1}(Nb)$.

Theorem 4: Let an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ be such that $\overline{\omega} \geq A[I-A]^{-1}(Nb)$. Then, for any RS $\left\langle p^*; (w^{\nu *}, r^{\nu *})_{\nu \in \mathcal{N}}, (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$ associated with a suitable $(\omega^{\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$, $p^* = \varsigma v$ holds for some $\varsigma > 0$, and there is no exploitation.

Proof. Note that for any $\overline{\omega} \geqq A[I-A]^{-1}(Nb)$, the social production activity $\sum_{\nu \in \mathcal{N}} x^{*\nu}$ of any RS is equal to $x^* = [I-A]^{-1}(Nb)$ and its associated equilibrium price vector p^* meets $p^* - p^*A > \mathbf{0}$, according to Proposition 1.

For the unique Frobenius eigenvector $\overline{p} > \mathbf{0}$ of A, consider $\overline{p}$ and L to be linearly dependent. Then, since $\overline{p}A$ and L are also linearly dependent, Lemma 1 implies that any RS price vector is characterized by labor value pricing. Then, for any $(\omega^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$ such that $v\omega^\nu \geqq vA\frac{x^*}{N}$ for any $\nu \in \mathcal{N}$, $\left\langle \overline{p}; (w,r), \left(\frac{x^*}{N}, \ldots, \frac{x^*}{N}\right)\right\rangle$ with $r \in [0, R)$ and $w > 0$ such that $\overline{p} = (1+r)\,\overline{p}A + wL$ constitutes an egalitarian RS. In this case, there should be a nation ν having $\overline{p}\omega^\nu > \overline{p}A\frac{x^*}{N}$ according to $\overline{\omega} \geq A[I-A]^{-1}(Nb)$ and $\overline{p} > \mathbf{0}$. However, $\frac{x^*}{N}$ is still an optimal activity for this agent.

Consider the case that $\overline{p}$ and L are linearly independent. In this case, we cannot apply Lemma 1 and Theorem 1, since an RS $\left\langle p^*; (w^{\nu*}, r^{\nu*})_{\nu\in\mathcal{N}}, (x^{*\nu})_{\nu\in\mathcal{N}}\right\rangle$ under $\overline{\omega} \geq A[I-A]^{-1}(Nb)$ may not be incomplete specialization. According to the definition of (MP^ν_t), we have $p^*(I-A)\,x^* = r^*p^*Ax^* + w^*Lx^*$, where $r^* \equiv \frac{\sum_{\nu\in\mathcal{N}} r^{\nu*}p^*Ax^{*\nu}}{p^*Ax^*}$ and $w^* \equiv \frac{\sum_{\nu\in\mathcal{N}} w^{\nu*}Lx^{*\nu}}{Lx^*}$. Therefore, if p^*A and L are linearly dependent, then Lemma 1 implies that $\overline{p}A$ and L are linearly dependent, which is a contradiction. Thus, let us focus on the case that p^*A and L are linearly independent. Then, whenever $r^* > 0$, $p^*(I-A)$ and L are linearly independent.

Suppose that $r^* > 0$. Note that x^* is a solution to the following program:

$$\min_{Lx\in[0,N]} Lx, \text{ subject to } p^*(I-A)\,x = p^*Nb; p^*Ax \leqq p^*Ax^*. \quad \text{(*')}$$

Then, there exists $x' \geq \mathbf{0}$ such that $p^*(I-A)\,x' = p^*Nb$, $Lx' < Lx^*$, and $p^*Ax' > p^*Ax^*$ because $p^*(I-A)$ and L are linearly independent, and $x^* = [I-A]^{-1}(Nb) > \mathbf{0}$. In fact, suppose that for any $x \in \mathbb{R}^n_+\setminus\{x^*\}$ such that $p^*(I-A)\,x = p^*Nb$, if $p^*Ax > p^*Ax^*$, then $Lx \geqq Lx^*$. This finding implies that for any $x \in \mathbb{R}^n_+\setminus\{x^*\}$ such that $p^*(I-A)\,x = p^*Nb$, if $Lx < Lx^*$, then $p^*Ax \leqq p^*Ax^*$. Thus, if there exists $x' \in \mathbb{R}^n_+\setminus\{x^*\}$ such that $Lx' < Lx^*$ and $p^*(I-A)\,x' = p^*Nb$, then $p^*Ax' \leqq p^*Ax^*$, which contradicts the fact that x^* is a solution to the program (*'). Therefore, for any $x \in \mathbb{R}^n_+\setminus\{x^*\}$ such that $p^*(I-A)\,x = p^*Nb$, $Lx \geqq Lx^*$ holds. However, since $p^*(I-A)$ and L are linearly independent, this finding implies that for any $x \in \mathbb{R}^n_+\setminus\{x^*\}$ such that $p^*(I-A)\,x = p^*Nb$, $Lx > Lx^*$ holds. Such a situation is possible only when x^* is in the boundary of $\mathbb{R}^n_+$. However, since $x^* > \mathbf{0}$, this is a contradiction. Thus, we must conclude that there exists $x' \geq \mathbf{0}$ such that $p^*(I-A)\,x' = p^*Nb$, $Lx' < Lx^*$, and $p^*Ax' > p^*Ax^*$. Then, define a convex combination $x^{*\prime} \equiv \epsilon x' + (1-\epsilon)\,x^*$ for sufficiently small positive ϵ. By definition, $p^*(I-A)\,x^{*\prime} = p^*Nb$, $Lx^{*\prime} < Lx^*$, and $p^*Ax^{*\prime} > p^*Ax^*$.

Since $\overline{\omega} \geq A[I-A]^{-1}(Nb)$ implies $p^*\overline{\omega} > p^*Ax^*$, we can have $p^*Ax^{*\prime} \leqq p^*\overline{\omega}$ for a sufficiently small positive ϵ. This finding implies that given a suitable assignment of $x^{*\prime}$ among the members of $\mathcal{N}$, there should be at least one nation $\nu \in \mathcal{N}$ such that $p^* (I - A) x^{*\prime\nu} = p^* b$, $p^* A x^{*\prime\nu} \leqq p^* \omega^\nu$, and $Lx^{*\prime\nu} < Lx^{*\nu}$. However, this is a contradiction, since $\left\langle p^*; (w^{\nu*}, r^{\nu*})_{\nu\in\mathcal{N}}, (x^{*\nu})_{\nu\in\mathcal{N}} \right\rangle$ is an RS. Therefore, $r^* \not> 0$ for an RS under $\overline{\omega} \geq A [I - A]^{-1} (Nb)$, meaning that $r^* = 0$ holds. Then, $p^* = w^* v$. As shown above, such an RS is egalitarian even for $\overline{\omega} \geq A [I - A]^{-1} (Nb)$. □

Theorem 4 implies that in any subsistence economy with $\overline{\omega} \geq A [I - A]^{-1} (Nb)$, no RS has exploitation and its equilibrium commodity prices are characterized by labor value pricing. This property holds regardless of whether the equilibrium interest rates are positive. It is particularly interesting when $\overline{p}$ and L are linearly dependent, since in such a case, an egalitarian RS with a positive equilibrium interest rate can exist even under a suitable unequal distribution of wealth, as the RS $\left\langle \overline{p}; (w, r), \left(\frac{x^*}{N}, \ldots, \frac{x^*}{N}\right) \right\rangle$ with $r > 0$ discussed in the proof of Theorem 4.

In summary, Theorems 2–4 together imply that regardless of whether economies involve inter-sector technical heterogeneity, any free trade equilibrium involves exploitation if and only if this equilibrium is derived from an unequal distribution of wealth and the corresponding equilibrium prices deviate from the labor values.

Corollary 1: For any economy $\left\langle \mathcal{N}, (A, L, u), (\omega^\nu)_{\nu\in\mathcal{N}} \right\rangle$ with $\overline{\omega} \geqq A [I - A]^{-1} (Nb)$, and any RS $\left\langle (p^*, w^*, r^*), (x^{*\nu})_{\nu\in\mathcal{N}} \right\rangle$ of this economy, this RS is inegalitarian if and only if $(p^* \omega^\nu)_{\nu\in\mathcal{N}}$ is unequal and there is no $\varsigma > 0$ such that $p^* = \varsigma v$.

In other words, even if an equilibrium price vector is associated with a positive interest rate, it does not involve exploitation under the unequal distribution of wealth whenever equilibrium prices are proportional to the labor values. Such a situation is possible according to Theorem 4. Therefore, Corollary 1 implies that positive equilibrium interest rates and the existence of exploitation are not necessarily equivalent—even under the unequal distribution of wealth.

5 Concluding Remarks

This paper introduced subsistence international economies with Leontief production techniques and examined the necessary and sufficient condition for the existence of equilibria with UEL, namely inegalitarian RSs. First, the presented findings showed that if the social endowments of aggregate capital goods are excessive relative to the minimal level necessary for the survival of the economy, then no inegalitarian RS exists regardless of whether the distribution of wealth among nations is unequal. Second, if the social endowments of aggregate capital goods are equal to the minimal level necessary for the survival of the economy, then inegalitarian RSs exist only under the condition that wealth distribution is unequal and essential technical differences exist among sectors. Such a condition implies that each nation has a strong motivation to participate in international trade based on the principle of comparative advantage. In other words, a richer nation finds its own comparative advantage when selecting a more capital-intensive production activity, while a poorer nation finds its own comparative advantage when selecting a more labor-intensive production activity. In summary, the existence of inegalitarian RSs is characterized by the unequal distribution of wealth among nations and deviation from the labor value pricing of commodities.

This characterization demonstrates an interesting contrast with the *Fundamental Marxian Theorem* (FMT) (Okishio, 1963; Morishima, 1973), which shows that the unequal distribution of financial capital and the positivity of profits are necessary and sufficient for the existence of exploitation in economies with labor markets. Unlike the FMT, this paper shows that UEL is not generated in the equilibrium with a common capital–labor ratio among sectors, even if the equilibrium interest rate is positive and wealth distribution among nations is unequal. However, this does not necessarily imply the violation of FMT in subsistence international economies, for two reasons. First, the premise of the FMT is based on economies with labor markets, while the characterization of inegalitarian RSs in this paper was established in international economies without international labor markets. Second, the FMT discusses the aggregate exploitation rate of the whole working class (the class of agents endowed with no financial capital), while the characterization of inegalitarian RSs in this paper could not refer to the exploitation status of nations with no financial capital, since such nations cannot survive in subsistence international economies. Thus, the main theorems of this paper do not satisfy the premise of the FMT.

The analysis of the existence of UEL in free trade equilibria presented herein is concerned only with the *temporary* features of international trade. However, if *intertemporal* features of international trade were also considered, as in the literature on dynamic Heckscher–Ohlin trade theory such as Chen (1992), Nishimura and Shimomura (2002, 2006), and Bond, Iwasa, and

Nishimura (2011, 2012),[6] the existence of UEL in dynamic free trade equilibria would have a quite different characterization. Indeed, our companion paper, Yoshihara and Kaneko (2014), shows that in subsistence international economies with infinite horizons, whenever an essential technical difference exists among sectors, inegalitarian RSs generically exist in every period, regardless of whether the initial endowments of aggregate capital goods are excessive. By contrast, in subsistence economies with finite horizons and no discount factor, there is no inegalitarian RS, regardless of whether the initial distribution of financial capital is unequal, whenever the initial endowments of aggregate capital goods are excessive.

Note also that this paper focused on international trade with incomplete specialization in Leontief production economies with no option of technical choice. In such an environment, the factor price equalization theorem holds, as shown in Theorem 1. However, it is well known that once a model is extended to allow technical choice, the factor price equalization theorem may not hold in general, as Metcalfe and Steedman (1972, 1973) discuss. Therefore, if subsistence international economies with an option set of multiple Leontief production techniques are considered, then we could not develop our analysis by relying on the factor price equalization theorem as this paper does, meaning that a new analytical technique for the subject would be necessary. This interesting open question is left for future research.

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