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A Full Characterization of Nash Implementation with Strategy Space Reduction

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Abstract

Noting that a full characterization of Nash-implementation is given using a canonical-mechanism and Maskin's theorem (Maskin, 1999) is shown using a mechanism with Saijo's type of strategy space reduction (Saijo, 1988), this paper fully characterizes the class of Nash-implementable social choice correspondences (SCCs) by mechanisms endowed with Saijo's message space specification - *s-mechanisms.* This class of SCCs is further shown to be equivalent to the class of Nash-implementable SCCs, though any further 'strategy space reduction' mechanism breaks this equivalent relationship down.

JEL classification: C72; D71; D82.
Keywords: Nash implementation, strategy space reduction, s-mechanisms, Condition μ_r^s, Condition M^s.

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1 Introduction

In Nash implementation theory, it is *Maskin's Theorem* (Maskin, 1999) which shows that when the planner faces at least three agents, a *social choice correspondence* (SCC) is implementable in (pure-strategy) Nash equilibria (henceforth, Nash-implementable) if it satisfies *Maskin monotonicity* and *no-veto power*; conversely, any Nash-implementable SCC is Maskin-monotonic. Two issues pertaining to this theorem stand out. First, it does not provide a complete characterization of Nash-implementable SCCs, since no-veto power is not necessary for Nash implementation. Second, a *canonical mechanism* proposed in this theorem, which requires each agent to report a preference profile, a feasible social outcome, and an integer, is not so attractive. This is because the message space of this mechanism is rather large and announcing all other agents' preferences is undesirable in terms of the informational efficiency of decentralized decision making (on this point see, for instance, Hurwicz, 1960).

Moore and Repullo (1990) address the first issue by providing, without any domain restriction, a necessary and sufficient condition, called *Condition* μ, for Nash implementability of SCCs in societies with more than two agents.[1] In contrast to the first issue, the issue of informational efficiency is addressed by Saijo (1988), who shows that a *mechanism with strategy-space reduction* (henceforth, *s-mechanism*) suffices to guarantee Maskin's Theorem. Note that, in s-mechanisms, each agent is requested to announce, in addition to a feasible social outcome and an integer, her own and her neighbor's preferences solely. Yet, as Moore and Repullo (1990) also use a canonical mechanism for showing the full characterization and Saijo (1988) does not discuss a full characterization of Nash implementation, it leaves unclear not only whether Moore and Repullo's result indispensably relies on canonical mechanisms but also whether s-mechanisms can Nash-implement any other SCC than Maskin-monotonic and no-veto power ones.

In this paper, we address the issue of what constitutes the necessary and sufficient condition for Nash implementation by s-mechanisms. We introduce a class of new conditions (labelled, $\{\text{Condition } \mu_r^s\}_{r=1,\ldots,n-2}$) which fully char-

[1] Note that, for two person societies, Moore and Repullo (1990) and Dutta and Sen (1991) independently provided necessary and sufficient conditions for Nash implementation, whereas even in societies with more than two agents, there are other works on complete characterizations of Nash implementation under some domain restrictions, such as Danilov (1992) and Yamato (1992).

acterize the class of SCCs Nash-implemantable by s-mechanisms. Each of Condition μ_r^s turns out to be equivalent to Condition μ. The same issue is addressed by introducing an alternative condition, *Condition* M^s, which is similar to Condition M appeared in Sjöstrom (1991); a new characterization of Nash-implementable SCCs via s-mechansims is provided by using Condition M^s. It is also shown that Condition M^s is equivalent to Condition μ_r^s for each $r = 1, \ldots, n-2$ (and so to Condition μ). Moreover, we show that s-mechanisms constitute the 'lower-bound of Nash-implementing mechanisms' in the sense that no further strategy space reduction can preserve the Moore and Repullo (1990) full characterization of Nash implementation.

It may be worth mentioning that all of our characterization results are obtained by restricting the class of available s-mechanisms to those satisfying *forthrightness*, which is a variation of those introduced in Dutta *et al.* (1995), Saijo *et al.* (1996), and Tatamitani (2001).[2] As a result, the outcome of an equilibrium message profile of our mechanisms is 'easy' to compute and the problem of *information smuggling* is avoided.

The paper is organized as follows. Section 2 describes the formal environment. Section 3 reports our main characterization result via Condition μ_r^s, whereas Section 4 reports an alternative characterization result via Condition M^s. Section 5 shows that the lower-bound property of s-mechanisms to reserve Nash implementation. Section 6 concludes.

2 Preliminaries

The set of (*social choice) environments* is $(N, X, \mathcal{R}^n)$, where $N \equiv \{1, ..., n\}$ is a set of $n \geq 3$ *agents*, $X \equiv \{x, y, z, ...\}$ is the set of attainable *alternatives* (or *outcomes)*, and $\mathcal{R}^n$ is the set of admissible *preference profiles* (or *states of the world*). Henceforth, we assume that the cardinality of X is $\#X \geq 2$. Let $\mathcal{R}(X)$ be the set of all complete preorders on X.[3] We assume that $\mathcal{R}^n \equiv \mathcal{R}_1 \times ... \times \mathcal{R}_n$ is a non-empty subset of the n-fold Cartesian product $\mathcal{R}^n(X) \equiv \underbrace{\mathcal{R}(X) \times .... \times \mathcal{R}(X)}_{n\text{-times}}$. An element of $\mathcal{R}^n$ is denoted by

[2] Note that the forthrightness condition is indispensable for showing Theorem 1, a main result of this paper.

[3] A complete preorder $R \in \mathcal{R}(X)$ is a complete and transitive binary relation. A relation R on X is *complete* if, for all $x, x' \in X$, $(x, x') \in R$ or $(x', x) \in R$; *transitive* if, for all $x, x', x'' \in X$, if $(x, x') \in R$ and $(x', x'') \in R$, then $(x, x'') \in R$.

$R \equiv (R_1, ..., R_n)$, where its ℓ-th component is $R_\ell \in \mathcal{R}_\ell$, for each $\ell \in N$. For any preference profile $R \in \mathcal{R}^n$ and any $\ell \in N$, let $R_{-\ell}$ be the list of elements of R for all agents except ℓ, i.e., $R_{-\ell} \equiv (R_1, ..., R_{\ell-1}, R_{\ell+1}, ..., R_n)$. Given a list $R_{-\ell}$ and $R_\ell \in \mathcal{R}_\ell$, we denote by $(R_{-\ell}, R_\ell)$ the preference profile consisting of these R_ℓ and $R_{-\ell}$. For any preference profile $R \in \mathcal{R}^n$ and any $S \subseteq N$, let R_{-S} be the list of elements of R for all agents in $N \backslash S$. Given a list R_{-S} and $R_S \in \times_{\ell \in S} \mathcal{R}_\ell$, we denote by (R_{-S}, R_S) the preference profile consisting of these R_S and R_{-S}. For any $(R_\ell, x) \in \mathcal{R}_\ell \times X$, agent ℓ's *weakly lower contour set* of R_ℓ at x is given by $L(R_\ell, x) \equiv \{y \in X | (x, y) \in R_\ell\}$. For each $\ell \in N$ and each $R_\ell \in \mathcal{R}_\ell$, $\max_{R_\ell} X \equiv \{x \in X | (x, y) \in R_\ell \text{ for all } y \in X\}$.

We also assume that N and X are fixed throughout the following discussion, so that the set of environments is boiled down to $\mathcal{R}^n$. A *social choice correspondence* (SCC) is a correspondence $F : \mathcal{R}^n \twoheadrightarrow X$ with $F(R) \neq \varnothing$ for all $R \in \mathcal{R}^n$.

A mechanism (or game-form) is a pair $\gamma \equiv (M, g)$, where $M \equiv M_1 \times ... \times M_n$, and $g : M \to X$ is the *outcome function*. Denote a generic message (or strategy) for agent ℓ by $m_\ell \in M_\ell$ and a generic message profile by $m = (m_1, ..., m_n) \in M$. For any $m \in M$ and $\ell \in N$, let $m_{-\ell} \equiv (m_1, ..., m_{\ell-1}, m_{\ell+1}, ..., m_n)$. Let $M_{-\ell} \equiv \times_{j \in N \backslash \{\ell\}} M_j$. Given $m_{-\ell} \in M_{-\ell}$ and $m_\ell \in M_\ell$, denote by $(m_\ell, m_{-\ell})$ the message profile consisting of these m_ℓ and $m_{-\ell}$. For any $m \in M$ and $S \subseteq N$, let $m_{-S} \equiv (m_\ell)_{\ell \in N \backslash S}$. Let $M_{-S} \equiv \times_{j \in N \backslash S} M_j$. Given $m_{-S} \in M_{-S}$ and $m_S \in M_S$, denote by (m_S, m_{-S}) the message profile consisting of these m_S and m_{-S}. Given $R \in \mathcal{R}^n$ and $\gamma = (M, g)$, the pair (γ, R) constitutes a (*non-cooperative*) *game*. Given a game (γ, R), $m \in M$ is a (pure strategy) *Nash equilibrium* of (γ, R) if and only if, for all $\ell \in N$, $(g(m), g(m'_\ell, m_{-\ell})) \in R_\ell$ for all $m'_\ell \in M_\ell$. Let $NE(\gamma, R)$ denote the set of Nash equilibria of (γ, R), whereas denote the set of *Nash equilibrium outcomes* of (γ, R) by $NA(\gamma, R) \equiv g(NE(\gamma, R))$.

A mechanism $\gamma = (M, g)$ *implements* F in *Nash equilibria*, or simply *Nash-implements* F, if and only if $NA(\gamma, R) = F(R)$ for all $R \in \mathcal{R}^n$. An SCC F is *Nash-implementable* if there is such a mechanism.

Moore and Repullo (1990) show that, under the society with more than two agents, the following condition is the necessary and sufficient condition for any SCC to be Nash-implemetable.

Condition μ (for short, μ). An SCC F satisfies Condition μ if there exists a set $Y \subseteq X$, and for all $R \in \mathcal{R}^n$ and for all $x \in F(R)$, there is a profile of sets

$(C_\ell(R,x))_{\ell\in N}$ such that $x \in C_\ell(R,x) \subseteq L(R_\ell,x) \cap Y$ for all $\ell \in N$,[4] and for any $R^* \in \mathcal{R}^n$:
(i) if $C_\ell(R,x) \subseteq L(R^*_\ell,x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for each $i \in N$, if $y \in C_i(R,x) \subseteq L(R^*_i,y)$ and $Y \subseteq L(R^*_\ell,y)$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Y \subseteq L(R^*_\ell,y)$ for all $\ell \in N$, then $y \in F(R^*)$.[5]

3 Main Result

Following Saijo (1988), we focus on mechanisms in which each agent reports her own preference $R_\ell \in \mathcal{R}_\ell$, her neighbor's preference $R_{\ell+1} \in \mathcal{R}_{\ell+1}$, an outcome $x \in Y \subseteq X$ and an integer $\diamond \in N$.

Definition 1. A mechanism $\gamma = (M,g)$ is s-mechanism if, for any $\ell \in N$, $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times Y \times N$, where $Y \subseteq X$ and $\ell+1 = 1$ if $\ell = n$.

Definition 2. An SCC F is Nash-implementable by an s-mechanism if there exists an s-mechanism $\gamma = (M,g)$ such that, for all $R \in \mathcal{R}^n$:
i) $F(R) = NA(\gamma,R)$; and
ii) for all $x \in F(R)$, if $m_\ell = (R_\ell, R_{\ell+1}, x, \diamond) \in M_\ell$ for all $\ell \in N$, with $\ell+1 = 1$ if $\ell = n$, then $m \in NE(\gamma,R)$ and $g(m) = x$.

In Definition 2, it is required not only that all F-optimal outcomes coincide with Nash equilibrium outcomes of the game defined by an s-mechanism for any state of the world, but also that such an s-mechanism satisfies *forthrightness*. It was originally introduced in economic environments by Dutta, Sen, and Vohra (1995) and Saijo, Tatamitani, and Yamto (1996), and it has desired implications. A mechanism satisfying forthrightness is simple in the sense that it is easy to compute the outcome of an equilibrium strategy profile. Moreover, if a mechanism fails to satisfy this condition, it is subject to information smuggling, that is, the strategy space can be reduced to an arbitrary smaller dimensional space. Thus, any Nash-implementable SCC seems to be Nash-implementable by s-mechanisms, while any SCC that is Nash-implementable by s-mechanisms seems to be Nash-implementable by a 'further strategy space reduction mechanism' like *self-relevant mechanisms*

[4] Weak set inclusion is denoted by $\subseteq$.

[5] We refer to the condition that requires only one of the conditions (i)–(iii) in Condition μ as Conditions μ(i)–μ(iii) each. Note that Condition μ implies Conditions μ(i)–μ(iii), but the converse is not true. We use similar conventions below

(Tatamitani, 2000), unless forthrightness is requested. This indicates that without forthrightness, there is no legitimate reason for characterizing the class of Nash-implementable SCCs by s-mechanisms. Hence, forthrightness should be requested in Definition 2.

Using the approach developed by Moore and Repullo (1990), we now introduce a class of conditions, labelled $\{$Condition $\mu_r^s\}$, to characterize Nash implementability by s-mechanisms. For each $r = 1, \ldots, n-2$, let us introduce the following.

Condition μ_r^s (for short, μ_r^s): An SCC F satisfies Condition μ_r^s if there exists a set $Y \subseteq X$, and for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $\left(C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)\right)_{\ell \in N}$ such that $x \in C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$, with $\ell + 1 = 1$ if $\ell = n$, and for all $R^* \in \mathcal{R}^n$:
(i) if $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i\left(R_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L(R_i^*, y)$ and $Y \subseteq L(R_\ell^*, y)$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Y \subseteq L(R_\ell^*, y)$ for all $\ell \in N$, then $y \in F(R^*)$,
(iv) for all $\ell \in N$ with $R^*_{-\{\ell+1,\ldots,\ell+r\}} = R_{-\{\ell+1,\ldots,\ell+r\}}$ and $x \in F(R^*)$, $C_\ell\left(R^*_{-\{\ell+1,\ldots,\ell+r\}}, x\right) = C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$.

Proposition 1. *An* SCC F on $\mathcal{R}^n$ *satisfies* Condition μ_r^s for each $r = 1, \ldots, n-2$ *if it is* Nash-implementable by an s-mechanism.

Proof. Let an SCC F on $\mathcal{R}^n$ be Nash-implementable by an s-mechanism. Let $\gamma = (M, g)$ be such an s-mechanism. Take any $r = 1, \ldots, n-2$. Define $Y \equiv g(M)$. For all $R \in \mathcal{R}^n$ and $x \in F(R)$, there exists an $m \in NE(\gamma, R)$ such that $g(m) = x$ and $m_\ell = \left(R_\ell, R_{\ell+1}, x, k^\ell\right)$ for all $\ell \in N$, by Definition 2(ii). Take any $\ell \in N$ and define

$$\begin{aligned} & F^{-1}\left(R_{-\{\ell,\ldots,\ell+r-1\}}, x\right) \\ \equiv\ & \left\{R'_{\{\ell,\ldots,\ell+r-1\}} \in \mathcal{R}_\ell \times \ldots \times \mathcal{R}_{\ell+r-1} | x \in F\left(R'_{\{\ell,\ldots,\ell+r-1\}}, R_{-\{\ell,\ldots,\ell+r-1\}}\right)\right\}. \end{aligned}$$

Then, for any $R'_{\{\ell,\ldots,\ell+r-1\}} \in F^{-1}\left(R_{-\{\ell,\ldots,\ell+r-1\}}, x\right)$, there exist

$$m'_{\{\ell,\ldots,\ell+r-1\}} = \left(\left(R'_\ell, R'_{\ell+1}, x, k^\ell\right), \ldots, \left(R'_{\ell+s}, R'_{\ell+s+1}, x, k^\ell\right), \ldots, \left(R'_{\ell+r-1}, R_{\ell+r}, x, k^\ell\right)\right),$$

where $1 \leq s \leq k-2$, and an $m'_{\ell-1} = \left(R_{\ell-1}, R'_\ell, x, k^{\ell-1}\right)$ such that

$$g\left(m'_{\{\ell,\ldots,\ell+r-1\}}, m'_{\ell-1}, m_{-\{\ell-1,\ell,\ldots,\ell+r-1\}}\right) = x \in NA\left(\gamma, \left(R'_{\{\ell,\ldots,\ell+r-1\}}, R_{-\{\ell,\ldots,\ell+r-1\}}\right)\right),$$

by Definition 2(ii). Therefore, for any $R'_{\{\ell,\ldots,\ell+r-1\}} \in F^{-1}\left(R_{-\{\ell,\ldots,\ell+r-1\}}, x\right)$, there is an

$$m'_{\{\ell,\ldots,\ell+k-1\}} = \left(\left(R'_\ell, R'_{\ell+1}, x, k^\ell\right), \ldots, \left(R'_{\ell+s}, R'_{\ell+s+1}, x, k^\ell\right), \ldots, \left(R'_{\ell+r-1}, R_{\ell+r}, x, k^\ell\right)\right)$$

such that $g\left(M_{\ell-1}, m'_{\{\ell,\ldots,\ell+r-1\}}, m_{-\{\ell-1,\ell,\ldots,\ell+r-1\}}\right) \subseteq L\left(R_{\ell-1}, x\right)$. Define

$$\begin{array}{ll} & C_{\ell-1}\left(R_{-\{\ell,\ldots,\ell+r-1\}}, x\right) \\ \equiv & \bigcup_{\substack{m'_{\{\ell,\ldots,\ell+r-1\}}=\left(\left(R'_\ell,R'_{\ell+1},x,k^\ell\right),\ldots,\left(R'_{\ell+r-1},R_{\ell+r},x,k^\ell\right)\right); \\ R'_{\{\ell,\ldots,\ell+r-1\}}\in F^{-1}\left(R_{-\{\ell,\ldots,\ell+r-1\}},x\right)}} g\left(M_{\ell-1}, m'_{\{\ell,\ldots,\ell+r-1\}}, m_{-\{\ell-1,\ell,\ldots,\ell+r-1\}}\right) \end{array} \tag{1}$$

for each $\ell-1 \in N$, with $\ell = 1$ if $\ell - 1 = n$. Then, by definition, $x \in C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L\left(R_\ell, x\right) \cap Y$ for all $\ell \in N$. Moreover, F satisfies μ^s_r(iv) by (1). Next, we show that F satisfies Conditions μ^s_r(i)-μ^s_r(iii). Take any $R^* \in \mathcal{R}^n$.

Suppose that $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L\left(R^*_\ell, x\right)$ for all $\ell \in N$. Then, it follows from (1) that $g\left(M_\ell, m_{-\ell}\right) \subseteq L\left(R^*_\ell, x\right)$ for all $\ell \in N$. We conclude that $g(m) = x \in NA\left(\gamma, R^*\right) = F\left(R^*\right)$. Hence, μ^s_r(i) holds.

For each $i \in N$, let $y \in C_i\left(R_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L\left(R^*_i, y\right)$ and $Y \subseteq L\left(R^*_\ell, y\right)$ for all $\ell \in N\backslash\{i\}$. It follows from (1) that there is an $m^*_{\{i+1,\ldots,i+r\}} \in M_{\{i+1,\ldots,i+r\}}$ such that $y \in g\left(M_i, m^*_{\{i+1,\ldots,i+r\}}, m_{-\{i,i+1,\ldots,i+r\}}\right) \subseteq C_i\left(R_{-\{i+1,\ldots,i+r\}}, x\right)$ so that $g\left(m^*_i, m^*_{\{i+1,\ldots,i+r\}}, m_{-\{i,i+1,\ldots,i+r\}}\right) = y$ for some $m^*_i \in M_i$. Moreover, $g(M) \subseteq L\left(R^*_\ell, y\right)$ for all $\ell \in N\backslash\{i\}$. It follows that $y \in NA\left(\gamma, R^*\right) = F\left(R^*\right)$. Hence, μ^s_r(ii) holds.

Finally, if $y \in Y \subseteq L\left(R^*_\ell, y\right)$ for all $\ell \in N$, then $y \in g(M) \subseteq L\left(R^*_\ell, y\right)$ for all $\ell \in N$. Thus, there is an $m^* \in M$ such that $g\left(m^*\right) = y$, which implies that $y \in NA\left(\gamma, R^*\right) = F\left(R^*\right)$. Hence, μ^s_r(iii) holds. We conclude that F satisfies μ^s. ■

To prove sufficiency of Condition μ^s_r we devise a class of s-mechanims which are similar but not identical to that used by Saijo (1988). Likely Saijo's mechanism, in our s-mechanisms agents make cyclic announcement of strategies and the preference profile, especially the deviator's preference relation, is determined without relying upon the deviator's announcement. While the proof of Saijo (1988) exploits in full the information coming from the cyclic announcement of strategies, we do not follow this course of action here as we can make use of the novelty of our Condition μ_r^s(iv).

Proposition 2. *For each* $r = 1, \ldots, n-2$, *an* SCC F on $\mathcal{R}^n$ *satisfying* μ_r^s *is* Nash-implementable by an s-mechanism.

Proof. Let $\gamma \equiv (M, g)$ be an s-mechanism. Suppose that F on $\mathcal{R}^n$ satisfies μ_r^s for some $r = 1, \ldots, n-2$. Fix any $m \in M$, $R \in \mathcal{R}^n$, and $x \in X$, and let $m_\ell = \left(R_\ell^\ell, R_{\ell+1}^\ell, x^\ell, k^\ell\right) \in M_\ell$, where $\ell + 1 = 1$ if $\ell = n$, and where the announcement of agent $\ell \in N$ about agent $\ell + 1$'s preferences is $R_{\ell+1}^\ell$. We say that the message profile $m \in M$ is:

(i) *consistent* with R and x if, for all $\ell \in N$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$ and $x^\ell = x$, where $\ell - 1 = n$ if $\ell = 1$;
(ii) m_{-i} *quasi-consistent* with x and R, where $i \in N$, if for all $\ell \in N$, $x^\ell = x$, and for all $\ell \in N \backslash \{i, i+1\}$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$, $R_i^{i-1} = R_i$, $R_{i+1}^{i+1} = R_{i+1}$, and $[R_i^i \neq R_i$ or $R_{i+1}^i \neq R_{i+1}]$, where $j - 1 = n$ if $j = 1$ for $j \in \{i, \ell\}$;
(iii) m_{-i} *consistent* with x and R, where $i \in N$, if for all $\ell \in N \backslash \{i\}$, $x^\ell = x \neq x^i$, and for all $\ell \in N \backslash \{i, i+1\}$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$, $R_i^{i-1} = R_i$ and $R_{i+1}^{i+1} = R_{i+1}$, where $j - 1 = n$ if $j = 1$ for $j \in \{i, \ell\}$.

Define the outcome function $g : M \to X$ as follows: For any $m \in M$,

Rule 1: m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, then $g(m) = x$.
Rule 2: For some $i \in N$, m_{-i} is quasi-consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, then $g(m) = x$.
Rule 3: For some $i \in N$, m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, and $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \neq Y$, with $i + 1 = 1$ if $i = n$, then

$$g(m) = \begin{cases} x^i & \text{if} \quad x^i \in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \\ x & \text{otherwise} \end{cases}.$$

Rule 4: Otherwise, $g(m) = x^{\ell^*(m)}$ where $\ell^*(m) \equiv \sum_{i \in N} k^i \pmod{n}$.[6]

Since F satisfies μ_r^s, it follows that, for any $R \in \mathcal{R}^n$ and any $x \in F(R)$, $x \in Y$. We show that $\gamma = (M, g)$ Nash-implements F. Take any $R \in \mathcal{R}^n$.

To show that $F(R) \subseteq NA(\gamma, R)$, let $x \in F(R)$ and suppose that, for all $\ell \in N$, $m_\ell = (R_\ell, R_{\ell+1}, x, \diamond)$, where $\diamond \in N$ is an arbitrary agent index. *Rule 1* implies that $g(m) = x$. By the definition of g we have that any deviation of agent $\ell \in N$ will get her to an outcome in $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$, so that

[6] If the remainder is *zero* the winner of the game is agent n.

$g(M_\ell, m_{-\ell}) \subseteq C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$. Since $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x)$, it follows that such deviations are not profitable, and so $m \in NE(\gamma, R)$. Furthermore, this guarantees the condition of Definition 2(ii).

Conversely, to show that $NA(\gamma, R) \subseteq F(R)$, let $m \in NE(\gamma, R)$. Consider the following cases.

Case 1: m falls into *Rule 1.*

Then, m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. Take any $\ell \in N$. Suppose that $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+k\}}, x\right) \neq Y$. For any $y \in C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \setminus \{x\}$, changing m_ℓ to $m_\ell^* = \left(R_\ell^\ell, R_{\ell+1}^\ell, y, \diamond\right) \in M_\ell$, agent ℓ can obtain $y = g(m_\ell^*, m_{-\ell})$ via *Rule 3.* In case that $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) = Y$, agent ℓ can attain any $y \in Y$ via *Rule 4.* Therefore, $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) = g(M_\ell, m_{-\ell})$ for all $\ell \in N$. As $m \in NE(\gamma, R)$ we have that $C_\ell\left(\bar{R}_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x)$ for all $\ell \in N$. μ_r^s(i) implies $x \in F(R)$.

Case 2: m falls into *Rule 2.*

Then m is m_{-i} quasi-consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. We proceed according the following sub-cases: 1) $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$, and 2) $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$.[7]

Sub-case 2.1. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$

As any $\ell \in N \setminus \{i\}$ can attain any $y \in Y \setminus \{x\}$ by inducing *Rule 4* and $m \in NE(\gamma, R)$, we have that $x \in \max_{R_\ell} Y$. Next, take any $y \in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \setminus \{x\}$. Suppose that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \neq Y$. By changing m_i to $m_i^* = \left(R_i^i, R_{i+1}^i, y, \diamond\right) \in M_i$, agent i can obtain $y = g(m_i^*, m_{-i})$, via *Rule 3.* In the case that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) = Y$, by changing m_i to $m_i^* = \left(R_i^i, R_{i+1}^i, y, k^i\right) \in M_i$, agent i can attain $y = g(m_i^*, m_{-i})$ with appropriately choosing k^i. It follows that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq g(M_i, m_{-i})$. Moreover, $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L(R_i, x)$ as $m \in NE(\gamma, R)$. Either μ_r^s(ii) or μ_r^s(iii) implies $x \in F(R)$.

Sub-case 2.2. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$

Let $R_i^i = R_i'$. We distinguish whether $x \in F\left(\bar{R}'\right)$ where $\bar{R}' \equiv \left(\bar{R}_{-i}, R_i'\right)$ or not. Suppose that $x \notin F\left(\bar{R}'\right)$. Then the same reasoning used above for *sub-case 2.1* carries over into this sub-case so that $x \in F(R)$. Otherwise, let $x \in F\left(\bar{R}'\right)$. Then, $i-1$ or i is the potential deviator. Agent $\ell \in N \setminus \{i-1, i\}$ can attain any $y \in Y \setminus \{x\}$ by inducing *Rule 4* so that $x \in \max_{R_\ell} Y$ as $m \in NE(\gamma, R)$. Consider agent $i-1$. Note that, by

[7] The sub-case $R_i^i = \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$ is not explicitly considered as it can be proved similarly to the *sub-case 2.2* shown below.

μ_r^s(iv), $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}},x\right)=C_{i-1}\left(\bar{R}'_{-\{i,\ldots,i+r-1\}},x\right)$ holds. Take any $y\in C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}},x\right)=C_{i-1}\left(\bar{R}'_{-\{i,\ldots,i+r-1\}},x\right)$ with $y\neq x$. Suppose that $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}},x\right)\neq Y$. By changing m_{i-1} to $m_{i-1}^*=\left(R_{i-1}^{i-1},R_i^{i-1},y,\diamond\right)\in M_{i-1}$, agent $i-1$ can obtain $y=g\left(m_{i-1}^*,m_{-(i-1)}\right)$ via *Rule 3*. In the case that $y\in C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}},x\right)=Y\setminus\{x\}$, by changing m_{i-1} to $m_{i-1}^*=\left(R_{i-1}^{i-1},R_i^{i-1},y,k^{i-1}\right)\in M_{i-1}$, agent $i-1$ can attain $y=g\left(m_{i-1}^*,m_{-(i-1)}\right)$ with appropriately choosing k^{i-1}. It follows that $C_{i-1}\left(\bar{R}_{-\{i,\ldots,i+r-1\}},x\right)\subseteq g\left(M_{i-1},m_{-(i-1)}\right)\subseteq L\left(R_{i-1},x\right)$ as $m\in NE\left(\gamma,R\right)$. Consider agent i. Again, take any $y\in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\setminus\{x\}$. Suppose that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\neq Y$. By changing m_i to $m_i^*=\left(R_i^i,R_{i+1}^i,y,\diamond\right)\in M_i$, agent i can obtain $y=g\left(m_i^*,m_{-i}\right)$ via *Rule 3*. In the case that $y\in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)=Y\setminus\{x\}$, by changing m_i to $m_i^*=\left(R_i^i,R_{i+1}^i,y,k^i\right)\in M_i$, agent i can attain $y=g\left(m_i^*,m_{-i}\right)$ with appropriately choosing k^i. It follows that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\subseteq g\left(M_i,m_{-i}\right)\subseteq L\left(R_i,x\right)$ as $m\in NE\left(\gamma,R\right)$. Therefore, $x\in F\left(R\right)$ by μ_r^s(i).

Case 3: m falls into Rule 3.

Then m is m_{-i} consistent with x and $\bar{R}\in\mathcal{R}^n$, where $x\in F\left(\bar{R}\right)$. Therefore, $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\neq Y$. First, we show that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\subseteq g\left(M_i,m_{-i}\right)$. For any $x^i\in C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\setminus\{x\}$, consider $m_i^*=\left(R_i^i,R_{i+1}^i,x^i,\diamond\right)$. Then, *Rule 3* implies that $g\left(m_{-i},m_i^*\right)=x^i$. On the other hand, to attain x agent i can induce *Rule 1* by changing m_i to $m_i^*=\left(\bar{R}_i,\bar{R}_{i+1},x,\diamond\right)$ so that $g\left(m_{-i},m_i^*\right)=x$. Hence, $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\subseteq g\left(M_i,m_{-i}\right)$.

Next, we claim that $g\left(M_\ell,m_{-\ell}\right)=Y$ for any $\ell\in N\setminus\{i\}$. We proceed according to whether $\#Y=2$ and $n=3$ or not.

Sub-case 3.1. not[$\#Y=2$ and $n=3$]

Take any $\ell\in N\setminus\{i\}$. Suppose that $\#Y>2$. By the definition of g, we have that $Y\subseteq g\left(M_\ell,m_{-\ell}\right)$ for any $\ell\in N\setminus\{i\}$. Otherwise, let $\#Y=2$. Then, $n>3$. Changing x to $x^\ell=x^i$, agent ℓ can make $\#\left\{\ell\in N|x^\ell=x\right\}\geq 2$ and $\#\left\{\ell\in N|x^\ell\neq x\right\}\geq 2$. As the outcome is determined by *Rule 4*, agent ℓ can attain any outcome in Y by appropriately choosing k^ℓ. Therefore, $Y\subseteq g\left(M_\ell,m_{-\ell}\right)$ for any $\ell\in N\setminus\{i\}$.

Sub-case 3.2. $\#Y=2$ and $n=3$

Then, let $N=\{i-1,i,i+1\}$ with $i+1=1$ if $i=n$ and $i-1=n$ if $i=1$. As $C_i\left(\bar{R}_{-\{i+1,\ldots,i+r\}},x\right)\neq Y$ it follows that $g\left(m\right)=x$. We proceed according to whether for some agents $\ell,\ell'\in N$, with $\ell\neq\ell'$, $\#\mathcal{R}_\ell\neq 1$ and $\#\mathcal{R}_{\ell'}\neq 1$ or not.

Sub-sub-case 3.2.1. For some $\ell,\ell'\in N$, with $\ell\neq\ell'$, $\#\mathcal{R}_\ell\neq 1$ and $\#\mathcal{R}_{\ell'}\neq 1$

In this case, agent $i-1$ (resp., $i+1$) can always induce the modulo game by appropriately changing the announcement of her own preference or that of her successor and by carefully choosing the outcome announcement. Finally, to attain x^i, agent $i-1$ (resp., $i+1$) has only to adjust the integer index so that agent i becomes the winner of the modulo game.

Sub-sub-case 3.2.2. For all $\ell, \ell' \in N$, with $\ell \neq \ell'$, $\#\mathcal{R}_\ell = 1$ or $\#\mathcal{R}_{\ell'} = 1$

Suppose that, for all $\ell^* \in \{i-1, i, i+1\}$, $\#\mathcal{R}_{\ell^*} = 1$. As m falls into *Rule 3*, it follows that $x \in F(R) = F(\bar{R})$. Otherwise, let us consider the case that, for some $\ell^* \in \{i-1, i, i+1\}$, $\#\mathcal{R}_{\ell^*} \neq 1$. If either $\#\mathcal{R}_{i-1} > 1$ or $\#\mathcal{R}_i > 1$, then agent $i-1$ can induce the modulo game by changing m_{i-1} into either $m^*_{i-1} = \left(R^{i-1}_{i-1}, \bar{R}_i, x, k^{i-1}\right)$ with $R^{i-1}_{i-1} \neq \bar{R}_{i-1}$ (if $\#\mathcal{R}_{i-1} > 1$) or $m^*_{i-1} = \left(\bar{R}_{i-1}, R^{i-1}_i, x^i, k^{i-1}\right)$ with $R^{i-1}_i \neq R^i_i$ (if $\#\mathcal{R}_i > 1$). To attain x^i, agent $i-1$ has only to choose an appropriate k^{i-1} so that $i = \ell^*\left(m_{-(i-1)}, m^*_{i-1}\right)$. Therefore, $Y \subseteq g\left(M_{i-1}, m_{-(i-1)}\right)$. Then, let $\#\mathcal{R}_{i-1} = \#\mathcal{R}_i = 1$. Agent $i-1$ can change m_{i-1} into $m^*_{i-1} = \left(\bar{R}_{i-1}, \bar{R}_i, x^i, k^{i-1}\right)$. Suppose that $x^i \notin F\left(\bar{R}_{i-1}, \bar{R}_i, R^i_{i+1}\right)$. Then, *Rule 4* applies and agent $i-1$ can attain x^i by adjusting k^{i-1} so that $i-1 = \ell^*\left(m_{-(i-1)}, m^*_{i-1}\right)$. Otherwise, let $x^i \in F\left(\bar{R}_{i-1}, \bar{R}_i, R^i_{i+1}\right)$. If $C_{i+1}\left(\bar{R}_i, R^i_{i+1}, x^i\right) = \{x^i\}$, *Rule 3* implies $g\left(m_{-(i-1)}, m^*_{i-1}\right) = x^i$. In the case that $C_{i+1}\left(\bar{R}_i, R^i_{i+1}, x^i\right) = Y$, the outcome is determined by *Rule 4*, so that by adjusting k^{i-1} agent $i-1$ can attain x^i. By similar reasoning, it can be shown that agent $i+1$ can attain $x^i \in Y$. Therefore, $Y \subseteq g(M_\ell, m_{-\ell})$ for $\ell \in \{i-1, i+1\}$.

In all the above sub-cases, we obtained $Y \subseteq g(M_\ell, m_{-\ell})$ for all $\ell \in N \backslash \{i\}$. As $m \in NE(\gamma, R)$ we have that $C_i\left(\bar{R}_{-\{i+1,\ldots,i+k\}}, x\right) \subseteq L(R_i, g(m))$ and $g(m) \in \max_{R_\ell} Y$ for any $\ell \in N \backslash \{i\}$, so that $g(m) \in F(R)$ by μ^s_r(ii).

Case 4: m falls into Rule 4.

Then, $Y \subseteq g(M_\ell, m_{-\ell})$ for all $\ell \in N$. Since $m \in NE(\gamma, R)$, it follows that $g(m) \in \max_{R_\ell} Y$ for $\ell \in N$. Therefore, $g(m) \in F(R)$ by μ^s_r(iii). ■

From the above propositions, we obtain the following main result.

Theorem 1. *An* SCC F on $\mathcal{R}^n$ *is* Nash-implementable by an s-mechanism *if and only if it satisfies* Condition μ^s_r *for each* $r = 1, \ldots, n-2$.

Furthermore, we can see that the class of SCCs Nash-implementable by s-mechanisms is not a proper subset of the class of Nash-implementable SCCs.

Lemma 1. *Let F be an* SCC *defined on $\mathcal{R}^n$. Then, for each $r = 1, \ldots, n-2$,* Condition μ^s_r *is equivalent to* Condition μ.

Proof. Take any $r = 1, \ldots, n-2$. Let F on $\mathcal{R}^n$ be an SCC satisfying Condition μ_r^s. Then, F is Nash-implementable by an s-mechanism via Theorem 1. Therefore, it is Nash-implementable. By Moore and Repullo (1990)'s result, it follows that F satisfies Condition μ. Conversely, let F be an SCC satisfying Condition μ. For any $\ell \in N$, $R \in \mathcal{R}^n$ and $x \in F(R)$, define the set $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+k\}}, x\right)$ as follows

$$C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \equiv \bigcup_{R'_{\{\ell+1,\ldots,\ell+r\}} \in F^{-1}\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)} C_\ell\left(\left(R_{-\{\ell+1,\ldots,\ell+r\}}, R'_{\{\ell+1,\ldots,\ell+r\}}\right), x\right). \tag{2}$$

We prove that F satisfies μ_r^s. Let $Y = g(M)$. Moreover, take any $R \in \mathcal{R}^n$ and $x \in F(R)$. It follows from (2) and μ that, for each $\ell \in N$, $x \in C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell, x) \cap Y$ and $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$ is well-defined. Moreover, it follows from (2) that F satisfies μ_r^s(iv). Next, we show that F meets μ_r^s(i)-μ_r^s(iii). Take any $R^* \in \mathcal{R}^n$.

Let $C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$. Since $x \in F(R)$, it follows from μ and (2) that $C_\ell(R, x) \subseteq C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)$ for all $\ell \in N$. Then, μ(i) implies that $x \in F(R^*)$, as we sought. Therefore, μ_r^s(i) holds. Let $y \in C_i\left(R_{-\{i+1,\ldots,i+r\}}, x\right) \subseteq L(R_i^*, y)$ for some $i \in N$ and $Y \subseteq L(R_\ell^*, y)$ for all $\ell \in N \setminus \{i\}$. As $y \in C_i\left(R_{-\{i+1,\ldots,i+r\}}, x\right)$, it follows from (2) and μ that $y \in C_i\left(\left(R_{-\{i+1,\ldots,i+r\}}, R'_{\{i+1,\ldots,i+r\}}\right), x\right) \subseteq C_i\left(R_{-\{i+1,\ldots,i+r\}}, x\right)$ for some $R'_{\{i+1,\ldots,i+r\}} \in F^{-1}\left(R_{-\{i+1,\ldots,i+r\}}, x\right)$. Then, μ(ii) implies that $y \in F(R^*)$. Thus, μ_r^s(ii) is satisfied. Let $y \in Y \subseteq L(R_\ell^*, y)$ for all $\ell \in N$. Then, μ(iii) implies that $y \in F(R^*)$, and so μ_r^s(iii) is met. We conclude that F satisfies μ_r^s if it satisfies μ. ■

From Theorem 1 and Lemma 1, the following corollaries are easily obtained.

Corollary 1. *An* SCC F on $\mathcal{R}^n$ *is* Nash-implementable by an s-mechanism *if and only if it is* Nash-implementable.

Corollary 2. *An* SCC F on $\mathcal{R}^n$ *is* Nash-implementable by an s-mechanism *if and only if it satisfies* Condition μ.

Note that we can also show that for any intermediate strategy space reduction mechanism between the canonical and the s-mechanisms, Nash implementation by such an intermediate one is equivalent to Nash implementation. Indeed, let us consider any intermediate strategy space reduction mechanism, say q-mechanism, with the strategy space $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times \ldots \times \mathcal{R}_{\ell+q} \times Y \times N$

for all $\ell \in N$, where $q = 2, \ldots, n-2$.[8] Then, it can be shown by a similar way to the proofs of Propositions 1 and 2, that an SCC satisfies Condition μ_r^s for each $r = q, q+1, \ldots, n-2$ if and only if it is Nash-implementable by q-mechanisms. Thus, because of Lemma 1, for each $q = 2, \ldots, n-2$, an SCC is Nash-implementable by q-mechanisms if and only if it is Nash-implementable.

4 An alternative characterization

Using the approach developed by Moore and Repullo (1990), we now introduce an alternative condition, labelled Condition M^s, to characterize implementability by s-mechanisms. The condition can be stated as follows.

Condition M^s (for short, M^s). An SCC F satisfies M^s if there exists a set $Z \subseteq X$, and for all $R \in \mathcal{R}^n$ and for all $x \in F(R)$, there is a profile of sets $(C_\ell^*(R_\ell, x))_{\ell \in N}$ such that $x \in C_\ell^*(R_\ell, x) \subseteq L(R_\ell, x) \cap Z$ for all $\ell \in N$; and for all $R^* \in \mathcal{R}^n$:
(i) if $C_\ell^*(R_\ell, x) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i^*(R_i, x) \subseteq L(R_i^*, y)$ and $Z \subseteq L(R_\ell^*, y)$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Z \subseteq L(R_\ell^*, y)$ for all $\ell \in N$, then $y \in F(R^*)$.

Instead of the profile $\left(C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)\right)_{\ell \in N}$ introduced in Condition μ_r^s, the above condition introduces the profile $(C_\ell^*(R_\ell, x))_{\ell \in N}$ which corresponds to the case $\left(C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)\right)_{\ell \in N}$ with $r = n-1$.[9] As for the profile $\left(C_\ell\left(R_{-\{\ell+1,\ldots,\ell+r\}}, x\right)\right)_{\ell \in N}$ we also show that $(C_\ell^*(R_\ell, x))_{\ell \in N}$ is well-defined and can be constructed by using the profile $(C_\ell(R, x))_{\ell \in N}$ given in Condition μ. Note that the profile $(C_\ell^*(R_\ell, x))_{\ell \in N}$ is similar to the profile of Condition M devised by Sjöstrom (1991). Finally, we show that Condition M^s is equivalent to Condition μ_r^s.

Before stating our next result, it may be worth mentioning here that Condition M^s do not include any condition of type of Condition μ_r^s(iv). For this reason - and in contrast to the proof of Proposition 2-, the proof of sufficiency of Condition M^s exploits in full the information coming from the

[8] According to this terminology, $q = 1$ corresponds to s-mechanisms while $q = n-1$ corresponds to canonical mechanisms. Both of the cases are excluded from the naming of q-mechanisms, since we are interested in intermediate strategy space reduction solely.

[9] The definition of Condition μ_r^s excludes this case.

cyclic announcement of strategies. This is done by constructing a mechanism which turn to be different from the one designed in Proposition 2. Note that the next result can be shown without imposing forthrightness.

Theorem 2. *An* SCC F *on* $\mathcal{R}^n$ *satisfies* M^s *if and only if it is* Nash-implementable by an s-mechanism*.*

Proof. Since the necessity of Condition M^s can be easily obtained by following the proof of the necessity of Condition μ given by Moore and Repullo (1990) we omit it here.

Conversely, suppose that F satisfies Condition M^s. We show that F is Nash-implementable via an s-mechanism. For, define the outcome function g as in Proposition 2 where *Rule 3* is replaced by the following one:

Rule 3:* For some $i \in N$, m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, and $C_i^*\left(R_i^{i-1}, x\right) \neq Z$, with $i-1=n$ if $i=1$, then

$$g(m) = \begin{cases} x^i & \text{if} \quad x^i \in C_i^*\left(R_i^{i-1}, x\right) \\ x & \qquad\text{otherwise} \end{cases}.$$

The proof follows the same arguments as those provided in the proof of Proposition 2 except for the *sub-case 2.2*, in which the case $x \in F\left(\bar{R}'\right)$ is considered. Therefore, we provide only the proof of this sub-case while we omit all others here.

Suppose that $m \in NE(\gamma, R)$ and m falls into *Rule 2* such that $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$. Let $R_i^i = R_i'$ and $x \in F\left(\bar{R}'\right)$ where $\bar{R}' \equiv \left(\bar{R}_{-i}, R_i'\right)$. Then, $i-1$ or i is the deviator. Agent $\ell \in N\backslash\{i-1, i\}$ can attain any $y \in Z\backslash\{x\}$ by inducing *Rule 4*, so that $x \in \max_{R_\ell} Z$ as $m \in NE(\gamma, R)$. Consider agent $i-1$. Take any $y \in C_{i-1}^*\left(\bar{R}_{i-1}, x\right) = C_{i-1}^*\left(R_{i-1}^{i-2}, x\right)$. Suppose that $C_{i-1}^*\left(\bar{R}_{i-1}, x\right) \neq Z$. By changing m_{i-1} to $m_{i-1}^* = \left(R_{i-1}^{i-1}, R_i^{i-1}, y, \diamond\right) \in M_{i-1}$, agent $i-1$ can obtain $y = g\left(m_{i-1}^*, m_{-(i-1)}\right)$ via *Rule 3*. In the case that $C_{i-1}^*\left(\bar{R}_{i-1}, x\right) = Z$, by changing m_{i-1} to $m_{i-1}^* = \left(R_{i-1}^{i-1}, R_i^{i-1}, y, k^{i-1}\right) \in M_{i-1}$, agent $i-1$ can attain $y = g\left(m_{i-1}^*, m_{-(i-1)}\right)$ with appropriately choosing k^{i-1}. It follows that $C_{i-1}^*\left(\bar{R}_{i-1}, x\right) \subseteq g\left(M_{i-1}, m_{-(i-1)}\right)$, so that $C_{i-1}^*\left(\bar{R}_{i-1}, x\right) \subseteq L(R_{i-1}, x)$ as $m \in NE(\gamma, R)$. Consider agent i. Again, take any $y \in C_i^*\left(\bar{R}_i, x\right) = C_i^*\left(R_i^{i-1}, x\right)$. Suppose that $C_i^*\left(\bar{R}_i, x\right) \neq Z$. By changing m_i to $m_i^* = \left(R_i^i, R_{i+1}^i, y, \diamond\right) \in M_i$, agent i can obtain $y = g(m_i^*, m_{-i})$ via *Rule 3*. In the case that $C_i^*\left(\bar{R}_i, x\right) = Z$, by changing m_i to $m_i^* = \left(R_i^i, R_{i+1}^i, y, k^i\right) \in M_i$, agent i can attain $y = g(m_i^*, m_{-i})$ with appropriately choosing k^i. It follows that $C_i^*\left(\bar{R}_i, x\right) \subseteq g(M_i, m_{-i})$, so that $C_i^*\left(\bar{R}_i, x\right) \subseteq L(R_i, x)$ as $m \in NE(\gamma, R)$. Therefore, $x \in F(R)$ by M^s(i). This completes the proof. ■

Lemma 2. *Let* F *be an* SCC *defined on* $\mathcal{R}^n$*. Then,* Condition M^s *is equivalent to* Condition μ.

Proof. Let F on $\mathcal{R}^n$ be an SCC. First, suppose that F satisfies M^s. Then, by Theorem 2, F is Nash-implementable by an s-mechanism, and so it is Nash-implementable. By Moore and Repullo (1990)'s result it follows that F satisfies μ. Conversely, suppose that F satisfies μ. Then, F is Nash-implementable. For any $\ell \in N$, $R \in \mathcal{R}^n$ and $x \in F(R)$, let

$$F^{-1}(R_\ell, x) \equiv \left\{R'_{-\ell} \in \mathcal{R}^n_{-\ell} | x \in F\left(R_\ell, R'_{-\ell}\right)\right\}$$

where $\mathcal{R}^n_{-\ell} \equiv \mathcal{R}_1 \times ... \times \mathcal{R}_{\ell-1} \times \mathcal{R}_{\ell+1} \times ... \times \mathcal{R}_n$. For any $\ell \in N$, $R \in \mathcal{R}^n$ and $x \in F(R)$, define the set $C^*_\ell(R_\ell, x)$ as follows

$$C^*_\ell(R_\ell, x) \equiv \cup_{R'_{-\ell} \in F^{-1}(R_\ell, x)} C_\ell\left(\left(R_\ell, R'_{-\ell}\right), x\right). \tag{3}$$

We prove that F satisfies M^s. Let $Z = Y$. Moreover, take any $R \in \mathcal{R}^n$ and $x \in F(R)$. It follows from (3) and μ that, for each $\ell \in N$, $x \in C^*_\ell(R_\ell, x) \subseteq L(R_\ell, x) \cap Z$ and $C^*_\ell(R_\ell, x)$ is well-defined. Next, we show that F meets M^s(i)-M^s(iii). For, take any $R^* \in \mathcal{R}^n$.

Let $C^*_\ell(R_\ell, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$. Since $x \in F(R)$, it follows from μ and (3) that $C_\ell(R, x) \subseteq C^*_\ell(R_\ell, x)$ for all $\ell \in N$. Then, μ(i) implies that $x \in F(R^*)$, as we sought. Thus, M^s(i) holds. Let $y \in C^*_i(R_i, x) \subseteq L(R^*_i, y)$ for some $i \in N$ and $Z \subseteq L(R^*_\ell, y)$ for all $\ell \in N \backslash \{i\}$. As $y \in C^*_i(R_i, x)$, it follows from (3) and μ that $y \in C_i\left(R_i, R'_{-i}, x\right) \subseteq C^*_i(R_i, x)$ for some $R'_{-i} \in F^{-1}(R_i, x)$. Then, μ(ii) implies that $y \in F(R^*)$. Thus, M^s(ii) is satisfied. Let $y \in Z \subseteq L(R^*_\ell, y)$ for all $\ell \in N$. Then, μ(iii) implies that $y \in F(R^*)$, and so M^s(iii) holds. We conclude that F satisfies M^s if it satisfies Condition μ. ■

Form Lemma 1 and Lemma 2 the corollary stated below is readily obtained.

Corollary 3. *Let* F *be an* SCC *defined on* $\mathcal{R}^n$*. Then,* Condition M^s *is equivalent to* Condition μ^s.

5 Characterizing s-Mechanisms as the Lower-Bound Strategy Space Reduction Mechanisms

The last two sections show that the 'strategy space reduction' from the canonical mechanisms up to s-mechanisms does not make any effect on the class of Nash implementable SCCs. The purpose of this section is to show that such a property can no longer hold if a further step of the 'strategy space reduction' is taken. Indeed, if the *self-relevant mechanism* defined in Tatamitani (2000) is taken as a further step of the strategy space reduction from s-mechanisms, it can be shown that the class of Nash-implementable SCCs by self-relevant mechanisms is a proper subset of the class of Nash implementable SCCs. However, there is another type of further strategy space reduction that is relevant for the issue at hand. In what follows, we consider a strategy space reduction mechanism in which each agent reveals only her neighbor's preferences in addition to an outcome and an integer (neighbor's preference mechanism, np-mechanism) and examine the cost of using this kind of mechanisms on implementability of SCCs.

Definition 3. A mechanism $\gamma = (M, g)$ is neighbor's preference mechanism (np-mechanism) if, for any $\ell \in N$, $M_\ell \equiv \mathcal{R}_{\ell+1} \times Y \times N$, where $Y \subseteq X$ and $\ell + 1 = 1$ if $\ell = n$.

Definition 4. An SCC F is Nash-implementable by an np-mechanism if there exists an np-mechanism $\gamma = (M, g)$ such that, for all $R \in \mathcal{R}^n$:
i) $F(R) = NA(\gamma, R)$; and
ii) for all $x \in F(R)$, if $m_\ell = (R_{\ell+1}, x, \diamond) \in M_\ell$ for all $\ell \in N$, with $\ell + 1 = 1$ if $\ell = n$, then $m \in NE(\gamma, R)$ and $g(m) = x$.

Using the approach developed by Moore and Repullo (1990), we now introduce a condition, Condition μ^{np}, which turns out to be necessary for SCCs that are implementable by np-mechanisms in the three or more agents case. Before describing the condition and prove its necessity, we need addition notation. Given $(R, x) \in \mathcal{R}^n \times X$, define $D(R, x) \equiv \left\{\ell \in N | F^{-1}\left(R_{-(\ell+1)}, x\right) \neq \varnothing\right\}$. The condition is stated as follows.

Condition μ^{np} (for short, μ^{np}): An SCC F satisfies Condition μ^{np} if there exists a set $Y \subseteq X$, and for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile

of sets $\left(C_\ell\left(R_{-\{\ell+1\}},x\right)\right)_{\ell\in N}$ such that $x\in C_\ell\left(R_{-\{\ell+1\}},x\right)\subseteq L\left(R_\ell,x\right)\cap Y$ for all $\ell\in N$, with $\ell+1=1$ if $\ell=n$, and for all $R^*\in\mathcal{R}^n$:
(i) if $C_\ell\left(R_{-\{\ell+1\}},x\right)\subseteq L\left(R^*_\ell,x\right)$ for all $\ell\in N$, then $x\in F\left(R^*\right)$;
(ii) for all $i\in N$, if $y\in C_i\left(R_{-\{i+1\}},x\right)\subseteq L\left(R^*_i,y\right)$ and $Y\subseteq L\left(R^*_\ell,y\right)$ for all $\ell\in N\backslash\{i\}$, then $y\in F\left(R^*\right)$;
(iii) if $y\in Y\subseteq L\left(R^*_\ell,y\right)$ for all $\ell\in N$, then $y\in F\left(R^*\right)$,
(iv) for all $\ell\in N$ with $R^*_{-\{\ell+1\}}=R_{-\{\ell+1\}}$ and $x\in F\left(R^*\right)$, $C_\ell\left(R^*_{-\{\ell+1\}},x\right)=C_\ell\left(R_{-\{\ell+1\}},x\right)$;
(v) if $x\notin F\left(R^*\right)$ and $D\left(R^*,x\right)\neq\varnothing$, then there exists an outcome $p\left(R^*,x\right)\in X$ such that:
(a) $p\left(R^*,x\right)\in C_\ell\left(R^*_{-\{\ell+1\}},x\right)$ for any $\ell\in D\left(R^*,x\right)$; and
(b) for all $R^{**}\in\mathcal{R}^n$, if $C_i\left(R^*_{-\{i+1\}},x\right)\subseteq L\left(R^{**}_i,p\left(R^*,x\right)\right)$ for all $i\in D\left(R^*,x\right)$, and $Y\subseteq L\left(R^{**}_\ell,p\left(R^*,x\right)\right)$ for all $\ell\in N\backslash D\left(R^*,x\right)$, then $p\left(R^*,x\right)\in F\left(R^{**}\right)$.

Proposition 3. *An* SCC F on $\mathcal{R}^n$ *satisfies* Condition μ^{np} *if it is* Nash-implementable by an *np*-mechanism.

Proof. Let an SCC F on $\mathcal{R}^n$ be Nash-implementable by an *np*-mechanism. Let $\gamma=(M,g)$ be such an *np*-mechanism. Define $Y\equiv g\left(M\right)$. For all $R\in\mathcal{R}^n$ and $x\in F\left(R\right)$, there exists an $m\in NE\left(\gamma,R\right)$ such that $g\left(m\right)=x$ and $m_\ell=\left(R_{\ell+1},x,k^\ell\right)$ for all $\ell\in N$, by Definition 4(ii). Take any $\ell\in N$. For any $R'_{\ell+1}\in F^{-1}\left(R_{-\{\ell+1\}},x\right)$, there exists $m'_\ell=\left(R'_{\ell+1},x,k^\ell\right)$ such that $g\left(m'_\ell,m_{-\ell}\right)=x\in NA\left(\gamma,\left(R'_{\ell+1},R_{-\{\ell+1\}}\right)\right)$, by Definition 4(ii). Thus, for any $R'_{\ell+1}\in F^{-1}\left(R_{-\{\ell+1\}},x\right)$, $g\left(M_\ell,m_{-\ell}\right)\subseteq L\left(R_\ell,x\right)$. Moreover, $g\left(m'_\ell,m_{-\ell}\right)=x\in NA\left(\gamma,R\right)$ also holds for any $R'_{\ell+1}\in F^{-1}\left(R_{-\{\ell+1\}},x\right)$ and any $m'_\ell=\left(R'_{\ell+1},x,k^\ell\right)$.

Define $C_\ell\left(R_{-\{\ell+1\}},x\right)\equiv g\left(M_\ell,m_{-\ell}\right)$. Then, $x\in C_\ell\left(R_{-\{\ell+1\}},x\right)\subseteq L\left(R_\ell,x\right)\cap Y$. Moreover, for any $R^*\in\mathcal{R}^n$ with $R^*_{-\{\ell+1\}}=R_{-\{\ell+1\}}$ and $x\in F\left(R^*\right)$, it follows from $g\left(M_\ell,m_{-\ell}\right)\subseteq L\left(R_\ell,x\right)$ that $C_\ell\left(R_{-\{\ell+1\}},x\right)=g\left(M_\ell,m_{-\ell}\right)=C_\ell\left(R^*_{-\{\ell+1\}},x\right)$, thus F satisfies μ^{np}(iv).

Next, we show that F satisfies Conditions μ^{np}(i)-μ^{np}(iii). Take any $R^*\in\mathcal{R}^n$.

Suppose that $C_\ell\left(R_{-\{\ell+1\}},x\right)\subseteq L\left(R^*_\ell,x\right)$ for all $\ell\in N$. Then, it follows from $C_\ell\left(R_{-\{\ell+1\}},x\right)\equiv g\left(M_\ell,m_{-\ell}\right)$ that $g\left(M_\ell,m_{-\ell}\right)\subseteq L\left(R^*_\ell,x\right)$ for all $\ell\in N$. We conclude that $g\left(m\right)=x\in NA\left(\gamma,R^*\right)=F\left(R^*\right)$. Hence, μ^{np}(i) holds.

For each $i \in N$, let $y \in C_i\left(R_{-\{i+1\}}, x\right) \subseteq L\left(R_i^*, y\right)$ and $Y \subseteq L\left(R_\ell^*, y\right)$ for all $\ell \in N\backslash\{i\}$. It follows from $C_i\left(R_{-\{i+1\}}, x\right) \equiv g\left(M_i, m_{-i}\right)$ that $y \in g\left(M_i, m_{-i}\right)$. Moreover, $g(M) \subseteq L\left(R_\ell^*, y\right)$ for all $\ell \in N\backslash\{i\}$. It follows that $y \in NA\left(\gamma, R^*\right) = F\left(R^*\right)$. Hence, μ^{np}(ii) holds.

If $y \in Y \subseteq L\left(R_\ell^*, y\right)$ for all $\ell \in N$, then $y \in g(M) \subseteq L\left(R_\ell^*, y\right)$ for all $\ell \in N$. Thus, there is an $m^* \in M$ such that $g\left(m^*\right) = y$, which implies that $y \in NA\left(\gamma, R^*\right) = F\left(R^*\right)$. Hence, μ^{np}(iii) holds.

Suppose that $x \notin F\left(R^*\right)$ and $D\left(R^*, x\right) \neq \varnothing$ and consider the strategy profile $m_\ell^* = \left(R_{\ell+1}^*, x, k^\ell\right) \in M_\ell$ for all $\ell \in N$. Let $p\left(R^*, x\right) \equiv g\left(m^*\right)$. Consider any $i \in D\left(R^*, x\right)$ and $R_{i+1}' \in F^{-1}\left(R_{-(i+1)}^*, x\right)$. Let $\left(R_{-(i+1)}^*, R_{i+1}'\right) = R'$. Then, from the previous discussion, it follows that there exists a profile $\left(C_\ell\left(R_{-(\ell+1)}', x\right)\right)_{\ell \in N}$, with $C_\ell\left(R_{-(\ell+1)}', x\right) \equiv g\left(M_\ell, m_{-\ell}'\right)$ for all $\ell \in N$, where $m_\ell' = m_\ell^*$ for all $\ell \in N\backslash\{i\}$ and $m_i' = \left(R_{i+1}', x, k^i\right)$. As it holds for any $i \in D\left(R^*, x\right)$ it follows that $g\left(m^*\right) \in Y$ and $g\left(m^*\right) \in C_i\left(R_{-(i+1)}^*, x\right) \equiv g\left(M_i, m_{-i}^*\right)$. Finally, let $C_i\left(R_{-\{i+1\}}^*, x\right) \subseteq L\left(R_i^{**}, p\left(R^*, x\right)\right)$ for all $i \in D\left(R^*, x\right)$ and $Y \subseteq L\left(R_\ell^{**}, p\left(R^*, x\right)\right)$ for all $\ell \in N\backslash D\left(R^*, x\right)$. Then, since $g\left(M_i, m_{-i}^*\right) \subseteq L\left(R_i^{**}, g\left(m^*\right)\right)$ for all $i \in D\left(R^*, x\right)$ and $g(M) \subseteq L\left(R_\ell^{**}, g\left(m^*\right)\right)$ for all $\ell \in N\backslash D\left(R^*, x\right)$ it follows that $p\left(R^*, x\right) = g\left(m^*\right) \in NA\left(\gamma, R^{**}\right) = F\left(R^{**}\right)$. Thus, μ^{np}(v) holds. We conclude that F satisfies μ^{np}. ■

The above proposition implies that Nash implementation by an np-mechanism is no equivalent to Nash implementation, since the existence of the punishment condition, Condition μ^{np}(v), makes Condition μ^{np} stronger than Condition μ. That is, Condition μ^{np} implies Condition μ, but the converse does not hold. Therefore, Proposition 3 implies that the class of Nash-implementable SCCs by np-mechanisms is a proper subset of the class of Nash-implementable SCCs.[10] Moreover, combined with the characterization result of Nash implementation by self-relevant mechanisms in Tatamitani (2000), Proposition 3 indicates that no more 'strategy-space-reduction mechanisms' than s-mechanisms can preserve the Moore and Repullo (1990) full characterization of Nash implementation. In other words, the class of s-mechanisms represents the *lower-bound* of 'mechanisms with strategy space

[10]To see it, for instance, consider classical economic environments as the domain of SCCs. Then, as shown in Saijo, Tatamitani, and Yamato (1999), the no-envy and efficient correspondence does not satisfy μ^{np}-(v), though it satisfies μ.

reduction' which can work for Nash implementation of the class of SCCs satisfying Condition μ.

6 Concluding Remarks

In this paper, we deal with the informational efficiency issue pertaining to Maskin's Theorem (Maskin, 1999). We focus on s-mechanisms in which each agent reports to the planner her own preference and her neighbor's preference solely, in addition to a feasible social outcome and an integer. We introduce a class of new conditions, labelled $\{\text{Condition } \mu_r^s\}_{r=1,\ldots,n-2}$, each of which fully characterizes the class of SCCs Nash-implemantable by s-mechanisms. Surprisingly, for each $r = 1, \ldots, n-2$, Condition μ_r^s is equivalent to Condition μ. This has two important implications for Nash implementation. First, the class of Nash-implementable SCCs is equivalent to the class of SCCs Nash-implementable by s-mechanisms. Second, even though our condition is stated in terms of the existence of certain sets, it can easily be checked in practice by the algorithm provided by Sjöström (1991).

Note that our results are in line with other well known results of Nash implementation in economic environments. In particular, the equivalent relationship between Nash implementation by s-mechanism and Nash implementation in general social choice environments is analogous to the equivalent relationship between Nash implementation by natural allocation mechanisms and Nash implementation by natural quantity^2 mechanisms (Saijo et al, 1996). Moreover, Tatamitani (2001) provides a full characterization of Nash implementation by self-relevant mechanisms, which together with Proposition 3 in this paper indicates that *any* further 'strategy space reduction' from s-mechanisms drastically decreases the class of Nash-implementable SCCs. This is parallel to the case of natural implementation in economic environments, in which the class of SCCs Nash-implementable by natural quantity mechanisms is much smaller than the Nash-implementable ones by natural quantity^2 mechanisms.

In contrast, whenever a small departure from the standard framework of implementation theory is considered the above relationship may break down. For example, Matsushima (2008) and Dutta and Sen (2009) introduce the notion of partial honesty in implementation theory and consider Nash implementation problems with partially-honest agents. A partially-honest agent is an agent who has preferences over message profiles and displays concerns for two dimensions in lexicographic order: (1) her outcome and (2) her truth-telling behavior. In the presence of partially honest agents, the equivalent relationship between Nash implementation and Nash implementation by s-mechanisms no longer holds, as Lombardi and Yoshihara (2011) show. This suggests that the equivalent relationship indispensably relies on the assumption that agents act purely to advance their own self-interest and are not inclined to attach (moral) rights and duties to their actions.

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