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A Full Characterization of Nash Implementation with Strategy Space Reduction

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A Full Characterization of Nash Implementation with Strategy Space Reduction*

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Abstract

Noting that a full characterization of Nash-implementation is given using a canonical-mechanism and Maskin's theorem (Maskin, 1999) is shown using a mechanism with Saijo's type of strategy space reduction (Saijo, 1988), this paper fully characterizes the class of Nash-implementable social choice correspondences (SCCs) by mechanisms with the strategy space reduction, which is further shown to be equivalent to the class of Nash-implementable SCCs.

JEL classification: C72; D71; D82.
Keywords: Nash implementation, strategy space reduction, s-mechanisms, Condition $\boldsymbol{\mu}^{\mathbf{s}}$

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1 Introduction

In Nash implementation theory, it is *Maskin's Theorem* (Maskin, 1999) which shows that when the planner faces at least three agents, a *social choice correspondence* (SCC) is implementable in (pure-strategy) Nash equilibria (henceforth, Nash-implementable) if it satisfies *Maskin monotonicity* and *no-veto power*; conversely, any Nash-implementable SCC is Maskin-monotonic. Two issues pertaining to this theorem stand out. First, it does not provide a complete characterization of Nash-implementable SCCs, since no-veto power is not necessary for Nash implementation. Second, a *canonical mechanism* proposed in this theorem, which requires each agent to report a preference profile, a feasible social outcome, and an integer, is not so attractive. This is because the message space of this mechanism is rather large and announcing all other agents' preferences is undesirable in terms of the informational efficiency of decentralized decision making (on this point see, for instance, Hurwicz, 1960).

Moore and Repullo (1990) address the first issue by providing, without any domain restriction, a necessary and sufficient condition, called *Condition* $\boldsymbol{\mu}$, for Nash implementability of SCCs in societies with more than two agents.[1] In contrast to the first issue, the issue of informational efficiency is addressed by Saijo (1988), which shows that proposing a *mechanism with strategy-space reduction* (henceforth, *s-mechanism*) would suffice to guarantee Maskin's Theorem. Note that, in s-mechanisms, each agent is requested to announce, in addition to a feasible social outcome and an integer, her own and her neighbor's preferences solely. Yet, as Moore and Repullo (1990) also use a canonical mechanism for showing the full characterization and Saijo (1988) does not discuss a full characterization of Nash implementation, it leaves unclear not only whether Moore and Repullo's result indispensably relies on canonical mechanisms but also whether s-mechanisms can Nash-implement any other SCC than Maskin-monotonic and no-veto power ones.

In this paper, we address the issue of what constitutes the necessary and sufficient condition for Nash implementation by s-mechanisms. We introduce a new condition (labelled, *Condition* $\boldsymbol{\mu}^{\mathbf{s}}$) which fully characterizes the class

[1]Note that, for two person societies, Moore and Repullo (1990) and Dutta and Sen (1991) independently provided necessary and sufficient conditions for Nash implementation, whereas even in societies with more than two agents, there are other works on complete characterizations of Nash implementation under some domain restrictions, such as Danilov (1992) and Yamato (1992).

of SCCs Nash-implemantable by s-mechanisms. Surprisingly, Condition $\boldsymbol{\mu}^{\mathbf{s}}$ is equivalent to Condition $\boldsymbol{\mu}$. This implies that the full characterization by Moore and Repullo (1990) works even if canonical mechanisms are excluded and the available class of mechanisms is restricted to that of s-mechanisms.

The paper is organized as follows. In section 2, we introduce notation and definitions. In Section 3, we state and prove our results.

2 Preliminaries

The set of (*social choice) environments* is $(N, X, \mathcal{R}^n)$, where $N \equiv \{1, ..., n\}$ is a set of $n \geq 3$ *agents*, $X \equiv \{x, y, z, ...\}$ is the set of attainable *alternatives* (or *outcomes)*, and $\mathcal{R}^n$ is the set of admissible *preference profiles* (or *states of the world*). Henceforth, we assume that the cardinality of X is $\#X \geq 2$. Let $\mathcal{R}(X)$ be the set of all complete preorders on X. We assume that $\mathcal{R}^n \equiv \mathcal{R}_1 \times ... \times \mathcal{R}_n$ is a non-empty subset of the n-fold Cartesian product $\mathcal{R}^n(X) \equiv \underbrace{\mathcal{R}(X) \times .... \times \mathcal{R}(X)}_{n\text{-times}}$. An element of $\mathcal{R}^n$ is denoted by $R \equiv (R_1, ..., R_n)$, where its ℓ-th component is $R_\ell \in \mathcal{R}_\ell$, for each $\ell \in N$. For any preference profile $R \in \mathcal{R}^n$ and any $\ell \in N$, let $R_{-\ell}$ be the list of elements of R for all agents except ℓ, i.e., $R_{-\ell} \equiv (R_1, ..., R_{\ell-1}, R_{\ell+1}, ..., R_n)$. Given a list $R_{-\ell}$ and $R_\ell \in \mathcal{R}_\ell$, we denote by $(R_{-\ell}, R_\ell)$ the preference profile consisting of these R_ℓ and $R_{-\ell}$. For any $(R_\ell, x) \in \mathcal{R}_\ell \times X$, agent ℓ's *weakly lower contour set* of R_ℓ at x is given by $L(R_\ell, x) \equiv \{y \in X | (x, y) \in R_\ell\}$. For each $\ell \in N$ and each $R_\ell \in \mathcal{R}_\ell$, $\max_{R_\ell} X \equiv \{x \in X | (x, y) \in R_\ell \text{ for all } y \in X\}$.

We also assume that N and X are fixed throughout the following discussion, so that the set of environments is boiled down to $\mathcal{R}^n$. A *social choice correspondence* (SCC) is a correspondence $F : \mathcal{R}^n \twoheadrightarrow X$ with $F(R) \neq \varnothing$ for all $R \in \mathcal{R}^n$. An SCC F is (*Maskin-*)*monotonic* if, for all $R, R' \in \mathcal{R}^n$ with $x \in F(R)$, we have that $x \in F(R')$ whenever $L(R_\ell, x) \subseteq L(R'_\ell, x)$ for all $\ell \in N$.[2] An SCC F satisfies *no-veto power* if, for all $R \in \mathcal{R}^n$, we have that $x \in F(R)$ whenever $x \in \max_{R_\ell} X$ for at least $n-1$ agents.

A mechanism (or game-form) is a pair $\gamma \equiv (M, g)$, where $M \equiv M_1 \times ... \times M_n$, and $g : M \to X$ is the *outcome function*. Denote a generic message (or strategy) for agent ℓ by $m_\ell \in M_\ell$ and a generic message profile by $m = (m_1, ..., m_n) \in M$. For any $m \in M$ and $\ell \in N$, let $m_{-\ell} \equiv (m_1, ..., m_{\ell-1}, m_{\ell+1}, ..., m_n)$. Let $M_{-\ell} \equiv \times_{j \in N \setminus \{\ell\}} M_j$. Given $m_{-\ell} \in M_{-\ell}$ and

[2] Weak set inclusion is denoted by $\subseteq$.

$m_\ell \in M_\ell$, denote by $(m_\ell, m_{-\ell})$ the message profile consisting of these m_ℓ and $m_{-\ell}$. Given $R \in \mathcal{R}^n$ and $\gamma = (M, g)$, (γ, R) constitutes a (*non-cooperative*) *game*. Given a game (γ, R), $m \in M$ is a (pure strategy) *Nash equilibrium* of (γ, R) if and only if, for all $\ell \in N$, $(g(m), g(m'_\ell, m_{-\ell})) \in R_\ell$ for all $m'_\ell \in M_\ell$. Let $NE(\gamma, R)$ denote the set of Nash equilibria of (γ, R), whereas denote the set of *Nash equilibrium outcomes* of (γ, R) by $NA(\gamma, R) \equiv g(NE(\gamma, R))$.

A mechanism $\gamma = (M, g)$ *implements* F in *Nash equilibria*, or simply *Nash-implements* F, if and only if $NA(\gamma, R) = F(R)$ for all $R \in \mathcal{R}^n$. An SCC F is *Nash-implementable* if there is such a mechanism.

Moore and Repullo (1990) show that, under the society with more than two agents, the following condition is the necessary and sufficient condition for any SCC to be Nash-implemetable.

Condition $\boldsymbol{\mu}$ (for short, $\boldsymbol{\mu}$): An SCC F satisfies $\boldsymbol{\mu}$ if there exists a set $Y \subseteq X$, and for all $R \in \mathcal{R}^n$ and for all $x \in F(R)$, there is a profile of sets $(C_\ell(R, x))_{\ell \in N}$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$, and for any $R^* \in \mathcal{R}^n$:
(i) if $C_\ell(R, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for each $i \in N$, if $y \in C_i(R, x) \subseteq L(R^*_i, y)$ and $Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Y \subseteq L(R^*_\ell, y)$ for all $\ell \in N$, then $y \in F(R^*)$.

3 Results

Following Saijo (1988), we focus on mechanisms in which each agent reports her own preference $R_\ell \in \mathcal{R}_\ell$, her neighbor's preference $R_{\ell+1} \in \mathcal{R}_{\ell+1}$, an outcome $x \in Y \subseteq X$, and an integer $k \in N$.

Definition 1: *A mechanism (M, g) is* s-mechanism *if, for any $\ell \in N$, $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times Y \times N$, where $Y \subseteq X$ and $\ell + 1 = 1$ if $\ell = n$.*

Definition 2: *An SCC F is* Nash-implementable by an s-mechanism *if there exists an s-mechanism (M, g) such that:*
i) for all $R \in \mathcal{R}^n$, $F(R) = NA(\gamma, R)$; and
ii) for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, if $m_\ell = \left(R_\ell, R_{\ell+1}, x, k^\ell\right) \in M_\ell$ for all $\ell \in N$, with $\ell + 1 = 1$ if $\ell = n$, then $m \in NE(\gamma, R)$ and $g(m) = x$.

We now introduce a condition, labelled **Condition $\boldsymbol{\mu}^{\mathbf{s}}$**, to characterize Nash implementability by s-mechanisms. The condition can be stated as follows.

Condition $\boldsymbol{\mu}^{\mathbf{s}}$ (for short, $\boldsymbol{\mu}^{\mathbf{s}}$): An SCC F satisfies condition $\boldsymbol{\mu}^{\mathbf{s}}$ if there exists a set $Y \subseteq X$, and for all $R \in \mathcal{R}^n$ and for all $x \in F(R)$, there is a profile of sets $\left(C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right)\right)_{\ell \in N}$ such that $x \in C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$, with $\ell + 1 = n$ if $\ell = n$, and for all $R^* \in \mathcal{R}^n$:
(i) if $C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i\left(R_{-\{i,i+1\}}, x\right) \subseteq L(R_i^*, y)$ and $Y \subseteq L(R_\ell^*, y)$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in Y \subseteq L(R_\ell^*, y)$ for all $\ell \in N$, then $y \in F(R^*)$.

Proposition 1. *An SCC F satisfies $\boldsymbol{\mu}^{\mathbf{s}}$ if it is Nash-implementable by an s-mechanism.*

Proof. Let an SCC F be Nash-implementable by an s-mechanism. Then, since it is Nash-implementable, it satisfies $\boldsymbol{\mu}$. Thus, there exists a set $Y \subseteq X$, and for all $R \in \mathcal{R}^n$ and for all $x \in F(R)$, there is a profile of sets $(C_\ell(R, x))_{\ell \in N}$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$. Moreover, for any $R^* \in \mathcal{R}^n$, $\boldsymbol{\mu}$(i)-(iii) are satisfied. Now, for each $R \in \mathcal{R}^n$ and each $x \in F(R)$, let $\left(C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right)\right)_{\ell \in N}$ be defined as $C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right) \equiv C_\ell(R, x)$ for each $\ell \in N$. Then, F satisfies $\boldsymbol{\mu}^{\mathbf{s}}$. ■

Proposition 2. *An SCC F satisfying $\boldsymbol{\mu}^{\mathbf{s}}$ is Nash-implementable by an s-mechanism.*

Proof. Let $\gamma \equiv (M, g)$ be an s-mechanism. Suppose that F satisfies $\boldsymbol{\mu}^{\mathbf{s}}$.

Fix any $m \in M$, $R \in \mathcal{R}^n$, and $x \in X$, and let $m_\ell = \left(R_\ell^\ell, R_{\ell+1}^\ell, x^\ell, k^\ell\right) \in M_\ell$, where $\ell + 1 = 1$ if $\ell = n$, and where the announcement of agent $\ell \in N$ about agent $\ell + 1$'s preferences is $R_{\ell+1}^\ell$. We say that the message profile $m \in M$ is:

(i) *consistent* with R and x if, for all $\ell \in N$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$ and $x^\ell = x$, where $\ell - 1 = n$ if $\ell = 1$;
(ii) m_{-i} *quasi-consistent* with x and R, where $i \in N$, if for all $\ell \in N$, $x^\ell = x$, and for all $\ell \in N \backslash \{i, i+1\}$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$, $R_i^{i-1} = R_i$, $R_{i+1}^{i+1} = R_{i+1}$, and $[R_i^i \neq R_i$ or $R_i^i \neq R_{i+1}]$, where $j - 1 = n$ if $j = 1$ for $j \in \{i, \ell\}$;
(iii) m_{-i} *consistent* with x and R, where $i \in N$, if for all $\ell \in N \backslash \{i\}$, $x^\ell = x \neq x^i$, and for all $\ell \in N \backslash \{i, i+1\}$, $R_\ell^\ell = R_\ell^{\ell-1} = R_\ell$, $R_i^{i-1} = R_i$ and $R_{i+1}^{i+1} = R_{i+1}$, where $j - 1 = n$ if $j = 1$ for $j \in \{i, \ell\}$.

Define the outcome function $g : M \to X$ as follows: For any $m \in M$,

Rule 1: m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, then $g(m) = x$.

Rule 2: For some $i \in N$, m_{-i} is quasi-consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, then $g(m) = x$.
Rule 3: For some $i \in N$, m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$, and $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \neq Y$, then

$$g(m) = \begin{cases} x^i & \text{if} \quad x^i \in C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \\ x & \text{otherwise} \end{cases}.$$

Rule 4: Otherwise, $g(m) = x^{\ell^*(m)}$ where $\ell^*(m) \equiv \sum_{i \in N} k^i \pmod{n}$.[3]

Since F satisfies $\boldsymbol{\mu}^{\mathbf{s}}$, it follows that, for any $R \in \mathcal{R}^n$ and any $x \in F(R)$, $x \in Y$. We show that $\gamma = (M, g)$ Nash-implements F. For, let $R \in \mathcal{R}^n$.

To show that $F(R) \subseteq NA(\gamma, R)$, let $x \in F(R)$ and suppose that, for all $\ell \in N$, $m_\ell = (R_\ell, R_{\ell+1}, x, \diamond)$, where $\diamond \in N$ is an arbitrary agent index. Since m is consistent with x and R and $x \in F(R)$, it follows from **Rule 1** that $g(m) = x$. Suppose that $\ell \in N$ deviates from m_ℓ to $m_\ell^* = \left(R_\ell^\ell, R_{\ell+1}^\ell, x^\ell, \diamond\right) \in M_\ell$ such that $(R_\ell, R_{\ell+1}, x) \neq \left(R_\ell^\ell, R_{\ell+1}^\ell, x^\ell\right)$. It follows from **Rules 2-3** that $g(M_\ell, m_{-\ell}) = C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right)$ if $C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right) \neq Y$. It is obvious that $g(M_\ell, m_{-\ell}) \subseteq C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right)$ if $C_\ell\left(R_{-\{\ell,\ell+1\}}, x\right) = Y$. Since F satisfies $\boldsymbol{\mu}^{\mathbf{s}}$, it follows that $g(M_\ell, m_{-\ell}) \subseteq L(R_\ell, x)$. As it holds for any $\ell \in N$, we have $m \in NE(\gamma, R)$ and so $x \in NA(\gamma, R)$. Furthermore, this guarantees the condition (ii) of **Definition 2**.

Conversely, to show that $NA(\gamma, R) \subseteq F(R)$, let $m \in NE(\gamma, R)$. Consider the following cases.

Case 1: m falls into **Rule 1**.

Then, m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. Take any $\ell \in N$. Suppose that $C_\ell\left(\bar{R}_{-\{\ell,\ell+1\}}, x\right) \neq Y$. For any $y \in C_\ell\left(\bar{R}_{-\{\ell,\ell+1\}}, x\right)$, changing m_ℓ for $m_\ell^* = \left(R_\ell^\ell, R_{\ell+1}^\ell, y, \diamond\right) \in M_\ell$ agent ℓ can obtain $y = g(m_i^*, m_{-i})$, by **Rule 3**. In case $C_i\left(R_{-\{i,i+1\}}, x\right) = Y$, agent ℓ can attain any $y \in Y$ by **Rule 4**. Thus, $C_\ell\left(\bar{R}_{-\{\ell,\ell+1\}}, x\right) = g(M_\ell, m_{-\ell})$ for all $\ell \in N$. As $m \in NE(\gamma, R)$, $C_\ell\left(\bar{R}_{-\{\ell,\ell+1\}}, x\right) \subseteq L(R_\ell, x)$ for all $\ell \in N$. Therefore, $x \in F(R)$ by $\boldsymbol{\mu}^{\mathbf{s}}$(i).

Case 2: m falls into **Rule 2**.

Then m is m_{-i} quasi-consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. We proceed according the following sub-cases: 1) $R_i^i \neq \bar{R}_i$ and $R_i^i \neq \bar{R}_{i+1}$, and 2) $R_i^i \neq \bar{R}_i$ and $R_i^i = \bar{R}_{i+1}$.[4]

[3] If the remainder is *zero* the winner of the game is agent n.

Sub-case 2.1. $R_i^i \neq \bar{R}_i$ and $R_i^i \neq \bar{R}_{i+1}$

Any $\ell \in N\backslash\{i\}$ can attain any $y \in Y\backslash\{x\}$ by inducing **Rule 4**, so that $x \in \max_{R_\ell} Y$ as $m \in NE(\gamma, R)$. Take any $y \in C_i\left(\bar{R}_{-\{i,i+1\}}, x\right)$. Suppose that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \neq Y$. By changing m_i for $m_i^* = \left(R_i^i, R_{i+1}^i, y, \diamond\right) \in M_i$ agent i can obtain $y = g\left(m_i^*, m_{-i}\right)$, by **Rule 3**. In the case that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) = Y$, by changing m_i for $m_i^* = \left(R_i^i, R_{i+1}^i, y, k^i\right) \in M_i$ agent i can attain $y = g\left(m_i^*, m_{-i}\right)$ by appropriately choosing k^i. It follows that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq g\left(M_i, m_{-i}\right)$. Moreover, as $m \in NE(\gamma, R)$, $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq L\left(R_i, x\right)$. Therefore, $x \in F(R)$ by either $\boldsymbol{\mu}^{\mathbf{s}}$(ii) or $\boldsymbol{\mu}^{\mathbf{s}}$(iii).

Sub-case 2.2. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$

Let $R_i^i = R_i'$. We distinguish whether $x \in F\left(\bar{R}'\right)$ where $\bar{R}' \equiv \left(\bar{R}_{-i}, R_i'\right)$ or not. Suppose that $x \notin F\left(\bar{R}'\right)$. Then the same reasoning used above for **sub-case 2.1** carries over into this sub-case, so that $x \in F(R)$. Otherwise, let $x \in F\left(\bar{R}'\right)$. Then, $i-1$ or i is the deviator. Agent $\ell \in N\backslash\{i-1, i\}$ can attain any $y \in Y\backslash\{x\}$ by inducing **Rule 4**, so that $x \in \max_{R_\ell} Y$ as $m \in NE(\gamma, R)$. Since $x \in F\left(\bar{R}\right)$, there exists $C_\ell\left(\bar{R}_{-\{\ell,\ell+1\}}, x\right) \subseteq Y$ for each $\ell \in N\backslash\{i-1, i\}$, and so $C_\ell\left(\bar{R}_{-\{\ell,\ell+1\}}, x\right) \subseteq L\left(R_\ell, x\right)$ by $Y \subseteq L\left(R_\ell, x\right)$ for each $\ell \in N\backslash\{i-1, i\}$. Observe that $\bar{R}_{-\{i,i+1\}} = \bar{R}'_{-\{i,i+1\}}$ and $\bar{R}_{-\{i-1,i\}} = \bar{R}'_{-\{i-1,i\}}$, so that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) = C_i\left(\bar{R}'_{-\{i,i+1\}}, x\right)$ and $C_{i-1}\left(\bar{R}_{-\{i-1,i\}}, x\right) = C_{i-1}\left(\bar{R}'_{-\{i-1,i\}}, x\right)$. Consider agent $i-1$ and take any $y \in C_{i-1}\left(\bar{R}_{-\{i-1,i\}}, x\right)$. Let $C_{i-1}\left(\bar{R}_{-\{i-1,i\}}, x\right) \neq Y$. By changing m_{i-1} into $m_{i-1}^* = \left(R_{i-1}^{i-1}, R_i^{i-1}, y, \diamond\right) \in M_{i-1}$ agent $i-1$ can obtain $y = g\left(m_{i-1}^*, m_{-(i-1)}\right)$, by **Rule 3**. In the case that $C_{i-1}\left(\bar{R}_{-\{i-1,i\}}, x\right) = Y$, by changing m_{i-1} for $m_{i-1}^* = \left(R_{i-1}^{i-1}, R_i^{i-1}, y, k^i\right) \in M_i$, agent $i-1$ can attain $y = g\left(m_{i-1}^*, m_{-(i-1)}\right)$ with appropriate choice of k^{i-1}. Therefore, $C_{i-1}\left(\bar{R}'_{-\{i-1,i\}}, x\right) \subseteq g\left(M_{i-1}, m_{-(i-1)}\right)$. By the same reasoning, we have that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq g\left(M_i, m_{-i}\right)$. Moreover, it follows from $m \in NE(\gamma, R)$ that $C_{i-1}\left(\bar{R}_{-\{i-1,i\}}, x\right) \subseteq L\left(R_{i-1}, x\right)$ and $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq L\left(R_i, x\right)$. Therefore, $x \in F(R)$ by $\boldsymbol{\mu}^{\mathbf{s}}$(i).

Case 3: m falls into **Rule 3**.

Then m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Therefore, $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \neq Y$. First, we show that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq g\left(M_i, m_{-i}\right)$. For any $x^i \in C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \setminus \{x\}$, consider $m_i^* = \left(R_i^i, R_{i+1}^i, x^i, \diamond\right)$.

[4] The sub-case $R_i^i = \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$ is not explicitly considered as it can be proved similarly to the **sub-case 2.2** shown below.

Then, **Rule 3** implies that $g(m_{-i}, m_i^*) = x^i$. On the other hand, to attain x agent i can induce **Rule 1** by changing m_i to $m_i^* = \left(\bar{R}_i, \bar{R}_{i+1}, x, \diamond\right)$ so that $g(m_{-i}, m_i^*) = x$. Hence, $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq g(M_i, m_{-i})$.

Next, we claim that $g(M_\ell, m_{-\ell}) = Y$ for any $\ell \in N \setminus \{i\}$. We proceed according to whether $\#Y = 2$ and $n = 3$ or not.

Sub-case 3.1. *not*[$\#Y = 2$ and $n = 3$]

Suppose that $\#Y > 2$. Take any $\ell \in N \setminus \{i\}$. Then, agent ℓ can induce the modulo game by choosing any $y \in Y \setminus \{x, x^i\}$ and changing m_ℓ into $m_\ell^* = \left(R_\ell^\ell, R_{\ell+1}^\ell, y, k^\ell\right)$. To attain y agent ℓ has only to adjust k^ℓ by which $\ell^*(m_{-\ell}, m_\ell^*) = \ell$. To attain x (resp., x^i) agent ℓ has only to adjust k^ℓ by which $\ell^*(m_{-\ell}, m_\ell^*) = j$ for $j \in N \setminus \{\ell, i\}$ (resp., $\ell^*(m_{-\ell}, m_\ell^*) = i$). Therefore, $Y \subseteq g(M_\ell, m_{-\ell})$ for any $\ell \in N \setminus \{i\}$. Otherwise, let $\#Y = 2$. Then, $n > 3$. Take any $\ell \in N \setminus \{i\}$. Choosing $x^\ell = x^i$, agent ℓ can make $\#\left\{\ell \in N | x^\ell = x\right\} \geq 2$ and $\#\left\{\ell \in N | x^\ell \neq x\right\} \geq 2$. As the outcome is determined by **Rule 4** agent ℓ can attain any outcome in Y by appropriately choosing k^ℓ. Therefore, $Y \subseteq g(M_\ell, m_{-\ell})$ for any $\ell \in N \setminus \{i\}$.

Sub-case 3.2. $\#Y = 2$ and $n = 3$

Then, let $N = \{i-1, i, i+1\}$ with $i+1 = 1$ if $i = n$ and $i - 1 = n$ if $i = 1$. As $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \neq Y$, it follows that $g(m) = x$. We proceed according to whether for some agents $\ell, \ell' \in N$, with $\ell \neq \ell'$, $\#\mathcal{R}_\ell \neq 1$ and $\#\mathcal{R}_{\ell'} \neq 1$ or not.

Sub-sub-case 3.2.1. For $\ell, \ell' \in N$, with $\ell \neq \ell'$, $\#\mathcal{R}_\ell \neq 1$ and $\#\mathcal{R}_{\ell'} \neq 1$

In this case, agent $i-1$ (resp., $i+1$) can always induce the modulo game by appropriately changing the announcement of her own preference or that of her successor and by carefully choosing the outcome announcement. Finally, to attain x^i, agent $i-1$ (resp., $i+1$) has only to adjust the integer index so that agent i becomes the winner of the modulo game.

Sub-sub-case 3.2.2. For some $\ell, \ell' \in N$, with $\ell \neq \ell'$, $\#\mathcal{R}_\ell = 1$ or $\#\mathcal{R}_{\ell'} = 1$

Suppose that, for all $\ell^* \in \{i-1, i, i+1\}$, $\#\mathcal{R}_{\ell^*} = 1$. As m falls into **Rule 3**, it follows that $x \in F(R) = F\left(\bar{R}\right)$. Otherwise, let us consider the case that, for some $\ell^* \in \{i-1, i, i+1\}$, $\#\mathcal{R}_{\ell^*} \neq 1$. If either $\#\mathcal{R}_{i-1} > 1$ or $\#\mathcal{R}_i > 1$, then agent $i-1$ can induce the modulo game by changing m_{i-1} into *either* $m_{i-1}^* = \left(R_{i-1}^{i-1}, \bar{R}_i, x, k^{i-1}\right)$ with $R_{i-1}^{i-1} \neq \bar{R}_{i-1}$ (if $\#\mathcal{R}_{i-1} > 1$), *or* $m_{i-1}^* = \left(\bar{R}_{i-1}, R_i^{i-1}, x^i, k^{i-1}\right)$ with $R_i^{i-1} \neq R_i^i$ (if $\#\mathcal{R}_i > 1$). To attain x^i, agent $i-1$ has only to choose an appropriate k^{i-1} so that $i = \ell^*\left(m_{-(i-1)}, m_{i-1}^*\right)$. Therefore, $Y \subseteq g\left(M_{i-1}, m_{-(i-1)}\right)$. Then, let $\#\mathcal{R}_{i-1} = \#\mathcal{R}_i = 1$. Agent $i-1$ can change m_{i-1} into $m_{i-1}^* = \left(\bar{R}_{i-1}, \bar{R}_i, x^i, k^{i-1}\right)$. Suppose that $x^i \notin F\left(\bar{R}_{i-1}, \bar{R}_i, R^i_{i+1}\right)$. Then, **Rule 4** applies, and agent $i-1$ can attain x^i by adjusting k^{i-1} so that $i-1 = \ell^*\left(m_{-(i-1)}, m^*_{i-1}\right)$. Suppose that $x^i \in F\left(\bar{R}_{i-1}, \bar{R}_i, R^i_{i+1}\right)$. If $C_{i+1}\left(\bar{R}_i, x^i\right) = \{x^i\}$, **Rule 3** implies $g\left(m_{-(i-1)}, m^*_{i-1}\right) = x^i$. In the case that $C_{i+1}\left(\bar{R}_i, x^i\right) = Y$, the outcome is determined by **Rule 4**, so that by adjusting k^{i-1} agent $i-1$ can attain x^i. By similar reasoning, it can be shown that agent $i+1$ can attain $x^i \in Y$. Therefore, $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for $\ell \in \{i-1, i+1\}$.

In all sub-cases, we obtained $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for all $\ell \in N\backslash\{i\}$. As $m \in NE(\gamma, R)$, we have that $C_i\left(\bar{R}_{-\{i,i+1\}}, x\right) \subseteq L\left(R_i, g(m)\right)$ and $g(m) \in \max_{R_\ell} Y$ for any $\ell \in N\backslash\{i\}$, so that $g(m) \in F(R)$ by $\boldsymbol{\mu}^{\mathbf{s}}$(ii).

Case 4: m falls into **Rule 4**.

Then the outcome is determined by the modulo game so that $g(m) = x^{\ell^*(m)}$, where agent $\ell^*(m) \in N$ is the winner of the modulo game. Thus, $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for $\ell \in N$. Since $m \in NE(\gamma, R)$, it follows that $g(m) \in \max_{R_\ell} Y$ for $\ell \in N$. Therefore, $g(m) \in F(R)$ by $\boldsymbol{\mu}^{\mathbf{s}}$(iii). ■

From the above propositions, we obtain the following main theorem.

Theorem. *An SCC F is Nash-implementable by an s-mechanism if and only if it satisfies $\boldsymbol{\mu}^{\mathbf{s}}$.*

Furthermore, we can see that the class of SCCs Nash-implementable by s-mechanisms is not proper subset of the class of Nash-implementable SCCs.

Lemma. *$\boldsymbol{\mu}^{\mathbf{s}}$ is equivalent to $\boldsymbol{\mu}$.*

Proof. From Proposition 1 and Moore and Repullo (1990), it is sufficient to show that $\boldsymbol{\mu}^{\mathbf{s}}$ implies $\boldsymbol{\mu}$. Let an SCC F satisfy $\boldsymbol{\mu}^{\mathbf{s}}$. Then, by Theorem, this F is Nash-implementable by an s-mechanism, so that it is Nash-implementable. Thus, by Moore and Repullo (1990), F satisfies $\boldsymbol{\mu}$. ■

From Theorem and Lemma, the following corollary holds:

Corollary. *An SCC F is Nash-implementable by an s-mechanism if and only if it is Nash-implementable.*

4 Concluding Remarks

In this paper, we deal with the informational efficiency issue pertaining to Maskin's Theorem (Maskin, 1999). We focus on s-mechanisms in which each agent reports to the planner her own preference and her neighbor's preference solely, in addition to a feasible social outcome and an integer. We introduce a new condition, labelled Condition $\boldsymbol{\mu}^{\mathbf{s}}$, which fully characterizes the class of SCCs Nash-implemantable by s-mechanisms. Surprisingly, Condition $\boldsymbol{\mu}^{\mathbf{s}}$ is equivalent to Condition $\boldsymbol{\mu}$. This has two important implications for Nash implementation. First, the class of Nash-implementable SCCs is equivalent to the class of SCCs Nash-implementable by s-mechanisms. Second, even though our condition is stated in terms of the existence of certain sets, it can easily be checked in practice by the agorithm provided by Sjöström (1991).

Note that our results are in line with other well known results of Nash implementation in economic environments. In particular, the equivalent relationship between Nash implementation by s-mechanism and Nash implementation by canonical mechanisms in general social choice environments is analogous to the equivalent relationship between Nash implementation by natural allocation mechanisms and Nash implementation by natural quantity^2 mechanisms (Saijo et al, 1996). Moreover, Tatamitani (2001) provides a full characterization of Nash implementation by self-relevant mechanisms, which together with this paper indicates that a further reduction of the strategy spaces of s-mechanisms drastically decreases the class of Nash-implementable SCCs. This is parallel to the case of natural implementation in economic environments, in which the class of SCCs Nash-implementable by natural quantity mechanisms is much smaller than the Nash-implementable ones by natural quantity^2 mechanisms.

In contrast, whenever we modify the standard framework of implementation theories to a more practical framework by introducing an element of perspectives from behavioral economics, the above mentioned relationship obtained in this paper would not preserve. To be more specific, Matsushima (2008) and Dutta and Sen (2009) introduce the notion of a partially honest agent as an element of behavioral economic perspectives, and consider Nash implementation problems with an assumption that there is at least one partially honest agent who not only has the standard self-interested preference on consequences but also has an intrinsic preference on truth-telling behavior. In such a framework, the equivalent relationship between Nash implementaion and Nash implemantaion by s-mechanisms no longer holds, as Lombardi and Yoshihara (2010) show. This suggests that the equivalent relationship indispensably relies on the standard assumption of self-interested behaviors.

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