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A New Insight into Three Bargaining Solutions in Convex Problems

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Abstract

We reconsider the three well-known solutions: the Nash, the egalitarian and the Kalai-Smorodinsky solutions, to the classical domains of convex bargaining problems. A new proof for the Nash solution that highlights the crucial role the axiom Contraction Independence plays is provided. We also give new axiomatic characterizations for both the egalitarian and the Kalai-Smorodinsky solutions. Our results focus on both contraction and expansion independence properties of bargaining problems and, as a consequence, some new insights on the three solutions from the perspective of rational choice may be derived.

1 Introduction

This paper reconsiders some well-known solutions to convex bargaining problems. Our purpose is two-fold. First, we provide a new proof for the Nash solution that highlights the crucial role the axiom Contraction Independence plays. Our proof method is proof-by-contradiction. Secondly, by employing similar proof methods as for the Nash solution, we provide new axiomatic characterizations for both the egalitarian and the Kalai-Smorodinsky solutions. Instead of using any monotonicity type axiom, which is commonly used in the literature for characterizing these two solutions (see, for example, Kalai (1977), Kalai and Smorodinsky (1975); see also Peters (1992) and Thomson (1994) for excellent surveys), we use variants of Contraction Independence and Expansion Independence to characterize the egalitarian and the Kalai-Smorodinsky solutions. Both Contraction Independence and Expansion Independence properties figure prominently in the theory of rational choice. Our new characterizations therefore may shed some new insights into the three well-known solutions to bargaining problems.

The remainder of the paper is organized as follows. Section 2 provides a basic framework for the subsequent analysis. Section 3 presents the axioms. Our main results and their proofs are contained in Section 4. Section 5 makes several concluding remarks.

2 Basic Model

For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, we write $\mathbf{x} > \mathbf{y}$ as $[x_i \geq y_i$ for all $i \in N$ and $\mathbf{x} \neq \mathbf{y}]$ and $\mathbf{x} \gg \mathbf{y}$ as $[x_i > y_i$ for all $i \in N]$.

Let π be a permutation of N. For all $\mathbf{x} = (x_i)_{i\in N} \in \mathbb{R}^n_+$, let $\pi(\mathbf{x}) = (x_{\pi(i)})_{i\in N}$. Π denotes the set of all permutations of N.

Let Σ be the set of all compact, convex, and comprehensive subsets of $\mathbb{R}^n_+$, each of which contains an interior point of $\mathbb{R}^n_+$. Elements in Σ are interpreted as *normalized bargaining problems*. For all $A \in \Sigma$ and any $\pi \in \Pi$, let $\pi(A) = \{\pi(\mathbf{a}) \mid \mathbf{a} \in A\}$. For all $A \in \Sigma$, A is a *symmetric problem* if $A = \pi(A)$ for all $\pi \in \Pi$.

For any $\mathbf{x} \in \mathbb{R}^n_+$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, let $\boldsymbol{\alpha}(\mathbf{x}) \equiv (\alpha_i x_i)_{i\in N}$. Given $A \in \Sigma$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$, let $\boldsymbol{\alpha}(A) \equiv \left\{\boldsymbol{\alpha}(\mathbf{x}) \in \mathbb{R}^n_+ \mid \mathbf{x} \in A\right\}$. For any A in $\mathbb{R}^n_+$, we define the *comprehensive hull of* A by

$$compA \equiv \left\{\mathbf{z} \in \mathbb{R}^n_+ \mid \exists \mathbf{x} \in A : \mathbf{z} \leq \mathbf{x}\right\}.$$

Let the *convex hull of* A be denoted by $conA$. The convex hull of $compA$ will be called the *convex and comprehensive hull* of A, and will be denoted by $concompA$.

A bargaining problem $A \in \Sigma$ is *strictly comprehensive* if and only if its boundary set constitutes the set of efficient utility points on A. Let us denote the set of strictly comprehensive problems by Σ^{sc}. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, $\mathbf{x}$ is *lexicographically greater than* $\mathbf{y}$ if there are permutation $\pi \in \Pi$ and $i \in N$ such that $x_{\pi(i)} > y_{\pi(i)}$ and $x_{\pi(j)} = y_{\pi(j)}$ for any $\pi(j) < \pi(i)$.

A *bargaining solution* F is a single-valued mapping from Σ to $\mathbb{R}^n_+$ such that for every bargaining problem $A \in \Sigma$, $F(A) \in A$. For given $F(A) \in A$, let $F_i(A) \in \mathbb{R}_+$ be its i-th component.

Definition 1: *A bargaining solution* $F^{NA} : \Sigma \to \mathbb{R}^n_+$ *is the* Nash solution *if for every* $A \in \Sigma$,

$$F^{NA}(A) = \arg\max_{(a_i)_{i \in N} \in A} \left(\prod_{i \in N} a_i \right).$$

Definition 2: *A bargaining solution* $F^K : \Sigma \to \mathbb{R}^n_+$ *is the* Kalai-Smorodinsky solution *if for every* $A \in \Sigma$, $F^K(A) \in A$ *implies that: (1) there is no other* $\mathbf{a} \in A$ *such that* $\mathbf{a} \gg F^K(A)$; *and (2) there exists* $\gamma \in (0,1)$ *such that* $F^K(A) = \gamma \cdot \mathbf{m}(A)$.

Definition 3: *A bargaining solution* $F^E : \Sigma \to \mathbb{R}^n_+$ *is the* egalitarian solution *if for every* $A \in \Sigma$, $F^E(A) \in A$ *implies that: (1) there is no other* $\mathbf{a} \in A$ *such that* $\mathbf{a} \gg F^E(A)$; *and (2)* $F^E_i(A) = F^E_j(A)$ *for all* $i, j \in N$.

3 Axioms

We consider the following axioms:

Efficiency: *For all* $A \in \Sigma$, *there is no* $\mathbf{x} \in A$ *such that* $\mathbf{x} > F(A)$.

Weak Efficiency: *For all* $A \in \Sigma$, *there is no* $\mathbf{x} \in A$ *such that* $\mathbf{x} \gg F(A)$.

Symmetry: *For all* $A \in \Sigma$, *if* A *is symmetric, then* $F_i(A) = F_j(A)$ *for all* $i, j \in N$.

Scale Invariance: *For all* $A, B \in \Sigma$*, and all* $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$*, if* $B = \boldsymbol{\alpha}(A)$*, then* $F(B) = \boldsymbol{\alpha}(F(A))$.

Contraction Independence: *For all* $A, B \in \Sigma$*, if* $A \supseteq B$ *and* $F(A) \in B$*, then* $F(B) = F(A)$.

Weak Contraction Independence: *For all* $A, B \in \Sigma$ *such that* $\mathbf{m}(A) = \mathbf{m}(B)$*, if* $A \supseteq B$ *and* $F(A) \in B$*, then* $F(B) = F(A)$.

Expansion Independence: *For all* $A, B \in \Sigma$*, if* $A \subseteq B$ *and* $F(A)$ *is efficient on* B*, then* $F(B) = F(A)$.

Weak Expansion Independence: *For all* $A, B \in \Sigma$ *such that* $\mathbf{m}(A) = \mathbf{m}(B)$*, if* $A \subseteq B$ *and* $F(A)$ *is efficient on* B*, then* $F(B) = F(A)$.

The first five axioms are standard ones discussed in the literature on convex bargaining problems (see, for example, Peters (1992) and Thomson (1994) for discussions). The last three are new and deserve further discussions. Weak Contraction Independence is weaker than Contraction Independence. It restricts its applicability to contraction situations in which the ideal point remains unchanged. Expansion Independence requires that, when a bargaining problem A is enlarged to another bargaining problem B, if the solution $F(A)$ to A continues to be efficient on B, then $F(A)$ should continue to be the solution to the bargaining problem B. The idea is that, even though there is an enlargement of "opportunities" from A to B, given that $F(A)$ is both efficient on A and on B, and that $F(A)$ is already the solution to the original problem A, any movement away from $F(A)$ will hurt at least one player, and thus the solution to the enlarged problem B should continue to be $F(A)$. This requirement suggests a *solidarity* type property embedded in the solution. This can also be seen as stating a certain inertia of the choice process. Weak Expansion Independence is weaker than Expansion Independence in that it restricts its applicability to situations where the ideal point remains unchanged.

We note that our Expansion Independence axiom is logically weaker than the following axiom, *Independence of Undominated Alternatives*, which is proposed in Thomson and Myerson (1980):

Independence of Undominated Alternatives: *For all* $A, B \in \Sigma$*, if* $A \subseteq B$ *and* $F(A)$ *is weakly efficient on* B*, then* $F(B) = F(A)$.

It is worth noting that Contraction Independence and Expansion Independence are logically implied by the *Monotonicity* axiom, which is introduced by Kalai (1977), together with Weak Efficiency, but the converse relation does not hold. In fact, the *monotone path solution*, which is proposed by Thomson and Myerson (1980) and which is characterized by Weak Efficiency and the monotonicity axiom of Kalai (1977), satisfies both Contraction Independence and Expansion Independence. On the other hand, we can construct a *non-monotone path solution* which satisfies Expansion Independence, Contraction Independence and Weak Efficiency, and which violates the monotonicity axiom of Kalai (1977).

4 Results and Their Proofs

This section presents our main results and their proofs follow.

Theorem 1: *A bargaining solution F is the* Nash Solution F^{NA} *if and only if it satisfies* **Efficiency**, **Symmetry**, **Scale Invariance**, *and* **Contraction Independence**.

Proof. It can be checked that if $F = F^{NA}$, then it satisfies the four axioms of Theorem 1. We therefore show that if a bargaining solution satisfies the four axioms of Theorem 1, then it must be the Nash solution.

Let F be a bargaining solution satisfying the four axioms of Theorem 1. For any $A \in \Sigma$, we first show that

Claim 1: For any $\mathbf{x}$ and $\mathbf{a}$ that are both efficient in A, and if $\mathbf{x} \in A$ is such that $\prod_{i \in N} x_i < \prod_{i \in N} a_i$, then $\mathbf{x} \neq F(A)$.

Let $\mathbf{x}$ and $\mathbf{a}$ be such that both are efficient in A and $\prod_{i \in N} x_i < \prod_{i \in N} a_i$. Suppose to the contrary that $\mathbf{x} = F(A)$. Consider $B \equiv \mathit{concomp}\{\mathbf{x}, \mathbf{a}\}$. By **Contraction Independence**, it follows that $\mathbf{x} \in F(B)$.

Now, by choosing $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$ appropriately, so that $\boldsymbol{\alpha}(\mathbf{a}) = (\beta, \ldots, \beta)$ for some $\beta \equiv \min_{i \in N} \{a_i\}$. Denote $B' \equiv \boldsymbol{\alpha}(B)$, $\mathbf{a}' \equiv \boldsymbol{\alpha}(\mathbf{a})$, and $\mathbf{x}' \equiv \boldsymbol{\alpha}(\mathbf{x})$. Note that $\mathbf{a}' \neq \mathbf{x}'$. By **Scale Invariance**, $F(B') = \mathbf{x}'$.

Consider the set $[\cup_{\pi \in \Pi} \pi(B')]$, and denote it by C. By construction, noting that $\prod_{i \in N} x_i < \prod_{i \in N} a_i$, C is symmetric, convex, and $C \supseteq B'$. Moreover, by the construction of B', both $\mathbf{a}'$ and $\mathbf{x}'$ are efficient on C. By **Efficiency** and **Symmetry**, it follows that $F(C) = \mathbf{a}'$. By **Contraction Independence**, $F(B') = \mathbf{a}'$, which is a contradiction. Therefore, $\mathbf{x} \neq F(A)$.

From **Claim 1**, we must then have that, for any $A \in \Sigma$,

$$F(A) \subseteq \left\{ \mathbf{a} \in A \mid \forall \mathbf{x} \in A : \prod_{i \in N} a_i \geq \prod_{i \in N} x_i \right\}.$$

Since the right-hand set is a singleton, the non-emptiness of F implies that $F = F^{NA}$. $\diamond$

Theorem 2: *A bargaining solution F over Σ is the* egalitarian Solution *if and only if it satisfies* **Weak Efficiency**, **Symmetry**, **Contraction Independence**, *and* **Expansion Independence**.

Proof. It can be checked that if $F = F^E$, then it satisfies the four axioms of Theorem 2. Therefore, we need only to show that if a solution satisfies the four axioms of Theorem 2, it must be the egalitarian solution.

Let F be a bargaining solution satisfying the four axioms of Theorem 2. By non-emptiness of F and **Weak Efficiency**, we need only to show the following claim:

Claim 2: For any $A \in \Sigma$, any $\mathbf{x}$ and $\mathbf{a}$ that are weakly efficient in A, if $[a_i = a_j$ for any $i, j \in N]$, but $x_i \neq x_j$ for some $i, j \in N$, then $\mathbf{x} \neq F(A)$.

Let $\mathbf{x}$ and $\mathbf{a}$ be such that both are weakly efficient in A and $[x_i \neq x_j$ for some $i, j \in N]$. Suppose to the contrary that $\mathbf{x} = F(A)$. Consider $B \equiv$ *comp* $\{\mathbf{x}\}$. Note that $B \subseteq A$. By **Contraction Independence**, $\mathbf{x} = F(B)$.

Consider the set *con* $[\cup_{\pi \in \Pi} \pi(B)]$, and denote it by C. By construction, C is a symmetric convex set having $C \supseteq B$. By the construction of B and C, $\mathbf{x}$ is efficient on C. Therefore, noting that $\mathbf{x} = F(B)$, $B \subseteq C$ and $\mathbf{x}$ is efficient on C, $\mathbf{x} = F(C)$ follows from **Expansion Independence**. Since C is symmetric, by **Weak Efficiency** and **Symmetry**, $F(C)$ must be weakly efficient and be the equal utility point, which is a contradiction. Therefore, $\mathbf{x} \neq F(A)$. This proves Claim 2 and thus Theorem 2. $\diamond$

Theorem 3: *A bargaining solution F over Σ is the* Kalai-Smorodinsky Solution *if and only if it satisfies* **Weak Efficiency**, **Symmetry**, **Scale Invariance**, *and* **Weak Contraction Independence**, *and* **Weak Expansion Independence**.

Proof. It can be checked that if $F = F^K$, then it satisfies the five axioms of Theorem 3. We therefore show that if a solution satisfies the five axioms of Theorem 3, it must be the Kalai-Smorodinsky solution.

Let F be a solution satisfying the five axioms of Theorem 3. By non-emptiness of F and **Weak Efficiency**, we need only to show the following claim:

Claim 3: For any $A \in \Sigma$, any $\mathbf{x}$ and $\mathbf{a}$ that are weakly efficient on A, if $[\frac{a_i}{m_i(A)} = \frac{a_j}{m_j(A)}$ for all $i, j \in N]$ and $[\frac{x_i}{m_i(A)} \neq \frac{x_j}{m_j(A)}$ for some $i, j \in N]$, then $\mathbf{x} \neq F(A)$.

Let $\mathbf{x}$ and $\mathbf{a}$ be such that both are weakly efficient on A and $[\frac{a_i}{m_i(A)} = \frac{a_j}{m_j(A)}$ for all $i, j \in N]$, and $[\frac{x_i}{m_i(A)} \neq \frac{x_j}{m_j(A)}$ for some $i, j \in N]$. Suppose to the contrary that $\mathbf{x} = F(A)$. Consider $con(\{\mathbf{x}\} \cup \{(m_i(A), \mathbf{0}_{-i}) \mid i \in N\} \cup \{\mathbf{0}\})$, and denote it by B. Note that $\mathbf{x}$ is efficient on B. By **Weak Contraction Independence**, $\mathbf{x} = F(B)$.

By choosing $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$ appropriately, we have $\boldsymbol{\alpha}(\mathbf{m}(A)) = (1, \ldots, 1)$. Let $B' \equiv \boldsymbol{\alpha}(B)$, $\mathbf{a}' \equiv \boldsymbol{\alpha}(\mathbf{a})$, and $\mathbf{x}' \equiv \boldsymbol{\alpha}(\mathbf{x})$. By **Scale Invariance**, $F(B') = \mathbf{x}'$.

Consider the set $con[\cup_{\pi\in\Pi}\pi(B')]$, and denote it by C. From the construction, C is a symmetric convex set having $C \supseteq B'$ and $\mathbf{m}(C) = \mathbf{m}(B') = \boldsymbol{\alpha}(\mathbf{m}(A))$. Moreover, by the construction of B', $\mathbf{x}'$ is efficient on C. By **Weak Expansion Independence**, $F(C) = \mathbf{x}'$. However, noting that C is symmetric, we have, by **Weak Efficiency** and **Symmetry**, that $F(C)$ must be the weakly efficient and equal utility point, which is a contradiction. Therefore, $\mathbf{x} \neq F(A)$. This proves Claim 3 and therefore Theorem 3. $\diamond$

Remark 1: *It can be verified that the egalitarian solution is also characterized by* **Weak Efficiency**, **Symmetry**, *and* **Independence of Undominated Alternatives**. *Note that if we use the axiom* **Independence of Undominated Alternatives**, *which is stronger than* **Expansion Independence**, *in the characterization of the egalitarian solution,* **Contraction Independence** *becomes superfluous and thus can be dropped out.*

Remark 2: *If $\#N = 2$, then the* Kalai-Smorodinsky Solution *is characterized by* **Efficiency**, **Symmetry**, **Scale Invariance**, *and* **Weak Expansion Independence**. *Thus,* **Weak Contraction Independence** *is no longer indispensable to characterize this solution in two person bargaining problems.*

To conclude this section, we make the following observations concerning the logical independence of the axioms used in each of Theorems 2 and 3.

Proposition 1: *The axioms*, **Weak Efficiency**, **Symmetry**, **Contraction Independence**, *and* **Expansion Independence** *are logically independent.*

Proof. It is fairly easy to see that there exists a solution which is not egalitarian and which violates one of the three axioms, Weak Efficiency, Symmetry and Expansion Independence, respectively, while satisfies the respectively remaining axioms in Theorem 2. Therefore, in what follows, we show that there exists a bargaining solution which satisfies **Weak Efficiency**, **Symmetry**, and **Expansion Independence**, and violates **Contraction Independence**. For this purpose, we consider the solution F^1 to be defined below. Given $\lambda \in [0,1]$, define the bargaining solution $F^{\lambda LE}$ as $F^{\lambda LE}(A) \equiv \lambda \cdot F^E(A) + (1-\lambda) \cdot F^L(A)$ for any $A \in \Sigma$, where $F^L : \Sigma \rightarrow \mathbb{R}^n_+$ is the lexicographic egalitarian solution defined as usual. Note that $F^{\lambda LE}(A) = F^E(A)$ if and only if $F^E(A)$ is efficient on $A \in \Sigma$. For instance, if $A \in \Sigma$ is symmetric or strictly comprehensive, then $F^E(A)$ is efficient on A. Let Σ^{sc} be the set of all bargaining problems in Σ each of which is also strictly comprehensive.

Now, consider the solution F^1 as follows. For some $\lambda \in (0,1)$, for all $A \in \Sigma$,

(1) if $A \in \Sigma^{sc}$ or $A = comp\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+$, then $F^1(A) = F^E(A)$;

(2) otherwise, $F^1(A) = F^{\lambda LE}(A)$.

It can be checked that F^1 satisfies **Symmetry** and **Weak Efficiency**. Next, we show that F^1 satisfies **Expansion Independence**. Let $A, B \in \Sigma$ be such that $A \subsetneq B$. There are two cases to be distinguished: **Case 1:** $A \in \Sigma^{sc}$ or $A = comp\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+$, and **Case 2:** neither $A \in \Sigma^{sc}$ nor $[A = comp\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+]$

Case 1: $A \in \Sigma^{sc}$ or $A = comp\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+$. In this case, we have $F^1(A) = F^E(A)$.

Case 1-1: $A \in \Sigma^{sc}$. If $F^1(A)$ is efficient on B, then it must be true that $F^1(B) = F^E(B)$, and therefore $F^1(A) = F^1(B)$. If $F^1(A)$ is not efficient on B, then the axiom **Expansion Independence** is trivially satisfied.

Case 1-2: $A = comp\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+$. If $F^1(A) = F^E(A)$ is efficient on B, then A must be symmetric, and B must not be the type of $comp\{\mathbf{y}\}$ for some $\mathbf{y} \in \mathbb{R}^n_+$. Then, if $B \in \Sigma^{sc}$, then $F^1(B) = F^E(B)$, so that $F^1(A) =$

$F^1(B)$ by the fact that $F^1(A)$ is efficient on B. If $B \notin \Sigma^{sc}$, then $F^1(B) = F^{\lambda LE}(B) = F^E(A) = F^1(A)$, since $F^1(A)$ is efficient on B. If, on the other hand, $F^1(A)$ is not efficient in B, then the axiom **Expansion Independence** is trivially satisfied.

Case 2: neither $A \in \Sigma^{sc}$ nor $[A = comp\,\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+]$. In this case, we have $F^1(A) = F^{\lambda LE}(A)$.

Case 2-1: $F^{\lambda LE}(A) = F^E(A)$. Note that this case is possible whenever $F^E(A)$ is efficient on A. Then, if $F^1(A)$ is also efficient on B, it must be true that $F^1(A) = F^E(B) = F^L(B) = F^{\lambda LE}(B)$. Therefore, $F^1(A) = F^1(B)$.

Case 2-2: $F^{\lambda LE}(A) \neq F^E(A)$. This case occurs whenever $F^E(A)$ is not efficient on A. Therefore, by definition, $F^1(A) = F^{\lambda LE}(A)$ is not efficient on A either. In this case, it is impossible for $F^{\lambda LE}(A)$ to be efficient on B. Therefore, the axiom **Expansion Independence** is trivially satisfied.

In summary, F^1 satisfies **Expansion Independence**.

We next show that F^1 violates **Contraction Independence**. Let $A \in \Sigma$ be neither $A \in \Sigma^{sc}$ nor $[A = comp\,\{\mathbf{x}\}$ for some $\mathbf{x} \in \mathbb{R}^n_+]$. Further, let $A \in \Sigma$ be such that $F^E(A)$ is not efficient on A. Then, $F^1(A) = F^{\lambda LE}(A) \neq F^E(A)$. Let $B \equiv comp\,\{F^1(A)\}$. Then, by definition, $F^1(B) = F^E(B) \neq F^{\lambda LE}(A)$. Noting that $B \subsetneq A$ and $F^1(A) \in B$, it must be true that F^1 violates **Contraction Independence**. $\diamond$

Proposition 2: For $\#N > 2$, *the axioms*, **Weak Efficiency**, **Symmetry**, **Scale Invariance**, **Weak Contraction Independence**, *and* **Weak Expansion Independence** *are logically independent.*

Proof. Again, it is relatively easy to see that there exists a bargaining solution that violates each of the axioms, **Weak Efficiency**, **Symmetry**, **Scale Invariance**, and **Weak Expansion Independence**, respectively, and that satisfies the other remaining axioms, respectively, of Theorem 3. Therefore, in what follows, we show that there exists a bargaining solution which satisfies **Weak Efficiency**, **Symmetry**, **Scale Invariance**, and **Weak Expansion Independence**, but violates **Weak Contraction Independence**. Let $\Sigma^u \equiv \{A \in \Sigma \mid \forall i \in N : m_i\,(A) = 1\}$. W.l.o.g., in the following discussion, we will focus on the case of $\#N = 3$. For $\#N = 3$, let

$$\triangle_3 \equiv con\,\{\mathbf{0}, (1,0,0)\,, (0,1,0)\,, (0,0,1)\,, (1,0,1)\,, (0,1,1)\}\,.$$

Since $\#N = 3$, $\triangle_3 \in \Sigma^u$. Note $F^K\,(\triangle_3) = F^E\,(\triangle_3) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, which is not efficient on $\triangle_3$. Consider the following solution, F^2, which is defined below: Let $\lambda \in (0,1)$ be given, for all $A \in \Sigma$,

(1) if $A \in \Sigma^u$ and,

(1-1) if $A \subseteq \triangle_3$ with $F^K(\triangle_3) = F^E(\triangle_3) \in A$, then $F^2(A) = F^{\lambda LE}(A)$;

(1-2) if otherwise, then $F^2(A) = F^K(A)$;

(2) if $A \notin \Sigma^u$, then $F^2(A) = \boldsymbol{\alpha}(F^2(B))$ for some $B \in \Sigma^u$ and some $\boldsymbol{\alpha} \in \mathbb{R}^n_{++}$ such that $\boldsymbol{\alpha}(B) = A$.

It is easy to see that F^2 satisfies **Symmetry**, **Scale Invariance**, and **Weak Efficiency**. By using a similar method as that in the proof of Proposition 1, we can check that F^2 satisfies **Weak Expansion Independence** and that F^2 violates **Weak Contraction Independence**. $\diamond$

5 Concluding Remarks

Our results on the characterizations of the three solutions are summarized in the following table.

Table 1

Axioms\ Solutions	NS	ES	KS
(E)	$\oplus$	$\times$	$\times$
(WE)	$\bigcirc$	$\oplus$	$\oplus$
(S)	$\oplus$	$\oplus$	$\oplus$
(SI)	$\oplus$	$\times$	$\oplus$
(CI)	$\oplus$	$\oplus$	$\times$
(WCI)	$\bigcirc$	$\bigcirc$	$\oplus$
(EI)	$\times$	$\oplus$	$\times$
(WEI)	$\times$	$\bigcirc$	$\oplus$

where

NS is for Nash Solution, ES for Egalitarian Solution, and KS for Kalai-Smorodinsky Solution

$\oplus$ stands for that the axiom is used for the characterization,

◯ stands for that the axiom is satisfied by the solution,

× stands for that the axiom is violated by the solution.

Clearly, all three solutions satisfy axioms Weak Efficiency, Symmetry and Weak Contraction Independence. The Nash solution satisfies all but Expansion Independence and Weak Expansion Independence, the egalitarian solution satisfies all but Efficiency and Scale Invariance, and the Kalai-Smorodinsky solution violates Efficiency, Contraction Independence and Expansion Independence while satisfies all the other axioms. It is also worth noting that Theorem 2 (*resp.* Theorem 3) constitutes a strengthening of the original characterization of the egalitarian solution (*resp.* the Kalai-Smorodinsky solution) by Kalai (1977) (*resp.* Kalai and Smorodinsky (1975)), since the combination of Contraction Independence and Expansion Independence (*resp.* Weak Contraction Independence and Weak Expansion Independence) is logically weaker than the monotonicity axiom (*resp.* the *weak monotonicity* axiom).

As far as contraction and expansion properties are concerned, it is interesting to note that the egalitarian solution satisfies all the contraction and expansion properties discussed in this paper, the Nash solution fails the two expansion properties while survives the two contraction properties, and the Kalai-Smorodinsky solution satisfies the weaker versions of contraction and expansion properties. The fact that the Kalai-Smorodinsky solution has some constrained contraction and expansion properties gives us some new insights on the rational choice property of this solution, which the previous literature does not provide since it is widely considered that it has no rational choice property.

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